Uniform Spacing of Zeros of Orthogonal Polynomials for Locally Doubling Measures

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Abstract

Recently it has been shown, that if a weight has the doubling property on its support $[-1,1]$, then the zeros of the associated orthogonal polynomials are uniformly spaced: if $\theta_{m,j}$ and $\theta_{m,j+1}$ are the places in $[0,\pi]$, for which $\cos\theta_{m,j}$ and $\cos\theta_{m,j+1}$ is the j-th and the $j+1$-th zero of the m-th orthogonal polynomial, then $\theta_{m,j}-\theta_{m,j+1}\sim\frac{1}{m}$. In this paper it is shown, that this result is also true in a local sense: if a weight has the doubling property in an interval of its support, then uniform spacing of the zeros is true inside that interval. The result contains as special cases some theorems of Last and Simon on local zero spacing of orthogonal polynomials.

1 Results

Let μ be a measure with compact support on the real line. The m-th associated orthonormal polynomial of degree m is denoted by $p_m = p_m(\mu, x)$. For a long while it has been well known that its zeros are distinct, single, and lie in the convex hull of the support. In the literature a lot of articles have dealt with the zeros of orthogonal polynomials and their asymptotic distribution, see e.g. [10, Chapter VI], [2, Chapter 5], [1, Chapter 2], [7] or [8, Chapter 1,8]. For establishing the distribution of the zeros relatively weak assumptions are needed, see e.g. [9, Chapter 2], but finer questions like the spacing between neighbouring zeros need stronger conditions. A relatively mild property, namely the doubling property will be used in this work. The measure μ is called doubling on an interval $[a,b]$ if for some constant L we have $\mu(2I)\le L\mu(I)$ for all intervals $2I\subseteq[a,b]$, where $2I$ is the interval twice the length of I and with midpoint at the midpoint of I. When using this terminology we tacitly will always assume that μ is not identically zero on $[a,b]$, and then the doubling property easily implies that $[a,b]$ must be part of the support of μ. Recently G. Mastroianni and V. Totik [5] proved that if the support of μ is the interval $[-1,1]$ and μ has

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the doubling property on it, then the zeros array themselves fairly regularly, namely if $x_{m,j} = \cos\theta_{m,j}$ and $x_{m,j+1} = \cos\theta_{m,j+1}$ are adjacent zeros of the m-th orthonormal polynomial then

$$\frac{1}{A}\left(\frac{\sqrt{1-x_{j,m}^2}}{m}+\frac{1}{m^2}\right) \le x_{m,j+1}-x_{m,j} \le A\left(\frac{\sqrt{1-x_{j,m}^2}}{m}+\frac{1}{m^2}\right)$$

or, in another form,

$$\frac{1}{A}\frac{1}{m} \le \theta_{m,j}-\theta_{m,j+1} \le A\frac{1}{m}$$

with some constant A depending only on the doubling constant of μ. Observe that this implies that the distance between neighbouring zeros lying in a fixed closed subinterval of $(-1,1)$ is $\sim \frac{1}{m}$.

In this paper we prove that this regular spacing of the zeros holds inside every interval on which the measure is doubling, i.e. the aforementioned uniform spacing is actually a consequence of a local property of the measure.

Theorem 1. *Let μ be a measure with compact support on the real line and with the doubling property on $[a,b]$. Then for every $\delta > 0$ there exists a constant A independent of m such that*

$$\frac{1}{Am} \le x_{m,j+1}-x_{m,j} \le \frac{A}{m}, \qquad j = k, k+1, \dots, l-1, \tag{1}$$

where $x_{m,k} < x_{m,k+1} < \dots < x_{m,l}$ are the zeros of p_m in $[a+\delta, b-\delta]$.

Remark 1. The assumptions of the theorem imply that for large m there are zeros in $[a+\delta, b-\delta]$, and their number actually tends to infinity with m. In fact, because of the compactness of the support the moment problem is determinate [2, II.2. Theorem 2.2], so if $x \in \operatorname{supp}(\mu)$ and $\hat\epsilon > 0$ then there is a zero of p_n in $(x-\hat\epsilon, x+\hat\epsilon)$ for large n [8, 1.2 11. Fact 1], which shows that the roots eventually fill $[a+\delta, b-\delta]$ for every $\delta > 0$.

Remark 2. It is clearly enough to prove the existence of a threshold m_0 such that for $m \ge m_0$ the theorem holds with a constant A'.

Remark 3. Monitoring the constants in the proof of this theorem and Lemma 6 it follows that A' depends only on δ, $\frac{\operatorname{diam}(\operatorname{supp}(\mu))}{b-a}$ and $\frac{\mu(\mathbb{R})}{\mu([a,b])}$.

This theorem is about the zeros lying inside $[a+\delta, b-\delta]$, i.e. about the zeros that do not lie too close to a or b. For zeros lying close to a or b the result may not be true, as is shown by any Jacobi weight and $[a,b] = [-1,1]$ (Jacobi weights are doubling, but around ± 1 their zero spacing is $\sim 1/m^2$).

The claim in the theorem can be formulated in the following way:

$$0 < \frac{1}{A} \le \liminf_{m\to\infty} m(x_{m,j+1}-x_{m,j}) \le \limsup_{m\to\infty} m(x_{m,j+1}-x_{m,j}) \le A < \infty. \tag{2}$$

From this form it immediately follows that the result above generalizes Y. Last and B. Simon's following two theorems:

$A \sim B$ means that the ratio of the two sides is bounded from below and from above by two positive constants.

Corollary 2 (Theorem 8.5 in [4]). *Suppose* $\mathrm{d}\mu$ *is purely absolutely continuous in a neighbourhood of the point* E_0*, and for some* $q > 0$,

$$0 < \liminf_{x\to E_0} \frac{w(x)}{|x - E_0|^q} \le \limsup_{x\to E_0} \frac{w(x)}{|x - E_0|^q} < \infty. \tag{3}$$

Then

$$\limsup_{m\to\infty} m|x_m^{(1)}(E_0) - x_m^{(-1)}(E_0)| < \infty,$$

where $x_m^{(1)}(E_0)$ *is the smallest zero and* $x_m^{(-1)}(E_0)$ *is the largest zero of* p_m *for which* $x_m^{(-1)}(E_0) \le E_0 < x_m^{(1)}(E_0)$.

Corollary 3 (Theorem 9.3 in [4]). *Suppose* $\mathrm{d}\mu = w\,\mathrm{d}x + \mathrm{d}\mu_s$*, where, for the singular part,* $\mu_s([x_0 - \delta, x_0 + \delta]) = 0$ *and, for the absolutely continous part,*

$$0 < \inf_{|y-x_0|\le\delta} w(x) \le \sup_{|y-x_0|\le\delta} w(x) < \infty. \tag{4}$$

Then for any $\epsilon < \delta$,

$$\inf_{|y-x_0|<\epsilon} \liminf_{m\to\infty} m|x_m^{(1)}(y) - x_m^{(-1)}(y)| > 0.$$

We should only remark that the assumptions (3) and (4) imply the doubling property, so Theorem 1 can be applied.

Before starting the next theorem we recall the definition of the m-th Christoffel function and Cotes numbers associated with the measure μ:

$$\lambda_m(\xi) := \min_{\substack{p(\xi)=1\\ \deg p\le m}} \int p^2(x)\,\mathrm{d}\mu(x),$$

where the infimum is taken for all polynomials of degree at most m taking the value 1 at ξ, and

$$\lambda_{m,k} := \lambda_m(x_{m,k})$$

respectively.

Theorem 4. *If* μ *is a measure with compact support on the real line and with the doubling property on* $[a, b]$*, then for every* $\delta > 0$ *there exists a constant* $B = B_\delta$ *such that*

$$\frac{1}{B} \le \frac{\lambda_{m,k}}{\lambda_{m,k+1}} \le B, \tag{5}$$

whenever $x_{m,k}$ *and* $x_{m,k+1} \in [a + \delta, b - \delta]$.

Theorem 1 and Theorem 4 together have a converse.

Theorem 5. *Let* μ *be a measure with compact support. If (1) and (5) hold on every interval* $[a+\delta, b-\delta] \subset \operatorname{supp}(\mu)$, $\delta > 0$ *(with some* A *and* B *in (1) and (5) that may depend on* δ*), then* μ *has the doubling property on every such interval.*

2 Preliminaries

For the upper estimate in Theorem 1 we need the following lemma:

Lemma 6. *Let μ be a measure with compact support on the real line and with the doubling property on $[a,b]$. Then for every $\delta > 0$ there is a constant D such that for $m > \frac{2}{\delta}$*

$$\frac{1}{D}\mu\left(\left[\xi - \tfrac{1}{m}, \xi + \tfrac{1}{m}\right]\right) \le \lambda_m(\xi) \le D\mu\left(\left[\xi - \tfrac{1}{m}, \xi + \tfrac{1}{m}\right]\right),$$

whenever $\xi \in [a+\delta, b-\delta]$.

Before proving this we cite two lemmas, that we shall use in this article.

Lemma 7 (Example 2 in [3])**.** *There exist positive constants C, c such that for every m there are polynomials P_m of degree at most m satisfying*

$$P_m(0) = 1, \qquad |P_m(x)| \le Ce^{-c\sqrt{m|x|}}, \quad x \in [-2,2]. \tag{6}$$

Lemma 8 (Lemma 2.1 in [6])**.** *The following conditions for a measure μ are equivalent:*

(i) μ has the doubling property on $[a,b]$: there is an $L = L([a,b])$ such that $\mu(2I) \le L\mu(I)$ for all intervals $2I \subset [a,b]$.

(ii) There is an s and a K such that $\mu(I) \le K\left(\frac{|I|}{|J|}\right)^s \mu(J)$ for all intervals $J \subset I \subset [a,b]$.

(iii) There is an $r > 0$ and a K such that $\mu(J) \le K\left(\frac{|J|}{|I|}\right)^r \mu(I)$ for all intervals $J \subset I \subset [a,b]$.

(iv) There is an $s > 0$ and a K such that

$$\mu(I) \le K\left(\frac{|I| + |J| + \mathrm{dist}\{I,J\}}{|J|}\right)^s \mu(J)$$

for arbitrary intervals I and $J \subset [a,b]$.

Now we are ready to verify Lemma 6. First we deal with the right-hand side. The idea is to find a suitable polynomial with which λ_m can be estimated from above. This polynomial will be fast decreasing on the support of the measure, so its integral is small outside of the doubling interval ('outside integral'), while inside of that interval we can estimate its integral by applying the doubling property ('inside integral').

As for the left-hand side we show it comes from the case when a measure has the doubling property on all its support.

Proof of Lemma 6. We may assume that the support of μ is a subset of $[-1,1]$. According to Lemma 7, there is a P_m polynomial of degree m with the properties in (6).

Using this we get for λ_m and for $\xi \in [a+\delta, b-\delta]$:

$$\lambda_m(\xi) = \min_{\substack{p(\xi)=1 \\ \deg p \le m}} \int p^2(x)\,\mathrm{d}\mu(x) \le \int P_m^2(x-\xi)\,\mathrm{d}\mu(x) \le \int C^2 e^{-2c\sqrt{m(x-\xi)}}\,\mathrm{d}\mu(x)$$
$$= \int_{\xi-\frac{1}{m}}^{\xi+\frac{1}{m}} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x)$$
$$+ \int_{a+\frac{\delta}{2}}^{\xi-\frac{1}{m}} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x) + \int_{\xi+\frac{1}{m}}^{b-\frac{\delta}{2}} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x)$$
$$+ \int_{-1}^{a+\frac{\delta}{2}} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x) + \int_{b-\frac{\delta}{2}}^{1} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x), \tag{7}$$

provided $m \ge \frac{2}{\delta}$

First we estimate the fourth and the fifth integrals ('outside integrals') of the right-hand side:

$$\int_{b-\frac{\delta}{2}}^{1} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x) \le \int_{b-\frac{\delta}{2}}^{1} C^2 e^{-2c\sqrt{m|(b-\frac{\delta}{2})-\xi|}}\,\mathrm{d}\mu(x)$$
$$= C^2 e^{-2c\sqrt{m|(b-\frac{\delta}{2})-\xi|}} \int_{b-\frac{\delta}{2}}^{1} \mathrm{d}\mu(x) \le C^2 e^{-2c\sqrt{m|(b-\frac{\delta}{2})-(b-\delta)|}}\,\mu([-1,1]) \tag{8}$$
$$= C^2 e^{-2c\sqrt{m\frac{\delta}{2}}}\,\mu([-1,1]).$$

Using the doubling property (Lemma 8 (ii))

$$\mu\left(\left[\xi-\tfrac{1}{m}, \xi+\tfrac{1}{m}\right]\right) \ge K\left(\frac{\left|\left[\xi-\frac{1}{m}, \xi+\frac{1}{m}\right]\right|}{|[a,b]|}\right)^s \mu([a,b])$$
$$= K\left(\frac{2}{b-a}\right)^s \mu([a,b])\,\frac{1}{m^s} \tag{9}$$

follows. Since for sufficiently large m

$$C^2 e^{-2c\sqrt{m\frac{\delta}{2}}}\,\mu([-1,1]) \le K\left(\frac{2}{b-a}\right)^s \mu([a,b])\,\frac{1}{m^s}$$

holds, by (8) and (9) the inequality

$$\int_{b-\frac{\delta}{2}}^{1} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x) \le \mu\left(\left[\xi-\tfrac{1}{m}, \xi+\tfrac{1}{m}\right]\right) \tag{10}$$

is also true for large m. The estimate of the fourth integral is similar.

Now we consider the second and the third integrals ('inside integrals') of the right-hand side of (7). Denote by T the integer, for which $\xi + \frac{T}{m} < b - \frac{\delta}{2} \le \xi + \frac{T+1}{m}$. Then

$$\int_{\xi+\frac{1}{m}}^{b-\frac{\delta}{2}} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x) \le \int_{\xi+\frac{1}{m}}^{\xi+\frac{T+1}{m}} C^2 e^{-2c\sqrt{m|x-\xi|}}\,\mathrm{d}\mu(x)$$
$$\le \sum_{i=1}^{T} \int_{\xi+\frac{i}{m}}^{\xi+\frac{i+1}{m}} C^2 e^{-2c\sqrt{m(\xi+\frac{i}{m}-\xi)}}\,\mathrm{d}\mu = \sum_{i=1}^{T} \int_{\xi+\frac{i}{m}}^{\xi+\frac{i+1}{m}} C^2 e^{-2c\sqrt{i}}\,\mathrm{d}\mu.$$

Again using the doubling property with some K and s (Lemma 8 (iv)) we have

$$\mu\left(\left[\xi+\tfrac{i}{m},\xi+\tfrac{i+1}{m}\right]\right) \le K\left(\frac{\frac{1}{m}+\frac{1}{m}+\frac{i-1}{m}}{\frac{1}{m}}\right)^s \mu\left(\left[\xi,\xi+\tfrac{1}{m}\right]\right)$$
$$\le K(i+1)^s \mu\left(\left[\xi,\xi+\tfrac{1}{m}\right]\right).$$

From this we obtain that

$$\sum_{i=1}^{T} \int_{\xi+\frac{i}{m}}^{\xi+\frac{i+1}{m}} C^2 e^{-2c\sqrt{i}}\,\mathrm{d}\mu \le \underbrace{K2^s\left(\sum_{i=1}^{\infty} i^s C^2 e^{-2c\sqrt{i}}\right)}_{\text{cons.}<\infty} \mu\left(\left[\xi,\xi+\tfrac{1}{m}\right]\right), \tag{11}$$

because $2i \ge i+1$.

Since the estimate of the second integral follows the same way, we do not detail it.

Collecting (8), (10) and (11) we get the required inequality of the right-hand side in Lemma 6.

In order to prove the lower estimate we recall that if μ is a doubling measure on $[-1,1]$, then for the Christoffel function there is a constant C independent of m and x such that

$$\frac{1}{C}\mu\left(\left[x-\left(\tfrac{\sqrt{1-x^2}}{m}+\tfrac{1}{m^2}\right), x+\left(\tfrac{\sqrt{1-x^2}}{m}+\tfrac{1}{m^2}\right)\right]\right) \le \lambda_m(x)$$

holds (see [6, (7.14)]). It is clear in the light of the doubling property that when we are of positive distance from ±1 and $\eta,\rho > 0$, then the last inequality is equivalent to the following one (maybe with a different C that may depend on η and ρ):

$$\frac{1}{C}\mu\left(\left[x-\tfrac{\eta}{m}, x+\tfrac{\eta}{m}\right]\right) \le \lambda_m(x), \qquad x\in[-1+\rho,1-\rho].$$

Simple linear transformation gives a similar inequality when μ is supported on an interval $[a,b]$ and is doubling there. Finally, if $[a,b]$ is a proper subset of the support of μ and μ is doubling there, then the lemma follows from the inequality $\lambda_m(x,\mu) \ge \lambda_m(x,\mu|_{[a,b]})$, if we apply the just mentioned inequality to the restricted measure $\mu|_{[a,b]}$.

□

3 Proofs

After these preparations the proof of Theorem 1, Theorem 4 and Theorem 5 is similar to those found in [5].

Proof of Theorem 1. First we deal with the upper estimate in (1). It can be assumed that $\mathrm{supp}(\mu) \subset [-1,1]$. Fix $m \geq \frac{2}{\delta}$ and let $x_j = x_{m,j}$, $x_{j+1} = x_{m,j+1} \in [a+\delta, b-\delta]$. We apply the Markoff inequality [2, I.5. (5.4)], that claims

$$\sum_{x_j<x} \lambda_{m,j} \leq \mu((-\infty,x)) \leq \mu((-\infty,x]) \leq \sum_{x_j \leq x} \lambda_{m,j}. \tag{12}$$

From this and Lemma 6 we get

$$\begin{aligned} &\mu([x_j,x_{j+1}]) \leq \lambda_{m,j} + \lambda_{m,j+1} \\ &\leq D\left(\mu\left(\left[x_j - \tfrac{1}{m}, x_j + \tfrac{1}{m}\right]\right) + \mu\left(\left[x_{j+1} - \tfrac{1}{m}, x_{j+1} + \tfrac{1}{m}\right]\right)\right). \end{aligned} \tag{13}$$

If $x_{j+1} - x_j \leq \frac{2}{m}$ then there is nothing to prove, so we may assume $x_{j+1} - x_j > \frac{2}{m}$. In this case

$$x_j + \tfrac{1}{m} < x_{j+1} - \tfrac{1}{m}.$$

Setting

$$\begin{aligned} I &= \left[x_j - \tfrac{1}{m}, x_{j+1} + \tfrac{1}{m}\right], \\ E_1 &= \left[x_j - \tfrac{1}{m}, x_j + \tfrac{1}{m}\right] \end{aligned}$$

and

$$E_2 = \left[x_{j+1} - \tfrac{1}{m}, x_{j+1} + \tfrac{1}{m}\right],$$

by the doubling property (Lemma 8 (i)), we have

$$\mu(I) \leq L\mu([x_j,x_{j+1}]) \leq DL\left(\mu(E_1) + \mu(E_2)\right),$$

where the last inequality follows from (13).

Again using the doubling property (Lemma 8 (iii))

$$\begin{aligned} \mu(E_1) &\leq K\left(\frac{|E_1|}{|I|}\right)^r \mu(I), \\ \mu(E_2) &\leq K\left(\frac{|E_2|}{|I|}\right)^r \mu(I) \end{aligned}$$

follows. Consequently, by simplifying with $\mu(I)$, the preceding inequalities imply

$$1 \leq DL\frac{K}{|I|^r}\left(|E_1|^r + |E_2|^r\right) \leq 2DL\frac{K}{|I|^r}\left(|E_1| + |E_2|\right)^r.$$

After a rearranging

$$x_{j+1} - x_j < |I| \leq (2DLK)^{\frac{1}{r}}\left(|E_1| + |E_2|\right) \leq \frac{4(2DLK)^{\frac{1}{r}}}{m}$$

is obtained, which was to be demonstrated.

Now, let us consider the inequality on the left-hand side of (1). The basis of the proof is the Remez inequality [6, (7.16)]: If μ is a doubling measure on $[-1,1]$, then for every $\Lambda > 0$ there is a constant $C = C_\Lambda$ such that for $|\arccos(E)| \le \frac{\Lambda}{m}$

$$\int_{-1}^{1} p_m^2 \,\mathrm{d}\mu \le C \int_{[-1,1]\setminus E} p_m^2 \,\mathrm{d}\mu, \tag{14}$$

where E consists of finitely many intervals. This implies by simple linear transformation that if μ is doubling on $[a,b]$, $\delta > 0$, $I \subset [a+\delta, b-\delta]$ is an interval of length $\le \frac{2}{m}$ and q_m is a polynomial of degree at most m, then

$$\int_{[a,b]} q_m^2 \,\mathrm{d}\mu \le C \int_{[a,b]\setminus I} q_m^2 \,\mathrm{d}\mu,$$

where C depends only on δ and the doubling constant of μ on $[a,b]$.

From here the proof is a literal repeat of the proof of Theorem 1 in [5]. In fact, we may assume that $x_{j+1} - x_j = \frac{\delta}{m}$, where $0 < \delta < 1/2$, otherwise we are done. Let $q_{m-2} = \frac{p_m}{(x-x_{j+1})(x-x_j)}$. Since $\deg(q_{m-2}) \le m-2$ we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}} p_m q_{m-2} \,\mathrm{d}\mu = \int_{-1}^{1} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x) \\ &= \int_{x_j}^{x_{j+1}} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x) \\ &+ \int_{[-1,1]\setminus[x_j,x_{j+1}]} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x) \\ &\ge \int_{x_j}^{x_{j+1}} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x) \\ &+ \int_{[a,b]\setminus[x_j,x_{j+1}]} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x) \end{aligned} \tag{15}$$

considering that $(x-x_{j+1})(x-x_j) \ge 0$ is positive outside $[x_j, x_{j+1}]$.

Let us deal with the last two integrals separately. As $x_{j+1} - x_j \le \frac{\delta}{m}$, we get for the first one:

$$\begin{aligned} &\int_{x_j}^{x_{j+1}} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x) \\ &= -\int_{x_j}^{x_{j+1}} q_{m-2}^2(x)|x-x_{j+1}||x-x_j|\,\mathrm{d}\mu(x) \ge -\frac{\delta^2}{m^2}\int_{x_j}^{x_{j+1}} q_{m-2}^2 \,\mathrm{d}\mu. \end{aligned}$$

In the case of the second integral we use the assumption $x_{j+1} - x_j \le \frac{\delta}{m} < \frac{1}{2m}$

and the Remez inequality:

$$\begin{aligned}
&\int_{[a,b]\setminus[x_j,x_{j+1}]} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x)\\
&\geq \int_{[a,b]\setminus\left[x_j-\frac{1}{m},x_j+\frac{1}{m}\right]} q_{m-2}^2(x)(x-x_{j+1})(x-x_j)\,\mathrm{d}\mu(x)\\
&\geq \frac{1}{(2m)^2}\int_{[a,b]\setminus\left[x_j-\frac{1}{m},x_j+\frac{1}{m}\right]} q_{m-2}^2\,\mathrm{d}\mu \geq \frac{1}{4Cm^2}\int_a^b q_{m-2}^2\,\mathrm{d}\mu\\
&\geq \frac{1}{4Cm^2}\int_{x_j}^{x_{j+1}} q_{m-2}^2\,\mathrm{d}\mu.
\end{aligned}$$

Using the last inequalities we continue (15):

$$\begin{aligned}
0 &\geq -\frac{\delta^2}{m^2}\int_{x_j}^{x_{j+1}} q_{m-2}^2\,\mathrm{d}\mu + \frac{1}{4Cm^2}\int_{x_j}^{x_{j+1}} q_{m-2}^2\,\mathrm{d}\mu\\
&= \left(\frac{1}{4C}-\delta^2\right)\left(\frac{1}{m^2}\right)\int_{x_j}^{x_{j+1}} q_{m-2}^2\,\mathrm{d}\mu.
\end{aligned}$$

This is possible only if $\frac{1}{4C}-\delta^2 \leq 0$, that is if $\delta \geq \frac{1}{2\sqrt{C}}$. This means that, necessarily, $x_{j+1}-x_j \geq \frac{1}{2\sqrt{C}}\frac{1}{m}$, so the lower estimate also holds.

□

Proof of Theorem 4. The theorem is a simple consequence of Theorem 1, Lemma 6 and the doubling property (Lemma 8 (i)) on $[a,b]$.

Theorem 1 shows that

$$\left[x_{k+1}-\tfrac{\hat{A}}{m}, x_{k+1}+\tfrac{\hat{A}}{m}\right] \supset \left[x_k-\tfrac{1}{m}, x_k+\tfrac{1}{m}\right]$$

holds for $\hat{A} := A+1$. Now by Lemma 6 and the doubling property we get the upper estimate :

$$\frac{\lambda_{m,k}}{\lambda_{m,k+1}} \leq \frac{D\mu\left(\left[x_{k+1}-\frac{\hat{A}}{m}, x_{k+1}+\frac{\hat{A}}{m}\right]\right)}{\frac{1}{D}\mu\left(\left[x_{k+1}-\frac{1}{m}, x_{k+1}+\frac{1}{m}\right]\right)} \leq D^2 L^{\lceil \log_2 \hat{A}\rceil}.$$

The proof of the lower estimate for the quotient $\frac{\lambda_{m,k}}{\lambda_{m,k+1}}$ is similar.

□

Poof of Theorem 5. As we mentioned above the proof follows the proof of Theorem 3 in [5], however it is technically somewhat simpler since we work far from the endpoints of $[a,b]$.

Fix δ. Applying Remark 1 to $\left[a+\frac{\delta}{2}, b-\frac{\delta}{2}\right]$ it follows that there is an m_1 such that whenever $m \geq m_1$ then there exists a zero of the m-th orthonormal polynomial on $\left[a+\frac{\delta}{2}, a+\delta\right]$ and on $\left[b-\delta, b-\frac{\delta}{2}\right]$, respectively.

$\lceil x\rceil$ denotes the least integer not less than x.

We have to prove that there is a constant L such that for every interval I for which $2I \subset [a+\delta, b-\delta]$

$$\mu(2I) \le L\mu(I)$$

holds. It can be easily seen that it is enough to prove this for intervals with length at most $\frac{8A}{m_1}$, where A is the constant in (1).

Let us choose m such that

$$\frac{4A}{m} < |I| \le \frac{8A}{m}, \tag{16}$$

so, if τ denotes the center of I, by (1) and the Remark 1, there is a k such that $x_{m,k} < \tau \le x_{m,k+1}$, moreover

$$[x_{m,k-1}, x_{m,k+1}] \subset I. \tag{17}$$

On the other hand since $2I \subset [a+\delta, b-\delta]$, there is a largest (smallest) zero to the left (right) of $2I$ by Remark 1, that is there are $x_{m,k-r}$ and $x_{m,k+s} \in \left[a+\frac{\delta}{2}, b-\frac{\delta}{2}\right]$ for which

$$[x_{m,k-r+1}, x_{m,k+s-1}] \subset 2I \subset [x_{m,k-r}, x_{m,k+s}]. \tag{18}$$

Note that (17) and (18) imply a lower and an upper estimate for the measure of I and $2I$ respectively. So if the quotient $\frac{\mu([x_{m,k-r}, x_{m,k+s}])}{\mu([x_{m,k-1}, x_{m,k+1}])}$ can be estimated above by a fix constant independent of I, we are done.

From (17) and the Markoff inequality (see (12)) we immediately obtain:

$$\mu(I) \ge \mu([x_{m,k-1}, x_{m,k+1}]) \ge \lambda_{m,k}. \tag{19}$$

Let us try to estimate the measure of $[x_{m,k-r}, x_{m,k+s}]$ by $\lambda_{m,k}$ too. Again using the Markoff inequality (see (12)), (1) and (5) we get

$$\mu([x_{m,k-r}, x_{m,k+s}]) \le \sum_{j=-r}^{s} \lambda_{m,k+j} \le \lambda_{m,k} \sum_{j=-r}^{s} B^{|j|}, \tag{20}$$

where $B = B_{\frac{\delta}{2}}$ is the constant in (5) for the interval $\left[a+\frac{\delta}{2}, b-\frac{\delta}{2}\right]$. According to (1), the left side of (16) and (18)

$$2|I| \ge x_{m,k+s-1} - x_{m,k} \ge (s-1)\tfrac{1}{Am} \ge (s-1)\tfrac{|I|}{8A^2}$$

from which we gain an upper estimate for s and, in a similar way, for r, namely $\max(s, r) \le 16A^2 + 1$. Putting this fact together with (20) we obtain

$$\mu(2I) \le \mu([x_{m,k-r}, x_{m,k+s}]) \le 2B^{16A^2+2}\lambda_{m,k}. \tag{21}$$

Comparing (19) and (21) we can infer the doubling property with the doubling constant $L = 2B^{16A^2+2}$, and this is already independent of I.

$\square$

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