

# An Elementary Proof of the General Poincaré Formula for $\lambda$ -additive Measures\*

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## Abstract

In a previous paper of ours [4], we presented the general formula for  $\lambda$ -additive measure of union of  $n$  sets and gave a proof of it. That proof is based on the fact that the  $\lambda$ -additive measure is representable. In this study, a novel and elementary proof of the formula for  $\lambda$ -additive measure of the union of  $n$  sets is presented. Here, it is also demonstrated that, using elementary techniques, the well-known Poincaré formula of probability theory is just a limit case of our general formula.

**Keywords:**  $\lambda$ -additive measure, Poincaré formula

## 1 Introduction

Since the fuzzy measures (monotone measures) play an important role in describing various phenomena, over time there has been a steady interest in them (see, e.g. [13, 14, 22, 10, 8]). One of the most widely applied classes of monotone measures is the class of  $\lambda$ -additive measures (Sugeno  $\lambda$ -measures) (see, e.g. [21, 11, 12, 2, 1, 17]). [21]. Although there are many theoretical and applied articles that discuss the  $\lambda$ -additive measure, the general form of  $\lambda$ -additive measure of the union of  $n$  sets has just recently been identified [4]. In [4], we proved that if  $X$  is a finite set,  $A_1, \dots, A_n \in \mathcal{P}(X)$ ,  $n \geq 2$ ,  $Q_\lambda$  is a  $\lambda$ -additive measure on  $X$ ,  $\lambda \in (-1, \infty)$  and  $\lambda \neq 0$ , then

$$Q_\lambda \left( \bigcup_{i=1}^n A_i \right) = \frac{1}{\lambda} \left( \prod_{k=1}^n \left( \prod_{1 \leq i_1 < \dots < i_k \leq n} (1 + \lambda Q_\lambda (A_{i_1} \cap \dots \cap A_{i_k})) \right)^{(-1)^{k-1}} - 1 \right), \quad (1)$$

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where  $\mathcal{P}(X)$  denotes the power set of  $X$ . Our proof in [4] is based on the fact that  $Q_\lambda$  is representable [2]; that is, one has  $Q_\lambda = h_\lambda \circ \mu$  for a uniquely determined additive measure  $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ , where  $h_\lambda : [0, 1] \rightarrow [0, 1]$  is a strictly increasing bijection given via

$$h_\lambda(x) = \begin{cases} \frac{(1+\lambda)^x - 1}{\lambda}, & \text{if } \lambda \neq 0 \\ x, & \text{if } \lambda = 0, \end{cases}$$

and  $\lambda \in (-1, \infty)$ . Here, we will prove the formula in Eq. (1) without utilizing the fact that  $Q_\lambda$  is representable. That is, we will give a novel and elementary proof of Eq. (1). Taking into account the fact that the fuzzy measures and the fuzzy measure related aggregation are important topics, it is worth mentioning that the formula in Eq. (1) may also be viewed as an aggregation related to the  $\lambda$ -additive measure, which is a fuzzy measure.

The well-known Poincaré formula of probability theory is

$$Pr\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} Pr(A_{i_1} \cap \dots \cap A_{i_k}), \quad (2)$$

where  $Pr$  is a probability measure on  $X$  and  $A_1, \dots, A_n \in \mathcal{P}(X)$ . Here, we will show that the Poincaré formula of probability theory given in Eq. (2) is a limit case of the general formula of  $\lambda$ -additive measure of the union of  $n$  sets given in Eq. (1). Namely, by using elementary techniques, we will prove that if  $X$  is a finite set,  $A_1, \dots, A_n \in \mathcal{P}(X)$ ,  $n \geq 2$ ,  $Q_\lambda$  is a  $\lambda$ -additive measure on  $X$  and  $\lambda \neq 0$ , then

$$\lim_{\lambda \rightarrow 0} Q_\lambda\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} Q_\lambda(A_{i_1} \cap \dots \cap A_{i_k}).$$

It is an acknowledged fact that the  $\lambda$ -additive measure is strongly connected with the belief- and plausibility measures of Dempster-Shafer theory (see, e.g. [22, 9, 5, 20, 7, 3, 16]), and with the theory of rough sets (see, e.g. [6, 24, 23, 15, 18, 19]). Hence, our formula for the  $\lambda$ -additive measure of the union of  $n$  sets may play an important role in these areas of computer science [4].

The rest of this paper is structured as follows. In Section 2, we will introduce our new result regarding the  $\lambda$ -additive measure of the union of  $n$  sets. Here, we will also prove that the Poincaré formula of probability theory is just a limit case of our novel formula; that is, our formula may be viewed as the generalization of the Poincaré formula. Lastly, in Section 3, we will give a short summary of our findings and highlight our future research plans including the possible application of our results in network science.

In this study, we will use the common notations  $\cap$  and  $\cup$  for the intersection and union operations over sets, respectively. Also, will use the notation  $\bar{A}$  for the complement of set  $A$ .

## 2 An elementary proof of the general Poincaré formula

Relaxing the additivity property of the probability measure, the following  $\lambda$ -additive measures were proposed by Sugeno in 1974 [21].

**Definition 1.** *The function  $Q_\lambda : \mathcal{P}(X) \rightarrow [0, 1]$  is a  $\lambda$ -additive measure (Sugeno  $\lambda$ -measure) on the finite set  $X$ , iff  $Q_\lambda$  satisfies the following requirements:*

$$(1) \quad Q_\lambda(X) = 1$$

$$(2) \quad \text{for any } A, B \in \mathcal{P}(X) \text{ and } A \cap B = \emptyset,$$

$$Q_\lambda(A \cup B) = Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B), \quad (3)$$

where  $\lambda \in (-1, \infty)$  and  $\mathcal{P}(X)$  is the power set of  $X$ .

Note that if  $X$  is an infinite set, then the continuity of function  $Q_\lambda$  is also required. From now on,  $\mathcal{P}(X)$  will denote the power set of the finite set  $X$  and  $Q_\lambda$  will always denote a  $\lambda$ -additive measure on  $X$ .

The calculation of the  $\lambda$ -additive measure of the union of two disjoint sets is given in Definition 1. The following well-known lemma (see Theorem 4.6 (2) in [22]) shows how the  $\lambda$ -additive measure of the union of two sets can be computed when these sets are not necessarily disjoint.

**Lemma 1.** *If  $X$  is a finite set and  $Q_\lambda$  is a  $\lambda$ -additive measure on  $X$ , then for any  $A, B \in \mathcal{P}(X)$ ,*

$$Q_\lambda(A \cup B) = \frac{Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A)Q_\lambda(B) - Q_\lambda(A \cap B)}{1 + \lambda Q_\lambda(A \cap B)}. \quad (4)$$

*Proof.* See the proof of Theorem 4.6 in [22]. □

**Remark 1.** Notice that if  $\lambda = 0$ , then Eq. (4) reduces to  $Q_\lambda(A \cup B) = Q_\lambda(A) + Q_\lambda(B) - Q_\lambda(A \cap B)$ , which has the same form as the probability measure of union of two sets.

Here, we will introduce a function and some quantities that we will utilize later on.

**Definition 2.** *The function  $p_{n,\lambda}^{(k)} : \mathcal{P}^n(X) \rightarrow \mathbb{R}$  is given by*

$$p_{n,\lambda}^{(k)}(A_1, \dots, A_n) = \prod_{1 \leq i_1 < \dots < i_k \leq n} (1 + \lambda Q_\lambda(A_{i_1} \cap \dots \cap A_{i_k})),$$

where  $X$  is a finite set,  $A_1, \dots, A_n \in \mathcal{P}(X)$ ,  $n \geq 2$ ,  $1 \leq k \leq n$ . For the sake of simplicity, we will also use the notation  $Z_{n,\lambda}^{(k)} = p_{n,\lambda}^{(k)}(A_1, \dots, A_n)$ .

Later, we will utilize the following quantity to identify the general formula for the  $\lambda$ -additive measure of the union of  $n$  sets.

**Definition 3.** The quantity  $Z_{n,\lambda}^{*(k)}$  is given by

$$Z_{n,\lambda}^{*(k)} = p_{n,\lambda}^{(k)}(A_1^*, \dots, A_n^*),$$

where  $X$  is a finite set,  $A_i^* = A_i \cap A_{n+1}$ ,  $A_i, A_{n+1} \in \mathcal{P}(X)$ ,  $n \geq 2$ ,  $1 \leq i \leq n$ ,  $1 \leq k \leq n$ .

Here, we will demonstrate how the  $\lambda$ -additive measure of the union of  $n$  general sets can be computed. That is, we will discuss the Poincaré formula for the  $\lambda$ -additive measure. First, we will discuss some key properties of the quantities that we introduced previously.

**Lemma 2.** If  $X$  is a finite set,  $A_1, \dots, A_n, A_{n+1} \in \mathcal{P}(X)$ ,  $A_i^* = A_i \cap A_{n+1}$  and  $1 \leq i \leq n$ , then

$$\begin{aligned} Z_{n,\lambda}^{*(k)} &= p_{n,\lambda}^{(k)}(A_1^*, \dots, A_n^*) = \\ &= \prod_{1 \leq i_1 < \dots < i_k \leq n} (1 + \lambda Q_\lambda(A_{i_1} \cap \dots \cap A_{i_k} \cap A_{n+1})), \end{aligned}$$

where  $n \geq 2$  and  $1 \leq k \leq n$ .

*Proof.* Exploiting the idempotent property of the set intersection operation, the lemma immediately follows from the definition of  $Z_{n,\lambda}^{*(k)}$ .  $\square$

The following lemma demonstrates a key connection between the  $Z_{n,\lambda}^{*(k)}$  and  $Z_{n,\lambda}^{(n)}$  quantities.

**Lemma 3.** Let  $X$  be a finite set and let  $A_1, \dots, A_n, A_{n+1} \in \mathcal{P}(X)$ ,  $A_i^* = A_i \cap A_{n+1}$  and  $1 \leq i \leq n$ . Then, for any  $n \geq 2$ ,  $1 \leq k \leq n$ , the quantity  $Z_{n,\lambda}^{*(k)}$  can be expressed in terms of  $Z_{n,\lambda}^{(k+1)}$  and  $Z_{n+1,\lambda}^{(k+1)}$  as follows:

$$Z_{n,\lambda}^{*(k)} = \begin{cases} \frac{Z_{n+1,\lambda}^{(k+1)}}{Z_{n,\lambda}^{(k+1)}}, & \text{if } k < n \\ Z_{n+1,\lambda}^{(n+1)}, & \text{if } k = n. \end{cases} \quad (5)$$

*Proof.* Here, we will distinguish two cases: (1)  $k < n$ , (2)  $k = n$ .

(1) Based on Lemma 2, the following relation holds:

$$\begin{aligned} Z_{n,\lambda}^{*(k)} &= p_{n,\lambda}^{(k)}(A_1^*, \dots, A_n^*) = \\ &= \prod_{1 \leq i_1 < \dots < i_k \leq n} (1 + \lambda Q_\lambda(A_{i_1} \cap \dots \cap A_{i_k} \cap A_{n+1})). \end{aligned} \quad (6)$$

Next, the right hand side of Eq. (6) can be written as

$$\begin{aligned} & \prod_{1 \leq i_1 < \dots < i_k \leq n} (1 + \lambda Q_\lambda (A_{i_1} \cap \dots \cap A_{i_k} \cap A_{n+1})) = \\ &= \frac{\prod_{1 \leq i_1 < \dots < i_{k+1} \leq n+1} (1 + \lambda Q_\lambda (A_{i_1} \cap \dots \cap A_{i_{k+1}}))}{\prod_{1 \leq i_1 < \dots < i_{k+1} \leq n} (1 + \lambda Q_\lambda (A_{i_1} \cap \dots \cap A_{i_{k+1}}))} = \frac{Z_{n+1, \lambda}^{(k+1)}}{Z_{n, \lambda}^{(k+1)}}. \end{aligned} \quad (7)$$

Notice that based on Definition 2,  $Z_{n, \lambda}^{(k+1)}$  exists only if  $k+1 \leq n$ ; that is, if  $k < n$ . This explains why we need to differentiate the two cases in Eq. (5).

(2) If  $k = n$ , then based on Definition 3 and Definition 2,

$$\begin{aligned} Z_{n, \lambda}^{*(k)} &= p_{n, \lambda}^{(k)}(A_1^*, \dots, A_n^*) = 1 + \lambda Q_\lambda (A_1^* \cap \dots \cap A_n^*) = \\ &= 1 + \lambda Q_\lambda ((A_1 \cap A_{n+1}) \cap \dots \cap (A_n \cap A_{n+1})) = \\ &= 1 + \lambda Q_\lambda (A_1 \cap \dots \cap A_n \cap A_{n+1}) = p_{n+1, \lambda}^{(n+1)}(A_1, \dots, A_{n+1}) = Z_{n+1, \lambda}^{(n+1)}. \end{aligned} \quad (8)$$

□

The following example demonstrates the usefulness of Lemma 3. In this example, we will show how the quantity  $Z_{3, \lambda}^{*(1)}$  can be expressed in terms of the quantities  $Z_{4, \lambda}^{(2)}$  and  $Z_{3, \lambda}^{(2)}$ .

**Example 1.**

$$\begin{aligned} Z_{3, \lambda}^{*(1)} &= (1 + \lambda Q_\lambda (A_1 \cap A_4)) (1 + \lambda Q_\lambda (A_2 \cap A_4)) (1 + \lambda Q_\lambda (A_3 \cap A_4)) \\ Z_{4, \lambda}^{(2)} &= (1 + \lambda Q_\lambda (A_1 \cap A_2)) (1 + \lambda Q_\lambda (A_1 \cap A_3)) (1 + \lambda Q_\lambda (A_1 \cap A_4)) \cdot \\ &\quad \cdot (1 + \lambda Q_\lambda (A_2 \cap A_3)) (1 + \lambda Q_\lambda (A_2 \cap A_4)) (1 + \lambda Q_\lambda (A_3 \cap A_4)) \\ Z_{3, \lambda}^{(2)} &= (1 + \lambda Q_\lambda (A_1 \cap A_2)) (1 + \lambda Q_\lambda (A_1 \cap A_3)) (1 + \lambda Q_\lambda (A_2 \cap A_3)) \end{aligned}$$

It can be seen from the expressions of  $Z_{3, \lambda}^{*(1)}$ ,  $Z_{4, \lambda}^{(2)}$  and  $Z_{3, \lambda}^{(2)}$  that the equation

$$Z_{3, \lambda}^{*(1)} = \frac{Z_{4, \lambda}^{(2)}}{Z_{3, \lambda}^{(2)}}$$

holds.

The next lemma shows how the  $\lambda$ -additive measure of set  $A_n$  can be expressed in terms of the  $Z_{n, \lambda}^{(1)}$  and  $Z_{n-1, \lambda}^{(1)}$  quantities.

**Lemma 4.** *If  $X$  is a finite set,  $A_1, \dots, A_n \in \mathcal{P}(X)$ ,  $n \geq 3$  and  $\lambda \neq 0$ , then*

$$Q_\lambda(A_n) = \frac{1}{\lambda} \left( \frac{Z_{n, \lambda}^{(1)}}{Z_{n-1, \lambda}^{(1)}} - 1 \right). \quad (9)$$

*Proof.* By utilizing the definitions of  $Z_{n,\lambda}^{(1)}$  and  $Z_{n-1,\lambda}^{(1)}$ , we have

$$\begin{aligned} \frac{Z_{n,\lambda}^{(1)}}{Z_{n-1,\lambda}^{(1)}} &= \frac{(1 + \lambda Q_\lambda(A_1)) \cdots (1 + \lambda Q_\lambda(A_{n-1})) (1 + \lambda Q_\lambda(A_n))}{(1 + \lambda Q_\lambda(A_1)) \cdots (1 + \lambda Q_\lambda(A_{n-1}))} = \\ &= 1 + \lambda Q_\lambda(A_n), \end{aligned}$$

from which Eq. (9) immediately follows.  $\square$

Now, we will state and prove a key theorem that allows us to compute the  $\lambda$ -additive measure of the union of  $n$  sets when the parameter  $\lambda$  is nonzero.

**Theorem 1.** *If  $X$  is a finite set,  $A_1, \dots, A_n \in \mathcal{P}(X)$ ,  $n \geq 2$ ,  $Q_\lambda$  is a  $\lambda$ -additive measure on  $X$  and  $\lambda \neq 0$ , then*

$$\begin{aligned} Q_\lambda \left( \bigcup_{i=1}^n A_i \right) &= \\ &= \frac{1}{\lambda} \left( \prod_{k=1}^n \left( \prod_{1 \leq i_1 < \dots < i_k \leq n} (1 + \lambda Q_\lambda(A_{i_1} \cap \dots \cap A_{i_k})) \right)^{(-1)^{k-1}} - 1 \right). \end{aligned} \quad (10)$$

*Proof.* By utilizing the definition of  $Z_{n,\lambda}^{(k)}$ , Eq. (10) can be written as

$$Q_\lambda \left( \bigcup_{i=1}^n A_i \right) = \frac{1}{\lambda} \left( \prod_{k=1}^n \left( Z_{n,\lambda}^{(k)} \right)^{(-1)^{k-1}} - 1 \right). \quad (11)$$

It can be shown by direct calculation that the formula in Eq. (11) holds for  $n = 2, 3$ ; that is,

$$\begin{aligned} Q_\lambda(A_1 \cup A_2) &= \frac{1}{\lambda} \left( \left( Z_{2,\lambda}^{(1)} \right) \left( Z_{2,\lambda}^{(2)} \right)^{-1} - 1 \right) \\ Q_\lambda(A_1 \cup A_2 \cup A_3) &= \frac{1}{\lambda} \left( \left( Z_{3,\lambda}^{(1)} \right) \left( Z_{3,\lambda}^{(2)} \right)^{-1} \left( Z_{3,\lambda}^{(3)} \right) - 1 \right). \end{aligned}$$

Here, we will apply induction; that is, we will prove that if Eq. (11) holds, then the equation

$$Q_\lambda \left( \bigcup_{i=1}^{n+1} A_i \right) = \frac{1}{\lambda} \left( \prod_{k=1}^{n+1} \left( Z_{n+1,\lambda}^{(k)} \right)^{(-1)^{k-1}} - 1 \right) \quad (12)$$

holds as well.

By making use of Lemma 1, the associativity of the set union operation and the distributivity of the set intersection operation over the set union operation, we

have

$$\begin{aligned}
 Q_\lambda \left( \bigcup_{i=1}^{n+1} A_i \right) &= Q_\lambda \left( \left( \bigcup_{i=1}^n A_i \right) \cup A_{n+1} \right) = \\
 &= \frac{1}{1 + \lambda Q_\lambda \left( \left( \bigcup_{i=1}^n A_i \right) \cap A_{n+1} \right)} \left( Q_\lambda \left( \bigcup_{i=1}^n A_i \right) + Q_\lambda(A_{n+1}) + \right. \\
 &\quad \left. + \lambda Q_\lambda \left( \bigcup_{i=1}^n A_i \right) Q_\lambda(A_{n+1}) - Q_\lambda \left( \left( \bigcup_{i=1}^n A_i \right) \cap A_{n+1} \right) \right) = \\
 &= \frac{1}{1 + \lambda Q_\lambda \left( \bigcup_{i=1}^n (A_i \cap A_{n+1}) \right)} \left( Q_\lambda \left( \bigcup_{i=1}^n A_i \right) + Q_\lambda(A_{n+1}) + \right. \\
 &\quad \left. + \lambda Q_\lambda \left( \bigcup_{i=1}^n A_i \right) Q_\lambda(A_{n+1}) - Q_\lambda \left( \bigcup_{i=1}^n (A_i \cap A_{n+1}) \right) \right).
 \end{aligned} \tag{13}$$

Now, by introducing  $A_i^* = A_i \cap A_{n+1}$  for all  $1 \leq i \leq n$ , Eq. (13) can be written as

$$\begin{aligned}
 Q_\lambda \left( \bigcup_{i=1}^{n+1} A_i \right) &= \frac{1}{1 + \lambda Q_\lambda \left( \bigcup_{i=1}^n A_i^* \right)} \left( Q_\lambda \left( \bigcup_{i=1}^n A_i \right) + Q_\lambda(A_{n+1}) + \right. \\
 &\quad \left. + \lambda Q_\lambda \left( \bigcup_{i=1}^n A_i \right) Q_\lambda(A_{n+1}) - Q_\lambda \left( \bigcup_{i=1}^n A_i^* \right) \right).
 \end{aligned} \tag{14}$$

Next, using the inductive condition and the fact that  $Z_{n,\lambda}^{(k)} = p_{n,\lambda}^{(k)}(A_1, \dots, A_n)$  holds by definition for any  $1 \leq k \leq n$ ,  $Q_\lambda(\bigcup_{i=1}^n A_i^*)$  can be written as

$$Q_\lambda \left( \bigcup_{i=1}^n A_i^* \right) = \frac{1}{\lambda} \left( \prod_{k=1}^n \left( p_{n,\lambda}^{(k)}(A_1^*, \dots, A_n^*) \right)^{(-1)^{k-1}} - 1 \right).$$

Since  $Z_{n,\lambda}^{*(k)} = p_{n,\lambda}^{(k)}(A_1^*, \dots, A_n^*)$  holds by definition for any  $1 \leq k \leq n$ , the previous equation can be rewritten in the following form:

$$Q_\lambda \left( \bigcup_{i=1}^n A_i^* \right) = \frac{1}{\lambda} \left( \prod_{k=1}^n \left( Z_{n,\lambda}^{*(k)} \right)^{(-1)^{k-1}} - 1 \right). \tag{15}$$

Recall that based on Lemma 3, we have the following equation

$$Z_{n,\lambda}^{*(k)} = \begin{cases} \frac{Z_{n+1,\lambda}^{(k+1)}}{Z_{n,\lambda}^{(k+1)}}, & \text{if } k < n \\ Z_{n+1,\lambda}^{(n+1)}, & \text{if } k = n. \end{cases} \tag{16}$$

Applying Eq. (16) to Eq. (15) yields

$$\begin{aligned} Q_\lambda \left( \bigcup_{i=1}^n A_i^* \right) &= \\ &= \frac{1}{\lambda} \left( \left( \prod_{k=1}^{n-1} \left( Z_{n+1,\lambda}^{(k+1)} \right)^{(-1)^{k-1}} \left( Z_{n,\lambda}^{(k+1)} \right)^{(-1)^k} \right) \left( Z_{n+1,\lambda}^{(n+1)} \right)^{(-1)^{n+1}} - 1 \right). \end{aligned} \quad (17)$$

Next, based on Lemma 4,

$$Q_\lambda(A_{n+1}) = \frac{1}{\lambda} \left( \frac{Z_{n+1,\lambda}^{(1)}}{Z_{n,\lambda}^{(1)}} - 1 \right). \quad (18)$$

Now, applying the inductive condition given in Eq. (11) and substituting the formulas for  $Q_\lambda(\bigcup_{i=1}^n A_i^*)$  and  $Q_\lambda(A_{n+1})$  given by Eq. (17) and Eq. (18), respectively, into Eq. (14) gives us

$$\begin{aligned} Q_\lambda \left( \bigcup_{i=1}^{n+1} A_i \right) &= \\ &= \frac{\frac{1}{\lambda} (Y_1 - 1) + \frac{1}{\lambda} (Y_2 - 1) + \lambda \frac{1}{\lambda} (Y_1 - 1) \frac{1}{\lambda} (Y_2 - 1) - \frac{1}{\lambda} (Y_3 - 1)}{Y_3}, \end{aligned} \quad (19)$$

where

$$Y_1 = \prod_{k=1}^n \left( Z_{n,\lambda}^{(k)} \right)^{(-1)^{k-1}}$$

$$Y_2 = \frac{Z_{n+1,\lambda}^{(1)}}{Z_{n,\lambda}^{(1)}}$$

$$Y_3 = \left( \prod_{k=1}^{n-1} \left( Z_{n+1,\lambda}^{(k+1)} \right)^{(-1)^{k-1}} \left( Z_{n,\lambda}^{(k+1)} \right)^{(-1)^k} \right) \left( Z_{n+1,\lambda}^{(n+1)} \right)^{(-1)^{n+1}}.$$

Simplifying Eq. (19) leads to

$$Q_\lambda \left( \bigcup_{i=1}^{n+1} A_i \right) = \frac{1}{\lambda} \left( \frac{Y_1 Y_2}{Y_3} - 1 \right). \quad (20)$$

Now, by substituting the definitions of  $Y_1$ ,  $Y_2$  and  $Y_3$  into Eq. (20), after simplifi-



cation we get

$$\begin{aligned}
 Q_\lambda \left( \bigcup_{i=1}^{n+1} A_i \right) &= \\
 &= \frac{1}{\lambda} \left( \frac{\left( \prod_{k=1}^n \left( Z_{n,\lambda}^{(k)} \right)^{(-1)^{k-1}} \right) \frac{Z_{n+1,\lambda}^{(1)}}{Z_{n,\lambda}^{(1)}}}{\left( \prod_{k=1}^{n-1} \left( Z_{n+1,\lambda}^{(k+1)} \right)^{(-1)^{k-1}} \right) \left( \prod_{k=1}^{n-1} \left( Z_{n,\lambda}^{(k+1)} \right)^{(-1)^k} \right) \left( Z_{n+1,\lambda}^{(n+1)} \right)^{(-1)^{n+1}} - 1} - 1 \right) = \\
 &= \frac{1}{\lambda} \left( \frac{Z_{n+1,\lambda}^{(1)}}{\left( \prod_{k=1}^{n-1} \left( Z_{n+1,\lambda}^{(k+1)} \right)^{(-1)^{k-1}} \right) \left( Z_{n+1,\lambda}^{(n+1)} \right)^{(-1)^{n+1}} - 1} - 1 \right) = \\
 &= \frac{1}{\lambda} \left( \frac{Z_{n+1,\lambda}^{(1)}}{\left( Z_{n+1,\lambda}^{(2)} \right) \left( Z_{n+1,\lambda}^{(3)} \right)^{-1} \cdots \left( Z_{n+1,\lambda}^{(n+1)} \right)^{(-1)^{n+1}} - 1} - 1 \right) = \\
 &= \frac{1}{\lambda} \left( \left( Z_{n+1,\lambda}^{(1)} \right) \left( Z_{n+1,\lambda}^{(2)} \right)^{-1} \left( Z_{n+1,\lambda}^{(3)} \right) \cdots \left( Z_{n+1,\lambda}^{(n+1)} \right)^{(-1)^n} - 1 \right) = \\
 &= \frac{1}{\lambda} \left( \prod_{k=1}^{n+1} \left( Z_{n+1,\lambda}^{(k)} \right)^{(-1)^{k-1}} - 1 \right).
 \end{aligned}$$

Notice that this formula is the same as the formula for  $Q_\lambda \left( \bigcup_{i=1}^{n+1} A_i \right)$  given in Eq. (12), which means that we have proved this theorem.  $\square$

On the one hand, Theorem 1 tells us how to compute the  $\lambda$ -additive measure of union of  $n$  sets in the case when  $\lambda$  is nonzero. On the other hand, it immediately follows from the definition of  $\lambda$ -additive measure that if  $\lambda = 0$ , then the  $\lambda$ -additive measure on the finite set  $X$  is a probability measure on  $X$ . Hence, if  $X$  is a finite set,  $A_1, \dots, A_n \in \mathcal{P}(X)$ ,  $n \geq 2$ ,  $Q_\lambda$  is a  $\lambda$ -additive measure on  $X$  and  $\lambda = 0$ , then  $Q_\lambda \left( \bigcup_{i=1}^n A_i \right)$  can be computed by using the Poincaré formula of probability theory:

$$Q_\lambda \left( \bigcup_{i=1}^n A_i \right) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} Q_\lambda (A_{i_1} \cap \dots \cap A_{i_k}). \quad (21)$$

The following theorem shows how the Poincaré formula of probability theory given in Eq. (21) may be viewed as a limit case of the general formula of  $\lambda$ -additive measure of the union of  $n$  sets given in Eq. (10).

**Theorem 2.** *If  $X$  is a finite set,  $A_1, \dots, A_n \in \mathcal{P}(X)$ ,  $n \geq 2$ ,  $Q_\lambda$  is a  $\lambda$ -additive measure on  $X$  and  $\lambda \neq 0$ , then*

$$\lim_{\lambda \rightarrow 0} Q_\lambda \left( \bigcup_{i=1}^n A_i \right) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} Q_\lambda (A_{i_1} \cap \dots \cap A_{i_k}). \quad (22)$$

*Proof.* Let  $\lambda \neq 0$ . Here, we will distinguish two cases. Namely, (1) when  $n$  is even; and (2) when  $n$  is odd.

(1) If  $n$  is even, then based on Theorem 1,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} Q_\lambda \left( \bigcup_{i=1}^n A_i \right) &= \lim_{\lambda \rightarrow 0} \left( \frac{1}{\lambda} \left( \frac{Z_{n,\lambda}^{(1)} Z_{n,\lambda}^{(3)} \cdots Z_{n,\lambda}^{(n-1)}}{Z_{n,\lambda}^{(2)} Z_{n,\lambda}^{(4)} \cdots Z_{n,\lambda}^{(n)}} - 1 \right) \right) = \\ &= \frac{\lim_{\lambda \rightarrow 0} \left( \frac{1}{\lambda} \left( Z_{n,\lambda}^{(1)} Z_{n,\lambda}^{(3)} \cdots Z_{n,\lambda}^{(n-1)} - Z_{n,\lambda}^{(2)} Z_{n,\lambda}^{(4)} \cdots Z_{n,\lambda}^{(n)} \right) \right)}{\lim_{\lambda \rightarrow 0} \left( Z_{n,\lambda}^{(2)} Z_{n,\lambda}^{(4)} \cdots Z_{n,\lambda}^{(n)} \right)}, \end{aligned} \quad (23)$$

where

$$Z_{n,\lambda}^{(k)} = \prod_{1 \leq i_1 < \cdots < i_k \leq n} (1 + \lambda Q_\lambda(A_{i_1} \cap \cdots \cap A_{i_k})),$$

$1 \leq k \leq n$ . Definition of  $Z_{n,\lambda}^{(k)}$  implies that

$$\lim_{\lambda \rightarrow 0} \left( Z_{n,\lambda}^{(2)} Z_{n,\lambda}^{(4)} \cdots Z_{n,\lambda}^{(n)} \right) = 1. \quad (24)$$

Let  $F(\lambda; A_1, \dots, A_n) = Z_{n,\lambda}^{(1)} Z_{n,\lambda}^{(3)} \cdots Z_{n,\lambda}^{(n-1)} - Z_{n,\lambda}^{(2)} Z_{n,\lambda}^{(4)} \cdots Z_{n,\lambda}^{(n)}$ . Applying the definition of  $Z_{n,\lambda}^{(k)}$ , after direct calculations we get

$$\begin{aligned} F(\lambda; A_1, \dots, A_n) &= \\ &1 + \sum_{1 \leq i \leq n} \lambda Q_\lambda(A_i) + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \lambda Q_\lambda(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \cdots \\ &\cdots + \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} \lambda Q_\lambda(A_{i_1} \cap \cdots \cap A_{i_{n-1}}) + G(\lambda) - \\ &-1 - \sum_{1 \leq i_1 < i_2 \leq n} \lambda Q_\lambda(A_{i_1} \cap A_{i_2}) - \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \lambda Q_\lambda(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}) - \cdots \\ &\cdots - \sum_{1 \leq i_1 < \cdots < i_n \leq n} \lambda Q_\lambda(A_{i_1} \cap \cdots \cap A_{i_n}) - H(\lambda), \end{aligned}$$

where  $G(\lambda)$  and  $H(\lambda)$  are at least second order polynomials of  $\lambda$  in which the constant term is equal to zero. Thus,

$$\lim_{\lambda \rightarrow 0} \left( \frac{1}{\lambda} G(\lambda) \right) = 0, \quad \lim_{\lambda \rightarrow 0} \left( \frac{1}{\lambda} H(\lambda) \right) = 0$$

and so

$$\begin{aligned}
 & \lim_{\lambda \rightarrow 0} \left( \frac{1}{\lambda} F(\lambda; A_1, \dots, A_n) \right) = \\
 & = \sum_{1 \leq i \leq n} Q_\lambda(A_i) - \sum_{1 \leq i_1 < i_2 \leq n} Q_\lambda(A_{i_1} \cap A_{i_2}) + \\
 & + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} Q_\lambda(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} Q_\lambda(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}) + \\
 & \dots \\
 & + \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} Q_\lambda(A_{i_1} \cap \dots \cap A_{i_{n-1}}) - \sum_{1 \leq i_1 < \dots < i_n \leq n} Q_\lambda(A_{i_1} \cap \dots \cap A_{i_n}).
 \end{aligned}$$

That is, we have the following equation:

$$\begin{aligned}
 & \lim_{\lambda \rightarrow 0} \left( \frac{1}{\lambda} F(\lambda; A_1, \dots, A_n) \right) = \\
 & = \lim_{\lambda \rightarrow 0} \left( \frac{1}{\lambda} \left( Z_{n,\lambda}^{(1)} Z_{n,\lambda}^{(3)} \dots Z_{n,\lambda}^{(n-1)} - Z_{n,\lambda}^{(2)} Z_{n,\lambda}^{(4)} \dots Z_{n,\lambda}^{(n)} \right) \right) = \\
 & = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} Q_\lambda(A_{i_1} \cap \dots \cap A_{i_k}). \tag{25}
 \end{aligned}$$

Now, by substituting the formulas in Eq. (24) and Eq. (25) into Eq. (23), we get

$$\lim_{\lambda \rightarrow 0} Q_\lambda \left( \bigcup_{i=1}^n A_i \right) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} Q_\lambda(A_{i_1} \cap \dots \cap A_{i_k}).$$

(2) In the case where  $n$  is an odd number, the theorem can be proved by following steps similar to those of case (1).  $\square$

This result tells us that our general formula for the  $\lambda$ -additive measure of the union of  $n$  sets may be viewed as the generalization of the Poincaré formula of probability theory.

### 3 Summary and future plans

The key findings of this study can be summarized as follows.

- (1) We presented the general formula for the  $\lambda$ -additive measure of the union of  $n$  sets in Eq.(1), and gave an elementary proof of it in Theorem 1.
- (2) Using elementary techniques, we demonstrated that the Poincaré formula of probability theory given in Eq. (2) is just a limit case of the general formula for the  $\lambda$ -additive measure of the union of  $n$  sets given in Eq. (1); that is, our formula may be viewed as a generalization of the Poincaré formula.

In the future, we should like to formulate a calculus of the  $\lambda$ -additive measure and generalize the Bayes theorem for  $\lambda$ -additive measures. We also plan to study how the  $\lambda$ -additive measure and the generalized Poincaré formula can be utilized in the fields of computer science, engineering and economics.

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