



# A note on quadratic Poisson brackets on $\mathfrak{gl}(n, \mathbb{R})$ related to Toda lattices

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## Abstract

It is well known that the compatible linear and quadratic Poisson brackets of the full symmetric and of the standard open Toda lattices are restrictions of linear and quadratic  $r$ -matrix Poisson brackets on the associative algebra  $\mathfrak{gl}(n, \mathbb{R})$ . We here show that the quadratic bracket on  $\mathfrak{gl}(n, \mathbb{R})$ , corresponding to the  $r$ -matrix defined by the splitting of  $\mathfrak{gl}(n, \mathbb{R})$  into the direct sum of the upper triangular and orthogonal Lie subalgebras, descends by Poisson reduction from a quadratic Poisson structure on the cotangent bundle  $T^*GL(n, \mathbb{R})$ . This complements the interpretation of the linear  $r$ -matrix bracket as a reduction of the canonical Poisson bracket of the cotangent bundle.

**Keywords** Integrable systems · Toda systems ·  $r$ -matrix Poisson brackets · Poisson reduction · bi-Hamiltonian systems

**Mathematics Subject Classification** 70H06 · 37J15 · 37K10

## 1 Introduction

The goal of this brief communication is to illuminate the group theoretic origin of a certain quadratic  $r$ -matrix structure on the associative algebra  $\mathcal{G} := \mathfrak{gl}(n, \mathbb{R})$ . This Poisson structure is associated with the QU factorization and it appeared in the theory of integrable systems [5, 6]. Like the corresponding linear  $r$ -matrix bracket, it can be restricted to the Poisson submanifolds consisting of symmetric and of tridiagonal

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symmetric matrices [6], thereby producing the bi-Hamiltonian structures of the full symmetric and of the usual (open) Toda lattices [1, 10]. It is well known (see e.g., [8]) that the linear  $r$ -matrix bracket on  $\mathcal{G}$  is a reduction of the canonical Poisson bracket of the cotangent bundle of the group  $G := \text{GL}(n, \mathbb{R})$ . Our observation is that  $T^*G$  carries also a quadratic Poisson bracket that descends to the relevant quadratic bracket on  $\mathcal{G}$  via the same reduction procedure which works in the linear case. The idea arises from [3, 4], where bi-Hamiltonian structures for spin Sutherland models were obtained by reducing bi-Hamiltonian structures on the cotangent bundle of  $\text{GL}(n, \mathbb{C})$ .

We now recall the necessary background information about linear and quadratic  $r$ -matrix Poisson brackets on  $\mathcal{G}$ . This is a specialization of general results found in [5, 6] (see also [9]). Let  $R$  be a linear operator on  $\mathcal{G}$  that solves the modified classical Yang–Baxter equation.<sup>1</sup> Decompose  $R$  as the sum of its anti-symmetric and symmetric parts,  $R_a$  and  $R_s$ , with respect to the non-degenerate bilinear form,

$$\langle X, Y \rangle := \text{tr}(XY), \quad \forall X, Y \in \mathcal{G}, \tag{1.1}$$

and suppose that  $R_a$  solves the same equation as  $R$ . For a smooth real function on  $\mathcal{G}$  let  $df$  denote its gradient defined using the trace form (1.1), and introduce the ‘left- and right-derivatives’  $\nabla f$  and  $\nabla' f$  by

$$\nabla f(L) := Ldf(L), \quad \nabla' f(L) := df(L)L. \tag{1.2}$$

Then the following formula defines a Poisson bracket on  $\mathcal{G}$ :

$$\{f, h\}_2 := \langle \nabla f, R_a \nabla h \rangle - \langle \nabla' f, R_a \nabla' h \rangle + \langle \nabla f, R_s \nabla' h \rangle - \langle \nabla' f, R_s \nabla h \rangle. \tag{1.3}$$

The Lie derivative of this quadratic  $r$ -matrix bracket along the vector field  $V(L) := \mathbf{1}_n$  is the linear  $r$ -matrix bracket,

$$\{f, h\}_1(L) = \langle L, [Rdf(L), dh(L)] + [df(L), Rdh(L)] \rangle, \tag{1.4}$$

and thus the two Poisson brackets are compatible. The Hamiltonians  $h_k(L) := \frac{1}{k} \text{tr}(L^k)$  are in involution with respect to both brackets. They enjoy the relation

$$\{f, h_k\}_2 = \{f, h_{k+1}\}_1, \quad \forall f \in C^\infty(\mathcal{G}), \tag{1.5}$$

and their Hamiltonian vector fields engender bi-Hamiltonian Lax equations:

$$\partial_{t_k}(L) := \{L, h_k\}_2 = \{L, h_{k+1}\}_1 = [R(L^k), L], \quad \forall k \in \mathbb{N}. \tag{1.6}$$

Turning to the example of our interest, let us decompose any  $X \in \mathcal{G}$  as

$$X = X_> + X_0 + X_<, \tag{1.7}$$

<sup>1</sup> For reviews on  $r$ -matrices and their use, one may consult, for example, [8, 10].

where  $X_>$ ,  $X_0$  and  $X_<$  are the strictly upper triangular, diagonal and strictly lower triangular parts of the matrix  $X$ , respectively. Denote  $\mathcal{A} < \mathcal{G}$  the Lie subalgebra of skew-symmetric matrices and  $\mathcal{B} < \mathcal{G}$  the subalgebra of upper triangular matrices. They enter the vector space direct sum

$$\mathcal{G} = \mathcal{A} + \mathcal{B}, \tag{1.8}$$

and, using the projections  $\pi_{\mathcal{A}}$  onto  $\mathcal{A}$  and  $\pi_{\mathcal{B}}$  onto  $\mathcal{B}$ , yield the  $r$ -matrix

$$R = \frac{1}{2}(\pi_{\mathcal{B}} - \pi_{\mathcal{A}}). \tag{1.9}$$

In terms of the triangular decomposition (1.7),

$$\pi_{\mathcal{A}}(X) = X_< - (X_<)^T, \quad \pi_{\mathcal{B}}(X) = X_> + X_0 + (X_<)^T, \tag{1.10}$$

and

$$\begin{aligned} R(X) &= \frac{1}{2}(X_> + X_0 - X_<) + (X_<)^T, \\ R_a(X) &= \frac{1}{2}(X_> - X_<), \\ R_s(X) &= \frac{1}{2}X_0 + (X_<)^T. \end{aligned} \tag{1.11}$$

This  $r$ -matrix  $R$  satisfies the conditions stipulated above, and we are going to derive its quadratic bracket (1.3) by reduction of a Poisson structure on  $T^*G$ .

**Remark 1.1** The matrix space  $\text{mat}(n \times n, \mathbb{R})$  is primarily an associative algebra, and the notation  $\mathfrak{gl}(n, \mathbb{R})$  is usually reserved for its induced Lie algebra structure. In this paper,  $\mathfrak{gl}(n, \mathbb{R})$  is understood to carry both algebraic structures, i.e., we identify  $\mathfrak{gl}(n, \mathbb{R})$  with  $\text{mat}(n \times n, \mathbb{R})$  when using the associative product. This should not lead to any confusion.

## 2 The $r$ -matrix brackets from Poisson reduction

We start with the manifold

$$\mathfrak{M} := G \times \mathcal{G} = \{(g, L) \mid g \in G, L \in \mathcal{G}\}, \tag{2.1}$$

which is to be viewed as a model of  $T^*G$  obtained via right translations and the identification  $\mathcal{G}^* \simeq \mathcal{G}$  given by the trace form. For smooth real functions  $F, H \in C^\infty(\mathfrak{M})$ , the following formulae define two compatible Poisson brackets:

$$\{F, H\}_1(g, L) = \langle \nabla_1 F, d_2 H \rangle - \langle \nabla_1 H, d_2 F \rangle + \langle L, [d_2 F, d_2 H] \rangle, \tag{2.2}$$

and

$$\begin{aligned} \{F, H\}_2(g, L) &= \langle R_a \nabla_1 F, \nabla_1 H \rangle \\ &\quad - \langle R_a \nabla'_1 F, \nabla'_1 H \rangle + \langle \nabla_2 F - \nabla'_2 F, r_+ \nabla'_2 H - r_- \nabla_2 H \rangle \\ &\quad + \langle \nabla_1 F, r_+ \nabla'_2 H - r_- \nabla_2 H \rangle - \langle \nabla_1 H, r_+ \nabla'_2 F - r_- \nabla_2 F \rangle, \end{aligned} \tag{2.3}$$

where

$$r_{\pm} := R_a \pm \frac{1}{2} \text{id}. \tag{2.4}$$

The derivatives are taken at  $(g, L)$ ,

$$\begin{aligned} \langle \nabla_1 F(g, L), X \rangle &= \left. \frac{d}{dt} \right|_{t=0} F(e^{tX} g, L), \quad \langle \nabla'_1 F(g, L), X \rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} F(g e^{tX}, L), \quad \forall X \in \mathcal{G}, \end{aligned} \tag{2.5}$$

and  $\nabla_2 F(g, L) := L d_2 F(g, L)$ ,  $\nabla'_2 F(g, L) := d_2 F(g, L)L$  with  $d_2 F$  denoting the gradient with respect to the second argument. The first bracket is just the canonical one. The second one is obtained by a change of variables from the Heisenberg double [7] of the Poisson–Lie group  $G$  equipped with the Sklyanin bracket that appears in the first two terms of (2.3). In the corresponding complex holomorphic case, this is explained in detail in [3]. The compatibility of the two brackets also follows by the same Lie derivative argument that works in the complex case [3].

We are interested in the restriction of the Poisson brackets (2.2) and (2.3) to those functions on  $\mathfrak{M}$  that are invariant with respect to the group

$$S := A \times B \quad \text{with} \quad A := O(n, \mathbb{R}), \quad B := \exp(\mathcal{B}), \tag{2.6}$$

whose factors correspond to the Lie algebras  $\mathcal{A}$  and  $\mathcal{B}$  in (1.8). That is,  $B$  consists of the upper triangular elements of  $G$  having positive diagonal entries. The action of  $S$  on  $\mathfrak{M}$  is given by letting any  $(a, b) \in A \times B$  act on  $(g, L) \in \mathfrak{M}$  by the diffeomorphism

$$(g, L) \mapsto (agb^{-1}, aLa^{-1}). \tag{2.7}$$

Due to the QU factorization,<sup>2</sup> every  $S$  orbit in  $\mathfrak{M}$  admits a unique representative of the form  $(\mathbf{1}_n, L)$ . Therefore, we may associate to any smooth,  $S$  invariant functions  $F, H$  on  $\mathfrak{M}$  unique smooth functions  $f, h$  on  $\mathcal{G}$  according to the rule

$$f(L) := F(\mathbf{1}_n, L), \quad h(L) := H(\mathbf{1}_n, L). \tag{2.8}$$

<sup>2</sup> That is, due to the fact that the matrix multiplication  $m : A \times B \rightarrow G$  is a diffeomorphism.

Provided that the invariant functions close under the Poisson brackets on  $\mathfrak{M}$ , we may define the reduced Poisson brackets on  $C^\infty(\mathcal{G})$  by setting

$$\{f, h\}_i^{\text{red}}(L) := \{F, H\}_i(\mathbf{1}_n, L), \quad i = 1, 2. \tag{2.9}$$

In other words, in this situation the Poisson brackets descend to the quotient space  $\mathfrak{M}/S \simeq \mathcal{G}$ . The closure is obvious for the first Poisson bracket, and for the second one we prove it below.

**Proposition 2.1** *If  $F$  and  $H$  are invariant with respect to the  $S$  action (2.7), then their second Poisson bracket (2.3) takes the simplified form*

$$\begin{aligned} 2\{F, H\}_2 &= \langle \nabla_2 F, \nabla_2' H \rangle - \langle \nabla_2 H, \nabla_2' F \rangle \\ &+ \langle \nabla_1 F, \nabla_2' H + \nabla_2 H \rangle - \langle \nabla_1 H, \nabla_2' F + \nabla_2 F \rangle. \end{aligned} \tag{2.10}$$

*This formula implies that the Poisson bracket of two  $S$  invariant functions is again  $S$  invariant.*

**Proof** The invariance of  $F$  with respect to the action of one parameter subgroups of  $A$  and  $B$  leads to the conditions

$$\langle \nabla_1' F, X \rangle = 0, \quad \forall X \in \mathcal{B} \quad \text{and} \quad \langle \nabla_1 F + \nabla_2 F - \nabla_2' F, Y \rangle = 0, \quad \forall Y \in \mathcal{A}. \tag{2.11}$$

The first condition means that  $\nabla_1' F(g, L)$  is strictly upper triangular, and since the same holds for  $H$  we get

$$\langle R_a \nabla_1' F, \nabla_1' H \rangle = 0. \tag{2.12}$$

By using the second condition in (2.11), we are going to show that the contributions containing  $R_a$  cancel from all other terms of (2.3) as well. To do this, it proves useful to employ the direct sum decomposition  $\mathcal{G} = \mathcal{A} + \mathcal{A}^\perp$ , where  $\mathcal{A}^\perp$  consists of the symmetric matrices in  $\mathcal{G}$ . Accordingly, we may decompose any element  $Z \in \mathcal{G}$  as

$$Z = Z^+ + Z^- \quad \text{with} \quad Z^+ \in \mathcal{A}, \quad Z^- \in \mathcal{A}^\perp. \tag{2.13}$$

Then the second condition in (2.11) means that

$$(\nabla_1 F)^+ = (\nabla_2' F - \nabla_2 F)^+. \tag{2.14}$$

By using this together with the anti-symmetry of  $R_a$  and that  $R_a$  maps  $\mathcal{A}$  into  $\mathcal{A}^\perp$  and  $\mathcal{A}^\perp$  into  $\mathcal{A}$ , we derive the equalities,

$$\begin{aligned} \langle R_a \nabla_1 F, \nabla_1 H \rangle &= \langle R_a (\nabla_1 H)^-, (\nabla_2 F - \nabla_2' F)^+ \rangle \\ &- \langle R_a (\nabla_1 F)^-, (\nabla_2 H - \nabla_2' H)^+ \rangle, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \langle \nabla_1 F, R_a(\nabla'_2 H - \nabla_2 H) \rangle &= \langle R_a(\nabla_1 F)^-, (\nabla_2 H - \nabla'_2 H)^+ \rangle \\ &+ \langle R_a(\nabla'_2 F - \nabla_2 F)^+, (\nabla_2 H - \nabla'_2 H)^- \rangle. \end{aligned} \tag{2.16}$$

By adding up (2.15) and the terms in (2.16) together with (minus one times) their counterparts having  $F$  and  $H$  exchanged, one precisely cancels  $\langle \nabla_2 F - \nabla'_2 F, R_a(\nabla'_2 H - \nabla_2 H) \rangle$  in (2.3). Then the formula (2.10) results directly from (2.3). Having derived (2.10), one sees that the right-hand side of this expression is invariant under the action (2.7) of  $S$ . Indeed, this is a consequence of the fact that the derivatives of invariant functions are equivariant, meaning for example that we have

$$\begin{aligned} \nabla_1 F(agb^{-1}, aLa^{-1}) &= a(\nabla_1 F(g, L))a^{-1}, \\ \nabla_2 F(agb^{-1}, aLa^{-1}) &= a(\nabla_2 F(g, L))a^{-1}. \end{aligned} \tag{2.17}$$

This and the conjugation invariance of the trace imply the claim. □

The following lemma will be important below.

**Lemma 2.2** *The  $S$  invariant function  $F$  on  $\mathfrak{M}$  and the function  $f$  on  $\mathcal{G}$  related by (2.8) satisfy the relations*

$$\begin{aligned} \nabla_1 F(\mathbf{1}_n, L) &= (r_+ - R_s)(\nabla' f(L) - \nabla f(L)), \\ d_2 F(\mathbf{1}_n, L) &= df(L), \end{aligned} \tag{2.18}$$

where  $R_s$  and  $r_+$  are given by (1.11) and (2.4).

**Proof** The second relation is obvious, and it implies the identities  $\nabla_2 F(\mathbf{1}_n, L) = \nabla f(L)$  and  $\nabla'_2 F(\mathbf{1}_n, L) = \nabla' f(L)$ . Since  $\nabla_1 F(\mathbf{1}_n, L) = \nabla'_1 F(\mathbf{1}_n, L)$  by (2.5), we see from (2.11) that

$$\nabla_1 F(\mathbf{1}_n, L) = (\nabla_1 F(\mathbf{1}_n, L))_{>}, \tag{2.19}$$

where we applied the triangular decomposition (1.7). Then, noting that the anti-symmetric part of any  $X \in \mathcal{G}$  is  $X^+ = \frac{1}{2}(X - X^T)$ , it follows from the equality (2.14) that

$$\begin{aligned} (\nabla_1 F(\mathbf{1}_n, L))_{>} &= 2((\nabla_1 F(\mathbf{1}_n, L))^+)_{>} \\ &= (\nabla' f(L) - \nabla f(L))_{>} - ((\nabla' f(L) - \nabla f(L))_{<})^T. \end{aligned} \tag{2.20}$$

Because  $r_+ X = X_{>} + \frac{1}{2} X_0$  and  $R_s X = \frac{1}{2} X_0 + (X_{<})^T$  by (1.11) and (2.4), the statement (2.18) is obtained by combining (2.19) and (2.20). □

We now prove our claim about the reduction origin of the quadratic bracket (2.3), which we could not find in the literature. For completeness, we also show that the linear  $r$ -matrix bracket (1.4) descends from (2.2), which is a classical result [8].

**Theorem 2.3** *The reductions (2.9) of the Poisson brackets (2.2) and (2.3) on the cotangent bundle  $\mathfrak{M} \equiv T^*\text{GL}(n, \mathbb{R})$  (2.1) defined by taking quotient by the action (2.7) of the group  $S$  (2.6) give the linear (1.4) and quadratic (1.3)  $r$ -matrix brackets on  $\mathcal{G} = \mathfrak{gl}(n, \mathbb{R})$ , respectively.*

**Proof** We have to evaluate the expressions (2.9) for  $f$  and  $h$  related to the  $S$  invariant functions  $F$  and  $H$  by (2.8). We start with the second bracket, relying on (2.10). Substitution of the relations (2.18) into (2.10) gives

$$\begin{aligned} &\langle \nabla_1 F, \nabla'_2 H + \nabla_2 H \rangle - \mathcal{E}(F, H) \\ &= \langle r_+(\nabla' f - \nabla f) - R_s(\nabla' f - \nabla f), \nabla' h + \nabla h \rangle - \mathcal{E}(f, h), \end{aligned} \tag{2.21}$$

where  $\mathcal{E}(F, H)$  stands for the terms obtained by exchanging the roles of  $F$  and  $H$ , and similarly for  $f$  and  $h$ . Writing  $r_+ = R_a + \frac{1}{2}\text{id}$ , we find

$$\frac{1}{2} \langle \nabla' f - \nabla f, \nabla' h + \nabla h \rangle - \mathcal{E}(f, h) = \langle \nabla' f, \nabla h \rangle - \langle \nabla f, \nabla' h \rangle. \tag{2.22}$$

The terms containing  $R_a$  and  $R_s$  contribute

$$\begin{aligned} &\langle R_a(\nabla' f - \nabla f), \nabla' h + \nabla h \rangle - \mathcal{E}(f, h) \\ &= 2\langle R_a \nabla' f, \nabla' h \rangle - 2\langle R_a \nabla f, \nabla h \rangle, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} &\langle R_s(\nabla f - \nabla' f), \nabla' h + \nabla h \rangle - \mathcal{E}(f, h) \\ &= 2\langle R_s \nabla f, \nabla' h \rangle - 2\langle R_s \nabla' f, \nabla h \rangle. \end{aligned} \tag{2.24}$$

Plugging these identities into (2.10), we obtain the result

$$\{f, h\}_2^{\text{red}} = \langle \nabla f, R_a \nabla h \rangle - \langle \nabla' f, R_a \nabla' h \rangle + \langle \nabla f, R_s \nabla' h \rangle - \langle \nabla' f, R_s \nabla h \rangle, \tag{2.25}$$

which reproduces the quadratic  $r$ -matrix bracket (1.3).

To continue, we evaluate (2.9) for  $i = 1$ . Substitution of (2.18) now gives, at the appropriate arguments,

$$\begin{aligned} \langle \nabla_1 F, d_2 H \rangle &= \langle (R_a + \frac{1}{2}\text{id} - R_s)[df, L], dh \rangle \\ &= \langle L, [df, Rdh] \rangle - \frac{1}{2} [df, dh]. \end{aligned} \tag{2.26}$$

Here,  $R = R_a + R_s$  and we used the standard invariance properties of the trace form (1.1). Consequently, we get

$$\begin{aligned} &\langle \nabla_1 F, d_2 H \rangle - \langle \nabla_1 H, d_2 F \rangle + \langle L, [d_2 F, d_2 H] \rangle \\ &= \langle L, [Rdf, dh] + [df, Rdh] \rangle. \end{aligned} \tag{2.27}$$

The right-hand side gives  $\{f, h\}_1^{\text{red}}$ , which coincides with the linear  $r$ -matrix bracket (1.4).  $\square$

**Remark 2.4** Let us recall [5, 6] that  $\mathcal{G}$  carries also a cubic  $r$ -matrix Poisson bracket which is compatible with the linear and quadratic ones. It can be obtained from the linear bracket by performing the densely defined change of variables  $L \mapsto L^{-1}$ , and then extending the result to the full of  $\mathcal{G}$ . For completeness, we note that the same change of variables is applicable on  $T^*G$ , too, and the so-obtained Poisson bracket then leads to the cubic bracket on  $\mathcal{G}$  by the reduction procedure described in the above.

### 3 Discussion

We explained that the quadratic  $r$ -matrix bracket (1.3) of the ‘generalized Toda hierarchy’ (1.6) on  $\mathfrak{gl}(n, \mathbb{R})$  is a reduction of a quadratic Poisson bracket on  $T^*\text{GL}(n, \mathbb{R})$ . This observation escaped previous attention, probably because the convenient form (2.3) of the relevant parent Poisson bracket came to light only recently [3]. The integrability of the system (1.6) was thoroughly studied in [2] (see also [5]), together with two other related hierarchies. These are of the form (1.6), but instead of  $R$  (1.11) use either  $R'$  given by  $R'(X) := \frac{1}{2}(X_{>} + X_0 - X_{<})$  or  $R'' := R_a$  (which gives the anti-symmetric part of  $R'$ , too). We can show that the quadratic  $r$ -matrix brackets obtained from (1.3) by replacing  $R$  with  $R'$  or  $R''$  are also reductions of the bracket (2.3) on  $\mathfrak{M}$ , similarly to how the linear  $r$ -matrix brackets descend [8] from (2.2). In the case of  $R'$ , one may use the group  $S' := A' \times B$ , where  $A'$  is the exponential of the strictly lower triangular subalgebra of  $\mathcal{G}$ . In the case of  $R''$ , the reduction group is  $S'' < (G \times G)$  having elements of the form  $(a, b) = (e^{X_0}e^{X_{<}}, e^{-X_0}e^{Y_{>}})$  which act in the same way as (2.7). (The notation refers to (1.7) with arbitrary  $X, Y \in \mathcal{G}$ .) To be precise, in these cases one needs to restrict the starting system to  $T^*\check{G}$ , where the leading principal minors of the elements of  $\check{G}$  are positive, otherwise the reduction procedure is identical to the presented case, even the crucial equations (2.10) and (2.18) keep their form for the corresponding invariant functions. The open Toda phase space is well known [10] to be a Poisson submanifold with respect to the linear  $r$ -matrix brackets for any of  $R, R'$  and  $R''$ . However, in contrast to the case of  $R$  (1.11), it is not a Poisson submanifold with respect to the quadratic brackets associated with  $R'$  and  $R''$ . It would be interesting to find the reduction origin of the modified quadratic  $r$ -matrix brackets of Suris [9, 10] that are free from this difficulty. Another open problem is to extend our treatment of the quadratic brackets to spectral parameter-dependent  $r$ -matrices.

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