Christoffel functions for weights with jumps*

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Abstract

The asymptotic behavior of Christoffel functions is established at points of discontinuity of the first kind.

1 The result

Let μ be a measure on the real line with compact support S. The Christoffel functions $\lambda_n(z,\mu)$ associated with μ are defined as

$$\lambda_n(z,\mu) = \inf_{P_n(z)=1} \int |P_n|^2 d\mu,$$

where the infimum is taken for all polynomials of degree at most n which take the value 1 at z. They play a fundamental role in the theory of orthogonal polynomials and in random matrix theory, see the papers [9] and [11] for their various use and properties. One of their most basic properties is that if $p_n(z) = \gamma_n z^n + \cdots$ are the orthonormal polynomials with respect to μ, then

$$\frac{1}{\lambda_n(z,\mu)} = \sum_{k=0}^{n} |p_k(z)|^2.$$

The aim of this paper is to establish the asymptotic behavior of $\lambda_n(x_0,\mu)$ at points x_0 where the density of μ has a jump singularity. To do so we shall need some basic notions from potential theory, see the books [5], [6] or [12] for the fundamentals of logarithmic potential theory. In particular, we need the notion of the equilibrium measure of S: it is the unique Borel-measure on S with total mass 1 which minimizes the energy

$$I(\nu) := \int\int \log \frac{1}{|z-t|} d\nu(z) d\nu(t) \tag{1}$$

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provided there is a measure on S for which this energy is finite (when all energies are infinite the set S is called polar and it does not have an equilibrium measure). This is the case if S contains a non-degenerated interval. Denote the equilibrium measure of S by μ_S. It is known that if I is an interval inside S then on I the equilibrium measure μ_S is absolutely continuous with respect to the Lebesgue-measure on $\mathbf{R}$, and we shall denote by ω_S its density: $d\mu_S(t) = \omega_S(t)dt$, $t \in I$. A general property of these densities that we shall often use is their monotonicity: if $I \subset S \subset S'$ then

$$\omega_S(t) \geq \omega_{S'}(t) \qquad \text{for } t \in I, \tag{2}$$

see [15, Lemma 4.1].

The quantity $\mathrm{cap}(S) = \exp(-I(\mu_S))$, where $I(\mu_S)$ is the minimal energy in (1), is called the logarithmic capacity of S. In general, the logarithmic capacity of a Borel-set is the supremum of the logarithmic capacities of its compact subsets.

We shall also need the so called **Reg** class from [13]: we say that $\mu \in \mathbf{Reg}$ if

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\mathrm{cap}(S)},$$

where γ_n is the leading coefficient of the aforementioned orthonormal polynomial $p_n(z)$. It is known [13, Theorem 3.2.1] that this is equivalent to the fact that for all $z \in S$ with the exception of points lying in a set of capacity zero

$$\limsup_{n\to\infty} \left(\frac{|Q_n(z)|}{\|Q_n\|_{L^2(\mu)}} \right)^{1/n} \leq 1 \tag{3}$$

for any polynomial sequence $\{Q_n\}$, $\deg(Q_n) \leq n$. When $\overline{\mathbf{C}} \setminus S$ is regular with respect to the Dirichlet-problem (see e.g. [12, Chapter 4]) then this is equivalent to the fact that

$$\limsup_{n\to\infty} \left(\frac{\|Q_n\|_{L^\infty(S)}}{\|Q_n\|_{L^2(\mu)}} \right)^{1/n} = 1 \tag{4}$$

for any polynomial sequence $\{Q_n\}$, $\deg(Q_n) \leq n$.

With these we state

Theorem 1 *Let $\mu \in \mathbf{Reg}$ with support $S \subset \mathbf{R}$, and let x_0 be a point in the interior of the support. Suppose that in a neighborhood of x_0 the measure μ is absolutely continuous: $d\mu(x) = w(x)dx$, and its density w has a singularity at x_0 of the first kind:*

$$\lim_{x\to x_0-0} w(x) = A, \qquad \lim_{x\to x_0+0} w(x) = B, \qquad A, B > 0. \tag{5}$$

Then

$$\lim_{n\to\infty} n\lambda_n(\mu, x_0) = \frac{1}{\omega_S(x_0)} \frac{A - B}{\log A - \log B}, \tag{6}$$

where $\omega_S(x_0)$ is the density of the equilibrium measure of S with respect to the Lebesgue measure.

We note without proof that the absolute continuity can be replaced by $\mu_s([x_0-\delta, x_0+\delta]) = o(\delta)$ where μ_s is the singular part of μ.

When A or B is 0, the limit in (6) is 0. This follows from the monotonicity of $\lambda_n(\mu, x)$ in μ and from (6) if we apply the latter to $d\mu(x) + \varepsilon dx$ and let ε tend to 0.

When $A = B$ the density is continuous, and in that case (or in the $A \to B$ case) the quantity $(A-B)/(\log A - \log B)$ should be interpreted as the common value A, i.e. if w is continuous at x_0 and $\mu \in \mathbf{Reg}$, then

$$\lim_{n\to\infty} n\lambda_n(\mu, x_0) = \frac{w(x_0)}{\omega_S(x_0)}. \tag{7}$$

This was proved in [14, Theorem 1] under the additional assumption that $\overline{\mathbf{C}} \setminus S$ is regular with respect to the Dirichlet problem. In the proof of [15, Theorem 3.1] it was mentioned that this latter condition can be dropped, but the proof outlined there was incomplete. Now Theorem 1 furnishes (7) in full generality (without the regularity assumption on S).

2 Proof

For simplicity we shall write

$$\gamma := \frac{A-B}{\log A - \log B}. \tag{8}$$

In what follows let $d\nu(x) = v(x)dx$ where

$$v(x) = \begin{cases} A & \text{if } x \in [-1, 0] \\ B & \text{if } x \in (0, 1] \end{cases} \tag{9}$$

or

$$v(x) = \begin{cases} B & \text{if } x \in [-1, 0] \\ A & \text{if } x \in (0, 1]. \end{cases} \tag{10}$$

Which of these two definitions is needed will be explained at the appropriate part of the proof. In any case Theorem 11 of [4] tells us that

$$\lim_{n\to\infty} n\lambda_n(\nu, 0) = \pi\frac{A-B}{\log A - \log B} =: \pi\gamma. \tag{11}$$

This is a key result, we shall deduce the theorem from it using the polynomial inverse image technique, see e.g. [16].

Without loss of generality we may assume $S \subset (-1/4, 1/4)$.

Fix a small $\eta > 0$ and choose $a > 0$ such that for $x \in (x_0 - a, x_0)$ we have

$$\frac{A}{1+\eta} \le w(x) \le (1+\eta)A, \tag{12}$$

and for $x \in (x_0, x_0 + a)$ we have

$$\frac{B}{1+\eta} \le w(x) \le (1+\eta)B. \tag{13}$$

The upper estimate

Choose $E = \cup_{j=1}^m [a_j, b_j] \subset [-1/4, 1/4]$ such that S lies in the interior of E and

$$\omega_E(x_0) > \frac{1}{1+\eta}\omega_S(x_0) \tag{14}$$

(the other direction $\omega_E(x_0) \le \omega_S(x_0)$ is automatic, c.f. (2)). This is possible, see e.g. [15, Lemma 4.1].

We call a polynomial T_N of exact degree N admissible if it has $N-1$ extrema that are all ≥ 1 in absolute value. Set $E_N = T_N^{-1}[-1,1]$ for an admissible T_N. It is known that sets of these type are dense among all sets consisting of finitely many intervals in the sense that if $E = \cup_{j=1}^m [\alpha_j, \beta_j]$ is a set consisting of finitely many intervals then for every ε there is an $E_N = \cup_{j=1}^m [\alpha_j', \beta_j']$ with $|\alpha_j - \alpha_j'| < \varepsilon$, $\beta_j - \beta_j'| < \varepsilon$ for all $j = 1, \ldots, m$. This automatically implies then that, besides this property we may also assume $E \subset E_N$ or, if we want, $E_N \subset E$. For all these see [16, Theorem 3.1] as well as the papers [2], [7], [8], [10] that contain this density theorem.

Let $\varepsilon > 0$ be so small that

$$S \subset \bigcup_{j=1}^m [a_j + 2\varepsilon, b_j - 2\varepsilon].$$

By the just formulated density theorem there is an admissible T_N such that for $E_N = T_N^{-1}[-1,1]$ we have

$$\bigcup_{j=1}^m [a_j + 2\varepsilon, b_j - 2\varepsilon] \subset E_N \subset \bigcup_{j=1}^m [a_j + \varepsilon, b_j - \varepsilon].$$

If T_N is replaced by $\mathcal{T}_k(T_N)$ with the classical Chebyshev polynomials $\mathcal{T}_k(x) = \cos(k \arccos x)$, then E_N does not change, but for large k all subintervals of E_N over which T_N is a 1–to–1 mapping onto $[-1,1]$ are shorter than ε, so by a translation of T_N by an amount $< \varepsilon$ we may assume that $S \subset E_N \subset E$ and $T_N(x_0) = 0$. In view of [16, (6)] we get in this case

$$\omega_{E_N}(x_0) = \frac{|T_N'(x_0)|}{N\pi\sqrt{1 - T_N^2(x_0)}} = \frac{|T_N'(x_0)|}{N\pi}. \tag{15}$$

There is a $0<b<a$ such that T_N is 1–to–1 on $[x_0-b,x_0+b]$, and

$$\frac{1}{1+\eta}|T_N'(x)|\le|T_N'(x_0)|\le(1+\eta)|T_N'(x)|$$

there. Note that by (12)–(13) we also have on $[x_0-b,x_0+b]\setminus\{x_0\}$ the inequality

$$w(x)\le(1+\eta)v(T_N(x))$$

with the v either from (9) or from (10) (depending on if T_N is increasing or decreasing on $[x_0-b,x_0+b]$).

Let for large n P_n be a polynomial of degree n such that $P_n(0)=1$ and

$$\int P_n^2d\nu=\int_{-1}^{1}P_n^2v\le\frac{1+\eta}{n}\pi\gamma \tag{16}$$

(see (11)), and with some small $\varepsilon>0$ set

$$R_n(x)=P_n(T_N(x))(1-(x-x_0)^2)^{[\varepsilon n]}.$$

This is a polynomial of degree at most $nN+2[\varepsilon n]$ such that $R_n(x_0)=1$. We can write

$$\begin{aligned}\int_{x_0-b}^{x_0+b}R_n^2d\mu &\le \frac{(1+\eta)^2}{|T_N'(x_0)|}\int_{x_0-b}^{x_0+b}P_n(T_N(x))^2v(T_N(x))|T_N'(x)|dx\\ &= \frac{(1+\eta)^2}{|T_N'(x_0)|}\int_{T_N([x_0-b,x_0+b])}P_n(u)^2v(u)du\le\frac{(1+\eta)^3}{|T_N'(x_0)|}\frac{\pi\gamma}{n}\end{aligned}$$

It follows from (16) via Nikolskii's inequality [3, Theorem 4.2.6] that $P_n=O(n)$ (actually $O(\sqrt{n})$) on $[-1,1]$, so on $S\setminus[x_0-b,x_0+b]$ we have $R_n=O(n(1-b^2)^{\varepsilon n})=o(1/n)$. These give (use R_n as test polynomials to estimate $\lambda_n(\mu,x_0)$ from above)

$$\begin{aligned}\limsup_{n\to\infty}(Nn+2[\varepsilon n])\lambda_{Nn+2[\varepsilon n]}(\mu,x_0)\le(N+2\varepsilon)\frac{(1+\eta)^3}{|T_N'(x_0)|}\pi\gamma &\le \frac{N+2\varepsilon}{N}\frac{(1+\eta)^4}{\omega_S(x_0)}\gamma\\ &\le \frac{1+2\varepsilon}{1}\frac{(1+\eta)^4}{\omega_S(x_0)}\gamma,\end{aligned}$$

where we used that by (2), (14) and (15)

$$\frac{N\pi}{|T_N'(x_0)|}=\frac{1}{\omega_{E_N}(x_0)}\le\frac{1}{\omega_E(x_0)}\le\frac{1+\eta}{\omega_S(x_0)}.$$

Since ε and η are arbitrarily small numbers, it follows from the preceding lim sup estimate and from the monotonicity of λ_n in n that

$$\limsup_{n\to\infty}n\lambda_n(\mu,x_0)\le\frac{1}{\omega_S(x_0)}\gamma=\frac{1}{\omega_S(x_0)}\frac{A-B}{\log A-\log B}.$$

The lower estimate in the case for regular sets

The proof of the lower estimate is simpler if we assume that $\overline{\mathbf{C}} \setminus S$ is regular with respect to the Dirichlet problem. In this subsection we assume that, and in the next subsection we shall deal with the general case.

Fix a small $\eta > 0$. Let

$$\liminf_{n\to\infty} n\lambda_n(\mu, x_0) = \gamma_0,$$

and select $\mathcal{N} \subset \mathbf{N}$ such that for $n \in \mathcal{N}$ there are P_n with $P_n(x_0) = 1$,

$$\int P_n^2 d\mu \le \frac{(1+\eta)\gamma_0}{n}. \tag{17}$$

Choose for this η the a as before (see (12)–(13)). By Nikolskii's inequality we have $P_n = O(n)$ on $[x_0 - a, x_0 + a]$ (with O depending on a).

The regularity of μ implies for every $\tau > 0$

$$\|Q_n\|_{L^\infty(S)} \le (1+\tau)^n \|Q_n\|_{L^2(\mu)}$$

for all polynomials Q_n of sufficiently large degree n. The regularity of S and the Bernstein-Walsh lemma (see e.g. [17, p. 77] or [12, Thm. 5.5.7]) give that for every $\tau > 0$ there is a $\delta > 0$ such that if $\mathrm{dist}(z, S) < \delta$, then

$$|Q_n(z)| \le (1+\tau)^n \|Q_n\|_{L^\infty(S)}.$$

Thus, there is a set $E \subset [-1/4, 1/4]$ consisting of finitely many intervals such that E contains S in its interior and

$$\|Q_n\|_{L^\infty(E)} \le (1+\tau)^{2n} \|Q_n\|_{L^2(\mu)} \tag{18}$$

for all polynomials Q_n of sufficiently large degree n.

Choose again an admissible T_N such $T_N(x_0) = 0$, and for $E_N = T_N^{-1}[-1,1]$ we have $S \subset E_N \subset E$. We can write $E_N = \cup_{j=1}^N [a_j, b_j]$, where the $[a_j, b_j]$'s are disjoint except perhaps for their endpoints, and T_N maps each $[a_j, b_j]$ in a 1–to–1 manner onto $[-1, 1]$. Thus, a branch of T_N^{-1} maps $[-1, 1]$ onto $[a_j, b_j]$. Let $x_0 \in [a_{j_0}, b_{j_0}]$ and let $b < \min\{x_0 - a_{j_0}, b_{j_0} - x_0\}$ be such that on $[x_0 - b, x_0 + b]$ we have

$$\frac{1}{1+\eta}|T_N'(x)| \le |T_N'(x_0)| \le (1+\eta)|T_N'(x)|.$$

Consider with some small $\varepsilon > 0$ the polynomial

$$R_n(x) = P_n(x)(1 - (x - x_0)^2)^{[\varepsilon n]}.$$

Its degree is $\le n + 2\varepsilon n$, and clearly $R_n(x_0) = 1$. On $[x_0 - a, x_0 + a] \setminus [a_{j_0}, b_{j_0}]$ we have

$$|R_n| \le Cn(1 - b^2)^{\varepsilon n}, \tag{19}$$

while on $E_N \setminus [x_0 - a, x_0 + a]$ we have for large n (apply (17)–(18) for P_n)

$$|R_n| \le \left((1+\tau)^2(1-a^2)^\varepsilon\right)^n, \tag{20}$$

and we choose $\tau > 0$ so small that

$$(1+\tau)^2(1-a^2)^\varepsilon < 1. \tag{21}$$

For an $x \in E_N$ let $\xi_j = \xi_j(x) \in [a_j, b_j]$, $j = 1, \ldots, N$ be the solutions of the equation

$$T_N(\xi) - T_N(x) = 0,$$

and set

$$R_n^*(x) = \sum_{j=1}^{N} R_n(\xi_j).$$

This is a symmetric polynomial of the ξ_j's, so it is a polynomial of their elementary symmetric polynomials, i.e. (in view of Viéte's formulae) of the coefficients of the polynomial $T_N(\xi) - T_N(x)$ (considered as a polynomial of ξ). Thus, $R_n^*(x)$ is a polynomial of $T_N(x)$: $R_n^*(x) = V_{n/N}(T_N(x))$, and here the degree of $V_{n/N}$ is at most $\deg(R_n^*)/N \le (1+2\varepsilon)n/N$ (c.f. [16, Sec. 5]).

Next, with the v from (9) if T_N is increasing on $[a_{j_0}, b_{j_0}]$ or with the v from (10) if T_N is decreasing on $[a_{j_0}, b_{j_0}]$ we can write

$$\begin{aligned}
\int_{T_N([x_0-b, x_0+b])} V_{n/N}(u)^2 v(u)\,du &= \int_{x_0-b}^{x_0+b} V_{n/N}(T_N(x))^2 v(T_N(x)) |T_N'(x)|\,dx \\
&\le (1+\eta)^2 |T_N'(x_0)| \int_{x_0-b}^{x_0+b} R_n^*(x)^2\,d\mu(x). \quad (22)
\end{aligned}$$

According to (19)-(21) and the fact that $R_n^*(x) = O(n)$ on $[x_0 - a, x_0 + a] \supset [a_{j_0}, b_{j_0}]$ we have here

$$R_n^*(x)^2 = R_n(x)^2 + O(\rho^n)$$

with some $\rho < 1$ independent of n (which may depend on a, b, τ, ε). Therefore, we can continue (22) as

$$\begin{aligned}
&\le (1+\eta)^2 |T_N'(x_0)| \int_{x_0-b}^{x_0+b} R_n(x)^2\,d\mu(x) + O(\rho^n) \\
&\le (1+\eta)^2 |T_N'(x_0)| \int_{x_0-b}^{x_0+b} P_n(x)^2\,d\mu(x) + O(\rho^n) \\
&\le (1+\eta)^2 |T_N'(x_0)| \frac{(1+\eta)\gamma_0}{n} + O(\rho^n).
\end{aligned}$$

On $[a_{j_0}, b_{j_0}] \setminus [x_0 - b, x_0 + b]$ the inequality

$$R_n^*(x)^2 = R_n(x)^2 + O(\rho^n) = O(\rho^n),$$

holds, so

$$\int_{[-1,1]\setminus T_N([x_0-b,x_0+b])} V_{n/N}(u)^2 v(u)du$$

$$= \int_{[a_{j_0},b_{j_0}]\setminus[x_0-b,x_0+b]} V_{n/N}(T_N(x))^2 v(T_N(x))|T_N'(x)|dx$$

$$\leq \int_{[a_{j_0},b_{j_0}]\setminus[x_0-b,x_0+b]} O(R_n^*(x)^2)dx = O(\rho^n).$$

All in all, for $n \in \mathcal{N}$ we can deduce

$$\int V_{n/N}^2 d\nu \leq (1+\eta)^2 |T_N'(x_0)| \frac{(1+\eta)\gamma_0}{n} + O(\rho^n).$$

Since here

$$\frac{|T_N'(x_0)|}{N\pi} = \omega_{E_N}(x_0) \leq \omega_S(x_0),$$

it follows that (use $V_{n/N}$ as test polynomials)

$$\limsup_{n\to\infty,\ n\in\mathcal{N}} \frac{(n+2[\varepsilon n])}{N} \lambda_{(n+2[\varepsilon n])/N}(\nu, 0) \leq (1+\eta)^3(1+2\varepsilon)\gamma_0\pi\omega_S(x_0).$$

Now $\varepsilon, \eta > 0$ are arbitrary, hence we can conclude

$$\liminf_{n\to\infty} n\lambda_n(\nu, 0) \leq \gamma_0\pi\omega_S(x_0), \tag{23}$$

and a comparison with (11) shows that we must have $\gamma_0 \geq \gamma/\omega_S(x_0)$.

This proves

$$\liminf_{n\to\infty} n\lambda_n(\nu, 0) \geq \frac{\gamma}{\omega_S(x_0)},$$

and the proof is complete.

■

The lower estimate in the general case

As before, fix a small $\eta > 0$, let

$$\liminf_{n\to\infty} n\lambda_n(\mu, x_0) = \gamma_0,$$

and select $\mathcal{N} \subset \mathbf{N}$ such that for $n \in \mathcal{N}$ there are polynomials P_n of degree n with $P_n(x_0) = 1$ and with property (17). Choose for this η again the number a so that (12)–(13) are satisfied. By Nikolskii's inequality [3, Theorem 4.2.6], we have $P_n = O(n)$ on $[x_0 - a, x_0 + a]$.

We say that a property holds quasi-everywhere on S if the set of points on S where it does not hold is of zero capacity. Now $\mu \in \mathbf{Reg}$ implies (see (3)) that for quasi-every $z \in S$ we have

$$\lim_{n\to\infty} \left(\sup_{Q_n \not\equiv 0} \frac{|Q_n(z)|}{\|Q_n\|_{L^2(\mu)}} \right)^{1/n} = 1. \tag{24}$$

For $\tau > 0$ and $M \in \mathbf{N}$

$$F_{M,\tau} := \left\{ z \in S \,\middle|\, \sup_{Q_n \not\equiv 0} \frac{|Q_n(z)|}{\|Q_n\|_{L^2(\mu)}} \leq (1+\tau)^n; \;\; n \geq M \right\},$$

are compact sets, $F_{M,\tau} \subset F_{M+1,\tau}$ and for a fix $\tau > 0$ their union for all M is $S \setminus H$, where H is of zero capacity (see (24)). Hence (see [12, Theorem 5.1.3,b]) $\mathrm{cap}(F_{M,\tau}) \to \mathrm{cap}(S)$ as $M \to \infty$. Choose $\tau > 0$ so that (21) is satisfied, and then for a fixed $\theta > 0$ choose M so large that $\mathrm{cap}(F_{M,\tau}) > \mathrm{cap}(S) - \theta$, and set $S'_\theta = F_{M,\tau}$. By Ancona's theorem [1] there are regular compact subsets $S_\theta \subseteq S'_\theta$ such that $\mathrm{cap}(S'_\theta \setminus S_\theta)$ is arbitrarily small, and then we choose such an S_θ for which $\mathrm{cap}(S_\theta) > \mathrm{cap}(S) - \theta$.

Now repeat the proof in the preceding subsection with S replaced by S_θ. There is again a set $E \subset [-1/4, 1/4]$ consisting of finitely many intervals such that E contains S_θ in its interior and

$$\|Q_n\|_{L^\infty(E)} \leq (1+\tau)^{2n} \|Q_n\|_{L^2(\mu)} \tag{25}$$

for all polynomials Q_n of sufficiently large degree n. Indeed, this follows from the definition of S_θ, from its regularity and from the Bernstein-Walsh lemma ([17, p. 77] or [12, Thm. 5.5.7]). The conclusion that the proof gives with this change is that (c.f. (23))

$$\liminf_{n\to\infty} n\lambda_n(\nu, 0) \leq \gamma_0 \pi \omega_{S_\theta}(x_0), \tag{26}$$

and a comparison with (11) shows that we must have $\gamma_0 \geq \gamma/\omega_{S_\theta}(x_0)$, i.e.

$$\liminf_{n\to\infty} n\lambda_n(\nu, 0) \geq \frac{\gamma}{\omega_{S_\theta}(x_0)}.$$

Here $\theta > 0$ is arbitrary, and, as $\theta \to 0$, we have $\omega_{S_\theta}(x_0) \to \omega_S(x_0)$ (see [15, Lemma 4.2]), so finally we can conclude

$$\liminf_{n\to\infty} n\lambda_n(\nu, 0) \geq \frac{\gamma}{\omega_S(x_0)},$$

and that completes the proof.

■

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