Amplitude truncation of Gaussian $1/f^{\alpha}$ noises: Results and problems

Zoltán Gingl[a)]
Department of Experimental Physics, University of Szeged, Dóm tér 9., Szeged, H-6725 Hungary

Shunya Ishioka
Department of Information Science, Kanagawa University at Hiratsuka, Tsuchiya 2946, Hiratsuka, Kanagara, 259-1293, Japan

Donghak Choi and Nobuko Fuchikami
Department of Physics, Tokyo Metropolitan University, Minami-Ohsawa, Hachioji, Tokyo, 192-0397, Japan

(Received 13 December 2000; accepted 14 March 2001; published 31 August 2001)

An interesting property of Gaussian $1/f$ noise was found experimentally a few years ago: The amplitude truncation does not change the power spectral density of the noise under rather general conditions. Here we present a brief theoretical derivation of this invariant property of band-limited Gaussian $1/f$ noise and include $1/f^{\alpha}$ noises also with $0\leq\alpha<2$. It is shown that when $\alpha\leq1$, a transformation of keeping only the sign of the zero-mean $1/f^{\alpha}$ noise does not alter the shape of the spectral density. The theoretical results are extended to truncation levels differing significantly from the mean value. Numerical simulation results are also presented to draw attention to unsolved problems of amplitude truncation using asymmetric levels.
[DOI: 10.1063/1.1378792]

Noise whereby the amplitude falls in inverse proportion to the frequency, "$1/f$ noise," is very common in several natural and even artificial systems. Although $1/f$ noise was discovered several decades ago, one of the most challenging problems in noise research is to understand the generality and special features of $1/f$ noise and to introduce new models and tools to treat this kind of fluctuation. This implies the need of extensive research into the properties of this noise as well, and new results in this field can help to understand it better. A strange invariant property of $1/f$ was found experimentally a few years ago: The power spectral density remains $1/f$ if the amplitude of the noise is truncated using two levels under rather general conditions. This kind of nonlinear transform can occur easily in overdriven measurements, saturating physical systems, and quantities with limited range. Moreover, invariances are usually very important features of physical phenomena and may help to answer several open questions about $1/f$ noise as well. In this paper we will show a theoretical explanation of this invariant property of band-limited Gaussian $1/f$ noise and include the treatment of $1/f^{\alpha}$ ($0\leq\alpha<2$) noises as well. Our theoretical results are verified by numerical simulations. We would like to draw attention to open problems of the amplitude truncation for some special cases unexplained by the theory. Some concluding remarks are presented about possible connection between this invariant property and the generality of $1/f$ noise.

I. INTRODUCTION

Several natural systems exhibit $1/f$ noise. Suprisingly, these systems are rather different. Just to name a few examples, $1/f$ noise can be found in semiconductors,[1–3] superconductors,[4] lasers,[5,6] astrophysical systems,[7] but it has also been reported that $1/f$ noise is present in biological systems,[8] traffic flow,[9] and even classical music.[10] Although $1/f$ noise was discovered several decades ago, the general occurrence of this phenomenon has not been explained yet, and many problems associated with the models and properties of $1/f$ noise remain unsolved. Research into the properties of this fluctuation may considerably contribute to the knowledge of this particular noise and can help to understand why it is so commonly observed.

An interesting property of Gaussian $1/f$ noise was found experimentally a few years ago as a result of investigations of $1/f$ noise driven stochastic resonance in a Schmitt-trigger:[11] The power spectral density (PSD) remains the same if the amplitude is truncated at certain levels under rather general conditions. This result can be considered as a significant contribution to the knowledge about $1/f$ noise, and shows that important properties of this kind of fluctuations can be found in spite of the fact that $1/f$ noise was discovered several decades ago. Since amplitude saturation can occur easily in real systems, this invariance may help to understand why $1/f$ noise is observed in such a wide range of physical, biological and other processes and can help to develop new models to understand the behavior of many systems better.

The above mentioned result was extended to $1/f^{\alpha}$ noises with $0\leq\alpha\leq2$ by experimental investigations and numerical simulations, but these results remained theoretically unexplained.[12] Here we briefly present our theoretical derivation[13] of the above-mentioned invariant property for certain cases. In addition, we extended the results to other conditions with the help of numerical simulations. Finally we draw attention to some unsolved problems associated with the amplitude truncation of $1/f^{\alpha}$ noises.

[a)]Electronic mail: gingl@physx.u-szeged.hu

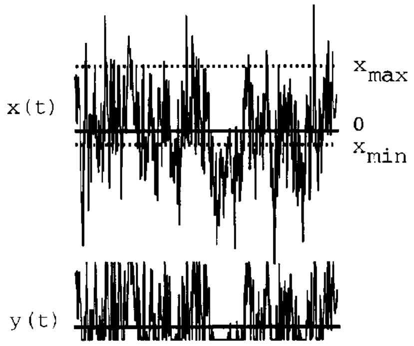

FIG. 1. An illustration of the amplitude truncation of a Gaussian $1/f$ noise. The upper signal $x(t)$ truncated at levels $x_{\min}$ and $x_{\max}$ gives the output signal $y(t)$.

II. AMPLITUDE TRUNCATION OF GAUSSIAN $1/f^{\alpha}$ NOISES

The amplitude truncation applied to Gaussian $1/f^{\alpha}$ noise is defined in the following way:

$$y(t)\quad \begin{cases} x_{\min}, & \text{if } x(t)\leqslant x_{\min} \\ x_{\max}, & \text{if } x(t)\geqslant x_{\max} \\ x(t), & \text{otherwise}. \end{cases} \tag{1}$$

Here $x(t)$ is the time-dependent amplitude of the noise, $y(t)$ is the amplitude of the truncated noise, $x_{\min}$ and $x_{\max}$ are the truncation levels. Figure 1 illustrates this transformation on a typical sample of $1/f$ noise. The magnitude of the spectrum is reduced by the same factor as the variance, which can be calculated easily from the probability density of the truncated signal. However, since Eq. (1) represents a strongly nonlinear transfrom, the shape of the PSDs can be rather different. In the following we show how the PSD of the truncated signal can be obtained theoretically as a function of the exponent α of the input signal for band-limited Gaussian $1/f^{\alpha}$ noises.

III. THEORETICAL RESULTS

Let us consider the case, when the mean value of input noise is zero and the upper and lower truncation levels are symmetric and very close to the mean value: $-x_{\min}=x_{\max}\approx 0$. In this case, we get an almost dichotomous output signal $y(t)$, which can be approximated by the following formula:

$$y(t)\propto \mathrm{sgn}[x(t)]=\begin{cases} +1, & \text{if } x(t)\geqslant 0 \\ -1, & \text{if } x(t)<0. \end{cases} \tag{2}$$

First we derive the relation between the correlation functions $R_x(t)$ and $R_y(t)$ of the input and output signals, respectively. The correlation function of the output signal is given by

$$\begin{aligned} R_y(t) &= 1\times P(y(0)y(t)=1)+(-1)\times P(y(0)y(t)=-1) \\ &= P(x(0)x(t)>0)-P(x(0)x(t)<0) \\ &= 2\times P(x(0)x(t)>0)-1, \end{aligned} \tag{3}$$

where $P(\cdot)$ is the probability that the condition of the argument is satisfied.

If we assume a stationary Gaussian process, the probability $P(x(0)x(t)>0)$ is obtained from the following joint probability:

$$\begin{aligned} P(x(t_1),x(t_2)) &= P(x(0),x(t_1-t_2)) \\ &= \frac{1}{A_0}\exp[-(x^2-2cxy+y^2)/B] \\ &\equiv f(x,y), \end{aligned} \tag{4}$$

where $x\equiv x(0)$, $y\equiv x(t_1-t_2)$, $c\equiv R_x(t_1-t_2)$, $A_0\equiv 2\pi R_x(0)(1-c^2)^{1/2}$, $B\equiv 2R_x(0)(1-c^2)$, $R_x(t)$ is the correlation function of $x(t)$. From Eq. (4), the probability that $x(t_1)x(t_2)>0$ is given as

$$\begin{aligned} P(xy>0) &= \int_0^{\infty}dx\int_0^{\infty}dyf(x,y)+\int_{-\infty}^{0}dx\int_{-\infty}^{0}dyf(x,y) \\ &= \frac{1}{2}+\frac{1}{\pi}\arcsin[R_x(t)]. \end{aligned} \tag{5}$$

Therefore, we obtain

$$R_y(t)=\frac{2}{\pi}\arcsin[R_x(t)]. \tag{6}$$

The relation (6) between the correlation functions leads to the relation between power spectral densities (PSD) applying the Wiener–Khinchine theorem, because we have assumed stationary processes

$$S_I(\omega)=2\int_0^{\infty}R_I(t)\cos(\omega t)dt \tag{7}$$

$$=-2\int_0^{\infty}R_I(t)\frac{\sin(\omega t)}{\omega}dt,\quad I=x,y. \tag{8}$$

It turns out that the PSD of the form $S_x(\omega)\sim 1/\omega^{\alpha}$ is transformed into PSD $S_y(\omega)\sim 1/\omega^{\beta}$.

We have investigated the relation of the exponents α and β for the following cases: $0<\alpha<1$, $\alpha=1$, and $1<\alpha<2$.

For $1<\alpha<2$, we have chosen the correlation function

$$R_x(t)=\begin{cases} 1-t^{\alpha-1} & t\leqslant 1 \\ 0 & t>1. \end{cases} \tag{9}$$

The corresponding PSD can be calculated using Eq. (8)

$$S_x(\omega)=2\frac{\alpha-1}{\omega^{\alpha}}\int_0^{\omega}\frac{\sin(z)}{z^{2-\alpha}}dz-\frac{2}{\omega^{\alpha}}\left[\Gamma(\alpha)\sin[\pi/2(\alpha-1)]-(\alpha-1)\int_{\omega}^{\infty}\frac{\sin(z)}{z^{3-\alpha}}dz\right]. \quad (10)$$

At high frequencies ($\omega\gg 1$), the second term can be neglected because the integrand $\sin x/x^{2-\alpha}$ becomes small for $x\geqslant\omega\geqslant 1$, so the spectrum becomes $\sim 1/\omega^{\alpha}$. The high-frequency condition $\omega\gg 1$ actually means that $\omega\gg 1/\tau_1$, where τ_1 is a correlation time of the signal. Using Eqs. (9) and (6), we obtain the correlation function of the output signal as

$$R_y(t)=\frac{2}{\pi}\arcsin[1-t^{\alpha-1}] \quad (t\leqslant 1)$$
$$\sim 1-2\frac{\sqrt{2}}{\pi}\sqrt{t^{\alpha-1}} \quad (t\ll 1),$$
$$R_y(t)=0 \quad (t>1). \quad (11)$$

The PSD of the transformed signal becomes

$$S_y(\omega)\sim\frac{1}{\omega^{\beta}},\quad \beta=\frac{\alpha+1}{2} \quad (\omega\gg 1), \quad (12)$$

in the high-frequency limit, because when $\omega\gg 1$, the main contribution to the integral (7) comes from small values of t, thus the approximation of (11) can be used.

In the case of $0<\alpha<1$, the correlation fuction is chosen as

$$R_x(t)=\begin{cases}1 & t\leqslant 1\\ \dfrac{1}{t^{1-\alpha}} & t>1.\end{cases} \quad (13)$$

Using Eq. (8), the corresponding PSD is obtained as

$$S_x(\omega)=2\int_1^{\infty}\frac{\sin(\omega t)}{t^{2-\alpha}\omega}(\alpha-1)dt-\frac{2}{\omega^{\alpha}}\left[\Gamma(\alpha)\cos(\pi/2\alpha)-(1-\alpha)\int_0^{\omega}\frac{\sin(z)}{z^{2-\alpha}}dz\right]. \quad (14)$$

At low frequencies ($\omega\ll 1$), the second term can be neglected, because $1<2-\alpha<2$ and we get $S_x(\omega)\sim 1/\omega^{\alpha}$. The dimensionless relation $\omega\ll 1$ actually corresponds to $\omega\ll 1/\tau_2$, where τ_2 is a typical time scale of the system above which the correlation function decays.

The correlation function

$$R_x(t)=\frac{1}{1+t^{1-\alpha}}, \quad (15)$$

also leads to the same form of PSD, because this expression of $R_x(t)$ may be replaced by (13) in the integral of Eq. (8) if $\omega\ll 1$.

Substituting Eq. (13) into (6) gives the correlation function of the output signal

$$R_y(t)=1 \quad (t<1),$$
$$R_y(t)=\frac{2}{\pi}\arcsin\left[\frac{1}{t^{1-\alpha}}\right] \quad (t\geqslant 1)$$
$$\sim\frac{2}{\pi}\frac{1}{t^{1-\alpha}} \quad (t\gg 1). \quad (16)$$

Using the approximation of Eq. (16) in (8), we obtain

$$S_y(\omega)\sim\frac{1}{\omega^{\beta}},\quad \beta=\alpha \quad (\omega\ll 1) \quad (17)$$

in the low-frequency limit.

For $1/f$ noise ($\alpha\approx 1$) the correlation function is approximated by the following formula:

$$R_x(t)=\begin{cases}1, & t\leqslant 1\\ \dfrac{1}{1+\log t}, & t>1.\end{cases} \quad (18)$$

Using this function the PSD is calculated as

$$S_x(\omega)=\frac{2}{\omega}\int_{\omega}^{\infty}\frac{\sin(z)}{(1+\log z/\omega)^2 z}dz. \quad (19)$$

Assuming $\omega\ll 1$, $1+\log(x/\omega)$ can be replaced by $\log(1/\omega)$ in the integrand and we get

$$S_x(\omega)\sim 2\frac{1}{\omega\left(\log\frac{1}{\omega}\right)^2}\int_0^{\infty}\frac{\sin(x)}{x}dx \quad (\omega\ll 1)$$
$$=2\frac{\pi}{2}\frac{1}{\omega\left(\log\frac{1}{\omega}\right)^2}. \quad (20)$$

The correlation function

$$R_x(t)=\frac{1}{1+\log(t+1)}, \quad (21)$$

can also be used to get the same approximated PSD as (20). The PSD of the output signal is given by the formula

$$S_y(\omega)\sim 2\frac{1}{\omega\left(\log\frac{1}{\omega}\right)^2} \quad (\omega\ll 1), \quad (22)$$

in the low-frequency limit.

The PSDs were calculated over the same frequency range for the input and output signals, and in summary, we have found that the exponent β of the output PSD depends on the exponent α of the input PSD as follows:

$$\beta=\begin{cases}\alpha, & \text{if } 0<\alpha\leqslant 1\\ \dfrac{\alpha+1}{2}, & \text{if } 1<\alpha<2\ .\end{cases} \quad (23)$$

The approximations used to derive this relation were verified by the numerical integration of Eq. (8), into which the proper correlation function was substituted. Figure 2 shows the results of the numerical integration for the cases of $\alpha=0.75$, $\alpha=1$, and $\alpha=1.25$.

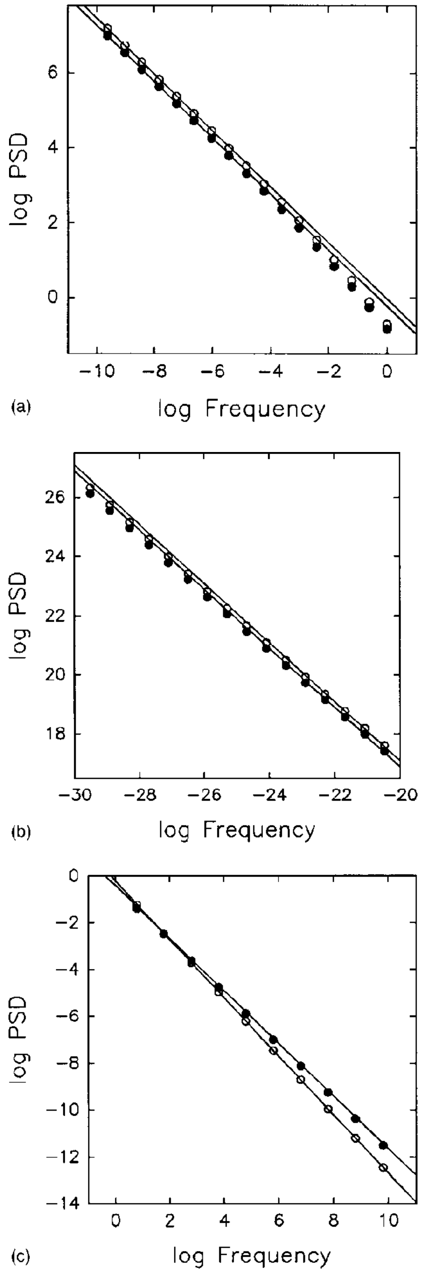

FIG. 2. PSD of the input (hollow circles) and output (filled circles) signals obtained by numerical integration for $\alpha=0.75$ (a), $\alpha=1$ (b), and $\alpha=1.25$ (c). The solid lines represent ideal $1/f^{\alpha}$ and $1/f^{\beta}$ spectra [see Eq. (23) and the text].

Our results have been confirmed by numerical simulations also. Gaussian noises with length of 2^{18} were generated, and the PSD was calculated by averaging 1000 samples. In Fig. 3 the numerical simulation result for $1/f$ noise ($\alpha=1$) is plotted and Fig. 4 shows how the numerical simulation confirms Eq. (23).

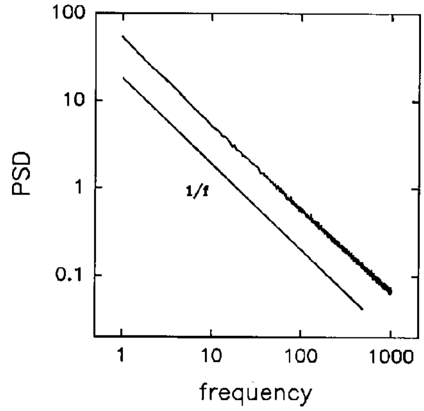

FIG. 3. PSD of Gaussian $1/f$ noise after amplitude truncation obtained by numerical simulation. The solid line represents perfect $1/f$ noise.

IV. GENERALIZATION OF THE THEORY TO OTHER TRUNCATION LEVELS

The above results for very close truncation levels can be generalized to distant truncation levels as well. Let us assume that we have a zero-mean Gaussian $1/f^{\alpha}$ noise and truncation levels $x_{\min}<0<x_{\max}$. For short time intervals (high frequencies) the PSD of the truncated signal is mainly determined by the noise amplitude behavior between the levels. Moreover, for time intervals much longer than the time required by the signal to pass between the two levels, the signal has a PSD similar to that obtained in the case of low levels. If $\alpha\leqslant 1$, this means that the spectrum has the same dependence both for low and high frequencies, for $\alpha>1$ the low-frequency part has exponent $\beta=(\alpha+1)/2$ below a certain

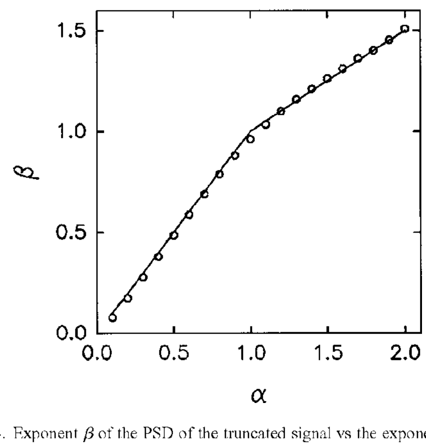

FIG. 4. Exponent β of the PSD of the truncated signal vs the exponent α of the PSD of the input signal. The solid line represents the theoretical result of Eq. (23), the circles are obtained by numerical simulations.

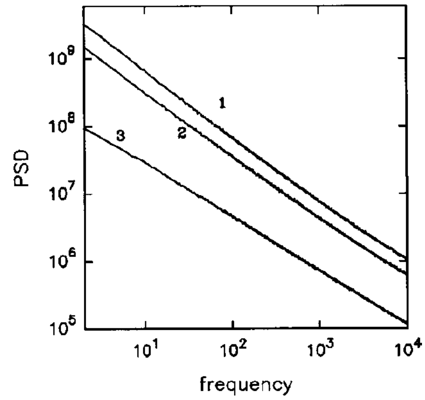

FIG. 5. Output PSDs for $1/f$ input noise with truncation levels located at 0, σ, and 2σ labeled with 1,2,3, respectively. The corresponding slopes are ~0.98, 0.91, and 0.8.

corner frequency, while for high frequencies $\beta=\alpha$ is expected. The corner frequency depends on the truncation levels, of course. Let us assume that the truncation is symmetric, i.e., $x_{max}=-x_{min}=U$. When the level is changed from U to a U, then the corner frequency changes from f_c to $a^{-2/(\alpha-1)}f_c$ for $1<\alpha<2$. This scaling property can be obtained from the self-affine character of the signal $x(t)$. Note that this argument does not hold for cases in which the mean of the input noise is not located between the truncation levels.

V. UNEXPLAINED BEHAVIOR AT ASYMMETRICAL TRUNCATION LEVELS EXCLUDING THE MEAN VALUE

We have carried out numerical simulations for cases when the mean value of the noise is not included in the interval defined by the upper and lower truncation levels. More precisely, the truncation levels are expressed as

$$x_{min}=x_0-\epsilon$$
$$x_{max}=x_0+\epsilon, \tag{24}$$

where x_0 and ϵ are both positive, $x_0>0$ and $\epsilon \ll x_0$.

Equation (23) is not valid in this case, however, the output PSD seems to follow a power law again with a modified value of β. Figure 5 illustrates the results for $1/f$ input noise, where x_0 equals σ (label 2), 2σ (label 3) together with the previous case (label 1). Here σ is the standard deviation of the simulated noise.

The theoretical result of Eq. (23) cannot be applied to these cases as the phenomenon is rather complex here.

VI. CONCLUSIONS AND OPEN QUESTIONS

In physical systems, measurements and data communications noise is always present, and several nonlinear transformations can occur including amplitude truncation. Simple examples are quantities with limited amplitude range, overdriven systems and systems with saturating transfer functions, e.g., $y(t)=x(t)^{1/n}$, where n is a large odd number.

Investigations of other nonlinear transforms of $1/f^{\alpha}$ noises are also very important, because it might help to understand these noises more precisely.

In this paper we have presented a theoretical explanation of the previously unsolved problem of the invariance of the PSD against the amplitude truncation for Gaussian $1/f^{\alpha}$ noises. Our theoretical results are extended to asymmetrical and distant truncation levels between which the mean value of the noise is located. Note here that the $1/f^{3/2}$ PSD of diffusion noise can be obtained using amplitude truncation of $1/f^2$ noise[14] in accordance with Eq. (23) even though $\alpha=2$ is not included in our theoretical derivation.

We have shown an open problem concerning the truncation levels both of which are above or under the mean value. Our numerical simulations show that the PSD of the truncated signals follows a power law again, but the exponent β of the output PSD has a theoretically unexplained dependence on the exponent α of the input PSD.

It is not yet clear, how this invariant property can help to understand the generality of $1/f$ noise. However, this result suggests a possible convergence from $1/f^2$ noise to $1/f$ noise via successive amplitude truncations.

ACKNOWLEDGMENT

The authors wish to thank L. B. Kiss for helpful discussions.

[1] H. E. Maes and S. H. Usmani, J. Appl. Phys. **54**, 1937 (1983).
[2] G. A. Garfunkel and M. B. Weissman, J. Appl. Phys. **57**, 634 (1985).
[3] N. B. Lukyanchikova, *Noise Research in Semiconductor Physics*, edited by B. K. Jones (Gordon and Breach, New York, 1996).
[4] L. B. Kiss and P. Svedlindh, IEEE Trans. Electron Devices **41**, 2112 (1994).
[5] R. J. Fronen and L. K. J. Vandamme, IEEE J. Quantum Electron. **24**, 724 (1988).
[6] R. J. Fronen, IEEE J. Quantum Electron. **25**, 1653 (1989).
[7] P. L. Nolan *et al.*, Astrophys. J. **246**, 494 (1981).
[8] T. Musha, H. Takeuchi, and T. Inoue, IEEE Trans. Biomed. Eng. **BME-30**, 194 (1983).
[9] T. Musha and H. Higuchi, Jpn. J. Appl. Phys. **15**, 1271 (1976).
[10] R. F. Voss and J. Clarke, Nature (London) **258**, 317 (1975).
[11] L. B. Kiss, Z. Gingl, Zs. Marton, J. Kertesz, F. Moss, G. Schmera, and A. Bulsara, J. Stat. Phys. **70**, 451 (1993).
[12] Z. Gingl and L. B. Kiss, in *Proceedings of the First International Conference on Unsolved Problems of Noise*, edited by Ch. R. Doering, L. B. Kiss, and M. F. Schlesinger (World Scientific, Singapore, 1996), p. 337.
[13] S. Ishioka, Z. Gingl, D. Choi, and N. Fuchikami, Phys. Lett. A **269**, 7 (2000).
[14] J. M. Richardson, Bell Syst. Tech. J. **29**, 117 (1950).