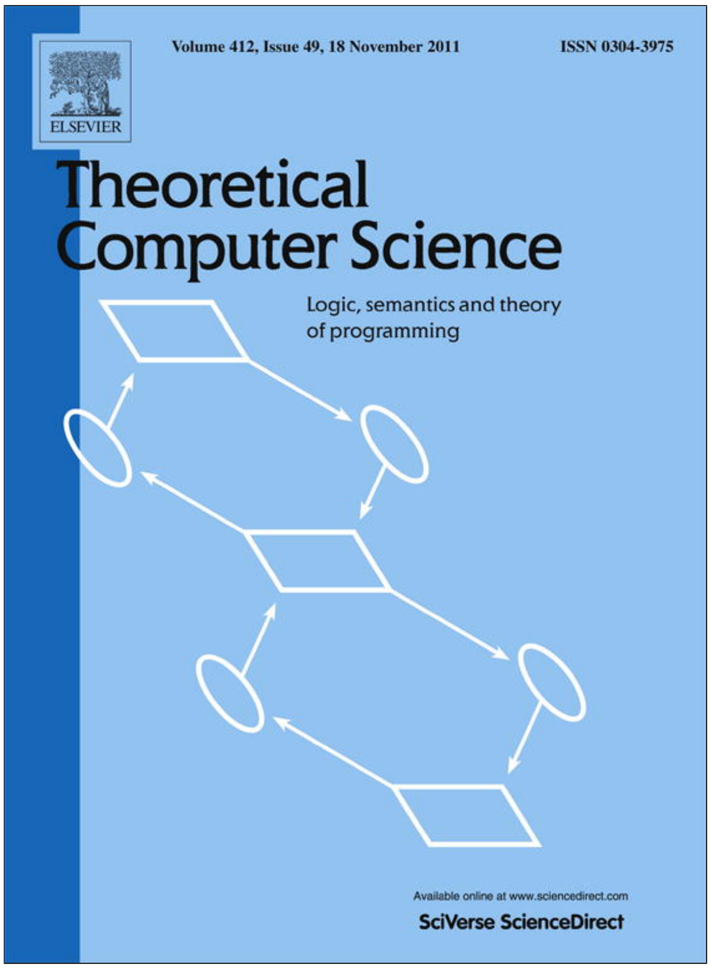
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A unifying theory of control dependence and its application to arbitrary program structures

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ABSTRACT

There are several similar, but not identical, definitions of control dependence in the literature. These definitions are given in terms of control flow graphs which have had extra restrictions imposed (for example, end-reachability).

We define two new generalisations of non-termination insensitive and non-termination sensitive control dependence called *weak and strong control-closure*. These are defined for all finite directed graphs, not just control flow graphs and are hence allow control dependence to be applied to a wider class of program structures than before.

We investigate all previous forms of control dependence in the literature and prove that, for the restricted graphs for which each is defined, vertex sets are closed under each if and only if they are either weakly or strongly control-closed. Low polynomial-time algorithms for producing minimal weakly and strongly control-closed sets over generalised control flow graphs are given.

This paper is the first to define an underlying semantics for control dependence: we define two relations between graphs called *weak and strong projections*, and prove that the graph induced by a set of vertices is a weak/strong projection of the original if and only if the set is weakly/strongly control-closed. Thus, all previous forms of control dependence also satisfy our semantics. Weak and strong projections, therefore, precisely capture the essence of control dependence in our generalisations and all the previous, more restricted forms. More fundamentally, these semantics can be thought of as *correctness criteria* for future definitions of control dependence.

1. Introduction

Control dependence is the relationship that exists between two vertices of a control flow graph (CFG) representing a program when one vertex determines whether or not the other can be executed. Informally, vertex v is said to control vertex w if v computes a value which determines whether w is executed or avoided. This fundamental concept in program analysis has been studied since the 1970s [15] and yet still produces new and surprising results. For example, recently

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[3,28,29], it was demonstrated that standard definitions of control dependence [16], in use for over two decades, were unsuitable for capturing control dependence in a wide class of reactive systems.

Control dependence is central to many program analysis and transformation techniques. For instance, it underpins work on program slicing [14,18,21,32], goto elimination [27] and compiler optimisations [16]. This paper focuses on the use of control dependence in program slicing [10,20,31], though the definitions and results concerning control dependence that the paper introduces also apply to other applications. The aim of program slicing is, given a chosen set of variables and chosen points in the program, to find a set of all statements which may affect the values of the variables at those points.

Slicing algorithms conventionally use two relations between statements in a program, or, more precisely, vertices in its control flow graph (CFG). These are *data dependence* and *control dependence*. Statement s is data dependent on statement t if t assigns a value to a variable v, say, which is referenced in s and there is a path from t to s with no intervening assignments to v.

Control dependence in program slicing can be understood by considering the program fragment represented by graph $G_{1(a)}$ in Fig. 1. Suppose we are interested in finding out which program statements contribute to the final values of variables x and y. The set of vertices of interest is therefore $\{g, h\}$ because these are the only vertices that correspond to statements that change the values of x and y. The vertices $\{\mathsf{start}, \mathsf{end}\}$ are also added since these are traditionally required in a CFG. This gives a starting set $\{g, h, \mathsf{start}, \mathsf{end}\}$.

We now see that predicates p_2 and p_3 both *control* which of g and h are executed, and in turn p_1 controls which of p_2 and p_3 is executed. To compute the vertices which control $\{g, h\}$, we thus build up a *closure* to finally arrive at the set $\{\mathsf{start}, g, h, p_2, p_3, p_1, \mathsf{end}\}$ which is *closed* under control dependence. These vertices are then reconnected to produce the slice $G_{1(c)}$. Vertices p_4 and k have been 'sliced away' and a new edge from p_2 to g has appeared.

Control dependence has a long history, throughout which authors have sought to capture the property for certain classes of program graphs of interest. The first authors to consider control dependence are widely regarded to be Denning and Denning in their seminal work on secure information flow [15], a topic which remains highly relevant to this day.

Weiser [33], was the first to express the Dennings' concept graph-theoretically in order to support slice construction. Subsequently, Ottenstein and Ottenstein [25] showed how Weiser's slicing could be captured as a graph reachability problem. Ferrante et al. [16] further developed these notions into a formal characterisation of the program dependence graph.[1]

In the 1990s, a generalisation of control dependence was defined by Bilardi and Pingali [6]. This generalisation is achieved by abstracting the notion of dominance to any set of paths. Generalised control dependence is thus 'parameterised' by this set. Instantiating different sets of paths yields different forms of control dependence. This, in effect, provides a framework for expressing different forms of control dependence. The value of the framework was demonstrated by using it to express those forms of control dependence known at the time including the weak control dependence of Podgurski and Clarke [26].

The program dependence graph, later extended to handle procedures as the System Dependence Graph [22], has formed the basis of many analyses, such as program slicing, since its introduction in the 1980s. However, more recently, building on the work of Podgurski and Clarke [26], Ranganath et al. [28,29] and Amtoft [3] developed new notions of control dependence for reactive systems. Ranganath et al. [29] showed that the definitions used up until then were inadequate to handle (increasingly prevalent) reactive systems, in which programs react to inputs continuously without termination. Such reactive programs are deliberately written to non-terminate. These programs, thus, have graphs that contain vertices from which end is not reachable.

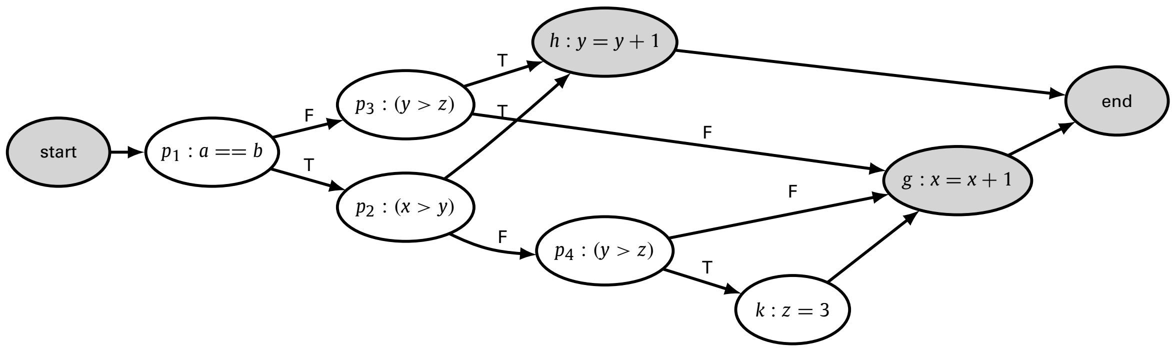
The wide range of applications of control dependence and its fundamental nature make it attractive to seek a simple, general characterisation that captures all previous definitions and which is also readily understood and from which it is easy to prove results for application areas. This paper introduces semantic definitions of control dependence, using a simple graph theoretic formulation, unhindered by specific restrictions on graph properties, such as constraints on the connectivity of the graph or presence or absence of certain structural features, such as special vertices.

1.1. Contributions of this paper

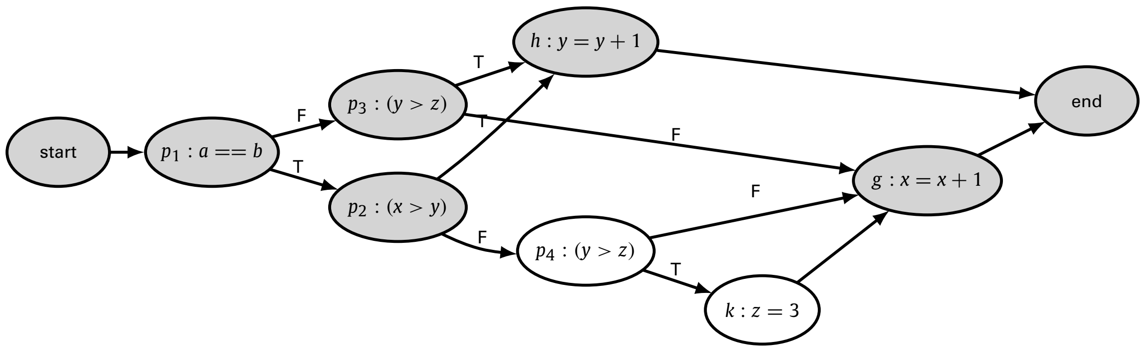
In this paper we develop a coherent theory of control dependence. There are four main contributions:

(1) We give two semantics for non-termination insensitive and non-termination sensitive control dependence by defining properties which must exist between graphs and graphs induced by subsets closed under control dependence in its different forms. We call these properties *weak* and *strong* projection.
(2) We introduce the notions of weak and strong control-closure and prove that graphs induced by sets satisfy our weak or strong semantics if and only if the sets are weakly or strongly control-closed respectively. Unlike conventional control dependence, weak and strong control-closure is defined not just for traditional control flow graphs but for all finite directed graphs.
(3) In program slicing, we require slices to be as small as possible. With this in mind, we give low order polynomial worst-case time complexity algorithms for computing the unique minimal weak and strong control-closed sets. These algorithms are functionally equivalent to previous ones but more generally applicable.

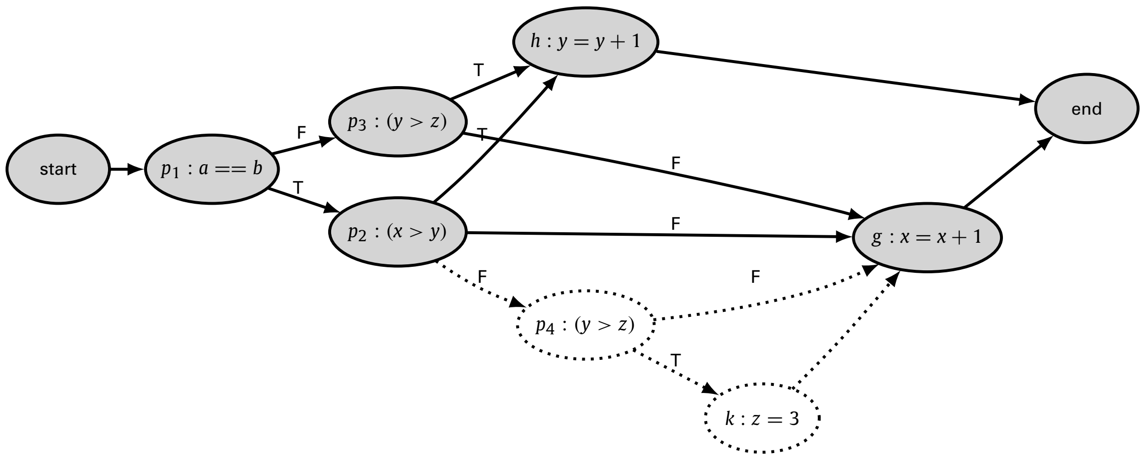
[1] Similar ideas are mentioned in [2,24,30,34].



(a) To slice $G_{1(a)}$ with respect to {start, h, g, end} . . . .



(b) . . . first, the statements (vertices) in the slice are computed: p_2 and p_3 are added because they control h and g and then p_1 is added because it controls p_2 and p_3. This gives the set {start, p_1, p_2, p_3, h, g, end} which is closed under control dependence.



(c) These statements are then rewired to produce the slice $G_{1(c)}$ above. Vertices p_4 and k have been 'sliced away' and a new edge from p_2 to g has appeared.

Fig. 1. The use of control dependence in program slicing.

(4) We believe that weak and strong projections capture the essence of control dependence. We demonstrate this by proving that the vertex sets closed under each form of control dependence in the literature induce graphs which are either weak or strong projections of the original. In so doing we provide a classification of all previous forms of control dependence into just two: weak and strong. Our algorithms can, thus, be used to compute sets closed under control dependence for a larger class of program structures.

1.2. Overview of the paper structure and results

This section provides an overview of the technical contributions of the paper and the structure within which they are presented in the remainder of the paper.

In Section 2, we define a generalised form of CFG upon which the rest of our theory is built. Our CFGs are finite, directed, labelled graphs. They are simple in the sense that we do not need variables, assignments and expressions (these are only needed for data-dependence). The edges are labelled with subsets of {T, F}. Our CFGs are deterministic in the sense that edges from the same vertex must be disjointly labelled. Many of the definitions of control dependence in the literature impose

Weak Control Dependence	Strong Control Dependence
$\xrightarrow{\text{W-controls}}$ [33]	
$\xrightarrow{\text{F-controls}}$ [16]	
$\xrightarrow{\text{WOD}}$ [3]	
	$\xrightarrow{\text{PC-weak}}$ [26]
	$\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ [29]

Fig. 2. Table showing the different forms of weak and strong control dependence.

constraints upon the types of graph for which they are defined. Our graphs encompass all those previously considered in the literature. They need not have a special end vertex which, when reached, represents successful termination. If it exists, the end vertex need not be reachable from all vertices. We allow all vertices to have out-degree zero and we allow predicates to have out-degree one. Leaving such 'incomplete' vertices corresponds to our program failing which we think of as reaching a state of silent non-termination. Intuitively, this can be thought of as the program appearing to do nothing but never returning to the operating system prompt. Incomplete vertices give rise to finite yet 'non-terminating' paths. Using the language of process algebra [23], we imagine the program infinitely engaging in (silent) τ-actions after reaching an incomplete vertex. It will be shown that graphs with incomplete vertices arise naturally in constructing minimal strong projections from deliberately non-terminating CFGs. Finally, our CFGs do not require a special start vertex.

We define a number of useful graph-theoretic concepts including V'-intervals and V'-paths, where V' is a set of vertices. A V'-interval is a path whose initial and final elements are both in V' but the intermediate ones are not. A V'-path is a path of length at least two whose final element lies in V', its first element may be in V', but none of its intermediate elements are in V'.

In Section 3, we describe the 'rewiring' problem: given a CFG G and a subset V' of the vertices of G (representing the statements in the slice of G), how do we connect the elements of V' and relabel the new edges in a 'sensible' way? We call this the graph *induced* by V' from G. Rewiring is achieved by connecting vertices v_1 and v_2 in G' if and only if there is a V'-interval connecting them[2] in G. Edge labels are formed by taking the union of 'corresponding' edges in the original.

In Section 4, we define weak and strong projections. These are semantic relations which exist between graphs and the graphs induced from them by subsets of vertices closed under termination insensitive and termination sensitive control dependence respectively. These projections are defined in terms of *walks*. A walk is very similar to a path, but elements which are predicates also include the boolean value in {T, F} representing the 'choice' that was taken at that predicate. A weak projection is a graph where every walk of the original, when restricted to the vertices of the projection, is a walk of the projection. This is analogous to the situation that arises in conventional slicing where if the original executes n times a statement that is also in the slice then the slice, when executed from the same initial state, also executes the statement n times. As is well known, in conventional slicing we may execute this statement more times in the slice than in the original program because, for example, non-terminating loops may have been sliced away. This is also the case with weak projections.

The graph induced from a CFG G by V' is not necessarily, itself, a CFG. In general, it may contain non-predicate vertices of out-degree greater than one, predicates of out-degree greater than two, or non-disjoint edge labellings. We prove (Proposition 20) that being a weak projection is no more than a by-product resulting from ensuring that the induced graph is indeed a CFG. In other words, if the induced graph from a CFG is a CFG then it must be a weak projection of the original too.

Weak projection captures the behaviour of slices produced using the weak forms of control dependence (see Fig. 2.) A *stronger* semantics is, however, required for slices produced using the strong forms of control dependence. With this aim, in Section 4.3, we define a strong projection. In the case of strong projection, every *maximal* walk of the original, when restricted to the vertices of the projection, gives rise to a *maximal* walk of the projection. Moreover, every walk of the projection arises in this way. For strong projection the number of times a walk passes through a vertex in the projection must be *equal* to the number of times the corresponding walk in the original passes though the vertex.

In Section 4.4, we observe that both weak and strong projections either may or may not preserve the termination conditions of the original. This is also true of the so-called non-termination sensitive control dependence of Ranganath et al. [29]. Termination and walk preservation are orthogonal conditions. The weak projection of a CFG G may terminate when G does not. Strong projections, on the other hand, are always non-termination preserving. If a weak or strong projection of a CFG contains end then it termination preserving. Strong projections containing end of a CFG G, thus, perfectly preserve the termination and non-termination of G.

In Section 5, we develop a theory of *weak and strong control-closure*, generalisations of non-termination sensitive and insensitive control dependence. We prove that these are properties of vertex sets which are necessary and sufficient to induce graphs which are weak and strong projections. Weak and strong control-closure are used both in the production of algorithms for producing minimal weak and strong projections and also in the proofs that classify the previous forms of control dependence as either weak or strong.

[2] This is not the first paper to define rewiring in this way. Earlier work [1,19] uses very similar definitions.

In Sections 5.1 and 5.2, we define weakly and strongly control-closure. In Section 5.3, we investigate graphs induced by weakly control-closed sets. The main result of Section 5.3 is Theorem 45 which states that the following three statements are all equivalent:

- The graph induced by V' from G is a CFG.
- V' is weakly control-closed in G.
- The graph induced by V' from G is a weak projection of G.

In Section 5.4, we investigate graphs induced by strongly control-closed sets. The main result of Section 5.4 is Theorem 49 which states that the graph induced by V' from G is a strong projection if and only if V' is strongly control-closed in G.

In Section 5.5, we prove Theorems 54 and 58 which state that given any set V' of vertices in a CFG, there are unique minimal weakly and strongly control-closed sets containing V'. These are the sets that are closed under control dependence which are required for slicing. This proves that for any vertex subset V' of CFG G, unique minimal weak and strong projections (slices) containing V' exist.

In Section 6, algorithms for computing minimal weak and strongly control-closed sets containing V' are defined and proved correct. This demonstrates that minimal weak and strong projections (slices) containing V' are computable. Furthermore, we show that these algorithms have worst-case time complexity $O(|V|^3)$, and $O(|V|^4)$ respectively where $|V|$ is the number of vertices of G. This is of a very similar order to the worst-case time complexity of the algorithms for computing the new control dependences of Ranganath et al. which is $O(|V|^4 \log |V|)$. It is likely that the efficiency of these algorithms can be improved, but this is a topic for future work and is beyond the scope of this paper.

In Section 7, we categorise the weak forms of control dependence in the literature.

They are:

- $\xrightarrow{\text{W-controls}}$: the control dependence of Weiser [33],
- $\xrightarrow{\text{F-controls}}$: the control dependence of Ferrante et al. [16] and
- $\xrightarrow{\text{WOD}}$: the weak order dependence of Amtoft [3].

The results of this section give the relationship between sets closed under the weak forms of control dependence mentioned above and weakly control-closed sets.

From these, we prove our main result, Theorem 67, which states that, indeed, all weak forms of control dependence in the literature induce weak projections.

In Section 8, we categorise the strong forms of control dependence in the literature. We call them *strong* because, as we show in this section, vertex sets closed under them induce strong projections. They are:

- the combination of $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ of Ranganath et al. [29].
- $\xrightarrow{\text{PC-weak}}$, the weak control dependence of Podgurski and Clarke [26].

The results of this section give the relationship between sets closed under the strong forms of control dependence mentioned above and strongly control-closed sets. From these, we can prove our main result of this section, Theorem 86, which shows that, indeed, both strong forms of control dependence in the literature induce strong projections. Sections 7 and 8, as well as semantically characterising current forms of control dependence, justify our claim that weak and strong projections capture the essence of control dependence.

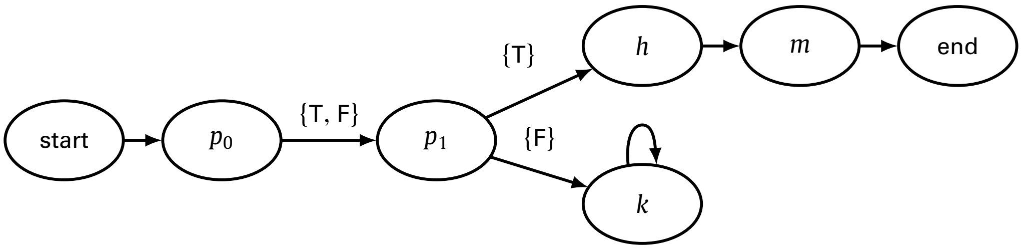
In Section 9, we conclude and give directions for future work. Section 10 is a glossary of all the main definitions and results of the paper. We encourage readers to refer to this whilst reading the paper.

2. Generalised CFGs for control dependence

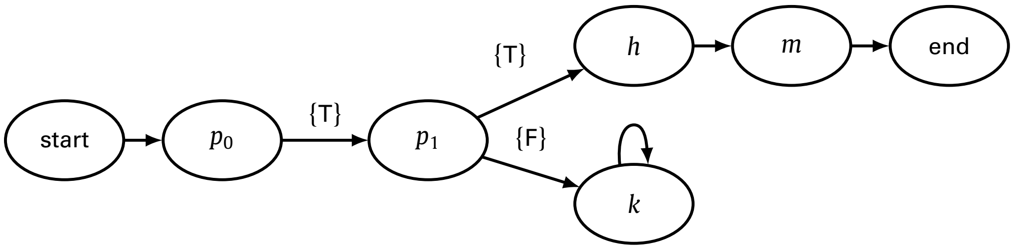
Our graphs, since we are exploring only control dependence, do not need the variables, assignments and expressions which would be required for data-dependence. We can therefore have a very simple definition of a CFG. Our CFGs are finite directed graphs whose vertices are either predicates or non-predicates. Non-predicates have out-degree of at most one and predicates of out-degree of at most two. The edges emerging from predicates are labelled with subsets of $\{T, F\}$. This allows for a predicate with one edge labelled $\{T, F\}$ to represent a predicate where both branches go to the same destination. The labellings of the edges from each predicate must be disjoint. In other words, our CFGs are deterministic.

There is at most one special non-predicate vertex called end of out-degree zero. Reaching end corresponds to termination. Unlike conventional CFGs, we allow other non-predicates to have out-degree zero and predicates (incomplete) to have edges whose union of labels is not $\{T, F\}$. Reaching such vertices corresponds to a program silently non-terminating. We imagine programs which reach such vertices apparently not performing any actions but also not returning to the 'operating system prompt'. This situation arises naturally when slicing away infinite loops when preserving termination properties. Unlike, conventional CFGs we do not insist on a special start vertex.

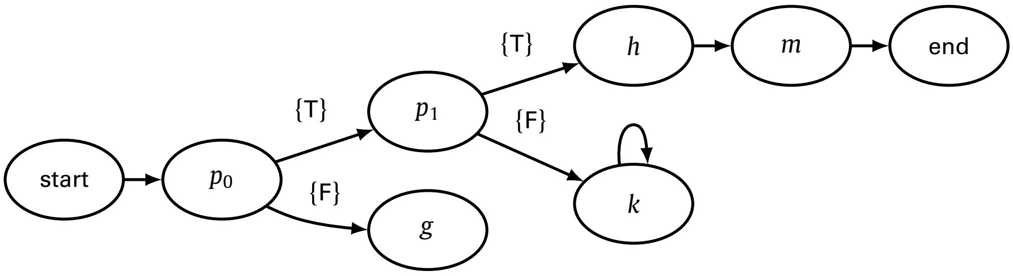
Definition 1 (*CFGs*)**.** A *control flow graph* (CFG) is a triple $G = (V, E, \beta)$ where (V, E) is a finite directed graph and the vertex set V is partitioned as $V = P \cup N$ (predicates and non-predicates) with $P \cap N = \emptyset$, and $\beta : E \to \mathcal{P}(\{T, F\})$ is the edge labelling function.



(a) $G_{3(a)}$: a CFG with a complete predicate p_0 of out-degree 1.



(b) $G_{3(b)}$: a CFG with an incomplete predicate p_0.



(c) $G_{3(c)}$: a CFG with a final non-end, non-predicate vertex g.

Fig. 3. Examples of CFGs: In Fig. 3(a) the predicate p_0, although of out-degree 1 is complete. Its successor is independent of evaluating p_0. In Fig. 3(b) the predicates are $\{p_0, p_1\}$. Vertex p_0 is incomplete since the union of the labels of its branches is not {T, F}. If predicate p_0 evaluates to F, a state representing silent non-termination is reached. In Fig. 3(c) the predicates are $\{p_0, p_1\}$ and both are complete. The other non-end vertices are non-predicates and have out-degree 1 except for g which is a final non-end vertex. After executing g, again all programs represented by $G_{3(c)}$ are deemed to silently non-terminate.

(1) • If $x \in P$ then the out-degree of x is at most 2.
  • If $x \in N$ then the out-degree of x is at most 1.
  • There is at most one end vertex. It has out-degree 0. (end $\in N$ is the only vertex which represents normal termination.)

(2) The edges are labelled by β where:
  • If $x \in P$ and $(x, y) \in E$ then $\beta(x, y) \neq \emptyset$.
  • If $x \in N$ and $(x, y) \in E$ then $\beta(x, y) = \emptyset$.
  (For clarity we omit the label $\emptyset$ from our diagrams.)

(3) Let p be a predicate. If $(p, y) \in E$ and $(p, z) \in E$ with $y \neq z$ then $\beta(p, y) \cap \beta(p, z) = \emptyset$. (In other words, our CFGs are deterministic.)

See Fig. 3 for examples of CFGs.

Definition 2 (*Complete Predicates of a CFG*)**.** A predicate is complete if and only if the union of the labels of its outgoing edges is {T, F}.

Definition 3 (*Complete CFGs*)**.** A CFG is complete if and only if all its predicates are complete.

Definition 4 (*Final Vertices of a CFG*)**.** A final vertex is either a non-predicate vertex of out-degree 0 or an incomplete predicate.

2.1. Specialised classes of graphs and CFGs

The previous forms of control dependence variously make references to a unique start or end of the program or CFG. The following definition gives notation for the required classes of specialised graphs and CFGs.

Definition 5 ({start, end}-*CFGs and Graphs*)**.** Let $G = (V, E)$ be a finite directed graph.

(1) If G has a unique distinguished vertex start $\in V$ and every $v \in V$ is reachable from start then G is a {start}-graph.
(2) If G has a unique distinguished vertex end $\in V$ that is reachable from every vertex $v \in V$ then G is an {end}-graph.
(3) If G is a {start}-graph and G is also an {end}-graph then G is a {start, end}-graph.

If $G = (V, E, \beta)$ is a CFG and the graph (V, E) is a {start}-graph then we call G a {start}-CFG, etc.

2.2. Useful graph-theoretic definitions

Definition 6 (*Paths of a Graph*). A path in a graph $G = (V, E)$ is a sequence of vertices $v_1, \ldots, v_i, v_{i+1}, \ldots$ with $(v_i, v_{i+1}) \in E$ for all i.

Definition 7 (*Proper Paths*). A path is *proper* if its initial and final vertices are distinct.

Definition 8 (*Prefixes*). A prefix of a path π is a path ρ such that there exists a path σ with $\pi = \rho\sigma$ (the concatenation of ρ and σ). Note π is a prefix of itself. Write $\rho \sqsubseteq \pi$. If $\rho \sqsubseteq \pi$ and $\rho \neq \pi$, ρ is called a 'proper' prefix of π.

Definition 9 (V'*-intervals and* V'*-paths*). Let $G = (V, E)$ be a graph and let $V' \subseteq V$.

- A V'-interval is a finite path of length > 1 in G where only the first and last elements are in V'.
- An $[l, m]$ V'-interval is a V'-interval that starts at $l \in V'$ and ends at $m \in V'$.
- A V'-path [17] is a finite path $v_1 \ldots v_m$ in G where $m > 1$, $v_m \in V'$ and $1 < i < m \Rightarrow v_i \notin V'$.

A V'-path is a path whose last element is in V' and whose first element may be in V' but none of the other elements are in V'. A V'-interval is a V'-path but not necessarily the converse.[3]

Definition 10 (*Complete Paths of a CFG*). A complete path is either an infinite path or a finite path whose last vertex is final.

Definition 11 (*Terminating Paths of a CFG*). A terminating path is a finite path whose last vertex is end.

Definition 12 (*Non-terminating Paths of a CFG*). A non-terminating path is a complete path which is either infinite or whose last vertex is not end.

2.3. Examples

In Fig. 3(c) the only terminating paths are those whose final vertex is end. The complete paths ending at g although finite are considered non-terminating because $g \neq$ end. Complete paths through k are infinite and hence non-terminating.

$G_{4(a)}$ in Fig. 4(a) is an example of a CFG where a predicate (p_1) is incomplete. Therefore p_1 is a final vertex and there are complete paths of $G_{4(a)}$ which end at p_1. These complete paths correspond to the situation where the predicate expression at p_1 evaluates to T.

$G_{4(b)}$ in Fig. 4(b) is an example of a CFG without an end vertex. All complete paths of $G_{4(b)}$, including the finite ones, are thus non-terminating.

$G_{4(c)}$ in Fig. 4(c) is not a CFG. If p_0 evaluates to T there is a choice of which edge to follow.

3. The induced graph

Given a program and a slicing criterion, a slicing algorithm produces a subset of the statements of the program, containing the slicing criterion. Normally, the slice produced is a valid and executable program. For a CFG $G = (V, E, \beta)$ the slicing criterion is a set $C \subseteq V$ of the vertices and a slicing algorithm will produce a possibly larger subset $C \subseteq V' \subseteq V$. The problem now is to make a valid CFG from V' and G.

In this section we show how to define a graph on such a subset by 'rewiring' the edges of the original CFG — this is called the *induced graph*.

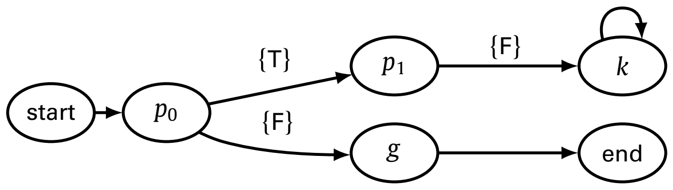
For any CFG $G = (V, E, \beta)$ and $V' \subseteq V$, we can construct a new graph on V' by connecting $x \in V'$ to $y \in V'$ if and only if there is a $[x, y]$ V'-interval (Definition 9). The new edge $(x, y) \in E'$ is labelled by the union of the labels of the edges $(x, x') \in E$ where x' is a successor of x from which there is a path in G to y.

Definition 13 (*The Induced Graph*). Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. The graph induced by V' from G has edge set $E' \subseteq V' \times V'$ where $(x, y) \in E'$ if and only if there is a V'-interval $x, \ldots, y$ in G. In the graph induced by V' from G,

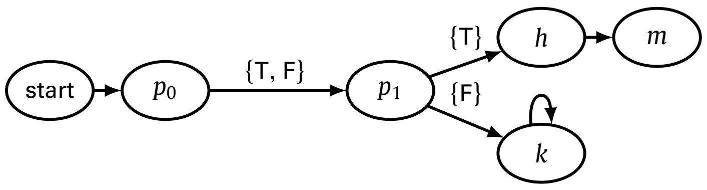
$$\beta'(x, y) = \bigcup_{x' \in K} \beta(x, x')$$

where $K = \{x' \in V \mid (x, x', \ldots, y) \text{ is a } V'\text{-interval}\}$.

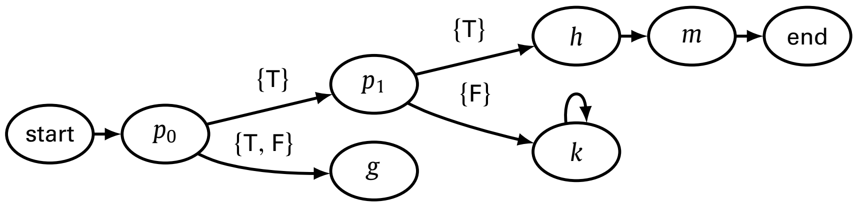
The predicates and non-predicates of the graph induced by V' from G are deemed to be $V' \cap P$ and $V' \cap N$, where P and N are the predicates and non-predicates of G respectively.



(a) $G_{4(a)}$: Predicate p_1 is not complete.



(b) $G_{4(b)}$: A CFG without end.



(c) $G_{4(c)}$: Not a CFG (non-deterministic).

Fig. 4. (a) and (b) are CFGs but (c) is not. In (a) the final vertices are p_1 and end. In (b) the only final vertex is m and the absence of end means that all complete paths are non-terminating.

Fig. 5 gives three examples of induced graphs, one of which is a CFG and two of which are not. Fig. 5(c) shows the graph induced by {start, h, g} from G_5. It has an edge (start, h) because there is a {start, h}-interval, startp_0p_1h, in G_5. Similarly, it has an edge (start, g). In general, the induced graph is not a CFG because the rewiring may increase the out-degree of vertices, and may destroy the necessary disjointness property of edge labels in a CFG. Clearly the graph induced by {start, h, g} from G_5 is not a CFG as it has a non-predicate vertex start of out-degree greater than 1. Similarly, the graph induced by $\{p_0, h, g, k\}$ from G_5 is not a CFG because it has a predicate vertex p_0 with non-disjoint edge labels.

In the next section we will show that for the graph induced by V' from G to be a CFG, V' must be *weakly control-closed* in G.

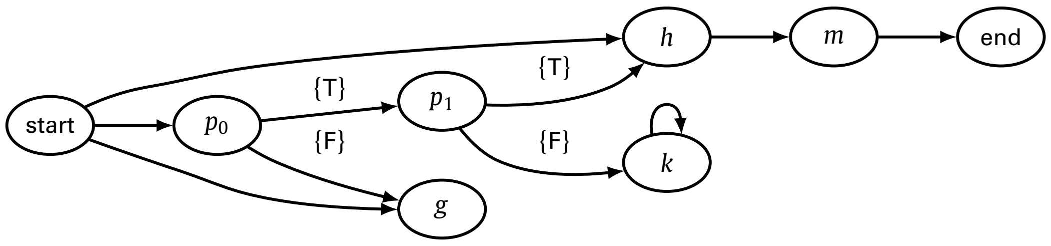
4. Weak and strong projections: a semantics of control dependence

In this section, we define weak and strong projections; properties that are preserved between graphs and graphs induced by sets closed under control dependence. The concepts of weak and strong projections are a semantics of non-termination insensitive and non-termination sensitive control dependence respectively. Later, in Sections 7 and 8, we demonstrate this by showing that all forms of control dependence in the literature are special cases of weak and strong projections. This provides strong evidence for our belief that these projections capture the underlying intention and essence of control dependence.

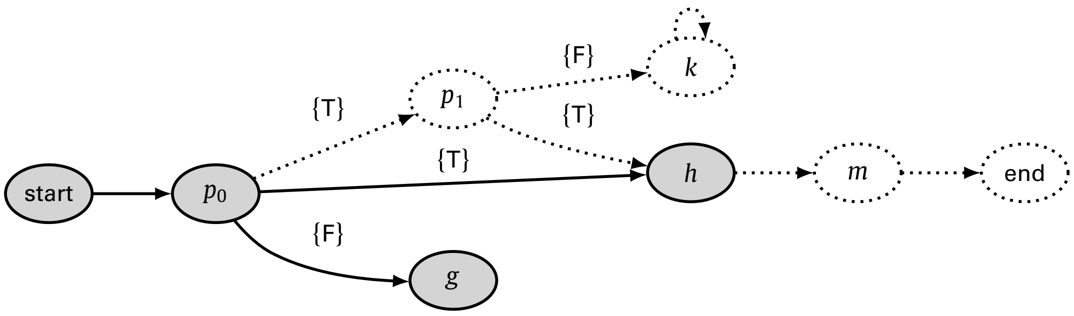
The semantic relationship between programs and their slices has been well studied [5,7–9,11–13]. Weak projection expresses a property analogous to the property of conventional slicing where if the program executes statement s n times then the slice, when executed from the same initial state, also executes s at least n times (if, of course, s is in the slice). In conventional slicing, s may execute more times in the slice than in the original program because, for example, non-terminating loops that prevent s being reached may have been sliced away. This is also true of weak projections. Strong projections, on the other hand, are analogous to the form of slice where for all initial states, s will execute *exactly* the same number of times in the slice as in the original.

We define weak and strong projections in terms of *walks* and give necessary and sufficient conditions for subsets of vertices of a CFG to induce weak and strong projections respectively. It turns out that the graph induced by V' from G is a weak projection of V' if and only if V' is weakly control-closed in G (Theorem 45). We go on to define a stronger condition, Strongly control-closed sets, and prove an analogous result, Theorem 49, for strong projections.

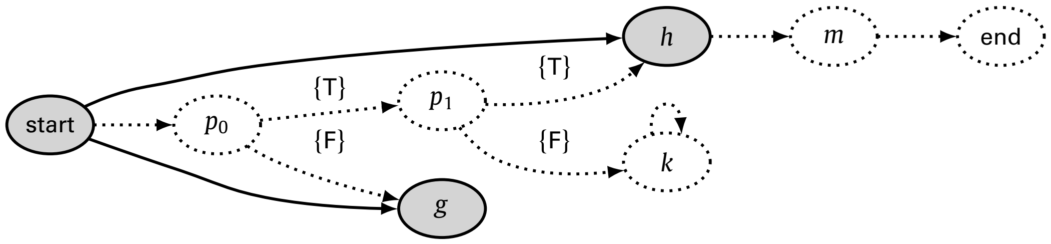
[3] Ranganath et al. [29] define a similar concept: the *first observable elements* from v, written $\text{obs}^1_{may}(v)$ is the set of *first* elements at the end of a V'-path from v.



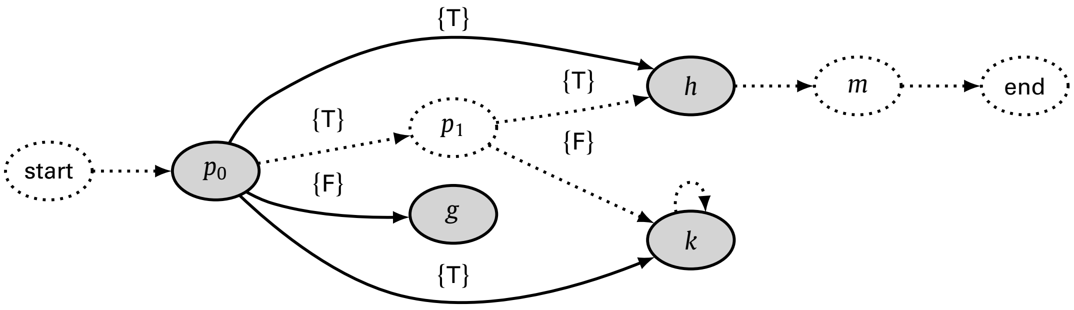
(a) CFG G_5.



(b) The graph induced by {start, p_0, h, g} from G_5 is a CFG.



(c) The graph induced by {start, h, g} from G_5 is not a CFG because it has a non-predicate vertex of out-degree greater than one.



(d) The graph induced by {p_0, h, g, k} from G_5 is not a CFG because it has a predicate vertex with non-disjoint edge labels.

Fig. 5. Inducing graphs from G_5 and different sets of vertices (Definition 13). Dotted edges and vertices represent those that are removed in producing the induced graph. Solid edges and labels represent those that remain in the induced graph.

4.1. *Walks of a* CFG

Conventional semantic slices are defined in terms of program executions, so we first define an analogous concept for CFGs using *walks*. A walk of a CFG is similar to a graph-theoretic path of a CFG, except that predicate vertices are replaced with a pair (p, B) where $\mathsf{B} \in \{\mathsf{T}, \mathsf{F}\}$ to represent evaluation of p. Notice, our paths and walks can start at *any* vertex in the graph.

Definition 14 (*Elements*)**.** Let $G = (V, E, \beta)$ be a CFG. An *element* w is either a vertex $v \in N \subseteq V$, or a pair (p, B) where $p \in P \subseteq V$ and $\mathsf{B} \in \{\mathsf{T}, \mathsf{F}\}$. Write $\bar{w}$ for the vertex component of an element and $\bar{\bar{w}}$ for the second (boolean) component of the pair when it exists.

Definition 15 (*Walks*)**.** Let $G = (V, E, \beta)$ be a CFG. A walk ω in G is a sequence $w_1, w_2, \ldots, w_i, \ldots$ of elements where:

(1) $\bar{\omega} = \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_i, \ldots$ is a path in G; and
(2) if w_i, w_{i+1} are consecutive elements of ω and $\bar{w}_i$ is a predicate vertex then $\bar{\bar{w}}_i \in \beta(\bar{w}_i, \bar{w}_{i+1})$.

For example the CFG $G_{4(b)}$ in Fig. 4 has a path $\pi_1 = p_0, p_1, h, m$ and there are two walks:

$$\omega_1 = (p_0, \mathsf{T}), (p_1, \mathsf{T}), h, m$$
$$\text{and } \omega_2 = (p_0, \mathsf{F}), (p_1, \mathsf{T}), h, m$$

which give rise to path π_1.

Observe that start, (p_0, F) is a walk of $G_{3(b)}$ in Fig. 3 although p_0 does not have a false branch. This walk cannot go any further. It is an example of a finite maximal walk caused by incomplete predicates.

Definition 16 ($\overrightarrow{G}$)**.** Let G be a CFG. $\overrightarrow{G}$ is the set of all walks in G.

4.2. *Weak projections of a* CFG

Restricting a path to a set of vertices means removing all the vertices not in the set.

Definition 17 (*Path Restriction*)**.** Let $G = (V, E)$ be a graph, let $V' \subseteq V$, and let π be a path in G. $\pi\downarrow V'$ is the subsequence of π obtained by removing all vertices v of π where $v \notin V'$. We say $\pi\downarrow V'$ is the *restriction* of π to V'.

Analogously, restricting a walk to a set of vertices means removing all the elements whose vertex component is not in the set.

Definition 18 (*Walk Restriction*)**.** Let $G = (V, E, \beta)$ be a CFG, let $V' \subseteq V$, and let ω be a walk in G. Define $\omega\downarrow V'$ to be the subsequence of ω obtained by removing all elements ω_i of ω where $\bar{\omega}_i \notin V'$. We say $\omega\downarrow V'$ is the *restriction* of ω to V'.

Note that $\overline{\omega\downarrow V'} = \bar{\omega}\downarrow V'$.

Definition 19 (*Weak Projections*)**.** Given a CFG $G = (V, E, \beta)$, a CFG $G' = (V', E', \beta')$ $(V' \subseteq V)$ is a *weak projection* of G if and only if and every walk of G when restricted to V', is a walk of G'. *i.e.*,

$$\omega \in \overrightarrow{G} \implies \omega\downarrow V' \in \overrightarrow{G'}.$$

Fig. 6 gives some examples of weak projections. Here $G_{6(b)}$ is a weak projection of $G_{6(a)}$. The walks of $G_{6(a)}$ are all the segments[4] of the three walks:

start, $(p_0, \mathsf{T}), (p_1, \mathsf{F}), k$, end
start, $(p_0, \mathsf{T}), (p_1, \mathsf{T}), h, m$, end
start, $(p_0, \mathsf{F}), g$, end

and the walks of $G_{6(b)}$ are all the segments of the two walks:

start, $(p_0, \mathsf{T}), h$
start, $(p_0, \mathsf{F}), g$.

Every walk of $G_{6(a)}$ when restricted to the vertices $\{\text{start}, p_0, h, g\}$ of $G_{6(b)}$ is a walk of $G_{6(b)}$. Similarly, $G_{6(c)}$ is a weak projection of $G_{6(a)}$.

Proposition 20. *Let $G = (V, E, \beta)$ be a* CFG *and $V' \subseteq V$. If the graph induced by V' from G is a* CFG *then the graph induced by V' from G is a weak projection of G.*

Proof. Let

$$\omega = \omega_1, \ldots, \omega_i, \omega_{i+1}, \ldots \text{ be a walk of } G$$
$$\text{and write } \omega\downarrow V' = \omega_{n_1}, \ldots, \omega_{n_i}, \omega_{n_{i+1}}, \ldots$$

where $1 \leqslant n_1 < n_2 < \cdots$. Then

$$\bar{\omega}_{n_i}, \bar{\omega}_{n_i+1}, \ldots, \bar{\omega}_{n_{i+1}}$$

is a V'-interval, and hence by Definition 13

$$\bar{\omega}_{n_1}, \ldots, \bar{\omega}_{n_i}, \bar{\omega}_{n_{i+1}}, \ldots$$

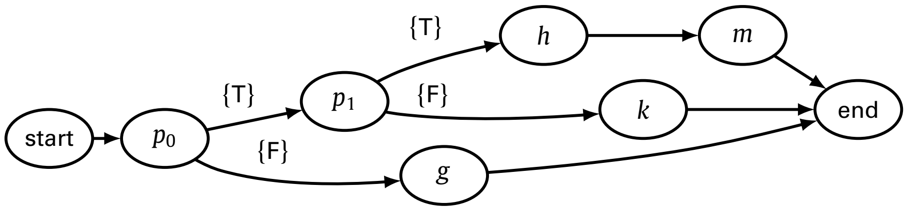
is a path in the graph induced by V' from G. Finally, again by Definition 13 we have $\beta(\bar{\omega}_{n_i}, \bar{\omega}_{n_i+1}) \subseteq \beta'(\bar{\omega}_{n_i}, \bar{\omega}_{n_{i+1}})$ and so $\omega\downarrow V'$ is a walk of the graph induced by V' from G. □

This shows that the mere act of ensuring that the induced graph is well-formed will also ensure it satisfies the semantic property of being a weak projection.

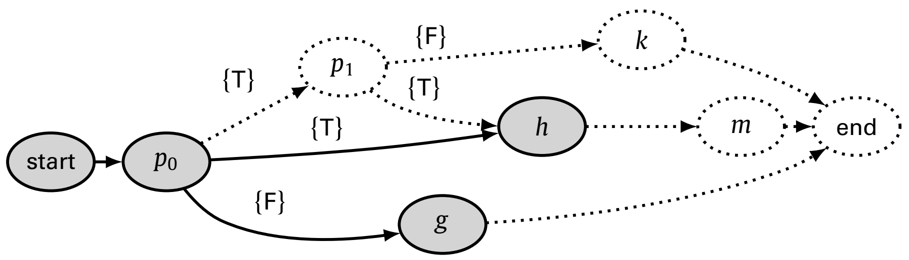
4.3. *Strong projections of a* CFG

A strong projection is a weak projection where also maximal walks project onto maximal walks.

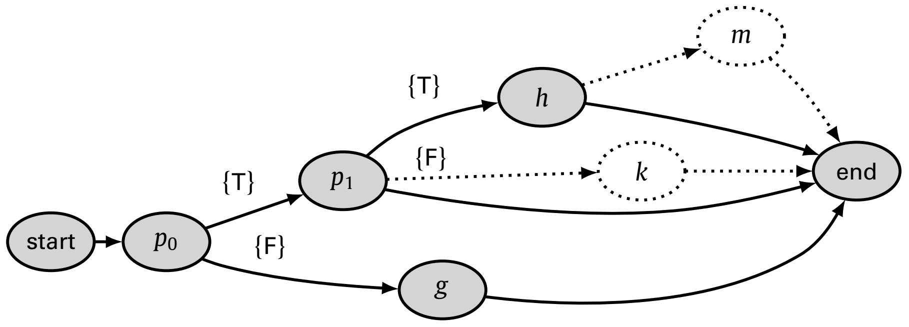
[4] A segment of a sequence is contiguous sequence of elements in the sequence. For example the sequence $\{u, v, w\}$ has segments $\{u, v, w\}, \{u, v\}, \{v, w\}, \{u\}, \{v\}$, and $\{w\}$ but not $\{u, w\}$.



(a) $G_{6(a)}$.



(b) $G_{6(b)}$ is a weak projection of $G_{6(a)}$.



(c) $G_{6(c)}$ is also a weak projection of $G_{6(a)}$.

Fig. 6. Two weak projections. Any walk of $G_{6(a)}$ when restricted to the vertices of $G_{6(b)}$ is a walk in $G_{6(b)}$. Ditto $G_{6(c)}$.

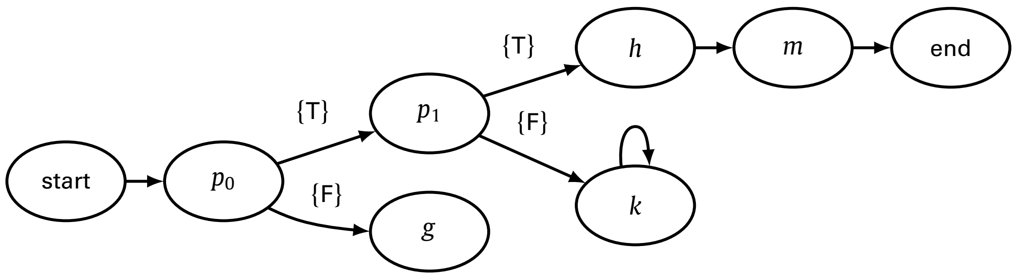


Fig. 7. G_7 has thirteen maximal walks (Definition 21): three of them end at g, four of them end with an infinite sequence of ks, and six of them end at end.

Definition 21 (*Maximal Walks*)**.** A maximal walk of G is a walk which is not a proper prefix of a walk of G.

For example the CFG G_7 in Fig. 7 has thirteen maximal walks. There are nine finite maximal walks:

start, $(p_0, \mathsf{F}), g$
$(p_0, \mathsf{F}), g$
g
start, $(p_0, \mathsf{T}), (p_1, \mathsf{T}), h, m$, end
$(p_0, \mathsf{T}), (p_1, \mathsf{T}), h, m$, end
$(p_1, \mathsf{T}), h, m$, end
h, m, end
m, end
end

and four infinite maximal walks:

$$\begin{array}{l} \mathsf{start}, (p_0, \mathsf{T}), (p_1, \mathsf{F}), k, k, \ldots \\ (p_0, \mathsf{T}), (p_1, \mathsf{F}), k, k, \ldots \\ (p_1, \mathsf{F}), k, k, \ldots \\ k, k, \ldots. \end{array}$$

The six walks ending with end correspond to executions which terminate normally. The other walks correspond to non-terminating programs, the finite ones to 'silently' non-terminating executions.

Proposition 22. *Let G be a* CFG*. If ω is a maximal walk of G then $\bar{\omega}$ is a complete path of G.*

Proof. If ω is infinite then $\bar{\omega}$ is infinite and thus is complete. If ω is finite then suppose that $\bar{\omega}$ is not complete. Let $\bar{\omega}$ end at v, say, and since $\bar{\omega}$ is not complete there exists an edge $(v, w) \in E$. But then ω can be extended to w, contradicting the maximality of ω. □

The converse is true only for CFGs where all predicates are complete.

Proposition 23. *Let G be a* CFG *and let ω be a walk in G. If $\bar{\omega}$ is complete and all predicates in G are complete then ω is maximal.*

Proof. If $\bar{\omega}$ is infinite then ω is infinite and thus is maximal. If $\bar{\omega}$ is finite then it ends at a final vertex. Now, all predicate vertices are complete, therefore all final vertices in G must have out-degree 0, and not 1. Thus $\bar{\omega}$ cannot be a prefix of any extending path and hence ω is maximal. □

The converse of Proposition 22 is not true for CFGs that contain incomplete predicates. For example in $G_{4(a)}$ in Fig. 4(a) the two walks

$$\omega_1 = (p_0, \mathsf{T}), (p_1, \mathsf{T})$$
$$\omega_2 = (p_0, \mathsf{T}), (p_1, \mathsf{F})$$

have $\bar{\omega}_1 = \bar{\omega}_2 = p_0, p_1$ which is complete because it ends at the final vertex p_1. However, ω_1 is maximal but ω_2 is not. Nevertheless, as this example implies, for every complete path π there exists a maximal walk ω with $\bar{\omega} = \pi$.

Proposition 24. *Let G be a* CFG*. If π is a complete path of G then there exists a maximal walk ω of G such that $\bar{\omega} = \pi$.*

Proof. Follows immediately from the definitions of maximal walks and complete paths. □

There may exist more than one maximal walk with the same complete path, for example where an edge is labelled $\{\mathsf{T}, \mathsf{F}\}$.

Definition 25 (*Strong projections*)**.** Let $G = (V, E, \beta)$ be a CFG and $V' \subseteq V$. A CFG $G' = (V', E', \beta')$ is a *strong projection* of CFG if and only if all maximal walks of G when restricted to V' give maximal walks of G'. *i.e.*,

$$\omega \in \overrightarrow{G} \text{ is maximal } \implies \omega{\downarrow}V' \in \overrightarrow{G'} \text{ and is maximal.}$$

So, for every walk in a strong projection, the number of times we visit a vertex in the projection is *identical* to the number of times we visit it in the corresponding walk in the original. It follows that strong projections have another pleasing property: *every* walk in a strong projection is the restriction of a walk of the original.

Lemma 26. *A strong projection is a weak projection.*

Proof. This follows immediately from the fact that every walk is the prefix of a maximal walk. □

Lemma 27. *Let the* CFG *$G' = (V', E', \beta')$ be a strong projection of the* CFG *$G = (V, E, \beta)$. For all $(x, y) \in E'$ there exists an $[x, y]$ V'-interval in G.*

Proof. Assume that $(x, y) \in E'$.

(1) If x is a predicate then without loss of generality we can assume that $\mathsf{T} \in \beta'(x, y)$. Let w be a maximal walk of G starting from (x, T).
Since G' is a strong projection of G, $w{\downarrow}V'$ is maximal in G'. The walk w will reach V' after (x, T), because otherwise $w{\downarrow}V'$ is just (x, T), which cannot be maximal since $\mathsf{T} \in \beta'(x, y)$. (This is where the proof would break down for weak projections.) Therefore let v' be the first V' vertex after (x, T) in w. Since $w{\downarrow}V'$ is a walk of G', there must be an edge $(x, v') \in E'$ with $\mathsf{T} \in \beta'(x, v')$. Since G' is a CFG we must have $y = v'$. Hence there is an $[x, y]$ V'-interval in G as required.
Moreover, it follows that if G' contains a walk with first element (x, T) and with second element having vertex component y, then G contains a walk with first element (x, T), with a later element having vertex component y, and with no intermediate element having a vertex component in V'. Similarly with F instead of T. (This stronger result is needed in order to use Lemma 27 for Proposition 28.)

(2) If x is a non-predicate then as above, let w be a maximal walk of G starting from x. Using the same argument as above, w will reach V' after x and by the uniqueness of the next element of non-predicate vertices we must have that y is the unique next element after x in $w{\downarrow}V'$. Hence there is an $[x, y]$ V'-interval in G as required. □

Proposition 28. *Let the* CFG *$G' = (V', E', \beta')$ be a strong projection of the* CFG *$G = (V, E, \beta)$. For all walks ω' of G' there exists a walk ω of G such that $\omega{\downarrow}V' = \omega'$.*

Proof. Follows from Lemma 27 by induction on the length of a finite prefix of a walk. □

In fact every path in a strong projection arises as the slice of a path in the original, *i.e.*, 'the path of the strong projection is (exactly) the slice of the path.'

Corollary 29. *Let the* CFG $G' = (V', E', \beta')$ *be a strong projection of the* CFG $G = (V, E, \beta)$*. For all complete paths* π' *of* G' *there exists a complete path* π *of* G *such that* $\pi \downarrow V' = \pi'$*.*

Proof. If π' is a complete path of G' then by Proposition 24 there is a maximal walk $\omega' \in \overrightarrow{G'}$ with $\bar{\omega}' = \pi'$. By Proposition 28 there exists $\omega \in \overrightarrow{G}$ with $\omega{\downarrow}V' = \omega'$ If ω is not maximal then simply take any maximal walk μ extending ω. G' is a strong projection therefore $\bar{\mu}{\downarrow}G' = \pi'$ for otherwise π' is not maximal. □

4.4. Weak and strong projections and non-termination

Consider the following definitions:

Definition 30 (*Terminating Walks of a CFG*)**.** Walk ω is a terminating walk if and only if the path $\bar{\omega}$ is a terminating path (Definition 11).

Definition 31 (*Non-terminating Walks of a CFG*)**.** Walk ω is a non-terminating walk if and only if the path $\bar{\omega}$ is a non-terminating path (Definition 12).

4.5. Strong projections preserve non-termination

This result is proved in the following lemma:

Lemma 32. *Let* $G = (V, E, \beta)$ *and* $G' = (V', E', \beta')$ *be* CFGs *and let* G' *be a strong projection of G. If* ω *is a non-terminating walk of G then* $\omega{\downarrow}V'$ *is a non-terminating walk of* G'*.*

Proof. Suppose $\omega{\downarrow}V'$ is a terminating walk of G'. Then by definition, the final element of $\omega{\downarrow}V'$ is end. Now, since $V' \subseteq V$ we must have end $\in V$. So end must be an element of ω and hence ω is terminating. Contradiction. □

4.5.1. Weak projections do not necessarily preserve termination

Consider $G_{6(b)}$ which is a weak projection of $G_{6(a)}$ in Fig. 6. The terminating walk start, $(p_0, \mathsf{F}), g,$ end restricts to the non-terminating walk start, $(p_0, \mathsf{F}), g$. This walk is non-terminating because it ends in a final non-end vertex g in $G_{6(b)}$.

4.5.2. Strong projections do not necessarily preserve termination

Consider $G_{8(b)}$ in Fig. 8. $G_{8(b)}$ is a strong projection of $G_{8(a)}$. The terminating path: start p_0 p_1 h m end in the original restricts to the path: start p_0 p_1 h in the induced graph. This path is non-terminating since it ends at a non-end final vertex, namely a non-predicate of out-degree zero.[5]

4.5.3. Weak projections do not necessarily preserve non-termination

The weak projection of a CFG G may terminate when G does not. To see this, consider Fig. 10. $G_{10(b)}$ is a weak projection of $G_{10(a)}$ but it does not preserve non-termination. Predicate vertex p_1, which can lead to non-termination in $G_{10(a)}$, does not exist in $G_{10(b)}$. The non-terminating path [start, $p_0, p_1, k, k, \ldots$] of $G_{10(a)}$ restricts to the path [start, p_0] in $G_{10(b)}$. The path [start, p_0] is neither terminating nor non-terminating in $G_{10(b)}$. It is clearly not the prefix of any non-terminating path of $G_{10(b)}$.

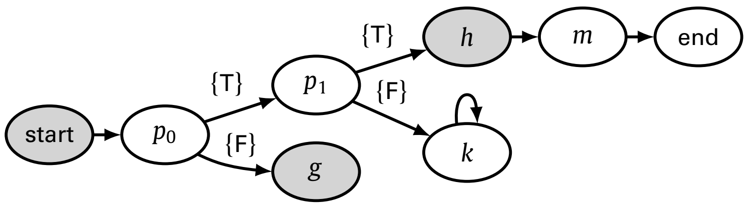
If a weak or strong projection contains end then it cannot introduce non-termination. This is stated formally in Lemma 33.

Lemma 33. *Let* $G = (V, E, \beta)$ *and* $G' = (V', E', \beta')$ *be* CFGs *containing* {end} *and let* G' *be a weak projection of G. If* ω *is a terminating walk of G then* $\omega{\downarrow}V'$ *is a terminating walk of* G'*.*

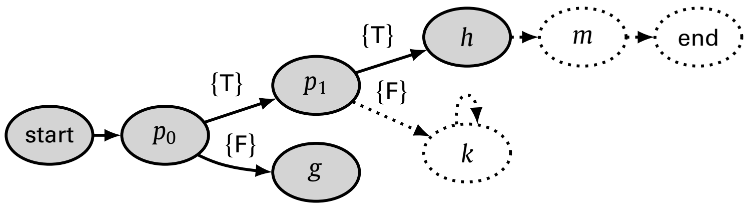
Proof. This follows immediately from the fact the end $\in V'$. □

Since a strong projection is a weak projection, Lemmas 32 and 33 guarantee that strong projections containing end of a CFG, g, preserve the termination conditions of g. See Fig. 9 for an example.

[5] In this example the smallest set containing {start, g, h} closed under $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ of Ranganath et al. (defined later) is also {start, g, h, p_0} inducing $G_{8(b)}$, showing that their so called 'non-termination sensitive control dependence' does not always preserve termination conditions either.

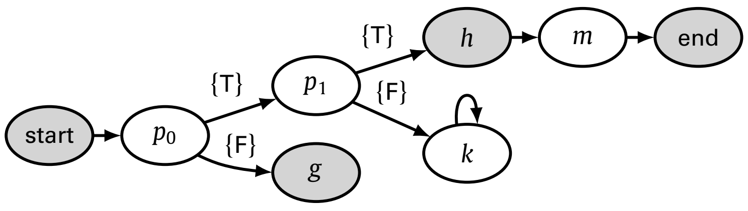


(a) $G_{8(a)}$.

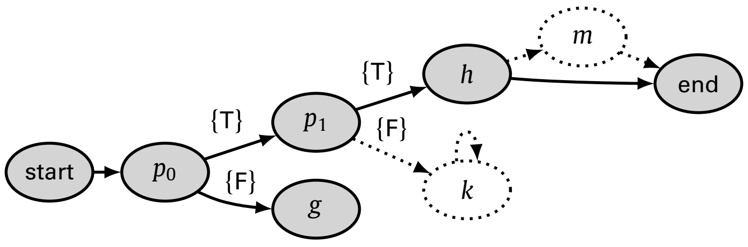


(b) $G_{8(b)}$.

Fig. 8. Strong projections not containing end do not necessarily preserve termination conditions: $G_{8(b)}$ (removed vertices and edges shown dotted) is a strong projection of $G_{8(a)}$ The terminating complete path: start p_0 p_1 h m end in the original induces the path: start p_0 p_1 h in the induced graph. This path is non-terminating since it ends at a non-end non-predicate of out-degree zero.



(a) $G_{9(a)}$.



(b) $G_{9(b)}$.

Fig. 9. Termination behaviour is preserved only when end is included in the strong projection. $G_{9(b)}$ is a strong projection of $G_{9(a)}$. Both have the same termination conditions.

5. Weak and strong control-closure: a generalisation of non-termination sensitive and non-termination insensitive control dependence

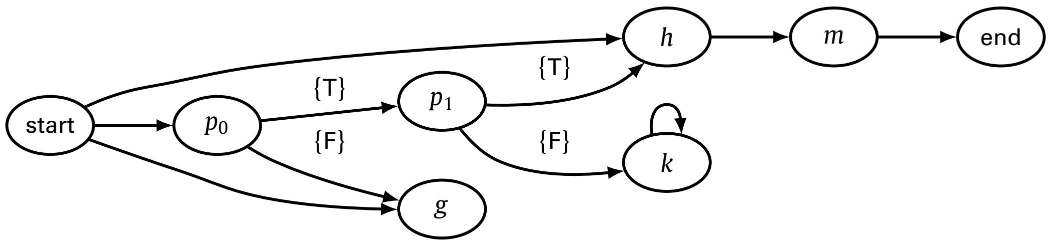
In this section we develop a theory of *weak and strong control-closure*. These are properties of vertex sets which are necessary and sufficient to induce graphs which are weak and strong projections. Weakly and strongly control-closed sets are used both in the production of algorithms for producing minimal weak and strong projections and also in the proofs that classify the previous forms of control dependence as either weak or strong. An advantage of weak and strong control-closure is that they are defined for *any* directed graph not just CFGs.

5.1. Weakly control-closed sets

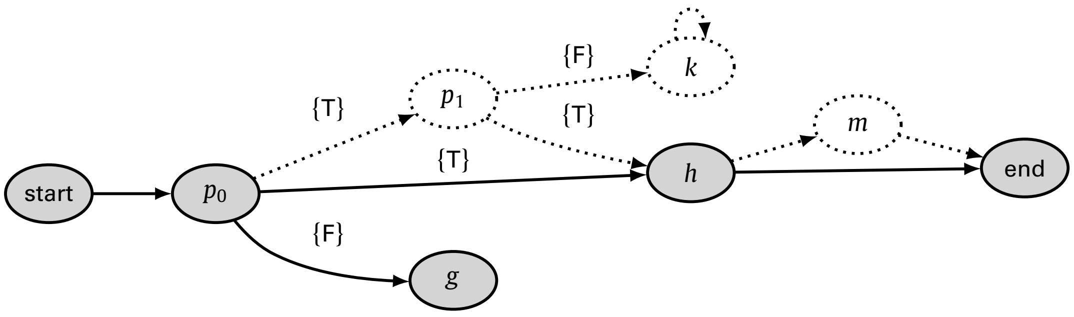
Informally, at this stage, a set is weakly control-closed if and only if it is closed under non-termination sensitive control dependence. Before we can define weakly control-closed sets, we need a preliminary definition:

Definition 34 (*V'-weakly Committing Vertices*)**.** Let G be a directed graph. A vertex v is V'-weakly committing in G if all V'-paths from v have the same end point. In other words, there is at most one element of V' that is 'first-reachable' from v.

For example, in Fig. 5 the predicate p_1 is $\{\text{start}, h, g\}$-weakly committing in G_5 since the only first-reachable vertex in $\{\text{start}, h, g\}$ from p_1 is h. Vertex p_0, on the other hand, is not $\{\text{start}, h, g\}$-weakly committing in G_5 since both h and g are first reachable from p_0.



(a) CFG $G_{10(a)}$.



(b) $G_{10(b)}$.

Fig. 10. $G_{10(b)}$ is a weak projection of $G_{10(a)}$ but it does not preserve non-termination. Predicate vertex p_1, which can lead to non-termination in $G_{10(a)}$, does not exist in $G_{10(b)}$.

Definition 35 (*Weakly Control-closed Sets*)**.** Let G be a directed graph and let $V' \subseteq V$. V' is weakly control-closed in G if and only if all vertices not in V' that are reachable from V' are V'-weakly committing in G.

In Fig. 5, $\{\text{start}, h, g\}$ is not weakly control-closed in G_5 because p_0 is reachable from $\{\text{start}, h, g\}$ but p_0 is not $\{\text{start}, h, g\}$-weakly committing since h and g are both first reachable from p_0. However, $\{\text{start}, p_0, h, g\}$ is weakly control-closed in G_5 because all of the vertices of G_5 that are not in $\{\text{start}, p_0, h, g\}$ and are reachable from $\{\text{start}, p_0, h, g\}$ are $\{\text{start}, p_0, h, g\}$-weakly committing. Later, we will show that sets closed under all weak forms of control dependence in the literature are weakly control-closed.

5.2. Strongly control-closed sets

Informally a set is strongly control-closed if and only if it is closed under non-termination sensitive control dependence.

Definition 36 (V'*-strongly Committing Vertices*)**.** Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. A vertex v is V'-strongly committing G if and only if it is V'-weakly committing in G and all complete paths in G from v contain an element of V'.

This means that *all* paths from v re-enter V' (and do so at the same vertex) whereas if v is only V'-weakly committing in G then some paths from v in G may never re-enter V'.

Definition 37 (V'*-avoiding Vertices*)**.** Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. A vertex v is V'-avoiding in G if and only if no vertex in V' is reachable in G from v.

Definition 38 (*Strongly Control-closed Sets*)**.** Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. V' is strongly control-closed in G if and only if every vertex in $V \backslash V'$ that is reachable in G from V' is V'-strongly committing or V'-avoiding in G.

In Fig. 7, the set $\{start, g, h, \text{end}\}$ is not strongly control-closed in G_7 because p_1 is reachable from $\{start, g, h, \text{end}\}$ but is neither $\{start, g, h, \text{end}\}$-strongly committing or $\{start, g, h, \text{end}\}$-avoiding. (Similarly p_0.)

In Fig. 7, the set $\{start, p_0, p_1, h, \text{end}\}$ is strongly control-closed in $G_{9(b)}$ because k and g are $\{start, p_0, p_1, h, \text{end}\}$-avoiding and m is $\{start, p_0, p_1, h, \text{end}\}$-strongly committing. Later, we will show that sets closed under all strong forms of control dependence in the literature are strongly control-closed.

5.3. Graphs induced by weakly control-closed sets

We now develop the theory of graphs induced by weakly control-closed sets. We show that for any subset V' of the vertices of a CFG, G, V' being weakly control-closed in G is both necessary and sufficient for the induced graph from V' not only to be well formed, but also to be a weak projection.

Lemma 39. *Let $G = (V, E, \beta)$ be a* CFG*, $V' \subseteq V$, and $v \in V'$. If V' is weakly control-closed in G then the out-degree of v in the graph induced by V' from G is at most the out-degree of v in G.*

Proof. Let $v_1, \ldots, v_n$ be all the successors of v in the graph induced by V'. Thus there are V'-paths, $v\gamma_i$, in G ending with v_i for each $i \leqslant n$. Let w_i be the first vertex of γ_i. We must have $i \neq j$ implies $w_i \neq w_j$, since if $w_i = w_j$ then either $w_i \notin V'$ in which case w_i would not be V'-weakly committing in G although it is reachable from V', implying that V' is not weakly control-closed in G, or $w_i \in V'$ and so $v_i = v_j$, contrary to our assumption. Thus v has n successors $w_1, \ldots, w_n$ in G, proving the result. □

Proposition 40. *Let $G = (V, E, \beta)$ be a* CFG *and $V' \subseteq V$. If V' is weakly control-closed in G then the graph induced by V' from G is a* CFG*.*

Proof. Let the graph induced by V' from G be (V', E', β'). By Lemma 39, the graph induced by G on V' satisfies conditions (1) and (2) of Definition 1. Suppose that part (3) of Definition 1 does not hold, then there exist edges $(x, y) \in E'$ and $(x, z) \in E'$ with $y \neq z$ but $\beta'(x, y) \cap \beta'(x, z) \neq \emptyset$. Assume without loss of generality that $\mathsf{T} \in \beta'(x, y) \cap \beta'(x, z)$. By Definition 13, there exists y_1 and z_1 in V such that $\mathsf{T} \in \beta(x, y_1)$ and $\mathsf{T} \in \beta(x, z_1)$ and V'-intervals $x, y_1 \ldots y$ and $x, z_1 \ldots z$. Since G is a CFG, and hence edges from the same predicate must be disjointly labelled, we must have $y_1 = z_1$. But the V'-paths $y_1 \ldots y$ and $z_1 \ldots z$ then contradict the hypothesis that V' is weakly control-closed in G. □

Proposition 41. *Let $G = (V, E, \beta)$ be a* CFG *and $V' \subseteq V$. If the graph induced by V' from G is a* CFG *then V' is weakly control-closed in G.*

Proof. Suppose not, then there exists $v \notin V'$ reachable from V' but not V'-weakly committing in G. Therefore exists $v' \in V'$ and V'-intervals

$$v', v'', \ldots, v, \ldots, l_1 \quad \text{and} \quad v', v'', \ldots, v, \ldots, l_2$$

with $l_1 \neq l_2$. Let the graph induced by V' from G be (V', E', β'). By Definition 13, the graph induced by V' from G will contain the edges $\{v', l_1\}$ and $\{v', l_2\}$. If $v' \in P$ (*i.e.*, it is a predicate) then the two edges would not have disjoint labelling, since $\beta'(v', l_1) \cap \beta'(v', l_2) \supseteq \beta(v', v'') \neq \emptyset$. If $v' \in N$ (*i.e.*, it is a non-predicate), then in the induced graph a non-predicate would have two successors. Both cases contradict the fact that the graph induced by V' from G is a CFG. □

The following three straightforward results show when a subset of the vertices includes the necessary distinguished vertices that the graph induced on that subset again belongs to the same restricted class.

Proposition 42. *Let $G = (V, E, \beta)$ be a $\{\mathsf{start}\}$-*CFG*. V' is weakly control-closed in G and* start $\in V'$ *if and only if the graph induced by V' from G is a $\{\mathsf{start}\}$-*CFG*.*

Proof. This follows immediately from Propositions 40 and 41, and the fact that if start $\in V'$, then every vertex in V' is reachable from start in the graph induced by V' from G, which is a consequence of Definition 13 and the analogous assertion in G. □

Proposition 43. *Let $G = (V, E, \beta)$ be an $\{\mathsf{end}\}$-*CFG*. V' is weakly control-closed in G and* end $\in V'$ *if and only if the graph induced by V' from G is an $\{\mathsf{end}\}$-*CFG*.*

Proof. Similar to the proof of Proposition 42. □

Proposition 44. *Let $G = (V, E, \beta)$ be a $\{\mathsf{start}, \mathsf{end}\}$-*CFG*. V' is weakly control-closed in G and $\{\mathsf{start}, \mathsf{end}\} \subseteq V'$ if and only if the graph induced by V' from G is a $\{\mathsf{start}, \mathsf{end}\}$-*CFG*.*

Proof. This is an immediate consequence of Propositions 42 and 43. □

We have proven that if V' is weakly control-closed in G then the graph induced from V' is a well formed CFG and conversely if the graph induced from $V' \subseteq V$ is a well-formed CFG then V' is weakly control-closed in G. A set being weakly control-closed in G is thus an equivalent to the graph induced by it from G being a well-formed CFG. In Section 7, we prove that weakly control-closed sets generalise the property that V' is closed under each of the weak forms of control dependence in the literature.

Theorem 45. *Let $G = (V, E, \beta)$ be a* CFG *and $V' \subseteq V$. The following are equivalent.*

(1) *The graph induced by V' from G is a* CFG*.*
(2) *V' is weakly control-closed in G.*
(3) *The graph induced by V' from G is a weak projection of G.*

Proof. (1) $\implies$ (2) by Proposition 41.
(2) $\implies$ (1) by Proposition 40.
(1) $\implies$ (3) by Proposition 20 and
(3) $\implies$ (1) because, by definition, a weak projection is a CFG. □

We have thus shown that for any subset V' of the vertices of a CFG, G, V' being weakly control-closed in G is both necessary and sufficient for the induced graph from V' not only to be well formed, but also to be a weak projection.

In the next section we define the corresponding relationships to Theorem 45 between *strong* control-closed sets and strong projections.

5.4. Graphs induced by strongly control closed sets

In this section, we investigate graphs induced by strongly control-closed sets. The main result of this section is that graph induced by V' from G is a strong projection of G if and only if V' is strongly control-closed in G.

Lemma 46. *If V' is strongly control-closed in G then V' is weakly control-closed in G.*

Proof. Result follows from the fact that a strongly committing vertex is weakly committing (Definition 36) and that a V'-avoiding vertex is vacuously V'-weakly committing. $\square$

An example of strong control-closure in terms of CFGs can be seen in Fig. 8. The smallest strongly control-closed set containing $\{\mathsf{start}, g, h\}$ is $\{\mathsf{start}, p_0, p_1, g, h\}$. Unlike in the weak case (see $G_{6(b)}$ in Fig. 6), vertex p_1 is included, since k is avoiding and h is strongly committing.

Proposition 47. *Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. If the graph induced by V' from G is a strong projection of G and hence a CFG, then V' is strongly control-closed in G.*

Proof. By Proposition 41, V' is weakly control-closed in G. Suppose $v \notin V'$ is reachable from V' but v is not V'-avoiding and v is not V'-strongly committing. Since v is reachable from V' there is a path $\rho = v_1 \dots v_{n-1}v_n$ $(v_n = v)$ from $v_1 \in V'$ to v with $v_2, \dots, v_n \in V \backslash V'$. Since v is not V'-avoiding in G, there is a V'-path $v\mu k$ from v to some vertex $k \in V'$. Since v is V'-weakly committing in G but not V'-strongly committing in G there exists a V'-avoiding path $vv_{n+1}v_{n+2}\dots$ say.

Now, $\rho v_{n+1}v_{n+2}\dots$ is a complete path and so by Proposition 24 there exists a maximal walk $\omega = \omega_1\omega_2\dots$ where $\bar{\omega}_i = v_i$. However, $\omega{\downarrow}V' = \omega_1$ and the existence of the V'-interval $\rho\mu k$ means that by Definition 13 there is an edge (v_1, k) in the graph induced by V' from G and if v_1 is a predicate then $\bar{\bar{\omega}}_1 \in \beta(v_1, v_2) \subseteq \beta'(v_1, k)$. Hence ω_1 is a prefix of any walk $\omega_1 l\dots$, where $\bar{l} = k$, *i.e.*, it is not maximal which contradicts that the graph induced by V' from G is a strong projection of G. $\square$

Proposition 48. *Let $G = (V, E, \beta)$ be a CFG and $V' \subseteq V$. If V' is strongly control-closed in G then the graph induced by V' from G is a strong projection of G.*

Proof. By Proposition 40 and Lemma 46, the graph induced by V' from G is a CFG and then, by Proposition 20, the graph induced by V' from G is a weak projection of G. Suppose it is not a strong projection. Then there exists a maximal walk ω of G such that $\omega{\downarrow}V'$ is a proper prefix of a maximal walk of the graph induced by V' from G. This means that $\omega{\downarrow}V'$ is finite, so write

$$\omega{\downarrow}V' = \omega_1, \dots, \omega_n.$$

Since $w_1 \dots w_n$ is not maximal in the graph induced by V' from G, there exists a walk $w_1 \dots w_n w'$ in the graph induced by V' from G. There must exist a predicate $p \notin V'$ on a path between w_n and w' in G from which some paths reach $w' \in V'$ and from which at least one path never re-enters V', since if all paths in G re-enter V' from w_n then w cannot be maximal G. By definition, p is not strongly V' avoiding or not strongly committing and hence V' is not strongly control-closed in G. $\square$

Theorem 49. *Let $G = (V, E, \beta)$ be a CFG and $V' \subseteq V$. The graph induced by V' from G is a strong projection of G if and only if V' is strongly control-closed in G.*

Proof. $(\Rightarrow)$ Proposition 47.
$(\Leftarrow)$ Proposition 48. $\square$

In Section 4, we defined weak and strong projections and in this section we have given necessary and sufficient conditions for a subset, V', of vertices of a CFG to induce weak and strong projections respectively. The conditions are that V' is weakly control-closed in G and V' is strongly control-closed in G respectively. In this section we prove that for any subset of vertices V' of a CFG, there are unique minimal sets V'' and V''' containing V' such that V'' is weakly control-closed in G and V''' is strongly control-closed in G. This implies that for any subset of vertices V' of a CFG, there are unique minimal sets V'' and V''' containing V' such that the graphs induced from G by V'' and V''' are weak and strong projections of G respectively.

5.5. Existence and uniqueness of minimal weakly control-closed sets

Good slices are small slices. Given a CFG $G = (V, E, \beta)$ and a slicing criterion $V' \subseteq V$ we want to find the smallest subset V'' of V containing V' such that the graph induced from G by V'' is a weak projection of G. By Theorem 45, V'' will be the smallest weakly control-closed set containing V'. In this subsection, we prove that such a set exists and is unique.

In order to do this we need the concept of a *weakly deciding* vertex. Informally, a vertex is weakly deciding over a set V' if it *decides* between any two vertices in V'. It is called weak because the choice does not guarantee reaching an element in V'. Having made the choice, however, there will be at least one path that reaches the vertex in V'. The choice guarantees that the *other* interesting vertex will definitely not be reached first.

Definition 50 (*Weakly Deciding Vertices*). Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. A vertex $v \in V$ is V'-weakly deciding in G if and only if there exist two finite proper V'-paths in G that both start at v and have no other common vertex. We write $\mathsf{WD}_G(V')$ for the set of all V'-weakly deciding vertices in G.

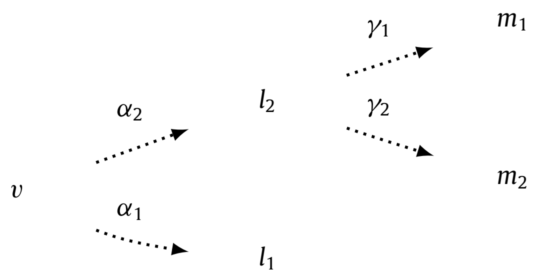


Fig. 11. Lemma 53: If $l_1 \in V'$ and $l_2 \in \mathsf{WD}_G(V')\backslash V'$ then there exist paths $l_2\gamma_1 m_1$ and $l_2\gamma_2 m_2$ where γ_1 and γ_2 are disjoint from each other and disjoint from V'. The vertices $m_1 \neq m_2$ are both in V'.

It is possible for a vertex to be neither V'-weakly committing in G, nor V'-weakly deciding in G. To see this, consider vertex start of $G_{6(a)}$ in Fig. 6. It is not {start, h, g}-weakly committing in $G_{6(a)}$ since h and g are both first reachable from start, nor {start, h, g}-weakly deciding in $G_{6(a)}$ since all proper V'-paths from start contain p_0 and are hence not disjoint.

There now follows a lemma which shows that V' is weakly control-closed if and only if all V'-weakly deciding vertices that are reachable from V' are in V'.

Lemma 51 (*Weak Control-closure in Terms of* WD)**.** *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. V' is weakly control-closed in G if and only if all V'-weakly deciding vertices in G that are reachable from V' are in V'.*

Proof. Suppose that V' is weakly control-closed in G and let $v \in \mathsf{WD}_G(V')\backslash V'$. Since v is V'-weakly deciding in G there must exist V'-paths $v\ldots v'$ and $v\ldots v''$ which share no common vertex after their common initial vertex v. Therefore v is not V'-weakly committing and because V' is weakly control-closed in G, v cannot be reachable from V'.

Conversely, let $v \notin V'$ be reachable from V' but not V'-weakly committing, so there exist V'-paths $v\ldots v'$ and $v\ldots v''$ for $v' \neq v''$. If w is the last common vertex on these paths then w is reachable from V' and $w\ldots v'$ and $w\ldots v''$ share no common vertex after w and $w \neq v'$, $w \neq v''$. Hence w is V'-weakly deciding in G and $w \notin V'$. □

There now follow a useful result stating that WD is monotonic.

Lemma 52 (WD *is Monotonic*)**.** *Let $G = (V, E)$ be a finite directed graph and $V_1 \subseteq V_2 \subseteq V$, then*

$$\mathsf{WD}_G(V_1) \subseteq \mathsf{WD}_G(V_2).$$

Proof. Suppose there is a vertex $v \in \mathsf{WD}_G(V_1)\backslash\mathsf{WD}_G(V_2)$. Clearly there are proper $[v, m_i]$ V_1-paths γ_i in G, for vertices $m_1, m_2 \in V_1 \subseteq V_2$, which share only v as a common vertex. Let δ_i be the prefix of γ_i ending at the first vertex in V_2 for each i; since $v \notin \mathsf{WD}_G(V_2)$, each δ_i is a proper path from v and the δ_i also share only v as a common vertex. Thus $v \in \mathsf{WD}_G(V_2)$, giving a contradiction. □

We now prove an important property of WD_G which we refer to as *idempotence*. This means that:

$$\mathsf{WD}_G(V' \cup \mathsf{WD}_G(V')) \subseteq V' \cup \mathsf{WD}_G(V').$$

This is equivalent to saying $V' \cup \mathsf{WD}_G(V')$ is closed with respect to WD_G *i.e.*, if we take the set of vertices that are weakly deciding on $V' \cup \mathsf{WD}_G(V')$ we will not get any new elements.

Lemma 53 (WD_G *is Idempotent*)**.** *Let G be a* CFG *with vertex set V and let $V' \subseteq V$. Then $\mathsf{WD}_G(V' \cup \mathsf{WD}_G(V')) \subseteq V' \cup \mathsf{WD}_G(V')$.*

Proof. Let $v \in \mathsf{WD}_G(V' \cup \mathsf{WD}_G(V'))$. Thus there are paths $v\alpha_1 l_1$ and $v\alpha_2 l_2$ with α_1 and α_2 disjoint from each other and from $V' \cup \mathsf{WD}_G(V')$ and with each $l_i \in V' \cup \mathsf{WD}_G(V')$. We will show that $v \in \mathsf{WD}_G(V')$, thus proving the Lemma. We consider three cases:

(1) If both $l_1, l_2 \in V'$ then by definition $v \in \mathsf{WD}_G(V')$, as required.

(2) If $l_1 \in V'$ and $l_2 \in \mathsf{WD}_G(V')\backslash V'$ then there exist paths $l_2\gamma_1 m_1$ and $l_2\gamma_2 m_2$ where γ_1 and γ_2 are disjoint from each other and disjoint from V'. The vertices $m_1 \neq m_2$ are both in V' (see Fig. 11). Either $m_1 \neq l_1$ or $m_2 \neq l_1$. Assume without loss of generality, that $m_1 \neq l_1$. If γ_1 is disjoint from α_1 then $v \in \mathsf{WD}_G(V')$, as $\alpha_1 l_1$ and $\alpha_2 l_2 \gamma_1 m_1$ are clearly disjoint.

Suppose that γ_1 and α_1 are not disjoint. Then there exists a vertex q which is the last point along γ_1 at which γ_1 and α_1 meet. So we can write $\gamma_1 = \pi_1 q \pi_2$ with π_2 disjoint from α_1. Similarly, we can write $\alpha_1 = \rho_1 q \rho_2$ where ρ_2 is chosen so that it has no occurrence of q in it. Then π_2 and ρ_2 are disjoint, and $l_1, m_1 \in V'$ and hence by definition $q \in \mathsf{WD}_G(V')$. As q lies on γ_1, $q \notin V' \cup \mathsf{WD}_G(V')$ giving a contradiction. So γ_1 is disjoint from α_1, and so $v \in \mathsf{WD}_G(V')$ as required.

(3) If both $l_1, l_2 \in \mathsf{WD}_G(V')\backslash V'$. Then there exist paths $l_1\gamma_{11}m_{11}$ and $l_1\gamma_{12}m_{12}$ where $\gamma_1 1$ and $\gamma_1 2$ are disjoint from each other and from V' and $m_{11} \neq m_{12}$ are both in V'. Similarly there exist paths $l_2\gamma_{21}m_{21}$ and $l_2\gamma_{22}m_{22}$ where $\gamma_2 1$ and $\gamma_2 2$ are disjoint from each other and from V' and $m_{21} \neq m_{22}$ are both in V' (see Fig. 12).

We prove first that α_2 and γ_{11} are disjoint. Suppose not and let w be the last element of α_2 which shares an element with γ_{11}. Therefore w weakly decides between m_{11} and l_2 and hence by Case (2), $w \in \mathsf{WD}_G(V')$. This contradicts our original assumption that α_2 is disjoint from $V' \cup \mathsf{WD}_G(V')$. Therefore α_2 and γ_{11} are disjoint which means that v weakly decides between m_{11} and l_2 and hence, again by Case (2), $v \in \mathsf{WD}_G(V')$ as required. □

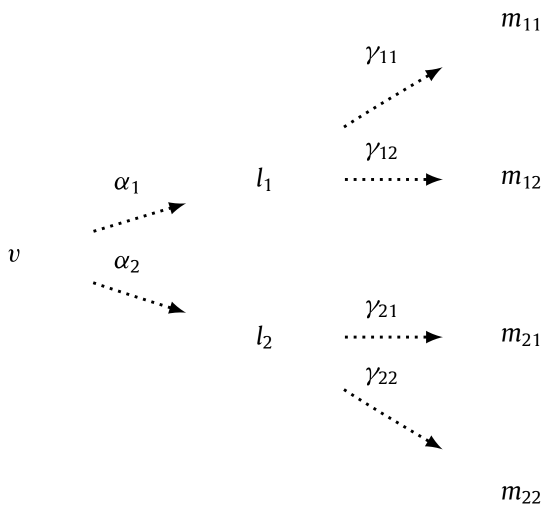


Fig. 12. Proof of Lemma 53.

Theorem 54. *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. There exists a unique minimal weakly control-closed subset of V that contains V'.*

Proof. Let W be the set of elements of $\text{WD}_G(V')$ that are reachable from V'. Then $V' \cup W$ is the unique minimal weakly control-closed subset of V that contains V'. Let v be reachable from $V' \cup W$ and $v \in \text{WD}_G(V' \cup W)$. Then by monotonicity $v \in \text{WD}_G(V' \cup \text{WD}_G(V'))$ and then by idempotence (Lemma 53) $v \in V' \cup \text{WD}_G(V')$. Now, v is reachable from V' since all elements of W are reachable from V'. Thus $v \in V' \cup W$ and so $V' \cup W$ is weakly control-closed by Lemma 51.

Let $K \subseteq V$ be such that W is not a subset of K. So by definition of W, there exists an element $z \in \text{WD}_G(V')$ reachable from V' but not in $V' \cup K$. By monotonicity, (Lemma 52), $z \in \text{WD}_G(V' \cup K)$. So by Lemma 51, $V' \cup K$ is not weakly control-closed in G, and so $V' \cup W$ is the unique minimal weakly control-closed subset of V that contains V'. □

5.6. Existence and uniqueness of minimal strongly control-closed sets

We now consider the analogous problem of proving the existence of the minimal strongly control-closed superset of a given set. We prove this to be well-defined in Lemma 57. In order to do this, we first need to define the function Γ which gives the set of vertices lying on complete paths not passing through V'.

Definition 55 (Γ). Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. We define $\Gamma(G, V')$ to be the set of all $x \in V$ that lie on a complete path in G which does not pass through V'.

We now define $\Theta(G, V', u)$, the set of first-reachable elements in V' from u.

Definition 56 (Θ). Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. Let H be the CFG obtained from G by deleting all edges (v', v) with $v' \in V'$. For any $u \in V$, we define $\Theta(G, V', u)$ to be the set of vertices in V' that are reachable in H from u.

Lemma 57. *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. If V' is not strongly control-closed then there is an edge (p, r) in G with:*

(1) *$p \in V \backslash V'$.*
(2) *p is reachable in G from V'.*
(3) *$|\Theta(G, V', r)| = 1$.*
(4) *$r \notin \Gamma(G, V')$.*
(5) *Either $|\Theta(G, V', p)| \geqslant 2$ or $p \in \Gamma(G, V')$.*

Furthermore, for any edge (p, r) in G satisfying these conditions, the vertex p lies in every strongly control-closed subset of V containing V'. From this, we show it follows that there is a unique minimal strongly control-closed superset of V'.

Proof. Since V' is not strongly control-closed, there is a vertex $p \in V \backslash V'$ that is reachable in G from V' that is neither V'-strongly committing nor V'-avoiding in G. Hence there is a path $p_0 = p, p_1 = r, \ldots, p_m \in V'$ in G. Choose p so that m is minimal. Thus r is V'-strongly committing and the conditions involving the functions Θ and Γ follow from this. Now assume that there is an edge (p, r) in G that satisfies the conditions given, but that there is a strongly control-closed set $W \supseteq V'$ not containing p. Since $|\Theta(G, V', r)| = 1$, there is a path $p_0 = p, p_1 = r, \ldots, p_m \in \Theta(G, V', r)$ in G, for some $m \geqslant 1$. From the condition on p, either of two possibilities may occur.

- $|\Theta(G, V', p)| \geqslant 2$ and so there is also a path $q_0 = p, q_1, \ldots, q_n \in V' \backslash \{p_m\}$ in G for some $n \geqslant 1$. Since $\Theta(G, V', p_i) = \{p_m\}$ for each $i \geqslant 1$, no $p_i = q_j$ unless $i = j = 0$. Since $\{p_m, q_n\} \subseteq V' \subseteq W$, but $p = p_0 = q_0 \notin W$, there exist minimal $k, l \geqslant 1$ such that $p_k \in W$ and $q_l \in W$, and so $p \in \mathrm{WD}_G(W)$, and so by Lemma 51, W is not weakly control-closed, and hence not strongly control-closed, giving a contradiction.
- $p \in \Gamma(G, V')$ and so there is a complete path $q_0 = p, q_1, \ldots$ in G such that every $q_j \notin V'$, and since $r = p_1 \notin \Gamma(G, V')$, no $p_i = q_j$ unless $i = j = 0$. Let $k \geqslant 1$ be minimal such that $p_k \in W$. If every $q_i \notin W$, then p is not W-strongly committing, giving a contradiction, and if $q_k \in W$ for minimal $l \geqslant 1$, then there is a $[p, p_k]$ W-path and a $[p, q_l]$ W-path in G, and so W is not weakly control-closed, and hence not strongly control-closed, giving a contradiction.

Thus we have shown if an edge (p, r) in V satisfies conditions (1)–(5), then p lies in every strongly control-closed superset of V'. □

Theorem 58. *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. There exists a unique minimal strongly control-closed superset of V' in G.*

Proof. We now prove the uniqueness of minimal strongly control-closed supersets of V' by induction on $|V| - |V'|$. We may assume that V' is not strongly control-closed, hence there is an edge (p, r) in V that satisfies conditions (1)–(5) of Lemma 57, and such that p lies in every strongly control-closed superset of V'. If $V' \cup \{p\}$ is strongly control-closed, then the uniqueness result follows immediately. Otherwise, by the inductive hypothesis, there is a unique minimal strongly control-closed superset of $V' \cup \{p\}$, and since every strongly control-closed superset of V' must contain p, the result follows. □

In this section we have proved that for any subset of vertices V' of a CFG, there are unique minimal weak and strongly control closed sets containing V' and hence unique minimal weak and strong projections containing V'. In Section 6, we give low polynomial algorithms for computing these sets.

6. Algorithms for computing minimal weakly and strongly control-closed sets

Given a CFG $G = (V, E, \beta)$ and a subset $V' \subseteq V$, we wish to compute the minimal superset of V' which is weakly control-closed and the minimal superset of V' which is strongly control-closed. In this section we define algorithms for computing each of these. (We showed that these sets exist in the previous section.) Since weakly and strongly control-closure is a necessary and sufficient condition for the induced graph to be a weak/strong projection, in effect, we have algorithms for producing minimal weak/strong projections (slices).

Our algorithms have worst-case time complexity $O(|V|^3)$ and $O(|V|^4)$ respectively. This is an improvement over worst-case time complexity of the algorithms for computing the new control dependences of Ranganath et al. which they give as $O(|V|^3 \times lg(|V|) \times \sum T_n) = O(|V|^4 \times lg(|V|))$. (Since they define T_n to be the number of successors of vertex, n so $O(\sum T_n) = O|V|)$). The authors believe that more efficient algorithms exist. This will be the subject of future work.

Definition 59. Let $G = (V, E)$ be a graph. We define $|G| = |V| + |E|$.

The computation of $\Theta(G, V', u)$ (Definition 56) has time complexity $O(|G|)$, since removing the appropriate edges from G to obtain H takes linear time, and the subsequent reachability problem can be computed by depth-first search and thus has time complexity $O(|V| + |E|)$.

For CFGs, since the maximum out-degree for each vertex is two, we have $O(|G|) = O(|V|)$. The computation of $\Theta(G, V', u)$, thus has time complexity $O(|V|)$.

6.1. An algorithm to compute the minimal weakly control-closed superset of V' in G

Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. We require an algorithm for computing the minimal weakly control-closed superset of V' in G. Before we give our algorithm, we state and prove the following result:

Lemma 60. *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. Suppose that V' is not weakly control-closed in G. Then there is an edge (p, r) in G such that*

(1) $|\Theta(G, V', r)| = 1$ and
(2) $|\Theta(G, V', p)| \geqslant 2$ and
(3) p is reachable in G from V'.

Furthermore, for any edge (p, r) satisfying (1)–(3), the vertex p lies in every weakly control-closed subset of V containing V'.

Proof. Since V' is not weakly control-closed, there is a vertex $p \in V \backslash V'$ that is reachable in G from V' and is not V'-weakly committing. Hence there is a path $p_0 = p, p_1 = r, \ldots, p_m \in V'$ in G. Choose p so that m is minimal. Thus r is V'-weakly committing and $|\Theta(G, V', r)| = 1$ and $|\Theta(G, V', p)| \geqslant 2$ follow from this. Now assume that an edge (p, r) is in G and satisfies the conditions given, but there is a weakly control-closed set $W \supseteq V'$ not containing p. Since $|\Theta(G, V', r)| = 1$, there is a path $p_0 = p, p_1 = r, \ldots, p_m \in \Theta(G, V', r)$ in G, for some $m \geqslant 1$. In addition, $|\Theta(G, V', p)| \geqslant 2$ and so there is also a path $q_0 = p, q_1, \ldots, q_n \in V' \backslash \{p_m\}$ in G for some $n \geqslant 1$. Since $\Theta(G, V', p_i) = \{p_m\}$ for each $i \geqslant 1$, no $p_i = q_j$ unless $i = j = 0$. Thus $p \in \mathrm{WD}_G(V') \subseteq \mathrm{WD}_G(W)$ by Lemma 52, and so W is not weakly control-closed, by Lemma 51, giving a contradiction.

Thus we have shown if an edge (p, r) in V satisfies (1)–(3) above, then p lies in every weakly control-closed superset of V'. □

Our algorithm is indicated by Lemma 60, which gives a condition stated in terms of the function Θ on vertices that must be added to the set V' in order to obtain a weakly control-closed set. We now give an algorithm for computing the minimal weakly control-closed superset of V' in G which we prove has time-complexity $O(|V|^3)$. Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. To compute the minimal weakly control-closed superset of V' in G proceed as follows:

Algorithm 61. (1) Assign $X = V'$.
(2) Choose any edge (p, v) in G with p reachable from V' and such that $|\Theta(G, X, v)| = 1$ and $|\Theta(G, X, p)| \geqslant 2$ hold, and assign $X = X \cup \{p\}$. If no such edge (p, v) exists then STOP.
(3) GOTO 2.

Theorem 62. *Algorithm 61 has time complexity $O(|V|^3)$, and the value of the set X when STOP is reached is the minimal weakly control-closed superset of V' in G.*

Proof. Step (2) cannot execute twice with the same value of p and is therefore executed at most $|V|$ times. For each execution of (2), not more than $|G| = |V|$ edges are tested, and for each testing, the time taken to compute Θ and is of $O(|V|)$, proving the total $O(|V|^3)$ time complexity. By Lemma 60, X is the unique smallest weakly control-closed set in G containing V' when STOP is reached. □

Later in the paper we will prove that weak control-closure subsumes all the previous definitions of weak control dependence in the literature. In other words, the problem of computing the set of vertices that transitively control a set of vertices (whichever definition in the literature we use) can be reduced to computing weakly control-closed sets. Furthermore weak control-closure is more general, in the sense that it is defined for graphs for which previous definitions of control dependence are not defined.

6.2. An algorithm to compute the minimal strongly control-closed superset of V' in G

We now consider the analogous problem of computing the minimal strongly control-closed superset of a given set. First we need an algorithm for computing $\Gamma(G, V')$ (Definition 55): the set of all $x \in V$ that lie on a complete path in G which does not pass through V'.

Algorithm 63 (*Algorithm for Computing $\Gamma(G, V')$*).

(1) Assign $X = V'$.
(2) Choose any edge (y, x) in G with $x \in X$, $y \notin X$. If no such edge exists, then STOP.
(3) Delete the edge (y, x) from G.
(4) If there is another edge (y, z) with $z \neq x$, then convert y to a non-predicate and GOTO (2).
(5) If y is a non-predicate then assign $X = X \cup \{y\}$.
(6) GOTO (2).

Theorem 64. *In Algorithm 63, the final value of $V \backslash X$ is precisely $\Gamma(G, V')$, and the algorithm has time complexity $O(|G|^2) = O(|V|^2)$.*

Proof. Each execution of (3) except the last deletes an edge from G, hence the number of iterations is bounded by $O(|G|)$. Also finding an edge with property (2) is $O(|G|)$. Steps (3)–(6) are all constant, proving the total time complexity bound $O(|G|^2) = O(|V|^2)$. For any execution of (2) and for the current value of X just before this execution, the edges deleted from G and the vertices added to X are not used in any complete path using only the vertices in $V \backslash X$. Thus the set of all such complete paths does not change throughout the whole execution, even as X changes. Since at the end of the execution there are no edges from $V \backslash X$ to X, the set $\Gamma(G, V')$ of vertices occurring on these paths is $V \backslash X$, proving the theorem. □

Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. To compute the minimal strongly control-closed superset of V', now proceed as follows:

Algorithm 65. (1) Assign $X = V'$.
(2) Find an edge (p, r) in G such that p is reachable in G from X and satisfying:
  (a) $|\Theta(G, X, r)| = 1$ and
  (b) $r \notin \Gamma(G, X)$ and
  (c) $|\Theta(G, X, p)| \geqslant 2$ or $p \in \Gamma(G, X)$.
  If no such edge exists, then STOP, else assign $X = X \cup \{p\}$.
(3) GOTO (2).

Theorem 66. *Algorithm 65 computes unique minimal strongly control-closed superset of V' and has worst-case time complexity $O(|V|^4)$.*

Proof. The proof that Theorem Algorithm 65 computes unique minimal strongly control-closed superset of V' is analogous to the proof of the corresponding weak case in Theorem 62 with Lemma 57 used in place of Lemma 60.

The proof Algorithm 65 has worst-case time complexity $O(|V|^4)$ is as follows: The main loop of Algorithm 65 is executed $O(|G|) = O(|V|)$ times. Computing Step (a) *i.e.*, computing $\Theta(G, X, r)$ is also $O(|V|)$. In computing Step (b) we first compute $\Gamma(G, X)$. This is $O(|V|^2)$ by Theorem 64. Each time $\Gamma(G, X)$ is recomputed we need to find an $r \notin \Gamma(G, X)$ satisfying (3). This search for r adds one to the order of magnitude for step (2) giving $O(|V|^3)$. Step (2) dominates the other steps in time complexity giving a total worst-case time complexity of $O(|V|^4)$ since step (2) is executed $O(|V|)$ times. □

7. The weak forms of control dependence

7.1. Summary

In the literature, there are three distinct forms of control dependence which we call *weak* because, as we show in this section, vertex sets closed under them induce weak projections. They are:

- $\xrightarrow{\text{W-controls}}$: the control dependence of Weiser [33],
- $\xrightarrow{\text{F-controls}}$: the control dependence of Ferrante et al. [16], and
- $\xrightarrow{\text{WOD}}$: Amtoft's weak order dependence [3].

There is also Podgurski and Clarke's' *strong control dependence* [26], but this is merely a paraphrasing of Weiser's.

The main results of this section give the relationship between sets closed under the weak forms of control dependence mentioned above and weakly control-closed sets. These can be summarised as follows:

Lemma 80 Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph. V' is closed under $\xrightarrow{\text{F-controls}}$ if and only if V' is closed under $\xrightarrow{\text{W-controls}}$.

Lemma 81 Let $G = (V, E)$ be a $\{\mathsf{start}, \mathsf{end}\}$-graph with $\{\mathsf{start}, \mathsf{end}\} \subseteq V' \subseteq V$. If V' is weakly control-closed in G then V' is closed under $\xrightarrow{\text{W-controls}}$.

Lemma 83 Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph with $V' \subseteq V$. If V' is closed under $\xrightarrow{\text{W-controls}}$ then V' is weakly control-closed in G.

Lemma 84 Let $G = (V, E)$ be a finite directed graph with $V' \subseteq V$. If V' is closed under $\xrightarrow{\text{WOD}}$ then V' is weakly control-closed in G.

Lemma 85 Let $G = (V, E)$ be a $\{\mathsf{start}\}$-graph with $\mathsf{start} \in V' \subseteq V$. If V' is weakly control-closed in G then V' is closed under $\xrightarrow{\text{WOD}}$.

From these, we can prove our main result which shows that, indeed, all weak forms of control dependence in the literature induce weak projections. These forms of control dependence have thus been *semantically* characterised for the first time. The characterisation is as follows:

Theorem 67 (*Main Theorem for Weak Control Dependence*)**.**

(1) *If G is a* $\{\mathsf{start}, \mathsf{end}\}$-CFG *with* $\{\mathsf{start}, \mathsf{end}\} \subseteq V'$, *then* V' *is closed under* $\xrightarrow{\text{W-controls}}$ *if and only if the induced graph induced by* V' *from G is a weak projection of G.*

(2) *If G is a* $\{\mathsf{start}, \mathsf{end}\}$-CFG *with* $\{\mathsf{start}, \mathsf{end}\} \subseteq V'$, *then* V' *is closed under* $\xrightarrow{\text{F-controls}}$ *if and only if the induced graph induced by* V' *from G is a weak projection of G.*

(3) *If G is a* $\{\mathsf{start}\}$-CFG *with* $\mathsf{start} \in V'$ *then* V' *is closed under* $\xrightarrow{\text{WOD}}$ *if and only if the induced graph induced by* V' *from G is a weak projection of G.*

Proof. (1) follows from Lemmas 81 and 83 and Theorem 45. (2) follows from (1) and Lemma 80 and, (3) follows from Lemmas 84 and 85 together with Theorem 45. □

It can be seen from Theorem 67, that each form of weak control dependence requires a restriction to the CFG for it to be characterised by a weak projection. As Theorem 45 showed, weak control closure, on the other hand, requires no such restriction and can thus be more generally applied.

7.2. Weiser's control dependence

In order to define Weiser's control dependence, we first need *forward domination*.

Definition 68 (*Forward Domination*)**.** Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and let $v, w \in V$. If every path from v to end passes through w then w *forward dominates* v.

Note, v forward dominates itself. *Forward domination* is the terminology of Podgurski and Clarke. Weiser calls it *inverse domination*. Ferrante et al. [16] call it *post domination*.

It is well known [26, Theorem 1] that if $v \neq \mathsf{end}$ is a vertex in a CFG, then the set of all vertices that forward dominate v always occur in the same order on any path from v to end.

Definition 69 (*Nearest Forward Dominator*)**.** We call the first such vertex apart from v the *nearest forward dominator* of v.

Note, v cannot be its own nearest forward dominator.

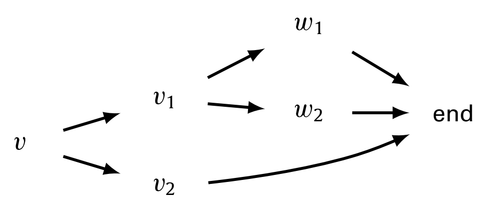

$$\begin{tikzcd} & & w_1 \arrow[dr] & \\ & v_1 \arrow[ur] \arrow[r] & w_2 \arrow[r] & \mathsf{end} \\ v \arrow[ur] \arrow[dr] & & & \\ & v_2 \arrow[uurr, bend right] & & \end{tikzcd}$$

Fig. 13. Ferrante et al. control dependence is not transitive: $v \xrightarrow{\text{F-controls}} v_1$ and $v_1 \xrightarrow{\text{F-controls}} w_1$ but $v \xrightarrow{\text{F-controls}} w_1$ is false.

Definition 70 (ND). Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and $v \in V$. $\mathsf{ND}(v)$ is the set of vertices which lie on a path from v to its nearest forward dominator b, excluding v and b themselves.

Implicit in this is Weiser's definition of control dependence:

Definition 71 ($\xrightarrow{\text{W-controls}}$). Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and $v, w \in V$, then $v \xrightarrow{\text{W-controls}} w$ if and only if $w \in \mathsf{ND}(v)$.

Note that in v cannot control itself using Weiser's definition.

7.3. *The control dependence of Ferrante et al.*

In defining the *program dependence graph*, control dependence was once again redefined [16].

Definition 72 ($\xrightarrow{\text{F-controls}}$). Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph, then $v \xrightarrow{\text{F-controls}} w$ if and only if v is not forward dominated by w, and there exists a path π from v to w such that for all vertices z occurring on π apart from v are forward dominated by w.

Ferrante et al. control dependence is not transitive, as is demonstrated by the example in Fig. 13 where $v \xrightarrow{\text{F-controls}} v_1$ and $v_1 \xrightarrow{\text{F-controls}} w_1$ but v does not control w_1. Also note that a vertex cannot control itself using this definition. This follows because every vertex forward dominates itself.

Lemma 73, in effect, allows an alternative definition of Ferrante et al. control dependence, which is often used:

Lemma 73. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and $v, w \in V$ with $w \neq v$. The vertex $v \xrightarrow{\text{F-controls}} w$ if and only if v has immediate successors v_1, v_2 such that w forward dominates v_1 but not v_2.*

Proof. Suppose that w is Ferrante control dependent on v. Then there is a path π from v to w on which all vertices except v are forward dominated by w, and there is a path $v\mu\mathsf{end}$ which does not pass through w. Let v_1 be the second vertex of π then $v\mu\mathsf{end}$ also does not pass through v_1 because w forward dominates v_1. Therefore the first vertex of μ must be some $v_2 \neq v_1$ and so v has immediate successors v_1, v_2 with w forward dominating v_1 but not v_2, as required.

Conversely, let v have immediate successors v_1, v_2 such that w forward dominates v_1 but not v_2. Since w forward dominates v_1, and end is reachable from v_1, there is a path $v_1\rho\mathsf{end}$ on which w occurs, and which hence has a prefix $v_1\nu w$ on which w does not occur except at the end. Thus every vertex on $v_1\nu$ is forward dominated by w. However since w does not forward dominate v_2, and there is an edge (v, v_2) and $w \neq v$, w does not forward dominate v either, and thus $v \xrightarrow{\text{F-controls}} w$ as required. □

7.4. *Amtoft's weak-order dependence*

Amtoft [3,29] observes that the traditional CFG is not well adapted to handle modern programming constructs which may intentionally fail to terminate, *e.g.*, reactive systems. Amtoft addresses this problem by define weak order dependence ($\xrightarrow{\mathsf{WOD}}$) as an extension of Ferrante et al. control dependence to handle CFGs which are not necessarily $\{\mathsf{end}\}$-graphs.

Definition 74 ($\xrightarrow{\mathsf{WOD}}$). Let $G = (V, E)$ be a finite directed graph with $v, b, c \in V$. Then $v \xrightarrow{\mathsf{WOD}} b, c$ if and only if:

(1) There is a path from v to b not containing c.
(2) There is a path from v to c not containing b.
(3) v has an immediate successor a such that either
- b is reachable from a, and all paths from a to c contain b; or
- c is reachable from a, and all paths from a to b contain c.

7.5. *The relationship between controlling predicates and weakly deciding predicates*

There is a strong connection between sets closed under the different forms of weak control dependence and the sets closed under WD_G, *i.e.*, those sets V' for which $\mathsf{WD}_G(V') \subseteq V'$.

7.6. Characterisation of sets closed under different weak forms of control dependence

Definition 75 (*The Reflexive Transitive Closure of a Relation*). . Given a relation r, the reflexive transitive closure of r, write r^* is defined to be the smallest reflexive, transitive relation containing r.

Definition 76 (*Closed Sets*). . Given a relation r, we define a set S to be closed under r to mean that if $s \in S$ and $t \xrightarrow{r} s$, then $t \in S$.

Lemma 77. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and let $p, v \in V$ with $p \neq v$, then $p \xrightarrow{\text{F-controls}^*} v$ if and only if p is $\{v, \mathsf{end}\}$-weakly deciding in G.*

Proof. ($\Longrightarrow$) Suppose that $p \xrightarrow{\text{F-controls}^*} v$. Thus there is a sequence $v = p_0, \ldots, p_m = p$ with $m \geqslant 1$ such that $p_i \xrightarrow{\text{F-controls}} p_{i-1}$. We prove $p \in \mathsf{WD}_G(\{v, \mathsf{end}\})$ by induction on m.

If $m = 1$ then $p \xrightarrow{\text{F-controls}} v$. Since end is reachable from every vertex, by Lemma 73, p has immediate successors p_1, p_2 such that there is a $[p_1, v]$ path $\alpha_1 v$ containing v only at the end and a $[p_2, \mathsf{end}]$ path, $\alpha_2\mathsf{end}$, not containing v, and every path from p_1 (and hence from any vertex in α_1) to end passes through v. Thus no vertex occurs on both paths $\alpha_1 v$ and $\alpha_2\mathsf{end}$, hence $p \in \mathsf{WD}_G(\{v, \mathsf{end}\})$ follows.

If $m > 1$, the inductive hypothesis gives us

$$p_{m-1} \in \mathsf{WD}_G(\{v, \mathsf{end}\}),$$

and since $p_m \xrightarrow{\text{F-controls}} p_{m-1}$, we have just shown that

$$p = p_m \in \mathsf{WD}_G(\{p_{m-1}, \mathsf{end}\})$$

follows. Thus by Lemma 52, we may replace $\{p_{m-1}, \mathsf{end}\}$ by its superset

$$\mathsf{WD}_G(\{v, \mathsf{end}\}) \cup \{v, \mathsf{end}\}$$

to get

$$\begin{aligned} p &\in \mathsf{WD}_G(\mathsf{WD}_G(\{v, \mathsf{end}\}) \cup \{v, \mathsf{end}\}) \\ &\subseteq \mathsf{WD}_G(\{v, \mathsf{end}\}) \cup \{v, \mathsf{end}\} \end{aligned}$$

by Lemma 53. Since clearly $p \notin \{v, \mathsf{end}\}$, the result $p \in \mathsf{WD}_G(\{v, \mathsf{end}\})$ follows, as required.
($\Longleftarrow$) Conversely, suppose that $w \in \mathsf{WD}_G(\{v, \mathsf{end}\})$ then there are paths $w\alpha_1 v$ and $w\alpha_2\mathsf{end}$ in V which are disjoint other than w. We will prove $w \xrightarrow{\text{F-controls}^*} v$ by induction on the length of the path α_1. If end is not reachable in $V\backslash\{v\}$ from any vertex on α_1, then $w \xrightarrow{\text{F-controls}} v$ is immediate. Thus we may write $\alpha_1 = \mu u \nu$ where there is a path $u\rho\mathsf{end}$ in V that does not pass through v. Assume u is the last vertex on α_1 from which end is reachable in $V\backslash\{v\}$, and so ν has no vertex in common with α_2. Therefore ρ and ν have no common vertex, and so $u \in \mathsf{WD}_G(\{v, \mathsf{end}\})$ and hence by induction $u \xrightarrow{\text{F-controls}^*} v$.

Finally, $w \in \mathsf{WD}_G(\{u, \mathsf{end}\})$ and hence by induction $w \xrightarrow{\text{F-controls}^*} u$ and thus $w \xrightarrow{\text{F-controls}^*} v$. □

Lemma 78. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph, then $p \xrightarrow{\text{W-controls}} v$ if and only if $p \in \mathsf{WD}_G\{v, \mathsf{end}\}$ and $p \neq v$.*

Proof. Let u be the nearest forward dominator of p in G. Suppose that $v \in \mathsf{ND}(p)$. Then $v \neq p$ by definition. Also there is a path $p\mu v\nu u$ on which u does not occur except at the end, and so v does not forward dominate p, hence there is also a path $v\sigma u$ not passing through v (see Fig. 14).

We can assume that $\mu v\nu u\tau$ and $\sigma u\tau$ have no repeated elements since if they do then the cycle can simply be removed. Also we can assume that τ and μ are disjoint since otherwise u would not be the nearest forward dominator of p.

We prove $p \in \mathsf{WD}_G\{v, \mathsf{end}\}$ by induction on $|\mu|$.

If $|\mu| = 1$ then $p \in \mathsf{WD}_G\{v, \mathsf{end}\}$ by definition. If $|\mu| > 1$, then if μ and $\sigma u\tau$ are disjoint then again, by definition, $p \in \mathsf{WD}_G\{v, \mathsf{end}\}$. If μ and $\sigma u\tau$ are not disjoint, let q be the first vertex along μ which is also in $\sigma u\tau$. Then q must be in σ since τ and μ are disjoint and $q \neq u$.

By the inductive hypothesis, $q \in \mathsf{WD}_G\{v, \mathsf{end}\}$ and also since $q \neq u$, $p \xrightarrow{\text{W-controls}} v$, so again by the inductive hypothesis, $p \in \mathsf{WD}_G\{q, \mathsf{end}\}$. So, by the idempotence of WD (Lemma 53), we have $p \in \mathsf{WD}_G\{v, \mathsf{end}\}$, as required.

Conversely, assume $p \in \mathsf{WD}_G\{v, \mathsf{end}\}$ and $p \neq v$. Then there are paths $p\mu\mathsf{end}$ and $p\nu v$ which share only p as a common vertex. There is also a path $v\pi\mathsf{end}$. Clearly u occurs on both paths $p\mu\mathsf{end}$ and $p\nu v\pi\mathsf{end}$, and the 'non-sharing' property implies that u occurs on $\pi\mathsf{end}$, proving $v \in \mathsf{ND}(p)$. □

Lemma 79. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and let $p, v \in V$ with $p \neq v$, then $p \xrightarrow{\text{F-controls}^*} v$ if and only if $p \xrightarrow{\text{W-controls}} v$.*

Proof. By Lemma 77 $p \xrightarrow{\text{F-controls}^*} v$ if and only if p is $\{v, \mathsf{end}\}$-weakly deciding in G. By Lemma 78 p is $\{v, \mathsf{end}\}$-weakly deciding in G if and only if $p \xrightarrow{\text{W-controls}} v$. □

From Lemma 79, it now follows immediately that:

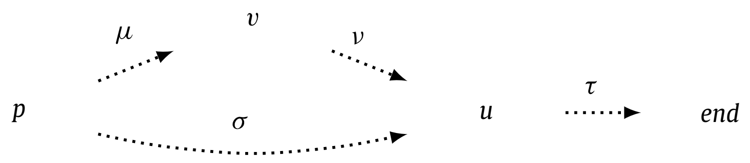


Fig. 14. Lemma 78.

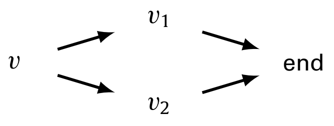


Fig. 15. Necessity that $\mathsf{start} \in V'$ in Lemma 81: Here v controls both v_1 and v_2, so $\{v_1, v_2\}$ is not closed under control dependence, but $\{v_1, v_2\}$ is weakly control-closed in G.

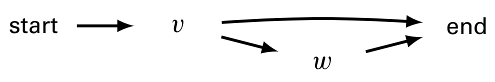


Fig. 16. Necessity that $\mathsf{end} \in V'$ in Lemma 81: Here if $V' = \{\mathsf{start}, w\}$ then V' is weakly control-closed in G but v controls w, so V' is not closed under control dependence.

Lemma 80. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and let $V' \subseteq V$. V' is closed under $\xrightarrow{\text{F-controls}}$ if and only if V' is closed under $\xrightarrow{\text{W-controls}}$.*

7.7. Sets closed under weak forms of control dependence and weak control closure

Lemma 81. *Let $G = (V, E)$ be a $\{\mathsf{start}, \mathsf{end}\}$-graph and let $\{\mathsf{start}, \mathsf{end}\} \subseteq V' \subseteq V$. If V' is weakly control-closed in G then V' is closed under $\xrightarrow{\text{F-controls}}$ (and by Lemma 80, $\xrightarrow{\text{W-controls}}$).*

Proof. Suppose not, then there are vertices v, w with $v \in V \backslash V'$ and $w \in V'$ such that $v \xrightarrow{\text{F-controls}} w$. Thus, by Lemma 73, v has distinct immediate successors z_1, z_2 such that there are paths $z_1\rho_1 w$ and $z_2\rho_2\mathsf{end}$, where w does not occur on either ρ_i and end is not reachable in $V\backslash\{w\}$ from z_1. Thus v is $\{\mathsf{end}, w\}$-weakly deciding in G. By Lemma 52 v is V'-weakly deciding in G. But v is reachable from $\mathsf{start} \in V'$ therefore by Lemma 51, V' is not weakly control-closed in G. □

Note that the condition that $\mathsf{start} \in V'$ really is necessary, as the example in Fig. 15 shows.

Furthermore, the condition that $\mathsf{end} \in V'$ cannot be dispensed with. In Fig. 16, let $V' = \{\mathsf{start}, w\}$. V' is weakly control-closed in G but v controls w so V' is not closed under control dependence.

Lemma 82. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and let $V' \subseteq V$. Suppose that $w \in V\backslash V' \cap \mathsf{WD}_G(V')$. Then there exists $w_1 \in V\backslash V'$ and $v \in V'$ such that w_1 is $\{v, \mathsf{end}\}$-weakly deciding in G.*

Proof. Since $w \in \mathsf{WD}_G(V')$, for $i \in \{1, 2\}$ there are proper V'-paths $w\rho_i z_i$ for vertices z_i such that no vertex lies on both paths $\rho_i z_i$. There is also a path $z_1\sigma\mathsf{end}$ through G which does not pass more than once through z_1. By replacing z_1 by z_2 if necessary, and using a suffix of $\sigma\mathsf{end}$, we may assume also that $\sigma\mathsf{end}$ does not pass through z_2. If no vertex on $\sigma\mathsf{end}$ also occurs on ρ_2, then the conclusion follows for $w_1 = w$ and $v = z_2$ by examining the paths $w\rho_1 z_1\sigma\mathsf{end}$ and $w\rho_2 z_2$. Otherwise we may write $\rho_2 = \alpha w_1\beta$ and $\sigma = \tau w_1\omega$, where $w_1 \in V$ and no vertex occurs on both β and ω, and the conclusion again follows for $v = z_2$ by examining the paths $w_1\beta z_2$ and $w_1\omega\mathsf{end}$. □

Lemma 83. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and let $V' \subseteq V$.*
If V' closed under $\xrightarrow{\text{F-controls}}$ then V' is weakly control-closed in G.

Proof. Suppose that V' is not weakly control-closed in G, then by Lemma 51, $V\backslash V'$ contains an element of $\mathsf{WD}_G(V')$. By Lemma 82, there exists $z \in V\backslash V'$ and $v \in V'$ such that $z \in \mathsf{WD}_G(\{v, \mathsf{end}\})$, and hence by Lemma 77 $z \xrightarrow{\text{F-controls}^*} v$. Hence V' is not closed under $\xrightarrow{\text{F-controls}}$. □

Lemma 84. *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$.*
If V' is closed under $\xrightarrow{\text{WOD}}$ then V' is weakly control-closed in G.

Proof. Suppose not, then there exists $y \in V\backslash V'$ which is reachable from V' and which is not weakly-committing. This implies there exist proper V'-paths $y\alpha_1 w_1$ and $y\alpha_2 w_2$ for distinct vertices $w_1 \neq w_2$. Let Ω be the set of all vertices from which w_2 is reachable in $V\backslash\{w_1\}$; thus $y \in \Omega$, but $w_1 \notin \Omega$. Thus we may write $y\alpha_1 w_1 = \beta v\gamma w_1$, where $v \in \Omega$ and no vertex in γw_1 lies in Ω. Hence $v \xrightarrow{\text{WOD}} w_1, w_2$ which contradicts that V' is closed under weak order dependence. □

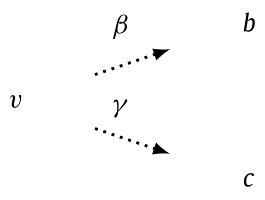


Fig. 17. Lemma 85.

Lemma 85. *Let $G = (V, E)$ be a* {start}*-graph with* start $\in V'$.
If V' is weakly control-closed in G then V' is closed under $\xrightarrow{\text{WOD}}$.

Proof. Suppose that $b, c \in V'$ and $v \xrightarrow{\text{WOD}} b, c$ holds for $v \in V$. We will assume that $v \notin V'$ and deduce a contradiction. From the definition of weak order dependence and by interchanging b and c if necessary, there are paths $v\beta b$ and $v\gamma c$ such that b does not occur on γc, nor c on βb, and all paths from the first vertex of βb to c pass through b before reaching c, which implies that no vertex occurs on both βb and γc. See Fig. 17. So $v \in \text{WD}_G(V')\backslash V'$. Vertex v is reachable from start $\in V'$ contradicting the assumption that V' is weakly control-closed in G (by Lemma 51). □

Note that the condition that start $\in V'$ really is necessary: in Fig. 15 $v \xrightarrow{\text{WOD}} v_1, v_2$ so $\{v_1, v_2\}$ is not closed under $\xrightarrow{\text{WOD}}$ but $\{v_1, v_2\}$ is weakly control-closed in G.

We have shown that all weak forms of control dependence in the literature are essentially the same: vertex sets closed under them all induce weak projections. We may call any relation on vertex sets that has this property *weak control dependence*. So $\xrightarrow{\text{W-controls}}$, $\xrightarrow{\text{F-controls}}$ and $\xrightarrow{\text{WOD}}$ are all examples of weak control dependence. In the next section, we turn our attention to the strong forms of control dependence.

8. The strong forms of control dependence

8.1. Summary

In the literature, there are two distinct forms of control dependence which we call *strong* because, as we show in this section, vertex sets closed under them induce strong projections. These are:

- the combination of $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ of Ranganath et al. [29].
- $\xrightarrow{\text{PC-weak}}$, the weak control dependence of Podgurski and Clarke [26].

The main results of this section give the relationship between sets closed under the strong forms of control dependence mentioned above and strongly control-closed sets. These can be summarised as follows:

Lemma 93 Let $G = (V, E, \beta)$ be a complete CFG. If $V' \subseteq V$ is closed under both $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ then V' is strongly control-closed in G.

Lemma 94 Let $G = (V, E)$ be {start}-graph and start $\in V' \subseteq V$. If V' is strongly control-closed in G then V' is closed under both $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$.

Lemma 96 Let $G = (V, E, \beta)$ be a complete {end}-CFG and $V' \subseteq V$. If V' is closed under $\xrightarrow{\text{PC-weak}}$ then V' is strongly control-closed in G.

Lemma 97 Let $G = (V, E)$ be a {start, end}-graph with start $\in V' \subseteq V$. If V' is strongly control-closed in G then V' is closed under $\xrightarrow{\text{PC-weak}}$.

From these, we can prove our main result which shows that, indeed, both strong forms of control dependence in the literature induce strong projections. These forms of control dependence have thus been *semantically* characterised for the first time. The characterisation is as follows:

Theorem 86 (*Main Theorem for Strong Control Dependence*).

(1) *If G is a complete* {start}*-CFG with* start $\in V'$ *then V' is closed under* $\xrightarrow{\text{DOD}}$ *and* $\xrightarrow{\text{NTSCD}}$ *if and only if the induced graph induced by V' from G is a strong projection of G.*
(2) *If G is a complete* {start, end}*-CFG with* start $\in V'$ *then V' is closed under* $\xrightarrow{\text{PC-weak}}$ *if and only if the induced graph induced by V' from G is a strong projection of G.*

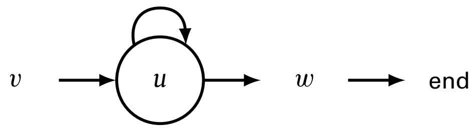


Fig. 18. Vertex w forward dominates v but does not strongly forward dominate v.

Proof. (1) follows from Lemmas 93, 94 and Theorem 49 and (2) follows from Lemmas 96, 97 and Theorem 49. □

It can be seen from Theorem 86, that each form of strong control dependence requires a restriction to the CFG for it to be characterised by a strong projection. As Theorem 49 showed, strong control-closure, on the other hand, requires no such restriction and can thus be more generally applied.

8.2. Ranganath's non-termination-sensitive control dependence $\xrightarrow{\text{NTSCD}}$

Ranganath et al. [29] observe that, despite being widely used, existing definitions and approaches to calculating control dependence are difficult to apply directly to modern program structures which make substantial use of exception processing and which may deliberately run indefinitely. A major motivation of Ranganath's work is that traditional forms of control require the end to be reachable from every vertex. They rightly claim that this is not a suitable restriction for such programs which are designed to non-terminate. So they, like us, allow CFGs where end is not necessarily reachable from every vertex.

For these sort of programs, they argue that the slice should non-terminate in all initial states when the original does. In order to compute this strong form of slice, Ranganath et al. define two new control dependence relations:

- Non-termination-sensitive control dependence, $\xrightarrow{\text{NTSCD}}$.
- Decisive order dependence (sometimes referred to as *direct order dependence*), $\xrightarrow{\text{DOD}}$.

In the definitions below, reproduced from their work [29], the term *maximal path* refers to a path that either is infinite or ends at end. The definitions still make sense for the more general class of graphs under investigation in this paper where a *maximal path* is taken as the path $\bar{\omega}$ of a maximal walk ω (see Definitions 15 and 21). By Proposition 23 both of these notions of *maximal path* coincide in CFGs where all predicates are complete.

Definition 87 ($\xrightarrow{\text{DOD}}$)**.** Let $G = (V, E)$ be a finite directed graph, then $v \xrightarrow{\text{DOD}} b, c$ if and only if:

(1) All maximal paths from v contain both b and c.
(2) v has an immediate successor from which all maximal paths contain b before any occurrence of c.
(3) v has an immediate successor from which all maximal paths contain c before any occurrence of b.

Definition 88 ($\xrightarrow{\text{NTSCD}}$)**.** Let $G = (V, E)$ be a finite directed graph, then $v \xrightarrow{\text{NTSCD}} w$ if and only if:

(1) v has at least two immediate successors.
(2) w occurs on all maximal paths from one of these immediate successors.
(3) there is a maximal path from another immediate successor which does not contain w.

8.3. Podgurski–Clarke weak control dependence

Definition 89 (*Strong Forward Domination*)**.** Let G be an {end}-graph. A vertex w *strongly forward dominates* a vertex v if and only if w forward dominates v and there exists an $n \in \mathbb{N}$ such that every path of length n from v contains w.

Strong forward domination is a properly stronger condition than forward domination as can be seen by the diagram in Fig. 18.

Definition 90 ($\xrightarrow{\text{PC-weak}}$)**.** Let $G = (V, E)$ be an {end}-graph, then $v \xrightarrow{\text{PC-weak}} u$ if and only if:

(1) v has at least two immediate successors w_1 and w_2.
(2) u strongly forward dominates w_1 but does not strongly forward dominate w_2.

In {end}-graphs $\xrightarrow{\text{PC-weak}}$ and $\xrightarrow{\text{NTSCD}}$ are equivalent. Ranganath et al. [29] prove a similar result (Theorem 3 (Coincidence Properties, II)). We shall show in Lemma 91 that $\xrightarrow{\text{NTSCD}}$ (by itself) is equivalent to Podgurski–Clarke weak control dependence in {end}-graphs.

Lemma 91. *Let $G = (V, E)$ be an {end}-graph. Then for all $u, v \in V$, $v \xrightarrow{\text{PC-weak}} u$ if and only if $v \xrightarrow{\text{NTSCD}} u$.*

Proof. Observe that since end is reachable from every vertex in V, a path in G is maximal if and only if it is either infinite or reaches end.

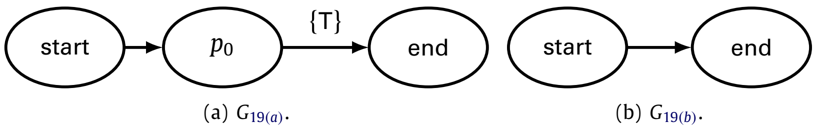


Fig. 19. The smallest set closed under $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ in $G_{19(a)}$ containing {start, end} is {start, end} but the induced graph, $G_{19(b)}$, is not a strong projection of $G_{19(a)}$ because the maximal walk, start, (p_0, F), of $G_{19(a)}$ restricts to the walk start of $G_{19(b)}$ which is not maximal in $G_{19(b)}$. Alternatively, we can see that {start, end} is not strongly control closed in $G_{19(a)}$ since p_0 is reachable from {start, end} in $G_{19(a)}$ but p_0 is neither {start, end}-strongly committing nor {start, end}-avoiding in $G_{19(a)}$.

We prove that for $u, w \in V$, u strongly forward dominates w if and only if every maximal path from w passes through u.

With this equivalence part 2 of Definition 88 becomes equivalent to the condition of w_1 in Definition 90. The contrapositive form of this equivalence makes part 3 of Definition 88 equivalent to the condition on w_2 in Definition 90.

Suppose that u strongly forward dominates v. Then for some $n \geqslant 0$ every path from v of length n contains u. Now let $v\mu$ be a maximal path. If $v\mu$ is infinite then it has a prefix of length n which contains u as required. If $v\mu$ is finite then it contains end and since strong forward domination implies forward domination, $v\mu$ again contains u.

Conversely, assume that every maximal path from w passes through u. Clearly u forward dominates w, but in addition every path $w\rho$ of length $|V|+1$ must pass through u, since $w\rho$ must pass more than once through at least one vertex, and so if $w\rho$ does not pass through u, then an infinite path from w exists which also does not pass through u, giving a contradiction, hence u strongly forward dominates w. □

Lemma 92. *Let $G = (V, E)$ be a finite directed graph and let $v \in V$. Let $d_1, \ldots, d_n \in V$ be all the immediate successors of v, and assume that $n \geqslant 2$. Let $e_1, \ldots, e_n \in V$ and assume that for each $i \leqslant n$, all maximal paths from d_i pass through e_i, and do so before passing through any e_j for $j \neq i$. Then either there exist e_k, e_l such that $v \xrightarrow{\text{DOD}} e_k, e_l$ or there exists e_l such that $v \xrightarrow{\text{NTSCD}} e_l$.*

Proof. If every vertex e_i occurs on every maximal path starting at any of the vertices d_j, then $v \xrightarrow{\text{DOD}} e_i, e_j$ for every $i \neq j$ follows. Otherwise, there exist i, j such that e_i does not occur on every maximal path starting at d_j, whereas necessarily e_i occurs on every maximal path starting at d_i. Hence $v \xrightarrow{\text{NTSCD}} e_i$ follows. □

8.4. The relationship between strongly control-closed sets and sets closed under both $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$

Lemma 93. *Let $G = (V, E, \beta)$ be a complete CFG. If V' is closed under both $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ then V' is strongly control-closed in G.*

Proof. For any $x \in V \setminus V'$, we first define S_x to be the set of all vertices in $V \setminus V'$ which are reachable from x via a path which only contains vertices in $V \setminus V'$. Observe that if (x, y) is an edge with $x, y \in V \setminus V'$, then $S_x \supseteq S_y$, with strict inclusion if there is no maximal path starting at x and not passing through V', since in that case $x \in S_x \setminus S_y$.

Suppose that V' is not strongly control-closed in G. Then there is a vertex $v \in V \setminus V'$ which is not V'-avoiding and is not strongly committing. Choose v satisfying this condition such that $|S_v|$ is minimal, and among all such v, such that the minimal length of any path from v to V' is minimal. This implies that v has two immediate successors d_1, d_2. We may assume that d_1 is closer to V' than v is.

Clearly d_1 is not V'-avoiding, and hence by the minimality assumption on v, the observation above on the sets S_x and the condition on d_1, d_1 is strongly committing and so every maximal path starting at d_1 passes through V', and enters V' at a single vertex e_1. Suppose the vertex d_2 is the initial vertex of a maximal path that does not pass through V'; then $v \xrightarrow{\text{NTSCD}} e_1$ holds, hence V' is not closed under $\xrightarrow{\text{NTSCD}}$. Thus we may suppose instead that every maximal path starting at d_2 (and hence v) passes through V'. If all maximal paths starting at d_2 always enter V' at e_1, hence since G is complete, v is strongly committing, giving a contradiction. Thus there is a maximal path starting at d_2 that enters V' at a vertex $e_2 \neq e_1$. If all such maximal paths enter V' at e_2, then the conclusion follows from Lemma 92. Otherwise, d_2 would not be either V'-avoiding or V' strongly committing, contradicting the minimality condition on v. □

Fig. 19 gives an example where the graph has incomplete predicates. V' is closed under both $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ but V' is not strongly control-closed in G and hence the induced graph is not a strong projection. This shows that we cannot drop the 'complete predicates' condition of Theorem 93.

Lemma 94. *Let $G = (V, E)$ be start-graph and start $\in V'$. If V' is strongly control-closed in G then V' is closed under both $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$.*

Proof. We consider two cases.

- First assume that V' is not closed under $\xrightarrow{\text{DOD}}$. Then $v \xrightarrow{\text{DOD}} b_1, b_2$ for some $b_1, b_2 \in V'$ and $v \in V \backslash V'$. Thus v has immediate successors x_1, x_2 such that for each i, all maximal paths from x_i contain b_i before b_{3-i}. By definition of $\xrightarrow{\text{DOD}}$, there are paths $x_i v_i$ on which b_{3-i} does not occur, and on which b_i occurs at the end, but not before; and by conditions (2) and (3) of the $\xrightarrow{\text{DOD}}$ definition, no vertex occurs on both paths. Hence each path $x_i v_i$ has a prefix $\tau_i a_i$ which is a V'-path for distinct $a_i \in V'$, and so $v \in \mathsf{WD}_G(V')$. Now, v is reachable from $\mathsf{start} \in V'$, so by Lemma 51 V' is not weakly control-closed, and hence not strongly control-closed.
- Suppose instead that V' is not closed under $\xrightarrow{\text{NTSCD}}$. Thus $v \xrightarrow{\text{NTSCD}} w$ holds for some $w \in V'$ and $v \in V \backslash V'$. Hence v has immediate successors x_1, x_2 such that all maximal paths from x_1 contain w, but there is a maximal and hence complete path $x_2\mu$ not passing through w. Thus v is not strongly committing. Since v is clearly not V'-avoiding, and is reachable from start, V' is not strongly control-closed. □

8.5. *The relationship between strongly control-closed sets and sets closed under Podgurski–Clarke weak control dependence*

Let $G = (V, E, \beta)$ be a $\{\mathsf{start}, \mathsf{end}\}$-CFG and $V' \subseteq V$. In this section we prove that V' is strongly control-closed in G if and only if V' is closed under Podgurski–Clarke weak control dependence.

Theorems 96 and 97 show that closure under $\xrightarrow{\text{PC-weak}}$ is equivalent to strong control-closure for vertex sets containing start, provided that end is reachable from all vertices.

Lemma 95. *Let $G = (V, E)$ be an $\{\mathsf{end}\}$-CFG, then for all $v, b, c \in V$, $v \xrightarrow{\text{DOD}} b, c$ never holds.*

Proof. Suppose that $v \xrightarrow{\text{DOD}} b, c$ holds. After interchanging b and c if necessary, there is a path $b\rho\mathsf{end}$ which does not pass through c. From condition (2) of Definition 87 there is a path $v\sigma b$ which also does not pass through c. However this contradicts condition (1) of Definition 87. □

Lemma 96. *Let $G = (V, E)$ be a complete $\{\mathsf{end}\}$-CFG and $V' \subseteq V$. If V' is closed under $\xrightarrow{\text{PC-weak}}$ then V' is strongly control-closed in G.*

Proof. This follows immediately from Lemmas 95, 91 and 93. □

Lemma 97. *Let $G = (V, E)$ be a $\{\mathsf{start}, \mathsf{end}\}$-graph and let $\mathsf{start} \in V' \subseteq V$. If V' is strongly control-closed in G then V' is closed under $\xrightarrow{\text{PC-weak}}$.*

Proof. Suppose V' is not closed under $\xrightarrow{\text{PC-weak}}$. Thus for some $u \in V'$ and $v \in V \backslash V'$, $v \xrightarrow{\text{PC-weak}} u$ holds. Thus by Lemma 91 and the fact that end is reachable from u, for vertices w_i there is a path $vw_1\rho_1 u$ and a maximal and hence complete path $vw_2\rho_2$ which does not pass through u, and every maximal path starting at a vertex on $w_1\rho_1 u$ passes through u. Clearly v is not V'-avoiding. Let x_1 be first element of V' to occur on $vw_1\rho_1$. If $vw_2\rho_2$ does not pass through V', then v is not V'-strongly committing. On the other hand, suppose that x_2 is the first element of V' to occur on $vw_2\rho_2$, then $x_1 \neq x_2$, since otherwise there would be a maximal path from x_1 not passing through u. Again, we have shown that v is not V'-strongly committing. Thus, since $\mathsf{start} \in V'$, V' is not strongly control-closed, proving the theorem. □

We have shown that both strong forms of control dependence in the literature are essentially the same: vertex sets closed under them all induce strong projections. We may call any relation on vertex sets that has the property *strong control dependence*. So $\xrightarrow{\text{PC-weak}}$ and the combination of $\xrightarrow{\text{NTSCD}}$ and $\xrightarrow{\text{DOD}}$ are both examples of strong control dependence.

9. Conclusions and future work

Authors have previously expressed control dependence as a relation between the vertices of a CFG. In an attempt to capture the *intention* of control dependence, we, on the other hand, define relations between CFGs and show that all previous forms of control dependence induce graphs which, indeed, satisfy these relations. Weak and strong projection can, thus, be thought of as a specification or a semantics of control dependence rather than an implementation. Furthermore, by introducing weak and strong control-closure, we have generalised control dependence and algorithms which compute sets closed under its different forms.

We believe these very natural relations can be considered as correctness criteria for future definitions of control dependence on more general structures and that authors of such new definitions will have a proof obligation based on them. The work we present here also has practical implications: we have defined reasonably efficient algorithms which can be used for slicing more general structures than those considered previously. Future research will include the following work:

(1) We will investigate the applicability of the theory to more general structures. Clearly, since the concept of a walk generalises to arbitrary finite labelled graphs, so do weak and strong projections. This may be useful, for example, in defining control dependence in graphs representing non-deterministic programs where non-predicate vertices may have out-degree greater than one and predicates may have non-disjoint edge labels.

(2) We will investigate the theoretical validity and practicability of combining the algorithms for the minimal weakly and strongly control-closed supersets of V' in G and described in this paper with data dependence to form weak and strong semantic slices of arbitrarily unstructured programs.
(3) Improvements to the algorithms for computing the minimal weakly and strongly control-closed supersets of V' in G will be investigated. It is believed that better than $O(V^3)$ worst-case time complexity algorithms may exist.
(4) We will investigate the application our generalised notions of control dependence to other structures for example extended finite state machines [4].

10. Glossary of definitions, results and algorithms

10.1. Definitions

CFGs (Definition 1) A *control flow graph* (CFG) is a triple $G = (V, E, \beta)$ where (V, E) is a finite directed graph and the vertex set V is partitioned as $V = P \cup N$ (predicates and non-predicates) with $P \cap N = \emptyset$, and $\beta : E \to \mathcal{P}(\{\mathsf{T}, \mathsf{F}\})$ is the edge labelling function.

(1) • If $x \in P$ then the out-degree of x is at most 2.
  • If $x \in N$ then the out-degree of x is at most 1.
  • There is at most one end vertex. It has out-degree 0. (end $\in N$ is the only vertex which represents normal termination.)
(2) The edges are labelled by β where:
  • If $x \in P$ and $(x, y) \in E$ then $\beta(x, y) \neq \emptyset$.
  • If $x \in N$ and $(x, y) \in E$ then $\beta(x, y) = \emptyset$.
  (For clarity we omit the label $\emptyset$ from our diagrams.)
(3) Let p be a predicate. If $(p, y) \in E$ and $(p, z) \in E$ with $y \neq z$ then $\beta(p, y) \cap \beta(p, z) = \emptyset$. (In other words, our CFGs are deterministic.)

Complete Predicates of a CFG (Definition 2) A predicate is complete if and only if the union of the labels of its outgoing edges is $\{\mathsf{T}, \mathsf{F}\}$.

Complete CFGs (Definition 3) A CFG is complete if and only if all its predicates are complete.

Final Vertices of a CFG (Definition 4) A final vertex is either a non-predicate vertex of out-degree 0 or an incomplete predicate.

{start, end}-CFGs and Graphs (Definition 5) Let $G = (V, E)$ be a finite directed graph.

(1) If G has a unique distinguished vertex start $\in V$ and every $v \in V$ is reachable from start then G is a {start}-graph.
(2) If G has a unique distinguished vertex end $\in V$ that is reachable from every vertex $v \in V$ then G is an {end}-graph.
(3) If G is a {start}-graph and G is also an {end}-graph then G is a {start, end}-graph.

If $G = (V, E, \beta)$ is a CFG and the graph (V, E) is a {start}-graph then we call G a {start}-CFG, etc.

Paths of a Graph (Definition 6) A path in a graph $G = (V, E)$ is a sequence of vertices $v_1, \ldots, v_i, v_{i+1}, \ldots$ with $(v_i, v_{i+1}) \in E$ for all i.

Note that paths can be empty (of length zero), of length one (consisting of a single vertex), or even infinite.

Proper Paths (Definition 7) A path is *proper* if its initial and final vertices are distinct.

Prefixes (Definition 8) A prefix of a path π is a path ρ such that there exists a path σ with $\pi = \rho\sigma$ (the concatenation of ρ and σ). Note π is a prefix of itself. Write $\rho \sqsubseteq \pi$. If $\rho \sqsubseteq \pi$ and $\rho \neq \pi$, ρ is called a 'proper' prefix of π.

V'-intervals and V'-paths (Definition 9) Let $G = (V, E)$ be a graph and let $V' \subseteq V$.

- A V'-interval is a finite path of length > 1 in G where only the first and last elements are in V'.
- An $[l, m]$ V'-interval is a V'-interval that starts at $l \in V'$ and ends at $m \in V'$.
- A V'-path [17] is a finite path $v_1 \ldots v_m$ in G where $m > 1$, $v_m \in V'$ and $1 < i < m \Rightarrow v_i \notin V'$.

Complete Paths of a CFG (Definition 10) A complete path is either an infinite path or a finite path whose last vertex is final.

Terminating Paths of a CFG (Definition 11) A terminating path is a finite path whose last vertex is end.

Non-terminating Paths of a CFG (Definition 12) A non-terminating path is a complete path which is either infinite or whose last vertex is not end.

The Induced Graph (Definition 13) Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. The graph induced by V' from G has edge set $E' \subseteq V' \times V'$ where $(x, y) \in E'$ if and only if there is a V'-interval $x, \ldots, y$ in G. In the graph induced by V' from G,

$$\beta'(x, y) = \bigcup_{x' \in K} \beta(x, x')$$

where $K = \{x' \in V \mid (x, x', \ldots, y) \text{ is a } V'\text{-interval}\}$.

The predicates and non-predicates of the graph induced by V' from G are deemed to be $V' \cap P$ and $V' \cap N$, where P and N are the predicates and non-predicates of G respectively.

Elements (Definition 14) Let $G = (V, E, \beta)$ be a CFG. An *element* w is either a vertex $v \in N \subseteq V$, or a pair (p, B) where $p \in P \subseteq V$ and $\mathsf{B} \in \{\mathsf{T}, \mathsf{F}\}$. Write $\bar{w}$ for the vertex component of an element and $\bar{\bar{w}}$ for the second (boolean) component of the pair when it exists.

Walks (Definition 15) Let $G = (V, E, \beta)$ be a CFG. A walk ω in G is a sequence $w_1, w_2, \ldots, w_i, \ldots$ of elements where:

(1) $\bar{\omega} = \bar{w_1}, \bar{w_2}, \ldots, \bar{w_i}, \ldots$ is a path in G; and
(2) if w_i, w_{i+1} are consecutive elements of ω and $\bar{w}_i$ is a predicate vertex then $\bar{\bar{w}}_i \in \beta(\bar{w}_i, \bar{w}_{i+1})$.

$\overrightarrow{G}$ **(Definition 16)** Let G be a CFG. $\overrightarrow{G}$ is the set of all walks in G.

Path Restriction (Definition 17) Let $G = (V, E)$ be a graph, let $V' \subseteq V$, and let π be a path in G. $\pi{\downarrow}V'$ is the subsequence of π obtained by removing all vertices v of π where $v \notin V'$. We say $\pi{\downarrow}V'$ is the *restriction* of π to V'.

Walk Restriction (Definition 18) Let $G = (V, E, \beta)$ be a CFG, let $V' \subseteq V$, and let ω be a walk in G. Define $\omega{\downarrow}V'$ to be the subsequence of ω obtained by removing all elements ω_i of ω where $\bar{\omega}_i \notin V'$. We say $\omega{\downarrow}V'$ is the *restriction* of ω to V'.

Weak Projections (Definition 19) Given a CFG $G = (V, E, \beta)$, a CFG $G' = (V', E', \beta')$ $(V' \subseteq V)$ is a *weak projection* of G if and only if and every walk of G when restricted to V', is a walk of G'. *i.e.*,

$$\omega \in \overrightarrow{G} \implies \omega{\downarrow}V' \in \overrightarrow{G'}.$$

Maximal Walks (Definition 21) A maximal walk of G is a walk which is not a proper prefix of a walk of G.

Strong Projections (Definition 25) Let $G = (V, E, \beta)$ be a CFG and $V' \subseteq V$. A CFG $G' = (V', E', \beta')$ is a *strong projection* of CFG if and only if all maximal walks of G when restricted to V' give maximal walks of G'. *i.e.*,

$$\omega \in \overrightarrow{G} \text{ is maximal} \implies \omega{\downarrow}V' \in \overrightarrow{G'} \text{ and is maximal.}$$

Terminating Walks of a CFG (Definition 30) Walk ω is a terminating walk if and only if the path $\bar{\omega}$ is a terminating path.

Non-terminating Walks of a CFG (Definition 31) Walk ω is a non-terminating walk if and only if the path $\bar{\omega}$ is a non-terminating path.

V'**-weakly Committing Vertices (Definition 34)** Let G be a directed graph. A vertex v is V'-weakly committing in G if all V'-paths from v have the same end point. In other words, there is at most one element of V' that is 'first-reachable' from v.

Weakly Control-closed Sets (Definition 35) Let G be a directed graph and let $V' \subseteq V$. V' is weakly control-closed in G if and only if all vertices not in V' that are reachable from V' are V'-weakly committing in G.

V'**-strongly Committing Vertices (Definition 36)** Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. A vertex v is V'-strongly committing G if and only if it is V'-weakly committing in G and all complete paths in G from v contain an element of V'.

V'**-avoiding Vertices (Definition 37)** Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. A vertex v is V'-avoiding in G if and only if no vertex in V' is reachable in G from v.

Strongly Control-closed Sets (Definition 38) Let $G = (V, E, \beta)$ be a CFG and let $V' \subseteq V$. V' is strongly control-closed in G if and only if every vertex in $V \backslash V'$ that is reachable in G from V' is V'-strongly committing or V'-avoiding in G.

Weakly Deciding Vertices (Definition 50) Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. A vertex $v \in V$ is V'-weakly deciding in G if and only if there exist two finite proper V'-paths in G that both start at v and have no other common vertex. We write $\mathsf{WD}_G(V')$ for the set of all V'-weakly deciding vertices in G.

The Set of Vertices $\Gamma(G, V')$ (Definition 55) Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. We define $\Gamma(G, V')$ to be the set of all $x \in V$ that lie on a complete path in G which does not pass through V'.

The Set of Vertices $\Theta(G, V', u)$ (Definition 56) Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. Let H be the CFG obtained from G by deleting all edges (v', v) with $v' \in V'$. For any $u \in V$, we define $\Theta(G, V', u)$ to be the set of vertices in V' that are reachable in H from u.

Forward Domination (Definition 68) Let $G = (V, E)$ be an $\{\mathsf{end}\}$-graph and let $v, w \in V$. If every path from v to end passes through w then w *forward dominates* v.

Nearest Forward Dominator (Definition 69) We call the first such vertex apart from v the *nearest forward dominator* of v.

ND (Definition 70) Let $G = (V, E)$ be an $\{\text{end}\}$-graph and $v \in V$. $\text{ND}(v)$ is the set of vertices which lie on a path from v to its nearest forward dominator b, excluding v and b themselves.

$\xrightarrow{\textbf{W-controls}}$ **(Definition 71)** Let $G = (V, E)$ be an $\{\text{end}\}$-graph and $v, w \in V$, then $v \xrightarrow{\text{W-controls}} w$ if and only if $w \in \text{ND}(v)$.

$\xrightarrow{\textbf{F-controls}}$ **(Definition 72)** Let $G = (V, E)$ be an $\{\text{end}\}$-graph, then $v \xrightarrow{\text{F-controls}} w$ if and only if v is not forward dominated by w, and there exists a path π from v to w such that for all vertices z occurring on π apart from v are forward dominated by w.

$\xrightarrow{\textbf{WOD}}$ **(Definition 74)** Let $G = (V, E)$ be a finite directed graph with $v, b, c \in V$. Then $v \xrightarrow{\text{WOD}} b, c$ if and only if:

(1) There is a path from v to b not containing c.
(2) There is a path from v to c not containing b.
(3) v has an immediate successor a such that either
- b is reachable from a, and all paths from a to c contain b; or
- c is reachable from a, and all paths from a to b contain c.

The Reflexive Transitive Closure of a Relation (Definition 75) Given a relation r, the reflexive transitive closure of r, write r^* is defined to be the smallest reflexive, transitive relation containing r.

Closed Sets (Definition 76) Given a relation r, we define a set S to be closed under r to mean that if $s \in S$ and $t \xrightarrow{r} s$, then $t \in S$.

$\xrightarrow{\textbf{DOD}}$ **(Definition 87)** Let $G = (V, E)$ be a finite directed graph, then $v \xrightarrow{\text{DOD}} b, c$ if and only if:

(1) All maximal paths from v contain both b and c.
(2) v has an immediate successor from which all maximal paths contain b before any occurrence of c.
(3) v has an immediate successor from which all maximal paths contain c before any occurrence of b.

$\xrightarrow{\text{NTSCD}}$ **(Definition 88)** Let $G = (V, E)$ be a finite directed graph, then $v \xrightarrow{\text{NTSCD}} w$ if and only if:

(1) v has at least two immediate successors.
(2) w occurs on all maximal paths from one of these immediate successors.
(3) there is a maximal path from another immediate successor which does not contain w.

Strong Forward Domination (Definition 89) Let G be an $\{\text{end}\}$-graph. A vertex w *strongly forward dominates* a vertex v if and only if w forward dominates v and there exists an $n \in \mathbb{N}$ such that every path of length n from v contains w.

$\xrightarrow{\textbf{PC-weak}}$ **(Definition 90)** Let $G = (V, E)$ be an $\{\text{end}\}$-graph, then $v \xrightarrow{\text{PC-weak}} u$ if and only if:

(1) v has at least two immediate successors w_1 and w_2.
(2) u strongly forward dominates w_1 but does not strongly forward dominate w_2.

10.2. Main results

Theorem (45). *Let $G = (V, E, \beta)$ be a* CFG *and $V' \subseteq V$. The following are equivalent.*

(1) *The graph induced by V' from G is a* CFG.
(2) *V' is weakly control-closed in G.*
(3) *The graph induced by V' from G is a weak projection of G.*

Theorem (49). *Let $G = (V, E, \beta)$ be a* CFG *and $V' \subseteq V$. The graph induced by V' from G is a strong projection of G if and only if V' is strongly control-closed in G.*

Theorem (54). *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. There exists a unique minimal weakly control-closed subset of V that contains V'.*

Theorem (58). *Let $G = (V, E)$ be a finite directed graph and let $V' \subseteq V$. There exists a unique minimal strongly control-closed superset of V' in G.*

Theorem (*67 Main Theorem for Weak Control Dependence*).

(1) *If G is a $\{\text{start}, \text{end}\}$-*CFG *with $\{\text{start}, \text{end}\} \subseteq V'$, then V' is closed under $\xrightarrow{\text{W-controls}}$ if and only if the induced graph induced by V' from G is a weak projection of G.*
(2) *If G is a $\{\text{start}, \text{end}\}$-*CFG *with $\{\text{start}, \text{end}\} \subseteq V'$, then V' is closed under $\xrightarrow{\text{F-controls}}$ if and only if the induced graph induced by V' from G is a weak projection of G.*

(3) *If G is a* {start}-CFG *with* start $\in V'$ *then* V' *is closed under* $\xrightarrow{\text{WOD}}$ *if and only if the induced graph induced by* V' *from* G *is a weak projection of* G.

Theorem (86 *Main Theorem for Strong Control Dependence*).

(1) *If G is a complete* {start}-CFG *with* start $\in V'$ *then* V' *is closed under* $\xrightarrow{\text{DOD}}$ *and* $\xrightarrow{\text{NTSCD}}$ *if and only if the induced graph induced by* V' *from* G *is a strong projection of* G.

(2) *If G is a complete* {start, end}-CFG *with* start $\in V'$ *then* V' *is closed under* $\xrightarrow{\text{PC-weak}}$ *if and only if the induced graph induced by* V' *from* G *is a strong projection of* G.

10.3. Algorithms

Algorithm (61 *To Compute the Minimal Weakly Control-Closed Superset of* V').

```
(1) Assign $X = V'$.
(2) Choose any edge $(p, v)$ in $G$ with $p$ reachable from $V'$ and such that $|\Theta(G, X, v)| = 1$ and $|\Theta(G, X, p)| \geqslant 2$ hold, and
    assign $X = X \cup \{p\}$. If no such edge $(p, v)$ exists then STOP.
(3) GOTO 2.
```

Theorem (62). *This algorithm has time complexity $O(|V|^3)$, and the value of the set X when STOP is reached is the minimal weakly control-closed superset of V' in G.*

Algorithm (65 *To compute the minimal strongly control-closed superset of* V').

```
(1) Assign $X = V'$.
(2) Find an edge $(p, r)$ in $G$ such that $p$ is reachable in $G$ from $X$ and satisfying:
    (a) $|\Theta(G, X, r)| = 1$ and
    (b) $r \notin \Gamma(G, X)$ and
    (c) $|\Theta(G, X, p)| \geqslant 2$ or $p \in \Gamma(G, X)$.
    If no such edge exists, then STOP, else assign $X = X \cup \{p\}$.
(3) GOTO (2).
```

Theorem (66). *This algorithm computes unique minimal strongly control-closed superset of V' and has worst-case time complexity $O(|V|^4)$.*

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