Iterated scaling limits for aggregation of random coefficient AR(1) and INAR(1) processes

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Abstract

By temporal and contemporaneous aggregation, doubly indexed partial sums of independent copies of random coefficient AR(1) or INAR(1) processes are studied. Iterated limits of the appropriately centered and scaled aggregated partial sums are shown to exist. The paper completes the results of Pilipauskaitė and Surgailis (2014) and Barczy, Nedényi and Pap (2015).

Keywords: random coefficient AR(1) processes, random coefficient INAR(1) processes, temporal aggregation, contemporaneous aggregation, idiosyncratic innovations.

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1. Introduction

The aggregation problem is concerned with the relationship between individual (micro) behavior and aggregate (macro) statistics. There exist different types of aggregation. The scheme of contemporaneous (also called cross-sectional) aggregation of random-coefficient AR(1) models was firstly proposed by Robinson (1978) and Granger (1980) in order to obtain the long memory phenomena in aggregated time series.

Puplinskaitė and Surgailis (2009, 2010) discussed aggregation of random-coefficient AR(1) processes with infinite variance and innovations in the domain

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of attraction of a stable law. Related problems for some network traffic models, M/G/∞ queues with heavy-tailed activity periods, and renewal-reward processes have also been examined. On page 512 in Jirak (2013) one can find many references for papers dealing with the aggregation of continuous time stochastic processes, and the introduction of Barczy et al. (2015) contains a detailed overview on the topic.

The aim of the present paper is to complete the papers of Pilipauskaitė and Surgailis (2014) and Barczy et al. (2015) by giving the appropriate iterated limit theorems for both the randomized AR(1) and INAR(1) models when the parameter $\beta = 1$, which case is not investigated in both papers.

Let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of non-negative integers, positive integers, real numbers and non-negative real numbers, respectively. The paper of Pilipauskaitė and Surgailis (2014) discusses the limit behavior of sums

$$S_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(j)}, \qquad t \in \mathbb{R}_+, \qquad N, n \in \mathbb{N}, \tag{1.1}$$

where $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, are independent copies of a stationary random-coefficient AR(1) process

$$X_k = \alpha X_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N}, \tag{1.2}$$

with standardized independent and identically distributed (i.i.d.) innovations $(\varepsilon_k)_{k\in\mathbb{N}}$ having $\mathbb{E}(\varepsilon_1) = 0$ and $\operatorname{Var}(\varepsilon_1) = 1$, and a random coefficient α with values in $[0, 1)$, being independent of $(\varepsilon_k)_{k\in\mathbb{N}}$ and admitting a probability density function of the form

$$\psi(x)(1 - x)^{\beta}, \qquad x \in [0, 1), \tag{1.3}$$

where $\beta \in (-1, \infty)$ and ψ is an integrable function on $[0, 1)$ having a limit $\lim_{x\uparrow 1} \psi(x) = \psi_1 > 0$. Here the distribution of X_0 is chosen as the unique stationary distribution of the model (1.2). Its existence was shown in Proposition 1 of Puplinskaitė and Surgailis (2009). We point out that they considered so-called idiosyncratic innovations, i.e., the innovations $(\varepsilon_k^{(j)})_{k\in\mathbb{N}}$,

$j \in \mathbb{N}$, belonging to $(X_k^{(j)})_{k \in \mathbb{Z}_+}$, $j \in \mathbb{N}$, are independent. In Pilipauskaitė and Surgailis (2014) they derived scaling limits of the finite dimensional distributions of $(A_{N,n}^{-1} S_t^{(N,n)})_{t \in \mathbb{R}_+}$, where $A_{N,n}$ are some scaling factors and first $N \to \infty$ and then $n \to \infty$, or vice versa, or both N and n increase to infinity, possibly with different rates. The iterated limit theorems for both orders of iteration are presented in the paper of Pilipauskaitė and Surgailis (2014), in Theorems 2.1 and 2.3, along with results concerning simultaneous limit theorems in Theorem 2.2 and 2.3. We note that the theorems cover different ranges of the possible values of $\beta \in (-1, \infty)$, namely, $\beta \in (-1, 0)$, $\beta = 0$, $\beta \in (0, 1)$, and $\beta > 1$. Among the limit processes is a fractional Brownian motion, lines with random slopes where the slope is a stable variable, a stable Lévy process, and a Wiener process. Our paper deals with the missing case when $\beta = 1$, for both two orders of iteration.

The paper of Barczy et al. (2015) discusses the limit behavior of sums (1.1), where $(X_k^{(j)})_{k \in \mathbb{Z}_+}$, $j \in \mathbb{N}$, are independent copies of a stationary random-coefficient INAR(1) process. The usual INAR(1) process with non-random-coefficient is defined as

$$X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \mathbb{N}, \tag{1.4}$$

where $(\varepsilon_k)_{k \in \mathbb{N}}$ are i.i.d. non-negative integer-valued random variables, $(\xi_{k,j})_{k,j \in \mathbb{N}}$ are i.i.d. Bernoulli random variables with mean $\alpha \in [0, 1]$, and X_0 is a non-negative integer-valued random variable such that X_0, $(\xi_{k,j})_{k,j \in \mathbb{N}}$ and $(\varepsilon_k)_{k \in \mathbb{N}}$ are independent. By using the binomial thinning operator $\alpha \circ$ due to Steutel and van Harn (1979), the INAR(1) model in (1.4) can be considered as

$$X_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N}, \tag{1.5}$$

which form captures the resemblance with the AR(1) model. We note that an INAR(1) process can also be considered as a special branching process with immigration having Bernoulli offspring distribution.

We will consider a certain randomized INAR(1) process with randomized thinning parameter α, given formally by the recursive equation (1.5), where

α is a random variable with values in $(0, 1)$. This means that, conditionally on α, the process $(X_k)_{k\in\mathbb{Z}_+}$ is an INAR(1) process with thinning parameter α. Conditionally on α, the i.i.d. innovations $(\varepsilon_k)_{k\in\mathbb{N}}$ are supposed to have a Poisson distribution with parameter $\lambda \in (0, \infty)$, and the conditional distribution of the initial value X_0 given α is supposed to be the unique stationary distribution, namely, a Poisson distribution with parameter $\lambda/(1 - \alpha)$. For a rigorous construction of this process see Section 4 of Barczy et al. (2015). The iterated limit theorems for both orders of iteration —that are analogous to the ones in case of the randomized AR(1) model— are presented in the latter paper, in Theorems 4.6-4.12. This paper deals with the missing case when $\beta = 1$, for both two orders of iteration. When first $N \to \infty$ and then $n \to \infty$, we use the technique that already appeared in the second proof of Theorem 4.6 of Barczy et al. (2015). We show convergence of finite dimensional distributions of Gaussian sequences by checking convergence of covariances. It turns out that in case of $\beta = 1$ these covariances can be computed explicitly. When first $n \to \infty$ and then $N \to \infty$, we apply a new approach. Using the ideas of the second proof of Theorem 4.9 of Barczy et al. (2015), it suffices to show weak convergence of sums of certain i.i.d. random variables scaled by the factor $N \log N$ towards a positive number. It will be a consequence of a classical limit theorem with a stable limit distribution for these sums scaled by the factor N and centered appropriately. One may wonder about the limit behavior if n and N converge to infinity simultaneously, not in an iterated manner. This question has not been covered for $\beta = 1$ for either models, but the authors of this paper are planning to do so. Another natural question, which remains open, is whether the finite-dimensional convergence can be replaced by the functional convergence in Skorokhod space.

2. Iterated aggregation of randomized INAR(1) processes with Poisson innovations

Let $\alpha^{(j)}$, $j \in \mathbb{N}$, be a sequence of independent copies of the random variable α, and let $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, be a sequence of independent copies of the process $(X_k)_{k\in\mathbb{Z}_+}$ with idiosyncratic innovations (i.e., the innovations $(\varepsilon_k^{(j)})_{k\in\mathbb{N}}$, $j \in \mathbb{N}$, belonging to $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, are independent) such that $(X_k^{(j)})_{k\in\mathbb{Z}_+}$ conditionally on $\alpha^{(j)}$ is a strictly stationary INAR(1) process with Poisson innovations for all $j \in \mathbb{N}$.

First we examine a simple aggregation procedure. For each $N \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N)} = (\widetilde{S}_k^{(N)})_{k\in\mathbb{Z}_+}$ given by

$$\widetilde{S}_k^{(N)} := \sum_{j=1}^{N} \big(X_k^{(j)} - \mathbb{E}(X_k^{(j)} \,|\, \alpha^{(j)})\big) = \sum_{j=1}^{N} \bigg(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}\bigg), \qquad k \in \mathbb{Z}_+.$$

The following two propositions are Proposition 4.1 and 4.2 of Barczy et al. (2015). We will use $\stackrel{\mathcal{D}_{\mathrm{f}}}{\longrightarrow}$ or $\mathcal{D}_{\mathrm{f}}$-lim for the weak convergence of the finite dimensional distributions.

2.1 Proposition. *If* $\mathbb{E}\big(\frac{1}{1-\alpha}\big) < \infty$, *then*

$$N^{-\frac{1}{2}} \widetilde{S}^{(N)} \stackrel{\mathcal{D}_{\mathrm{f}}}{\longrightarrow} \widetilde{\mathcal{Y}} \qquad \text{as} \quad N \to \infty,$$

where $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\widetilde{\mathcal{Y}}_0 \widetilde{\mathcal{Y}}_k) = \operatorname{Cov}\left(X_0 - \frac{\lambda}{1-\alpha}, X_k - \frac{\lambda}{1-\alpha}\right) = \lambda\,\mathbb{E}\Big(\frac{\alpha^k}{1-\alpha}\Big), \quad k \in \mathbb{Z}_+. \quad (2.1)$$

2.2 Proposition. *We have*

$$\left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{S}_k^{(1)}\right)_{t\in\mathbb{R}_+} = \left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} (X_k^{(1)} - \mathbb{E}(X_k^{(1)} \,|\, \alpha^{(1)}))\right)_{t\in\mathbb{R}_+} \stackrel{\mathcal{D}_{\mathrm{f}}}{\longrightarrow} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B$$

as $n \to \infty$, *where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion, independent of* α.

In the forthcoming theorems we assume that the distribution of the random variable α, i.e., the mixing distribution, has a probability density described in (1.3). We note that the form of this density function indicates $\beta > -1$. Furthermore, if α has such a density function, then for each $\ell \in \mathbb{N}$ the expectation $\mathbb{E}((1-\alpha)^{-\ell})$ is finite if and only if $\beta > \ell - 1$.

For each $N, n \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N,n)} = (\widetilde{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$\widetilde{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \big(X_k^{(j)} - \mathbb{E}(X_k^{(j)} \mid \alpha^{(j)})\big), \qquad t \in \mathbb{R}_+.$$

2.3 Theorem. *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (n \log n)^{-\frac{1}{2}} N^{-\frac{1}{2}} \, \widetilde{S}^{(N,n)} = \sqrt{2\lambda\psi_1} B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

Proof of Theorem 2.3. Since $\mathbb{E}((1-\alpha)^{-1}) < \infty$, the condition in Proposition 2.1 is satisfied, meaning that

$$N^{-\frac{1}{2}} \widetilde{S}^{(N)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \widetilde{\mathcal{Y}} \qquad \text{as} \quad N \to \infty,$$

where $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ is a stationary Gaussian process with zero mean and covariances

$$\mathbb{E}(\widetilde{\mathcal{Y}}_0 \widetilde{\mathcal{Y}}_k) = \operatorname{Cov}\left(X_0 - \frac{\lambda}{1-\alpha}, X_k - \frac{\lambda}{1-\alpha}\right) = \lambda \, \mathbb{E}\Big(\frac{\alpha^k}{1-\alpha}\Big), \qquad k \in \mathbb{Z}_+.$$

Therefore, it suffices to show that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{\mathcal{Y}}_k = \sqrt{2\lambda\psi_1} B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process. This follows from the continuity theorem if for all $t_1, t_2 \in \mathbb{N}$ we have

$$\operatorname{Cov}\left(\frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\lfloor nt_1 \rfloor} \widetilde{\mathcal{Y}}_k, \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\lfloor nt_2 \rfloor} \widetilde{\mathcal{Y}}_k\right) \to 2\lambda\psi_1 \min(t_1, t_2), \qquad (2.2)$$

as $n \to \infty$. By (2.1) we have

$$\operatorname{Cov}\left(\frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt_1\rfloor}\widetilde{\mathcal{Y}}_k, \frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt_2\rfloor}\widetilde{\mathcal{Y}}_k\right) = \frac{\lambda}{n\log n}\,\mathbb{E}\left(\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{\alpha^{|k-\ell|}}{1-\alpha}\right)$$
$$= \frac{\lambda}{n\log n}\int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{a^{|k-\ell|}}{1-a}\psi(a)(1-a)\,\mathrm{d}a.$$

First we derive

$$\frac{1}{n\log n}\int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}\,\mathrm{d}a \to 2\min(t_1,t_2), \tag{2.3}$$

as $n \to \infty$. Indeed, if we suppose that $t_2 > t_1$, then

$$\int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}\,\mathrm{d}a = \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\frac{1}{|k-\ell|+1}$$
$$= (\lfloor nt_1\rfloor+1)(H(\lfloor nt_1\rfloor)-1) + 2 - \lfloor nt_1\rfloor + \lfloor nt_1\rfloor(H(\lfloor nt_2\rfloor)-1)$$
$$+ \big(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1\big)\left(H(\lfloor nt_2\rfloor) - H(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1)\right)$$
$$= (\lfloor nt_1\rfloor+1)(\log(\lfloor nt_1\rfloor) + O(1)) + 2 - \lfloor nt_1\rfloor + \lfloor nt_1\rfloor(\log\lfloor nt_2\rfloor + O(1))$$
$$+ \big(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1\big)\left(\log(\lfloor nt_2\rfloor) - \log(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor + 1) + O(1)\right),$$

where $H(n)$ denotes the n-th harmonic number, and it is well known that $H(n) = \log n + O(1)$ for every $n \in \mathbb{N}$. Therefore, convergence (2.3) holds. Consequently, (2.2) will follow from

$$I_n := \frac{1}{n\log n}\int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}|\psi(a)-\psi_1|\,\mathrm{d}a \to 0$$

as $n \to \infty$. Note that for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for every $a \in (1-\delta_\varepsilon, 1)$ it holds that $|\psi(a)-\psi_1| < \varepsilon$. Hence

$$I_n n\log n \leqslant \int_0^{1-\delta_\varepsilon} \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}(\psi(a)+\psi_1)\,\mathrm{d}a$$
$$+ \int_{1-\delta_\varepsilon}^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}|\psi(a)-\psi_1|\,\mathrm{d}a$$
$$\leqslant \int_0^{1-\delta_\varepsilon}\frac{2\lfloor nt_1\rfloor}{\delta_\varepsilon}(\psi(a)+\psi_1)\,\mathrm{d}a + \varepsilon\int_{1-\delta_\varepsilon}^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}\,\mathrm{d}a,$$

meaning that for every $\varepsilon > 0$ by (2.3) we have $\limsup_{n\to\infty} |I_n| \leqslant 0 + 4\varepsilon \min(t_1, t_2)$, resulting that $\lim_{n\to\infty} I_n = 0$, which completes the proof. □

2.4 Theorem. *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \frac{1}{\sqrt{nN\log N}} \widetilde{S}^{(N,n)} = \sqrt{\lambda\psi_1} B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

Proof of Theorem 2.4. By Proposition 2.2 of the current paper and the second proof of Theorem 4.9 of Barczy et al. (2015) it suffices to show that

$$\frac{1}{N\log N} \sum_{j=1}^{N} \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2} \xrightarrow{\mathcal{D}} \lambda\psi_1, \qquad N \to \infty.$$

Let us apply Theorem 7.1 of Resnick (2007) with

$$X_{N,j} := \frac{1}{N} \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2},$$

meaning that

$$N\,\mathbb{P}(X_{N,1} > x) = N\,\mathbb{P}\left(\frac{\lambda(1+\alpha)}{(1-\alpha)^2} > Nx\right) = N \int_{1-\widetilde{h}(\lambda,Nx)}^{1} \psi(a)(1-a)\mathrm{d}a,$$

where $\widetilde{h}(\lambda, x) = (1/4 + \sqrt{1/16 + x/(2\lambda)})^{-1}$. Note that for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for every $a \in (1-\delta_\varepsilon, 1)$ it holds that $|\psi(a) - \psi_1| < \varepsilon$. Then,

$$N \int_{1-\widetilde{h}(\lambda,Nx)}^{1} |\psi(a) - \psi_1|(1-a)\mathrm{d}a \leqslant N\varepsilon \frac{(\widetilde{h}(\lambda,Nx))^2}{2} \leqslant \frac{\varepsilon\lambda}{x}$$

for every $x > 0$ and large enough N. Therefore, for every $x > 0$ we have

$$\begin{aligned}
&\lim_{N\to\infty} N\,\mathbb{P}(X_{N,1} > x) = \lim_{N\to\infty} N \int_{1-\widetilde{h}(\lambda,Nx)}^{1} \psi_1(1-a)\mathrm{d}a \\
&= \lim_{N\to\infty} N\psi_1 \frac{(\widetilde{h}(\lambda,Nx))^2}{2} = \lim_{N\to\infty} \frac{\psi_1}{2} \frac{N}{\left(\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{Nx}{2\lambda}}\right)^2} = \frac{\psi_1\lambda}{x} =: \nu([x,\infty)),
\end{aligned}$$

where ν is obviously a Lévy-measure. By the decomposition

$$N\,\mathbb{E}\left(X_{N,1}^2 \mathbb{1}_{\{|X_{N,1}|\leqslant\varepsilon\}}\right) = N \int_0^{1-\widetilde{h}(\lambda,N\varepsilon)} \left(\frac{\lambda(1+a)}{N(1-a)^2}\right)^2 \psi(a)(1-a)\mathrm{d}a = I_N^{(1)} + I_N^{(2)},$$

where

$$I_N^{(1)} := N \int_0^{1-\delta_\varepsilon} \left(\frac{\lambda(1+a)}{N(1-a)^2} \right)^2 \psi(a)(1-a)\mathrm{d}a \leqslant \frac{1}{N} \lambda^2 \frac{2^2}{\delta_\varepsilon^4} 1 \to 0$$

as $N \to \infty$, and

$$\begin{aligned} I_N^{(2)} &:= N \int_{1-\delta_\varepsilon}^{1-\widetilde{h}(\lambda, N\varepsilon)} \left(\frac{\lambda(1+a)}{N(1-a)^2} \right)^2 \psi(a)(1-a)\mathrm{d}a \\ &\leqslant \frac{8\psi_1\lambda^2}{N} \int_{1-\delta_\varepsilon}^{1-\widetilde{h}(\lambda, N\varepsilon)} \frac{\mathrm{d}a}{(1-a)^3} = \frac{4\psi_1\lambda^2}{N} \left[\widetilde{h}(\lambda, N\varepsilon)^{-2} - \delta_\varepsilon^{-2} \right] \leqslant 8\psi_1\lambda^2\varepsilon \end{aligned}$$

for large enough N values, so it follows that

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} N\, \mathbb{E}\left(X_{N,1}^2 \mathbb{1}_{\{|X_{N,1}| \leqslant \varepsilon\}} \right) = 0.$$

Therefore, by applying Theorem 7.1 of Resnick (2007) with the choice $t = 1$ we get that

$$\begin{aligned} &\sum_{j=1}^N \left[\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2} - \mathbb{E}\left(\frac{\lambda(1+\alpha)}{N(1-\alpha)^2} \mathbb{1}_{\left\{ \frac{\lambda(1+\alpha)}{N(1-\alpha)^2} \leqslant 1 \right\}} \right) \right] \\ &= \sum_{j=1}^N \Bigg[\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2} - \frac{\lambda\psi_1}{N} \int_0^{1-\sqrt{\frac{2\lambda}{N}}} \frac{2}{(1-a)^2}(1-a)\mathrm{d}a \\ &\quad + \frac{\lambda\psi_1}{N} \int_0^{1-\sqrt{\frac{2\lambda}{N}}} \frac{2}{(1-a)^2}(1-a)\mathrm{d}a - \frac{\lambda\psi_1}{N} \int_0^{1-\widetilde{h}(\lambda,N)} \frac{2}{(1-a)^2}(1-a)\mathrm{d}a \\ &\quad + \frac{\lambda\psi_1}{N} \int_0^{1-\widetilde{h}(\lambda,N)} \frac{2}{(1-a)^2}(1-a)\mathrm{d}a - \frac{\lambda\psi_1}{N} \int_0^{1-\widetilde{h}(\lambda,N)} \frac{1+a}{(1-a)^2}(1-a)\mathrm{d}a \\ &\quad + \frac{\lambda\psi_1}{N} \int_0^{1-\widetilde{h}(\lambda,N)} \frac{1+a}{(1-a)^2}(1-a)\mathrm{d}a - \frac{\lambda}{N} \int_0^{1-\widetilde{h}(\lambda,N)} \frac{1+a}{(1-a)^2}\psi(a)(1-a)\mathrm{d}a \Bigg] \\ &=: \frac{\lambda}{N} \sum_{j=1}^N J_{j,N}^{(0)} + \lambda J_N^{(1)} + \lambda J_N^{(2)} + \lambda J_N^{(3)} \stackrel{\mathcal{D}}{\longrightarrow} X_0, \end{aligned}$$

where by (5.37) of Resnick (2007)

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_0}) = \exp\left\{ \int_1^\infty (\mathrm{e}^{\mathrm{i}\theta x} - 1) \frac{\psi_1\lambda\mathrm{d}x}{x^2} + \int_0^1 (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x) \frac{\psi_1\lambda\mathrm{d}x}{x^2} \right\}, \quad \theta \in \mathbb{R}.$$

We show that

$$\frac{|J_N^{(1)}| + |J_N^{(2)}| + |J_N^{(3)}|}{\log N} \to 0, \qquad N \to \infty,$$

resulting

$$\frac{1}{\log N}\sum_{j=1}^{N}\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2} = \frac{1}{\log N}\sum_{j=1}^{N}\left[\frac{\lambda(1+\alpha^{(j)})}{N(1-\alpha^{(j)})^2} - \frac{\lambda\psi_1}{N}\int_0^{1-\sqrt{\frac{2\lambda}{N}}}\frac{2}{1-a}\mathrm{d}a\right]$$
$$+\frac{2\lambda\psi_1}{\log N}\left(-\log\left(\sqrt{\frac{2\lambda}{N}}\right)\right) \xrightarrow{\mathcal{D}} 0\cdot X_0 + \lambda\psi_1 = \lambda\psi_1, \qquad N\to\infty.$$

Indeed,

$$\frac{J_N^{(1)}}{\log N} = \frac{\psi_1}{\log N}\int_{1-\sqrt{\frac{2\lambda}{N}}}^{1-\widetilde{h}(\lambda,N)}\frac{2}{1-a}\mathrm{d}a = \frac{2\psi_1}{\log N}\log\left(\sqrt{\frac{2\lambda}{N}}\left(\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{N}{2\lambda}}\right)\right)$$

converges to 0 as $N\to\infty$. Moreover,

$$\frac{J_N^{(2)}}{\log N} = \frac{\psi_1}{\log N}\int_0^{1-\widetilde{h}(\lambda,N)}\frac{1-a}{(1-a)^2}(1-a)\mathrm{d}a = \frac{\psi_1}{\log N}\left(1-\frac{1}{\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{N}{2\lambda}}}\right)$$

converges to 0 as $N\to\infty$. Finally,

$$\begin{aligned}\left|\frac{J_N^{(3)}}{\log N}\right| &= \left|\frac{1}{\log N}\int_0^{1-\widetilde{h}(\lambda,N)}\frac{1+a}{1-a}(\psi_1-\psi(a))\mathrm{d}a\right| \\ &\leqslant \frac{1}{\log N}\int_0^{1-\delta_\varepsilon}\frac{2}{\delta_\varepsilon}(\psi_1+\psi(a))\mathrm{d}a + \frac{1}{\log N}\int_{1-\delta_\varepsilon}^{1-\widetilde{h}(\lambda,N)}\frac{2}{1-a}\,\varepsilon\,\mathrm{d}a \\ &\leqslant \frac{1}{\log N}\frac{2}{\delta_\varepsilon}(\psi_1+\delta_\varepsilon^{-1}) + \frac{2\varepsilon}{\log N}\left[\log\delta_\varepsilon + \log\left(\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{N}{2\lambda}}\right).\right],\end{aligned}$$

One can easily see that for all $\varepsilon > 0$, we get $\limsup_{N\to\infty}|J_N^{(3)}/\log N| \leqslant 0+\varepsilon$, resulting that $\lim_{N\to\infty} J_N^{(3)}/\log N = 0$, which completes the proof. □

3. Iterated aggregation of randomized AR(1) processes with Gaussian innovations

Let $\alpha^{(j)}$, $j\in\mathbb{N}$, be a sequence of independent copies of the random variable α, and let $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j\in\mathbb{N}$, be a sequence of independent copies of the process $(X_k)_{k\in\mathbb{Z}_+}$ with idiosyncratic Gaussian innovations (i.e., the innovations $(\varepsilon_k^{(j)})_{k\in\mathbb{Z}_+}$, $j\in\mathbb{N}$, belonging to $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j\in\mathbb{N}$, are independent) having zero mean and variance $\sigma^2\in\mathbb{R}_+$ such that $(X_k^{(j)})_{k\in\mathbb{Z}_+}$ conditionally on $\alpha^{(j)}$ is a strictly stationary AR(1) process for all $j\in\mathbb{N}$. A rigorous construction of this random-coefficient process can be given similarly as in case of the randomized INAR(1) process detailed in Section 4 of Barczy et al. (2015).

First we examine a simple aggregation procedure. For each $N \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N)} = (\widetilde{S}_k^{(N)})_{k \in \mathbb{Z}_+}$ given by

$$\widetilde{S}_k^{(N)} := \sum_{j=1}^{N} X_k^{(j)}, \qquad k \in \mathbb{Z}_+.$$

The following two propositions are the counterparts of Proposition 2.1 and 2.2, and can be proven similarly as the two concerning the randomized INAR(1) process.

3.1 Proposition. *If* $\mathbb{E}\big(\frac{1}{1-\alpha^2}\big) < \infty$, *then*

$$N^{-\frac{1}{2}} \widetilde{S}^{(N)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \widetilde{\mathcal{Y}} \qquad \text{as} \quad N \to \infty,$$

where $(\widetilde{\mathcal{Y}}_k)_{k \in \mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\widetilde{\mathcal{Y}}_0 \widetilde{\mathcal{Y}}_k) = \operatorname{Cov}(X_0, X_k) = \sigma^2 \, \mathbb{E}\Big(\frac{\alpha^k}{1-\alpha^2}\Big), \qquad k \in \mathbb{Z}_+.$$

3.2 Proposition. *We have*

$$\Bigg(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{S}_k^{(1)}\Bigg)_{t \in \mathbb{R}_+} = \Bigg(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(1)}\Bigg)_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \frac{\sigma}{1-\alpha} B$$

as $n \to \infty$, *where* $B = (B_t)_{t \in \mathbb{R}_+}$ *is a standard Brownian motion, independent of* α.

Again, we assume that the distribution of the random variable α has a probability density described in (1.3). Note that for each $\ell \in \mathbb{N}$ the expectation $\mathbb{E}((1-\alpha^2)^{-\ell})$ is finite if and only if $\beta > \ell - 1$.

For each $N, n \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N,n)} = (\widetilde{S}_t^{(N,n)})_{t \in \mathbb{R}_+}$ given by

$$\widetilde{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(j)}, \qquad t \in \mathbb{R}_+.$$

3.3 Theorem. *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (n\log n)^{-\frac{1}{2}} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \sqrt{\sigma^2\psi_1} B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

Proof of Theorem 3.3. Since $\mathbb{E}((1-\alpha^2)^{-1}) < \infty$, the condition in Proposition 3.1 is satisfied, meaning that

$$N^{-\frac{1}{2}} \widetilde{S}^{(N)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \widetilde{\mathcal{Y}} \qquad \text{as} \quad N \to \infty,$$

where $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ is a stationary Gaussian process with zero mean and covariances

$$\mathbb{E}(\widetilde{\mathcal{Y}}_0 \widetilde{\mathcal{Y}}_k) = \operatorname{Cov}(X_0, X_k) = \sigma^2\, \mathbb{E}\Big(\frac{\alpha^k}{1-\alpha^2}\Big), \qquad k \in \mathbb{Z}_+.$$

Therefore, it suffices to show that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \frac{1}{\sqrt{n\log n}} \sum_{k=1}^{\lfloor nt\rfloor} \widetilde{\mathcal{Y}}_k = \sqrt{\sigma^2\psi_1} B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process. This follows from the continuity theorem, if for all $t_1, t_2 \in \mathbb{N}$ we have

$$\operatorname{Cov}\left(\frac{1}{\sqrt{n\log n}} \sum_{k=1}^{\lfloor nt_1\rfloor} \widetilde{\mathcal{Y}}_k, \frac{1}{\sqrt{n\log n}} \sum_{k=1}^{\lfloor nt_2\rfloor} \widetilde{\mathcal{Y}}_k\right) \to \sigma^2\psi_1 \min(t_1, t_2), \qquad n \to \infty.$$

It is known that

$$\operatorname{Cov}\left(\frac{1}{\sqrt{n\log n}} \sum_{k=1}^{\lfloor nt_1\rfloor} \widetilde{\mathcal{Y}}_k, \frac{1}{\sqrt{n\log n}} \sum_{k=1}^{\lfloor nt_2\rfloor} \widetilde{\mathcal{Y}}_k\right) = \frac{\sigma^2}{n\log n}\, \mathbb{E}\left(\sum_{k=1}^{\lfloor nt_1\rfloor} \sum_{\ell=1}^{\lfloor nt_2\rfloor} \frac{\alpha^{|k-\ell|}}{1-\alpha^2}\right)$$

$$= \frac{\sigma^2}{n\log n} \int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor} \sum_{\ell=1}^{\lfloor nt_2\rfloor} \frac{a^{|k-\ell|}}{1-a^2} \psi(a)(1-a)\mathrm{d}a$$

$$= \frac{\sigma^2}{n\log n} \int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor} \sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}\psi(a)\mathrm{d}a - \frac{\sigma^2}{n\log n} \int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor} \sum_{\ell=1}^{\lfloor nt_2\rfloor} \frac{a^{|k-\ell|+1}}{1+a}\psi(a)\mathrm{d}a$$

It was shown in the proof of Theorem 2.3 that

$$\frac{\sigma^2}{n\log n} \int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor} \sum_{\ell=1}^{\lfloor nt_2\rfloor} a^{|k-\ell|}\psi(a)\mathrm{d}a \to 2\sigma^2\psi_1 \min(t_1, t_2), \qquad n \to \infty.$$

We are going to prove that

$$\frac{\sigma^2}{n\log n}\int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} \frac{a^{|k-\ell|+1}}{1+a}\psi(a)\mathrm{d}a - \frac{\sigma^2}{n\log n}\int_0^1 \sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor} \frac{a^{|k-\ell|}}{1+a}\psi(a)\mathrm{d}a$$

converges to 0 as $n\to\infty$, which proves our theorem. Indeed, if $t_2 > t_1$, then

$$\left|\sum_{k=1}^{\lfloor nt_1\rfloor}\sum_{\ell=1}^{\lfloor nt_2\rfloor}\left(\frac{a^{|k-\ell|+1}}{1+a} - \frac{a^{|k-\ell|}}{1+a}\right)\right| = \frac{1}{1+a}\left|\sum_{k=1}^{\lfloor nt_1\rfloor}\left(a^k - (a+1) + a^{\lfloor nt_2\rfloor - k+1}\right)\right|$$

$$= \frac{1}{1+a}\left|\frac{a(a^{\lfloor nt_1\rfloor}-1)}{a-1} - (a+1)\lfloor nt_1\rfloor + \frac{a^{\lfloor nt_2\rfloor+1} - a^{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor+1}}{a-1}\right| \leqslant 4\lfloor nt_2\rfloor,$$

and as $\psi(a)$, $a\in(0,1)$ is integrable,

$$\frac{\sigma^2}{n\log n}\int_0^1 4\lfloor nt_2\rfloor\psi(a)\mathrm{d}a \to 0, \qquad n\to\infty.$$

This completes the proof. □

3.4 Theorem. *If* $\beta = 1$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\ \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \frac{1}{\sqrt{nN\log N}}\,\widetilde{S}^{(N,n)} = \sqrt{\frac{\sigma^2\psi_1}{2}}B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

The proof is similar to the INAR(1) case since the only difference is a missing $1+\alpha$ factor in the numerator and the constants.

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