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SPECTRAL BEHAVIOUR OF QUASIANALYTIC CONTRACTIONS

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Abstract. We pose, and answer partially, questions in connection with the spectral behaviour of quasianalytic contractions. These problems are related to the hyperinvariant subspace problem in the class of asymptotically non-vanishing contractions.

Let $\mathcal{H}$ be a separable, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of bounded, linear operators acting on $\mathcal{H}$. For any $T \in \mathcal{L}(\mathcal{H})$, $\operatorname{Lat} T$ means the lattice of all invariant subspaces of T, while $\operatorname{Hlat} T$ stands for the lattice of all hyperinvariant subspaces of T. We recall that a subspace (i.e. closed linear manifold) $\mathcal{M}$ is hyperinvariant for T, if it is invariant for every operator C in the commutant $\{T\}' = \{A \in \mathcal{L}(\mathcal{H}) : AT = TA\}$ of T, i.e. $C\mathcal{M} \subset \mathcal{M}$. The Invariant Subspace Problem (ISP) asks whether $\operatorname{Lat} T$ is non-trivial, i.e. $\operatorname{Lat} T \neq \{\{0\}, \mathcal{H}\}$, for every $T \in \mathcal{L}(\mathcal{H})$. The Hyperinvariant Subspace Problem (HSP) asks whether $\operatorname{Hlat} T$ is non-trivial, provided T is not a scalar multiple of the identity operator I. These problems are arguably the most challenging open questions in operator theory. They can be reduced to the case when T is an absolutely continuous (a.c.) contraction, that is $\|T\| \leq 1$ and T splits into the orthogonal sum $T = U \oplus T_c$, where U is a unitary operator whose spectral measure is a.c. with respect to the normalized Lebesgue measure m on the unit circle $\mathbb{T}$, and T_c is completely non-unitary, i.e. the restriction of T_c to any non-zero invariant subspace is not unitary.

We focus on the particular case when the a.c. contraction T is asymptotically non-vanishing (a.n.v.), that is when $\lim_{n\to\infty} \|T^n h\| > 0$ holds for some vector $h \in \mathcal{H}$. Then a non-trivial unitary asymptote can be associated with T, which is a pair (X, V), where V is an a.c. unitary operator acting on a non-zero Hilbert space $\mathcal{K}$, X is a bounded linear transformation from $\mathcal{H}$ into $\mathcal{K}$, $\|Xh\| = \lim_{n\to\infty} \|T^n h\|$ holds for all $h \in \mathcal{H}$, $XT = XV$ and $\vee_{n=1}^{\infty} V^{-n} X\mathcal{H} = \mathcal{K}$. The pair (X, V) is unique up to isomorphism; for details see Chapter IX in [NFBK], and [Ker13].

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The residual set $\omega(T)$ of T is the measurable support of the spectral measure E of V, which means that $E(\alpha) = 0$ if and only if $m(\alpha \cap \omega(T)) = 0$. Clearly, $m(\omega(T)) > 0$.

Another important spectral invariant is connected with the Sz.-Nagy–Foias functional calculus $\Phi_T \colon H^\infty \to \mathcal{L}(\mathcal{H})$ for T. For the definition and basic properties of Φ_T see Chapter III in [NFBK]. We note that Φ_T is monotone, that is $f_2 \overset{a}{\prec} f_1$ (i.e. $|f_2(z)| \le |f_1(z)|$ for all z in the open unit disc $\mathbb{D}$) implies $f_2(T) \overset{a}{\prec} f_1(T)$ (i.e. $\|f_2(T)h\| \le \|f_1(T)h\|$ for all $h \in \mathcal{H}$). Taking a decreasing sequence $F = \{f_n\}_{n=1}^\infty$ in H^∞, let us consider the limit function $\varphi_F(\zeta) = \lim_{n\to\infty} |f_n(\zeta)|$ defined for a.e. $\zeta \in \mathbb{T}$, the measurable set $N_F = \{\zeta \in \mathbb{T} : \varphi_F(\zeta) > 0\}$, and the hyperinvariant subspace $\mathcal{H}_0(T,F) = \{h \in \mathcal{H} : \lim_{n\to\infty} \|f_n(T)h\| = 0\}$. The quasianalytic spectral set $\pi(T)$ of T is the largest measurable set on $\mathbb{T}$ with the property that $\mathcal{H}_0(T,F) = \{0\}$ whenever $m(N_F \cap \pi(T)) > 0$. The a.n.v. contraction T is called quasianalytic, if $\pi(T) = \omega(T)$. (Note that $\pi(T) \subset \omega(T)$ always holds.) The class of quasianalytic contractions was introduced and studied in the papers [Ker01], [Ker11], [Ker13], [KT12] and [KSz]. Though (ISP) and (HSP) are open for a.n.v. contractions, the following result of [Ker01] shows that these questions are settled in the non-quasianalytic case.

Proposition 1. *If T is an a.n.v. contraction which is not quasianalytic, then T has a non-trivial hyperinvariant subspace.*

By this proposition, it becomes crucial to determine the spectral behaviour of quasianalytic contractions. Namely, if an a.n.v. contraction T does not meet this behaviour, then T is not quasianalytic, and so $\operatorname{Hlat} T$ is non-trivial.

If the contraction T is quasianalytic, then it is of class C_{10}, which means that $\lim_{n\to\infty} \|T^{*n}h\| = 0 < \lim_{n\to\infty} \|T^n h\|$ holds for all non-zero $h \in \mathcal{H}$; see Theorem 8 in [KSz]. Under this asymptotic behaviour, there is a connection between the spectrum $\sigma(T)$ of T and the spectrum $\sigma(V)$ of V. First we note that $\sigma(V)$ is the essential support of $\omega(T)$: $\sigma(V) = \operatorname{es}(\omega(T))$, which is the complement of the largest open set $\mathcal{O}$ on $\mathbb{T}$ with the property $m(\mathcal{O} \cap \omega(T)) = 0$. It can be easily proved that $\sigma(V)$ is neatly contained in $\sigma(T)$, that is $\sigma(V) \subset \sigma(T)$ and $m(\sigma(V) \cap \sigma') > 0$ holds for every non-empty closed subset σ' of $\sigma(T)$ with the property that $\sigma(T) \setminus \sigma'$ is also closed. More importantly, this is the only constraint on the spectrum of a C_{10}-contraction, even in the cyclic case, that is when $\vee_{n=0}^\infty T^n h = \mathcal{H}$ holds with some vector $h \in \mathcal{H}$; see Chapter IX in [NFBK]. Are there any other constraints if T is quasianalytic? More precisely, we pose the following problem.

Question 1. Given a measurable set $\omega_0 \subset \mathbb{T}$ of positive measure and a compact subset σ of the closed unit disc $\mathbb{D}^-$ such that $\operatorname{es}(\omega_0)$ is neatly contained in σ, does a quasianalytic contraction T exist with the properties $\sigma(T) = \sigma$ and $\omega(T) = \omega_0$?

In the C_{10} class the construction starts by producing a C_{10}-contraction T satisfying the conditions $\omega(T) = \omega_0$ and $\sigma(T) = \operatorname{es}(\omega_0)$, as a restriction of a bilateral weighted shift W to an appropriately chosen invariant subspace. However, if $m(\mathbb{T}\setminus\omega_0) > 0$ then W is necessarily non-quasianalytic; otherwise $\pi(T) = \pi(W) = \mathbb{T}$ would happen. Furthermore, the coincidence $\sigma(T) = \operatorname{es}(\omega(T))$ is ensured by the condition $\sum_{n=1}^\infty n^p \|T^{-n}\| < \infty$ with some integer p. However, this relation implies the existence of an operator $C \in \{T\}'$ and a non-zero continuous function f on $\mathbb{T}$ such that $XC = f(V)X$ and the set $\{\zeta \in \omega(T) : f(\zeta) = 0\}$ has positive measure, less than $m(\omega(T))$; see Lemma IX.2.11 and its proof in [NFBK]. Hence we can present a non-zero vector $h \in \mathcal{H}$ such that Xh is not cyclic for the commutant $\{V\}'$, which is impossible if T is quasianalytic; see Theorem 16 in [Ker01]. Therefore, we have to find another approach to provide a quasianalytic contraction T, if it exists at all, such that its spectrum $\sigma(T)$ is a proper subset of $\mathbb{T}$. First of all, the following simpler question should be answered.

Question 2. Do we have for every closed arc J of positive measure on $\mathbb{T}$ and for every $c > 0$ a quasianalytic contraction T satisfying the conditions $\sigma(T) = \pi(T) = J$ and $\|T^{-1}\| > c$?

We know that the a.c. contraction T has shift-type invariant subspaces if $\omega(T) = \mathbb{T}$. Namely, $\mathcal{H} = \vee \operatorname{Lat_s} T$, where $\operatorname{Lat_s} T$ consists of those invariant subspaces $\mathcal{M}$ where $T|\mathcal{M}$ is similar to the simple unilateral shift $S \in \mathcal{L}(H^2)$, $Sf = \chi f$ $(\chi(\zeta) = \zeta)$; see Theorem IX.3.6 in [NFBK]. Any quasianalytic contraction can be related to such a contraction having a rich invariant subspace lattice.

Theorem 2. *For every quasianalytic contraction T_1, there exists a quasianalytic contraction T_2 with $\pi(T_2) = \mathbb{T}$ such that $\{T_2\}' \supset \{T_1\}'$ and so $\operatorname{Hlat} T_2 \subset \operatorname{Hlat} T_1$.*

Proof. By Theorem 3 of [KT12] there exist a compact set $K \subset \pi(T_1)$ and a continuous function f on $\mathbb{D}^-$ such that f is analytic (even univalent) on $\mathbb{D}$, $f^{-1}(\mathbb{T}) = K$ and $m(f(\alpha)) = 0$ for every Borel subset α of K of zero measure. Then $\pi(T_2) = \mathbb{T}$ holds for the a.c. contraction $T_2 = f(T_1)$ by Corollary 2.5 of [Ker11] (see also Lemma 5 in [KT12]). It is obvious that $\{T_2\}' \supset \{T_1\}'$. □

Therefore, the (HSP) for a.n.v. contractions can be reduced to the case, when T is quasianalytic and $\pi(T) = \mathbb{T}$. Clearly, $\mathbb{T}$ is neatly contained in $\sigma(T)$ exactly when $\sigma(T)$ is connected. Thus, in this particular class, Question 1 has the following modified form.

Question 3. Given a connected, compact subset σ of $\mathbb{D}^-$, containing $\mathbb{T}$, does there exist a quasianalytic contraction T satisfying the conditions $\sigma(T) = \sigma$ and $\pi(T) = \mathbb{T}$?

The preceding two questions are related. Let $\mathbb{D}_+ := \{z \in \mathbb{D} : \operatorname{Im} z > 0\}$, $\mathbb{T}_+ := \{\zeta \in \mathbb{T} : \operatorname{Im} \zeta \geq 0\}$, and for any $K \subset \mathbb{C}$ let $K^2 := \{z^2 : z \in K\}$.

Theorem 3. *A positive answer for Question 2 implies an affirmative answer for Question 3 in the special case, when $\sigma = K^2$ with a connected, compact set K such that $\mathbb{T}_+ \subset K \subset \mathbb{D}_+^-$.*

Proof. Let K be a connected, compact set such that $\mathbb{T}_+ \subset K \subset \mathbb{D}_+^-$ and $K^2 = \sigma$. We apply the technique used in Section IX.2 of [NFBK] to obtain a quasianalytic contraction $\widetilde{T}$ satisfying the conditions $\sigma(\widetilde{T}) = K$ and $\pi(\widetilde{T}) = \mathbb{T}_+$.

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a dense sequence in K. For every $n \in \mathbb{N}$, let us consider the connected, open set $\Omega_n = \{z \in \mathbb{C} : \operatorname{dist}(z, K) < 1/n\}$, and select a point $\lambda_n' \in \mathbb{D} \cap \Omega_n$ so that $|\lambda_n - \lambda_n'| < 1/(2n)$. Let $\Gamma_n \subset (\Omega_n \cap \mathbb{D}) \cup \{-1, 1\}$ be a simple rectifiable curve, with endpoints -1 and 1, such that the simply connected domain G_n bounded by $\mathbb{T}_+ \cup \Gamma_n$ is contained in Ω_n and $\lambda_n' \in G_n$. There exists a conformal mapping $f_n \colon \mathbb{D} \to G_n$, having continuous extension onto $\mathbb{D}^-$, such that $f_n(0) = \lambda_n'$. Let us consider the closed arc $J_n = f_n^{-1}(\mathbb{T}_+)$. By our assumption there exists a quasianalytic contraction $T_n \in \mathcal{L}(\mathcal{H}_n)$ such that $\sigma(T_n) = \pi(T_n) = J_n$

and $\|T_n^{-1}\| > n$. Then $\widetilde{T}_n = f_n(T_n)$ is also a quasianalytic contraction with the properties $\sigma(\widetilde{T}_n) = \pi(\widetilde{T}_n) = \mathbb{T}_+$; see Proposition IX.2.4 in [NFBK] and Corollary 2.5 in [Ker11].

Setting $\widetilde{T} = \sum_{n=1}^{\infty} \oplus \widetilde{T}_n$, we may verify that $\sigma(\widetilde{T}) = K$ and $\pi(\widetilde{T}) = \mathbb{T}_+$. Indeed, for every $n \in \mathbb{N}$, there exists a unit vector $e_n \in \mathcal{H}_n$ such that $\|T_n e_n\| < 1/n$. Since $f_n(z) - \lambda_n^{'} = z g_n(z)$, where $\|g_n\|_\infty \leq 2$, it follows that $\|\widetilde{T}_n e_n - \lambda_n^{'} e_n\| = \|g_n(T_n) T_n e_n\| \leq 2/n$. Taking into account that each $\lambda \in K$ is a cluster point of the sequence $\{\lambda_n^{'}\}_{n=1}^{\infty}$, we infer that $K \subset \sigma(\widetilde{T})$. On the other hand, if $\lambda \notin K$ then $\delta_0 = \operatorname{dist}(\lambda, G_{n_0}) > 0$ for some $n_0 \in \mathbb{N}$. Thus, $\|(\widetilde{T}_n - \lambda I)^{-1}\| \leq \|1/(f_n - \lambda)\|_\infty \leq 1/\delta_0$ holds whenever $n \geq n_0$. Since $\lambda \notin \sigma(\widetilde{T}_n)$ for all n, it follows that $\lambda \notin \sigma(\widetilde{T})$. Therefore, $\sigma(\widetilde{T}) = K$. Finally, $\pi(\widetilde{T}) = \cap_{n=1}^{\infty} \pi(\widetilde{T}_n) = \mathbb{T}_+$ is obvious. Now, $T = \widetilde{T}^2$ is a quasianalytic contraction satisfying the conditions $\sigma(T) = K^2 = \sigma$ and $\pi(T) = \mathbb{T}_+^2 = \mathbb{T}$. □

Remark 4. Not every connected, compact set $\mathbb{T} \subset \sigma \subset \mathbb{D}^-$ can be represented as $\sigma = K^2$ with a connected, compact set $\mathbb{T}_+ \subset K \subset \mathbb{D}_+^-$. Indeed, let $\rho\colon [0,1) \to [0,\infty)$ and $\varphi\colon [0,1) \to [0,\infty)$ be strictly increasing, continuous functions satisfying the conditions $\rho(0) = \varphi(0) = 0$, $\lim_{t\to 1-} \rho(t) = 1$, $\lim_{t\to 1-} \varphi(t) = \infty$, and take the connected compact set $\sigma = \mathbb{T} \cup \{\rho(t) e^{i\varphi(t)} : t \in [0,1)\}$.

The a.n.v. contraction T is called asymptotically cyclic if its unitary asymptote V is cyclic. In that case the commutant $\{T\}'$ of T can be identified with a subalgebra $\mathcal{F}(T)$ of $L^\infty(\mathbb{T})$, called the functional commutant of T. See [Ker11] and [KSz] for the study of $\mathcal{F}(T)$. Clearly, the (ISP) can be reduced to the case when T is asymptotically cyclic. Therefore, it is important to know the spectral behaviour in this setting, too. We recall that $\mathcal{L}_0(\mathcal{H})$ stands for the set of asymptotically cyclic, quasianalytic contractions on $\mathcal{H}$, while $\mathcal{L}_1(\mathcal{H}) = \{T \in \mathcal{L}_0(\mathcal{H}) : \pi(T) = \mathbb{T}\}$. The same commutants arise in these two classes. Namely, the following statement was proved in [KT12].

Proposition 5. *For every $T_0 \in \mathcal{L}_0(\mathcal{H})$, there exists $T_1 \in \mathcal{L}_1(\mathcal{H})$ such that $\{T_0\}' = \{T_1\}'$ and so* $\operatorname{Hlat} T_0 = \operatorname{Hlat} T_1$.

This proposition makes it especially important to answer the following question.

Question 4. What are the possible spectra of the contractions belonging to $\mathcal{L}_1(\mathcal{H})$?

We know that for every $0 \leq \delta < 1$ there is a contraction $T_\delta \in \mathcal{L}_1(\mathcal{H})$ such that $\sigma(T_\delta) = \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$; see Example 5.8 in [Ker11] and Example 24 in [KSz]. Now we show that the spectrum can be the unit circle $\mathbb{T}$ too. The following theorem also gives a positive answer for Question 2 in the special case, when the arc J is the whole circle $\mathbb{T}$.

Theorem 6. *For every $c > 1$, there is a contraction $T \in \mathcal{L}_1(\mathcal{H})$ such that $\sigma(T) = \mathbb{T}$ and $\|T^{-1}\| \geq c$.*

Proof. We present a bilateral weighted shift with the prescribed properties. (For their study see [Shi74] and Section 7 in [KSz].) Let $\beta\colon \mathbb{Z} \to [1,\infty)$ be a sequence such that $\beta(n) = 1$ for all $n \geq 0$ and $\beta(-n) = e^{\varphi(n)}$ for $n \in \mathbb{N}$, where the increasing sequence $\varphi\colon \mathbb{N} \to [1,\infty)$ with $\lim_{n\to\infty} \varphi(n) = \infty$ is specified later. Let $\sum_{n=-\infty}^{\infty} \widehat{f}(n) \chi^n$ stand for the Fourier series of the function $f \in L^2(\mathbb{T})$, where

$\chi(\zeta) = \zeta \quad (\zeta \in \mathbb{T})$. We consider the Hilbert space

$$L^2(\beta) = \{f \in L^2(\mathbb{T}) : \|f\|_\beta^2 := \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \beta(n)^2 < \infty\}$$

and the asymptotically cyclic C_{10}-contraction $T_\beta \in \mathcal{L}(L^2(\beta))$, defined by $T_\beta f = \chi f$.

Now we specify the sequence φ so that T_β is quasianalytic with $\sigma(T_\beta) = \mathbb{T}$. Select a strictly decreasing sequence $\{q_k\}_{k=1}^\infty$ of real numbers in $(0,1)$ such that $\lim_{k\to\infty} q_k = 0$, and then select a strictly increasing sequence $\{p_k\}_{k=1}^\infty$ of positive integers satisfying the conditions $p_1 = 1$ and $\sum_{n=p_k+1}^{p_{k+1}} \frac{1}{n} \geq 1/q_k$ for every $k \in \mathbb{N}$. Setting $c > 1$, let $\varphi(1) := c$, and for any $k \in \mathbb{N}$ and $p_k < n \leq p_{k+1}$ let $\varphi(n) := \varphi(p_k) + (n - p_k)q_k$. Clearly, φ is increasing. It can be verified by induction that $\varphi(p_k) \geq p_k q_k$ holds for every $k \in \mathbb{N}$. Thus $\varphi(n) \geq n q_k$ if $p_k < n \leq p_{k+1}$, whence

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\log \beta(-n)}{n^2} &= \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} = c + \sum_{k=1}^{\infty} \sum_{n=p_k+1}^{p_{k+1}} \frac{\varphi(n)}{n^2} \\ &\geq \sum_{k=1}^{\infty} q_k \sum_{n=p_k+1}^{p_{k+1}} \frac{1}{n} \geq \sum_{k=1}^{\infty} 1 = \infty\end{aligned}$$

follows. We conclude that T_β is quasianalytic, and so $T_\beta \in \mathcal{L}_1(L^2(\beta))$; see Proposition 31 in [KSz].

Since the sequence $\{q_k\}_{k=1}^\infty$ is decreasing, it follows that

$$\left\|T_\beta^{-n}\right\| = \sup\left\{\frac{\beta(j-n)}{\beta(j)} : j \in \mathbb{Z}\right\} = e^{\varphi(n)} \quad \text{for all } n \in \mathbb{N}.$$

In particular, we get $\|T_\beta^{-1}\| = e^{\varphi(1)} = e^c \geq c$. Furthermore, for every $k \in \mathbb{N}$ and $p_k < n \leq p_{k+1}$ we have

$$0 \leq \frac{\varphi(n)}{n} = \frac{\varphi(p_k)}{n} + \frac{n - p_k}{n} q_k \leq \frac{\varphi(p_k)}{p_k} + q_k.$$

Hence $\lim_{n\to\infty} \varphi(n)/n = 0$ holds, if $\lim_{k\to\infty} \varphi(p_k)/p_k = 0$. The inequality

$$\frac{1}{q_k} \leq \sum_{n=p_k+1}^{p_{k+1}} \frac{1}{n} \leq \int_{p_k}^{p_{k+1}} \frac{\mathrm{d}x}{x} = \ln \frac{p_{k+1}}{p_k}$$

yields that

$$\frac{p_k}{p_{k+1}} \leq e^{-\frac{1}{q_k}} \leq \frac{1}{2}.$$

Applying the recursive formula

$$\frac{\varphi(p_{k+1})}{p_{k+1}} = \frac{\varphi(p_k)}{p_k} \cdot \frac{p_k}{p_{k+1}} + \left(1 - \frac{p_k}{p_{k+1}}\right) q_k$$

we can check by induction that $\varphi(p_k)/p_k \leq 2c$ holds for all $k \in \mathbb{N}$. The previous inequalities imply that

$$\frac{\varphi(p_{k+1})}{p_{k+1}} \leq 2ce^{-\frac{1}{q_k}} + q_k,$$

whence $\lim_{k\to\infty} \varphi(p_k)/p_k = 0$ immediately follows. Therefore, the spectral radius of T_β^{-1} is

$$r\big(T_\beta^{-1}\big) = \lim_{n\to\infty} \big\|T_\beta^{-n}\big\|^{\frac{1}{n}} = \lim_{n\to\infty} e^{\varphi(n)/n} = 1.$$

In view of the circular symmetry of $\sigma(T_\beta)$ we obtain that $\sigma(T_\beta) = \mathbb{T}$. □

Relying on this statement we can provide contractions in $\mathcal{L}_1(\mathcal{H})$ with more sophisticated spectra.

Examples 7. (a) Given any $\delta \in (0,1)$, let us consider the domain $\Omega_\delta = \{z = re^{it} : \sqrt{\delta} < r < 1 \text{ and } 0 < t < \pi\}$. Let η_δ be a conformal mapping of $\mathbb{D}$ onto Ω_δ, and set $\vartheta_\delta = \eta_\delta^2$. If $T \in \mathcal{L}_1(\mathcal{H})$ is an operator with $\sigma(T) = \mathbb{T}$, then $T_\delta = \vartheta_\delta(T) \in \mathcal{L}_1(\mathcal{H})$ and $\sigma(T_\delta) = \mathbb{T} \cup \delta\mathbb{T} \cup [\delta, 1]$. Observe that $\mathbb{D} \setminus \sigma(T)$ is not connected.
(b) We recall that a domain $\Omega \subset \mathbb{C}$ is called a circular comb domain, if it is of the form $\Omega = \mathbb{D} \setminus \{r\zeta : \zeta \in H, \rho(\zeta) < r < 1\}$, where $H \subset \mathbb{T}$ is countable and $\rho\colon H \to (0,1)$. Let K be a Cantor-type compact set on $\mathbb{T}$ of positive measure. In view of Theorem 3 of [KT12] we know that there exists a compact set $\widetilde{K} \subset K$ and a conformal mapping f of $\mathbb{D}$ onto a circular comb domain Ω such that f can be continuously extended onto $\mathbb{D}^-$, $f^{-1}(\mathbb{T}) = \widetilde{K}$, and $m(f(\alpha)) = 0$ whenever $\alpha \subset \widetilde{K}$ is of measure zero. If $T \in \mathcal{L}_1(\mathcal{H})$ with $\sigma(T) = \mathbb{T}$, then the spectrum of $\widetilde{T} = f(T) \in \mathcal{L}_1(\mathcal{H})$ is $\sigma(\widetilde{T}) = \mathbb{T} \cup \{r\zeta : \zeta \in H, \rho(\zeta) \le r < 1\}$, where H is dense in $\mathbb{T}$.

Questions 1–4, in their full generality, remain open.

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