

# A note on rational $L^p$ approximation on Jordan curves\*

Vilmos Totik<sup>†</sup>

July 18, 2013

## Abstract

The precise asymptotics for the error of best rational approximation of meromorphic functions in integral norm is shown to be a consequence of a result of Gonchar and Rakhmanov. This reproves and extends a recent result of Baratchart, Stahl and Yattselev.

Let  $T$  be a rectifiable Jordan curve,  $G$  and  $O$  the interior and exterior domains of  $T$ , respectively, with respect to  $\overline{\mathbf{C}}$ . Let  $A(G)$  denote the set of functions  $f$  such that

- $f$  vanishes at infinity and admits holomorphic and single-valued continuation from infinity to an open neighborhood of  $\overline{O}$ ,
- $f$  admits meromorphic, possibly multi-valued, continuation along any arc in  $G \setminus E_f$  starting from  $T$ , where  $E_f$  is a finite set of points in  $G$ ,
- $E_f$  is non-empty, the meromorphic continuation of  $f$  from infinity has a branch point at each element of  $E_f$ .

Examples of such functions are algebraic functions with branch points. See the paper [1] for other examples, motivation and history.

In the recent landmark paper L. Baratchart, H. Stahl and M. Yattselev [1] have developed the theory of rational approximation of functions  $f \in A(G)$  in the  $L^2(s_T)$  norm on  $T$ , where  $s_T$  is the arc measure on  $T$ , and where the approximation is done from the set  $\mathcal{R}_n(G)$  of rational functions  $p_{n-1}/q_n$  of degree  $((n-1), n)$  which have all their poles in  $G$ . Let the error of best approximation

---

\*AMS Classification: 41A20

Key Words: rational approximation, Jordan curves, meromorphic functions, condenser capacity

<sup>†</sup>Supported by the European Research Council Advanced Grant No. 267055

in  $L^p(s_T)$  be denoted by  $\rho_{n,p}(f, O)$ . The theory in [1] gave, besides a lot of information on the best approximants, the  $p = 2$  case of the asymptotic formula

$$\lim_{n \rightarrow \infty} \rho_{n,p}^{1/2n}(f, O) = \exp \left( -\frac{1}{\text{cap}(K_T, T)} \right) \quad (1)$$

(see below for the definition of the minimal condenser capacity  $\text{cap}(K_T, T)$ ). For  $p = \infty$  the same formula follows from a result of A. A. Gonchar and E. A. Rakhmanov [2, Theorem 1']. As a consequence, (1) has been established for all  $2 \leq p \leq \infty$ .

In this note we derive (1) for all  $1 \leq p < \infty$  directly from the  $p = \infty$  case proven in [2, Theorem 1'].

To have a basis of discussion, let  $g_G(z, \zeta)$  denote the Green's function of  $G$  with pole at  $\zeta \in G$ , and if  $K \subset G$  is a compact set, then consider the minimal energy

$$I_G(K) := \inf_{\omega} I_G(\omega) := \inf_{\omega} \int \int g_G(z, t) d\omega(z) d\omega(t),$$

where the infimum is taken for all unit Borel-measures on  $K$ . In the case when  $K$  is not polar (has positive logarithmic capacity) there is a unique minimizing measure  $\omega_{K,T}$ , called the Green equilibrium measure of  $K$  (with respect to  $\Omega$ ).  $\text{cap}(K, T) := 1/I_G(K)$  is called the condenser capacity of the condenser  $(K, T)$ .

Next, we need the notion of a set of minimal condenser capacity. We say that a compact  $K \subset G$  is admissible for  $f \in A(G)$  if  $\overline{\mathbf{C}} \setminus K$  is connected, and  $f$  has a meromorphic and single-valued extension there. The collection of all admissible sets for  $f$  is denoted by  $\mathcal{K}_f(G)$ . A compact  $K_T \in \mathcal{K}_f(G)$  is said to be a set of minimal condenser capacity for  $f$  if

- $\text{cap}(K_T, T) \leq \text{cap}(K, T)$  for any  $K \in \mathcal{K}_f(G)$ ,
- $K_T \subseteq K$  for any  $K \in \mathcal{K}_f(G)$  for which  $\text{cap}(K, T) = \text{cap}(K_T, T)$ .

See [1] for the existence and unicity of such a  $K_T$ . The set  $K_T$  of minimal condenser capacity is the complement of the "largest" (regarding capacity) domain containing  $O$  on which  $f$  is single-valued and meromorphic. It turns out (see [1, Theorem S]) that  $K_T = E_0 \cup E_1 \cup (\cup_j \gamma_j)$ , where  $\cup_j \gamma_j$  is a finite union of open analytic arcs,  $E_0 \subset E_f$ , each point in  $E_0$  is the endpoint of exactly one  $\gamma_j$ , while  $E_1$  consist of those finitely many points where at least three arcs  $\gamma_j$  meet.

These definitions explain the notation in (1), and with these we claim

**Theorem 1** (1) holds for all  $1 \leq p \leq \infty$ .

**Proof.** The  $p = \infty$  case is covered by the Gonchar-Rakhmanov theorem from [2], so it is left to show

$$\liminf_{n \rightarrow \infty} \rho_{n,1}^{1/2n}(f, O) \geq \exp \left( -\frac{1}{\text{cap}(K_T, T)} \right). \quad (2)$$

Let  $G_1 \supset G_2 \supset \dots$  be a nested sequence of Jordan domains with boundaries  $T_1, T_2, \dots$  such that  $T_{j+1} \subset G_j$ , each  $T_j$  lies outside  $\overline{G}$ , the maximal distance from a point of  $T_j$  to  $T$  is less than  $1/j$  and  $\text{length}(T_j) \rightarrow \text{length}(T)$  (say some level line of the conformal mapping of  $O$  onto the exterior of the unit disk suffices as  $T_j$ ). Then there is a compact set  $K \subset G$  and a  $j_0$  such that  $K_{T_j} \subset K$  for  $j \geq j_0$  (see Lemma 2 below), and for  $z, t \in K$  we have  $g_{G_j}(z, t) \leq g_G(z, t) + \eta_j$  where  $\eta_j \rightarrow 0$  (see Lemma 3 below). If  $r \in \mathcal{R}_n(G)$  is any rational function from  $\mathcal{R}_n(G)$  and if we apply Cauchy's formula for  $(f - r_n)(z)$ ,  $z \in T_j$ , in  $O$  using integration on  $T$ , we obtain

$$\sup_{z \in T_j} |f(z) - r_n(z)| \leq \|f - r_n\|_{L^1(s_T)} \frac{1}{\text{dist}(T_j, T)},$$

so

$$\liminf_{n \rightarrow \infty} \rho_{n,1}^{1/2n}(f, O) \geq \liminf_{n \rightarrow \infty} \rho_{n,\infty}^{1/2n}(f, O_j) = \exp\left(-I_{G_j}(\omega_{K_{T_j}, T_j})\right),$$

where the equality follows by the aforementioned Gonchar-Rakhmanov theorem. Here for  $j \geq j_0$  we have

$$I_{G_j}(\omega_{K_{T_j}, T_j}) \leq I_{G_j}(\omega_{K_{T_j}, T})$$

by the definition of the Green equilibrium measure  $\omega_{K_{T_j}, T_j}$ , and clearly  $g_{G_j}(z, t) \leq g_G(z, t) + \eta_j$ ,  $t \in K$  and  $K_{T_j} \subseteq K$  imply

$$I_{G_j}(\omega_{K_{T_j}, T}) \leq I_G(\omega_{K_{T_j}, T}) + \eta_j.$$

Finally, by the fact that  $K_T$  is the set of minimal condenser capacity for  $G$ , so it maximizes the energies  $I_G(\omega_{K_S, T})$  for all  $S \subset G$ , it follows that

$$I_G(\omega_{K_{T_j}, T}) \leq I_G(\omega_{K_T, T}).$$

Putting all these together we get

$$\liminf_{n \rightarrow \infty} \rho_{n,1}^{1/2n}(f, O) \geq \exp(-I_G(\omega_{K_T, T})) e^{-\eta_j} = \exp\left(-\frac{1}{\text{cap}(K_T, T)}\right) e^{-\eta_j},$$

which proves (2) if we let  $j \rightarrow \infty$ . ■

The proof above used the following two quite plausible facts.

**Lemma 2** *There is a compact set  $K \subset G$  and a  $j_0$  such that  $K_{T_j} \subset K$  for  $j \geq j_0$ .*

**Lemma 3** *For  $z, t \in K$  we have  $g_{G_j}(z, t) \leq g_G(z, t) + \eta_j$  where  $\eta_j \rightarrow 0$ .*

**Proof of Lemma 2.** Let  $H_a = \{z \mid \Re z > a\}$ , and fix a neighborhood  $S$  around  $T$  to which  $f$  has a single-valued analytic continuation.

Assume to the contrary that there is a sequence of points  $P_j \in K_{T_j}$ ,  $j = 1, 2, \dots$ , such that

$$\liminf_{j \rightarrow \infty} \text{dist}(P_j, \overline{\mathbb{C}} \setminus G) = 0.$$

We may assume that here the liminf is actually a limit and  $P_j \rightarrow P \in T$  (select a subsequence). Select a  $\tilde{P}_j \in T_j$  with  $\text{dist}(P_j, \tilde{P}_j) \rightarrow 0$ . Fix a  $z_0 \in G$  and let  $\varphi^*$ ,  $\varphi_j^*$  be the conformal maps that map the unit disk onto  $G$ ,  $G_j$  such that  $\varphi^*(0) = \varphi_j^*(0) = z_0$  and  $(\varphi^*)'(0) > 0$ ,  $(\varphi_j^*)'(0) > 0$ . It is known (see e.g. [3, Theorem 6.12 and Exercise 6.3/4]) that  $\varphi_j^* \rightarrow \varphi^*$  uniformly on the closed unit disk, therefore  $(\varphi_j^*)^{-1}(P_j) \rightarrow (\varphi^*)^{-1}(P)$ ,  $(\varphi_j^*)^{-1}(\tilde{P}_j) \rightarrow (\varphi^*)^{-1}(P)$ . Combine these with some fixed mapping of the unit disk onto the right-half plane  $H_0$  to deduce the following: if  $\varphi_j$ ,  $\varphi$  are conformal maps of  $G_j$ ,  $G$  onto  $H_0$  such that  $\varphi_j(z_0) = \varphi(z_0) = 1$ ,  $\varphi_j(\tilde{P}_j) = 0$ ,  $\varphi(P) = 0$ , then  $\varphi_j \rightarrow \varphi$  uniformly on compact subsets of  $G$  and  $\varphi_j(P_j) \rightarrow \varphi(P) = 0$ . Therefore, there is an  $a > 0$  such that  $\varphi_j(E_f) \subset \overline{H_a}$  for all large  $j$  and at the same time  $\varphi_j(P_j) \notin \overline{H_a}$ . Hence, if  $B_j := \varphi_j(K_{T_j})$ , then

$$B_j = \varphi_j(K_{T_j}) \not\subset \overline{H_a} \quad \text{for } j \geq j_0 \quad (3)$$

with some  $j_0$ . We may also assume  $a > 0$  to be so small and  $j_0$  so large that  $\varphi_j(G \setminus S) \subset H_a$  for  $j \geq j_0$  (note that  $\varphi(G \setminus S)$  is a compact subset of  $H_0$ ). Fix a  $j \geq j_0$ , and with this  $j$  we get a contradiction as follows.

Consider the mapping

$$z = x + iy \rightarrow z' = \max(x, a) + iy$$

(the projection onto  $\overline{H_a}$ ) and set  $B'_j = \{z' \mid z \in B_j\}$ . Then

$$g_{H_0}(z, w) = \log \left| \frac{z + \overline{w}}{z - w} \right| \leq \log \left| \frac{z' + \overline{w'}}{z' - w'} \right| = g_{H_0}(z', w') \quad (4)$$

(just note that the imaginary parts are the same, while the real parts increase resp. decrease when we go from  $z + \overline{w}$  resp.  $z - w$  to  $z' + \overline{w'}$  resp.  $z' - w'$ ).

We need

**Lemma 4** *There is a Borel-mapping  $\Phi : B'_j \rightarrow B_j$  such that  $\Phi(x)' = x$  for all  $x \in B'_j$ . For every Borel-measure  $\mu$  on  $B'_j$  this generates a Borel-measure  $\nu$  on  $B_j$  via  $\nu(E) = \mu(\Phi^{-1}[E])$  for all Borel-sets  $E \subset B_j$  (here  $\Phi^{-1}[E]$  is the complete inverse image of  $E$ ) such that*

$$\int \log \left| \frac{z + \overline{w}}{z - w} \right| d\nu(z) d\nu(w) = \int \log \left| \frac{\Phi(u) + \overline{\Phi(v)}}{\Phi(u) - \Phi(v)} \right| d\mu(u) d\mu(v).$$

With this lemma at hand we continue the proof of Lemma 2. We have

$$\begin{aligned} I_{H_0}(\nu) &= \int \log \left| \frac{z + \bar{w}}{z - w} \right| d\nu(z) d\nu(w) = \int \log \left| \frac{\Phi(u) + \overline{\Phi(v)}}{\Phi(u) - \overline{\Phi(v)}} \right| d\mu(u) d\mu(v) \\ &\leq \int \log \left| \frac{u + \bar{v}}{u - v} \right| d\mu(u) d\mu(v) = I_{H_0}(\mu), \end{aligned}$$

where, at the second inequality, we used (4).

Let  $\Omega_j$  be the unbounded component of  $\overline{\mathbf{C}} \setminus B'_j$  and  $\text{Pc}(B'_j) : \overline{\mathbf{C}} \setminus \Omega_j$  be the so called polynomial convex hull of  $B'_j$ . Next we show that  $\text{Pc}(B'_j)$  is an admissible set for the function  $F := f(\varphi_j^{-1})$  in  $H_0$ . To see this let  $\Gamma$  be a polygonal curve in  $\Omega_j \cap H_0$  starting and ending at the origin, i.e.  $\Gamma$  is a closed curve that lies in the right-half plane  $H_0$  except for the point  $0 \in \Gamma$ , and  $\Gamma$  does not intersect  $\text{Pc}(B'_j)$ . Let  $F^*$  be the continuation of  $F$  along (a neighborhood of)  $\Gamma$  as we traverse  $\Gamma$  once from 0 to 0. We need to show that after traversing  $\Gamma$  we get back to the same function element, i.e.  $F^* = F$  in a neighborhood of the origin.

By assumption,  $F$  has a continuation to the strip  $H_0 \setminus \overline{H_a}$  which we denote by  $F_0$ . Also, by the assumption on  $K_{T_j}$ ,  $F$  has a single-valued continuation  $F_1$  to the set  $\overline{\mathbf{C}} \setminus B_j$ . Note that necessarily  $F_1 = F_0$  on the set  $(H_0 \setminus \overline{H_a}) \setminus B_j$ . We may assume that  $\Gamma$  does not contain a vertical segment, and for some small  $\varepsilon > 0$  let  $Q_1, \dots, Q_m$  be the points of  $\Gamma$  (in the order of the traverse) that lie on the line  $\Re z = a - \varepsilon$ . Let here  $\varepsilon > 0$  be so small that  $\overline{H_{a-\varepsilon}} \cap \Gamma \cap B_j = \emptyset$  (there is such an  $\varepsilon > 0$  since the preceding relation is true with  $\varepsilon = 0$ ). Then the points  $Q_1, \dots, Q_m$  lie outside  $B_j$ , and let  $D_k \subset H_0 \setminus \overline{H_a}$  be a small disk around  $Q_k$  not intersecting  $B_j$ . Note that, as we have just remarked,  $F_1 \equiv F_0$  on all these disks. Now we can easily prove by induction that  $F^* \equiv F_0 \equiv F_1$  on each  $D_k$ . Indeed, for  $k = 1$  the equality  $F^* \equiv F_0$  is true by the monodromy theorem in  $H_0 \setminus \overline{H_a}$ . Now assume that we already know the claim for  $D_k$ . The portion  $\Gamma_k$  of  $\Gamma$  in between the points  $Q_k$  and  $Q_{k+1}$  either lies in  $H_{a-\varepsilon}$  or in  $H_0 \setminus \overline{H_{a-\varepsilon}}$ . In the former case the continuation of  $F^* \equiv F_1$  along  $\Gamma_k$  is the same as  $F_1$  (note that  $\Gamma_k$  does not intersect  $B_j$ ), hence on  $D_{k+1}$  we have  $F^* \equiv F_1 \equiv F_0$ . On the other hand, if  $\Gamma_k$  lies in  $H_0 \setminus \overline{H_{a-\varepsilon}}$ , then the continuation  $F^* \equiv F_0$  along  $\Gamma_k$  is the same as  $F_0$  by the monodromy theorem in  $H_0 \setminus \overline{H_a}$ , hence in this case we have again  $F^* \equiv F_0 \equiv F_1$  on  $D_{k+1}$ , by which the induction has been carried out. Another application of the monodromy theorem along the portion of  $\Gamma$  from  $Q_m$  to 0 shows that, indeed, as we get back to the origin, with  $F^*$  we arrive back to the same function element  $F_0$  that we started with.

We have thus shown that  $\text{Pc}(B'_j)$  is an admissible set for  $f(\varphi_j^{-1})$  in  $H_0$ , hence  $K_j^* := \varphi_j^{-1}(\text{Pc}(B'_j))$  is an admissible set for  $f$  in  $G_j$ , and  $K_j^*$  lies in  $\varphi_j^{-1}(\overline{H_a})$ . If we define the measure  $\mu$  on  $B'_j$  by stipulating  $\mu(E) = \omega_{K_j^*, T_j}(\varphi_j^{-1}(E))$  for all Borel-sets  $E \subset B'_j$ ,  $\nu$  is the associated measure via Lemma 4, and finally  $\omega$  is the measure defined by  $\omega(E) = \nu(\varphi_j(E))$ , then  $\omega$  is supported on  $K_{T_j}$ , and has total mass 1 because  $\omega_{K_j^*, T_j}$  is supported on the outer boundary of  $K_j^*$  (see

[1, Sec. 7.1.3]), and hence the interior of  $\text{Pc}(B'_j)$  has zero  $\mu$ -measure. Now we obtain from Lemma 4 and from the conformal invariance of the Green's function

$$I_{G_j}(\omega) = I_{H_0}(\nu) \leq I_{H_0}(\mu) = I_{G_j}(\omega_{K_j^*, T_j}),$$

which implies

$$I_{G_j}(K_{T_j}) \leq I_{G_j}(\omega) \leq I_{G_j}(\omega_{K_j^*, T_j}) = I_{G_j}(K_j^*).$$

Therefore, by the extremality of  $K_{T_j}$  for  $G_j$ , we must have equality here, and then, by the definition of the set  $K_{T_j}$  of minimal condenser capacity, we must have  $K_{T_j} \subseteq K_j^* \subseteq \varphi_j^{-1}(\overline{H_a})$ , which contradicts (3).

This contradiction proves the claim in Lemma 3. ■

**Proof of Lemma 4.** In this proof we use the special structure of the sets  $K_{T_j}$  described before Theorem 1.

For  $z \in H_a \cap B'_j = H_a \cap B_j$  set  $\Phi(z) = z$ , and for  $z = a + iy \in B'_j \cap \{x = a\}$  let  $\Phi(z) = x(z) + iy \in B_j$  be the point in  $B_j$  with the smallest possible  $x$ -coordinate  $x(z)$ . In the latter case  $\Phi(z) \in H_0 \setminus H_a$ , and clearly  $\Phi(z)' = z$  for all  $z \in B'_j$ , so it is left to verify that  $\Phi$  is a Borel-map. To this it is sufficient to show that for a dense set of  $B < C$  and for a dense set of  $A \in [0, a)$  the inverse image  $\Phi^{-1}[R]$  is a Borel-set, where  $R = [0, A] \times [B, C]$ . To get this note that if the boundary of  $R$  does not contain either endpoints of an open analytic arc  $\gamma \subset B_j$  which is not a vertical or horizontal segment, then  $\partial R \cap \gamma$  is a finite set. Therefore, in this case  $R \cap \gamma$  consists of a finite number of analytic arcs, and hence  $(R \cap \gamma)'$  is the union of finitely many closed segments on  $\partial H_a$ . Since  $B_j$  is the union of finitely many points and finitely many open analytic arcs, it follows that  $(R \cap B_j)'$  consists of a finite number of closed segments on  $\partial H_a$  provided  $\partial R$  does not contain any of the endpoints of these arcs. Since  $\Phi^{-1}[R] = (R \cap B_j)'$ , we are done. ■

**Proof of Lemma 3.** Let  $\varepsilon > 0$  and select a Jordan curve  $\sigma$  separating  $K$  and  $T$  so that  $g_G(z, \tau) \leq \varepsilon$  for all  $z \in \sigma$ ,  $\tau \in K$ . (There is such a  $\sigma$ : if  $\sigma_1$  separates  $T$  and  $K$  then  $g_G(z, t) \leq M$  for all  $z \in \sigma_1$ ,  $t \in K$  with some constant  $M$ . Map now the strip in between  $T$  and  $\sigma_1$  into a ring  $R = \{1 \leq |z| \leq r\}$  by a conformal map  $\varphi$ . Then the three-circle-theorem gives

$$g_G(z, t) \leq M \frac{\log |\varphi(z)|}{\log r},$$

so

$$\sigma = \left\{ z \left| |\varphi(z)| = \exp \left( \varepsilon \frac{\log r}{M} \right) \right. \right\}$$

suffices for small  $\varepsilon$ .) Now  $g_{G_j}(z, \tau) \searrow g_G(z, \tau)$  for all  $z \in \sigma$  and  $\tau \in K$ , so, by Dini's theorem, this convergence is uniform in  $z \in \sigma$  for all fixed  $\tau \in K$ , i.e.  $g_{G_j}(\zeta, \tau) < 2\varepsilon$  for  $j \geq j_\tau$  and all  $\zeta \in \sigma$ ,  $\tau \in K$ . Then  $g_{G_{j_\tau}}(z, t) < 2\varepsilon$  is true for all  $z \in \sigma$  and  $t \in K$  lying sufficiently close to some  $\zeta \in \sigma$  and  $\tau \in K$ , and by compactness of  $\sigma$  we get  $g_{G_{j_\tau}}(z, t) < 2\varepsilon$  for all  $z \in \sigma$  and  $t$  lying sufficiently close to  $\tau$ . Then for the same values  $g_{G_j}(z, t) < 2\varepsilon$  automatically holds for  $j \geq j_\tau$  because the Green's function  $g_{G_j}$  decrease. Finally, by the compactness of  $K$  there is a  $j_0$  such that this inequality holds for all  $z \in \sigma$ ,  $t \in K$  and  $j \geq j_0$ .

As a consequence,  $g_{G_j}(z, t) - g_G(z, t) \leq 2\varepsilon$  for  $z \in \sigma$ ,  $t \in K$  and  $j \geq j_0$ , and then, by the maximum theorem, this inequality persists for all  $t \in K$  and  $z$  lying inside  $\sigma$ . ■

## References

- [1] L. Baratchart, H. Stahl and M. Yattselev, Weighted extremal domains and best rational approximation, *Advances in Math.*, **229**(2012), 357-407.
- [2] A.A. Gonchar and E.A. Rakhmanov, Equilibrium distributions and the degree of rational approximation of analytic functions, (Russian) *Mat. Sb.*, **134**(176)(1987), 306-352; English transl. in *Math. USSR Sb.*, **62**(1989), 305-348.
- [3] Ch. Pommerenke, *Boundary Behavior of Conformal Mappings*, Grundlehren der mathematischen Wissenschaften, **299**, Springer Verlag, Berlin, Heidelberg New York, 1992.

Bolyai Institute  
MTA-SZTE Analysis and Stochastics Research Group  
University of Szeged  
Szeged  
Aradi v. tere 1, 6720, Hungary  
and  
Department of Mathematics and Statistics  
University of South Florida  
4202 E. Fowler Ave, CMC342  
Tampa, FL 33620-5700, USA  
*totik@mail.usf.edu*