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A minimax problem for sums of translates on the torus

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Abstract

We extend some equilibrium-type results first conjectured by Ambrus, Ball and Erdélyi [**2**], and then proved recently by Hardin, Kendall and Saff [**14**]. Similarly to them, we too work on the torus $\mathbb{T} \simeq [0, 2\pi)$ (unit circle), but a motivation comes from an analogous setup on the unit interval, investigated earlier by Fenton [**12**].

The problem is to minimize—with respect to the arbitrary translates $y_0 = 0, y_j \in \mathbb{T}$, $j = 1, \dots, n$—the maximum of the sum function $F := K_0 + \sum_{j=1}^n K_j(\cdot - y_j)$, where the K_j's are certain fixed "kernel functions". If they are concave on $\mathbb{T}$, except for having possible singularities or cusps at zero, then the translates by y_j will have singularities at y_j (while in between these nodes the sum function F still behaves regularly). So one can consider the maxima m_i on each subinterval between the nodes y_j, and look for the minimization of $\max F = \max_i m_i$. Also the dual question of maximization of $\min_i m_i$ arises.

Hardin, Kendall and Saff considered one single *even* kernel, $K_j = K$ for $j = 0, \dots, n$, and Fenton considered the case of the interval $[-1, 1]$ with *two* fixed kernels $K_0 = J$ and $K_j = K$ for $j = 1, \dots, n$. Here we build up a systematic treatment of the situation when *all the kernel functions can be different* without assuming them to be even. As an application we generalize a result of Bojanov [**6**] about Chebyshev type polynomials with prescribed zero order.

1. *Introduction*

The present work deals with an ambitious extension of an equilibrium-type result, conjectured by Ambrus, Ball and Erdélyi [**2**] and recently proved by Hardin, Kendall and Saff [**14**]. To formulate this equilibrium result, it is convenient to identify the circle (or one dimensional torus) $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and $[0, 2\pi)$, and call a function $K : \mathbb{T} \to \mathbb{R} \cup \{-\infty, \infty\}$ a *kernel*. The setup of [**2**] and [**14**] requires that the kernel function is *convex* and has values in $\mathbb{R} \cup \{\infty\}$. However, due to historical reasons we shall suppose that the kernels are *concave* and have values in $\mathbb{R} \cup \{-\infty\}$, the transition between the two settings is a trivial multiplication by -1. Accordingly, we take the liberty to reformulate the results of [**14**] after a multiplication by -1, so in particular for concave kernels, see Theorem 1.1 below.

The setup of our investigation is therefore that some *concave* function $K : \mathbb{T} \to \mathbb{R} \cup \{-\infty\}$ is fixed, meaning that K is concave on $[0, 2\pi)$. Then K is necessarily either finite valued (i.e., $K : \mathbb{T} \to \mathbb{R}$) or it satisfies $K(0) = -\infty$ and $K : (0, 2\pi) \to \mathbb{R}$ (the degenerate situation when K is constant $-\infty$ is excluded), and K is upper semi-continuous on $[0, 2\pi)$, and continuous on $(0, 2\pi)$; furthermore, K is necessarily differentiable a.e. and its derivative K' is non-increasing.

The kernel functions are extended periodically to $\mathbb{R}$ and we consider the sum of translates function

$$F(y_0, \dots, y_n, t) := \sum_{j=0}^{n} K(t - y_j).$$

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The points $y_0, \ldots, y_n$ are called *nodes*. Then we are interested in solutions of the minimax problem

$$\inf_{y_0,\ldots,y_n\in[0,2\pi)} \sup_{t\in[0,2\pi)} \sum_{j=0}^{n} K(t-y_j) = \inf_{y_0,\ldots,y_n\in[0,2\pi)} \sup_{t\in[0,2\pi)} F(y_0,\ldots,y_n,t),$$

and address questions concerning existence and uniqueness of solutions, as well as the distribution of the points $y_0, \ldots, y_n$ (mod 2π) in such extremal situations.

In [**2**] it was shown that for $K(t) := -|\mathrm{e}^{it}-1|^{-2} = -\frac{1}{4}\sin^{-2}(t/2)$, (which comes from the Euclidean distance $|\mathrm{e}^{it}-\mathrm{e}^{is}| = 2\sin((t-s)/2)$ between points of the unit circle on the complex plane), $\max F$ is minimized exactly for the regular, i.e., equidistantly spaced, configuration of points, i.e., when $y_j = 2\pi j/(n+1)$ $(j = 0,\ldots,n)$ and $\{\mathrm{e}^{iy_j} : j = 0,\ldots,n\}$ forms a regular $(n+1)$-gon on the circle. (The authors in [**2**] mention that the concrete problem stems from a certain extremal problem, called "strong polarization constant problem" by [**1**].)

Based on this and natural heuristical considerations, Ambrus, Ball and Erdélyi conjectured that the same phenomenon should hold also when $K(t) := -|\mathrm{e}^{it}-1|^{-p}$ $(p > 0)$, and, moreover, even when K is any concave kernel (in the above sense). Next, this was proved for $p = 4$ by Erdélyi and Saff [**10**]. Finally, in [**14**] the full conjecture of Ambrus, Ball and Erdélyi was indeed settled for symmetric (even) kernels.

THEOREM 1.1 (Hardin, Kendall, Saff). *Let K be any concave kernel function such that $K(t) = K(-t)$ and K is non-decreasing on $(0,\pi)$. For any $0 = y_0 \le y_1 \le \ldots \le y_n < 2\pi$ write $\mathbf{y} := (y_1,\ldots,y_n)$ and $F(\mathbf{y},t) := K(t) + \sum_{j=1}^{n} K(t-y_j)$. Let $\mathbf{e} := (\frac{2\pi}{n+1},\ldots,\frac{2\pi n}{n+1})$ (together with 0 the equidistant node system in $\mathbb{T}$).*
(a) *Then*

$$\inf_{0=y_0\le y_1\le\ldots\le y_n<2\pi} \sup_{t\in\mathbb{T}} F(\mathbf{y},t) = \sup_{t\in\mathbb{T}} F(\mathbf{e},t),$$

i.e., the smallest supremum is attained at the equidistant configuration.
(b) *Furthermore, if K is strictly concave, then the smallest supremum is attained at the equidistant configuration* only.

Although this might seem as the end of the story, it is in fact not. The equilibrium phenomenon, captured by this result, is indeed much more general, when we interpret it from a proper point of view. However, to generalize further, we should first analyze what more general situations we may address and what phenomena we can expect to hold in the formulated more general situations. Certainly, regularity in the sense of the nodes y_j distributed *equidistantly* is a rather strong property, which is intimately connected to the use of one single and fixed kernel function K. However, this regularity obviously entails *equality of the "local maxima"* (suprema) m_j for all $j = 0,1,\ldots,n$, and this is what is usually natural in such equilibrium questions.

We say that the configuration of points $0 = y_0 \le y_1 \le \cdots \le y_n \le y_{n+1} = 2\pi$ *equioscillates*, if

$$m_j(y_1,\ldots,y_n) := \sup_{t\in[y_j,y_{j+1}]} F(y_1\ldots,y_n,t) = \sup_{t\in[y_i,y_{i+1}]} F(y_1,\ldots,y_n,t) =: m_i(y_1,\ldots,y_n)$$

holds for all $i,j \in \{0,\ldots,n\}$. Obviously, with one single and fixed kernel K, if the nodes are equidistantly spaced, then the configuration equioscillates. In the more general setup, this —as will be seen from this work— is a good replacement for the property that a point configuration is equidistant.

To give a perhaps enlightening example of what we have in mind, let us recall here a remarkable, but regrettably almost forgotten result of Fenton (see [**12**]), in the analogous, yet also somewhat different situation, when the underlying set is not the torus $\mathbb{T}$, but the unit interval $\mathbb{I} := [0, 1]$. In this setting the underlying set is not a group, hence defining translation $K(t - y)$ of a kernel K can only be done if we define the basic kernel function K not only on $\mathbb{I}$ but also on $[-1, 1]$. Then for any $y \in \mathbb{I}$ the translated kernel $K(\cdot - y)$ is well-defined on $\mathbb{I}$, moreover, it will have analogous properties to the above situation, provided we assume $K|_{\mathbb{I}}$ and also $K|_{[-1,0]}$ to be concave. Similarly, for any node systems the analogous sum F will have similar properties to the situation on the torus.

From here one might derive that under the proper and analogous conditions, a similar regularity (i.e., equidistant node distribution) conclusion can be drawn also for the case of $\mathbb{I}$. But this is *not the only* result of Fenton, who indeed did dig much deeper.

Observe that there is one rather special role, played by the *fixed* endpoint(s) $y_0 = 0$ (and perhaps $y_{n+1} = 1$), since perturbing a system of nodes the respective kernels are translated—but not the one belonging to $K_0 := K(\cdot - y_0)$, since y_0 is fixed. In terms of (linear) potential theory, $K = K(\cdot - y_0) =: K_0$ is a fixed *external field*, while the other translated kernels play the role of a certain "gravitational field", as observed when putting (equal) point masses at the nodes. The potential theoretic interpretation is indeed well observed already in [**10**], where it is mentioned that the *Riesz potentials* with exponent p on the circle correspond to the special problem of Ambrus, Ball and Erdélyi. From here, it is only a little step further to separate the role of the varying mass points, as generating the corresponding gravitational fields, from the stable one, which may come from a similar mass point and law of gravity—or may come from anywhere else.

Note that this potential theoretic external field consideration is far from being really new. To the contrary, it is the fundamental point of view of studying weighted polynomials (in particular, orthogonal polynomial systems with respect to a weight), which has been introduced by the breakthrough paper of Mhaskar and Saff [**18**] and developed into a far-reaching theory in [**22**] and several further treatises. So in retrospect we may interpret the factual result of Fenton as an early (in this regard, not spelled out and very probably not thought of) external field generalization of the equilibrium setup considered above.

THEOREM 1.2 (Fenton). *Let $K : [-1, 1] \to \mathbb{R} \cup \{-\infty\}$ be a kernel function in $\mathrm{C}^2(0, 2\pi)$ which is concave and which is monotone both on $(-1, 0)$ and $(0, 1)$ with $K'' < 0$ and $D_\pm K(0) = \pm\infty$ that is, the left- and right-hand side derivatives of K at 0 are $-\infty$ and $+\infty$, respectively. Let $J : (0, 1) \to \mathbb{R}$ be a concave function and put $J(0) := \lim_{t\to 0} J(t)$, $J(1) := \lim_{t\to 1} J(t)$ which could be $-\infty$ as well. For $\mathbf{y} = (y_1, \ldots, y_n) \in [0, 1]^n$ consider*

$$F(\mathbf{y}, t) := J(t) + \sum_{j=0}^{n+1} K(t - y_j).$$

where $y_0 := 0$, $y_{n+1} := 1$. Then the following are true:

(a) *There are $0 = w_0 \le w_1 \le \cdots \le w_n \le w_{n+1} = 1$ such that with $\mathbf{w} = (w_1, \ldots, w_n)$*

$$\inf_{0\le y_1\le\cdots\le y_n\le 1}\ \max_{j=0,\ldots,n}\ \sup_{t\in[y_j,y_{j+1}]} F(\mathbf{y}, t) = \sup_{t\in[0,1]} F(\mathbf{w}, t).$$

(b) *The sum of translates function of $\mathbf{w}$ equioscillates, i.e.,*

$$\sup_{t\in[w_j,w_{j+1}]} F(\mathbf{w}, t) = \sup_{t\in[w_i,w_{i+1}]} F(\mathbf{w}, t)$$

for all $i, j \in \{0, \ldots, n\}$.

(c) *We have*

$$\inf_{0\le y_1\le\cdots\le y_n\le 1}\max_{j=0,\ldots,n}\sup_{t\in[y_j,y_{j+1}]}F(\mathbf{y},t)=\sup_{0\le y_1\le\cdots\le y_n\le 1}\min_{j=0,\ldots,n}\sup_{t\in[y_j,y_{j+1}]}F(\mathbf{y},t).$$

(d) *If* $0\le z_1\le\cdots\le z_n\le 1$ *is a configuration such that the sum of translates function* $F(\mathbf{z},\cdot)$ *equioscillates, then* $\mathbf{w}=\mathbf{z}$.

This gave us the first clue and impetus to the further, more general investigations, which, however, were executed for the torus setup. As regards Fenton's setup, i.e., similar questions on the interval, we plan to return to them in a subsequent paper. The two setups are rather different in technical details, and we found it difficult to explain them simultaneously—while in principle they should indeed be the same. Such an equivalency is at least exemplified also in this paper, when we apply our results to the problem of Bojanov on so-called "restricted Chebyshev polynomials": In fact, the original result of Bojanov (and our generalization of it) is formulated on an interval. So in order to use our results, valid on the torus, we must work out both some corresponding (new) results on the torus itself, and also a method of transference (working well at least in the concrete Bojanov situation). The transference seems to work well in symmetric cases, but becomes intractable for non-symmetric ones. Therefore, it seems that to capture full generality, not the transference, but direct, analogous arguments should be used. This explains our decision to restrict current considerations to the case of the torus only.

Nevertheless, as for generality of the results, the reader will see that we indeed make a further step, too. Namely, we will allow not only an external field (which, for the torus case, would already be an extension of Theorem 1.1, analogous to Theorem 1.2), but we will study situations when *all the kernels*, fixed or translated, may as well be different. (Definitely, this makes it worthwhile to work out subsequently the analogous questions also for the interval case.) It is not really easy to interpret this situation in terms of physics or potential theory anymore. However, one may argue that in physics we do encounter some situations, e.g., in sub-atomic scales, when different forces and laws can be observed simultaneously: strong kernel forces, electrostatic and gravitational forces etc. In any case, the reader will see that the generality here is clearly a powerful one: e.g., the above mentioned new solution (and generalization and extension to the torus) of Bojanov's problem of restricted Chebyshev polynomials requires this generality. Hopefully, in other equilibrium type questions the generality of the current investigation will prove to be of use, too.

In this introduction it is not yet possible to precisely formulate our results, because we need to discuss a couple of technical details first, to be settled in Section 2. One such, but not only technical, matter is the loss of symmetry with respect to the *ordering* of the nodes. Indeed, in case of a fixed kernel to be translated (even if the external field is different), all permutations of the nodes $y_1,\ldots,y_n$ are equivalent, while for different kernels $K_1,\ldots,K_n$ we of course must distinguish between situations when the ordering of the nodes differ. Also, the original extremal problem can have *different interpretations* according to consideration of *one fixed order* of the kernels (nodes), or *simultaneously all possible orderings* of them. We will treat *both* type of questions, but the *answers will be different*. This is not a technical matter: We will see that, e.g., it can well happen that in some prescribed ordering of the nodes (i.e., the kernels) the extremal configuration has equioscillation, while in some other ordering that fails.

We shall progress methodologically, defining notation, properties and discussing details step by step. Our main result will only be formulated later in Section 11. In the next section (Section 2) we will first introduce the setup precisely, hoping that the reader will be satisfied with the motivation provided by this introduction. In subsequent sections we will discuss various aspects—such as continuity properties in Section 3, limits and approximations in

Section 4, concavity, distributions of local extrema, etc—without providing more motivation or explanation, hoping that the final results will justify also the otherwise seemingly unmotivated technical terms in this course of investigation. Finally, in Section 13 we shall describe, how Bojanov's results (and extensions of it) can be derived via our equilibrium results.

2. *The setting of the problem*

For given 2π-periodic kernel functions $K_0, \dots, K_n : \mathbb{R} \to [-\infty, \infty)$ we are interested in solutions of minimax problems like

$$\inf_{y_0,\dots,y_n \in [0,2\pi)} \sup_{t\in[0,2\pi)} \sum_{j=0}^{n} K_j(t-y_j),$$

and address questions concerning existence and uniqueness of solutions, as well as the distribution of the points $y_0, \dots, y_n \pmod{2\pi}$ in such extremal situations. In the case when $K_0 = \cdots = K_n$ similar problems were studied by Fenton [**12**] (on intervals), Hardin, Kendall and Saff [**14**] (on the unit circle). For twice continuously differentiable kernels an abstract framework for handling of such minimax problems was developed by Shi [**23**], which in turn is based on the fundamental works of Kilgore [**15**], [**16**], and de Boor, Pinkus [**9**] concerning interpolation theoretic conjectures of Bernstein and Erdős. Apart from the fact that we do not pose any smoothness conditions on the kernels (as required by the setting of Shi), it will turn out that Shi's framework is not applicable in this general setting (cf. Example 5.13 and Section 9). The exact references will be given at the relevant places below, but let us stress already here that we do not assume the functions K_j to be smooth (in contrast to [**23**]), and that they may be different (in contrast to [**12**] and [**14**]).

For convenience we use the identification of the unit circle (torus) $\mathbb{T}$ with the interval $[0, 2\pi)$ (with addition mod 2π), and consider 2π-periodic functions also as functions on $\mathbb{T}$; we shall use the terminology of both frameworks, whichever comes more handy. So that we may speak about concave functions on $\mathbb{T}$ (i.e., on $[0, 2\pi)$), just as about arcs in $[0, 2\pi)$ (i.e., on $\mathbb{T}$); this shall cause no ambiguity. We also use the notation

$$d_{\mathbb{T}}(x,y) = \min\big\{|x-y|, 2\pi - |x-y|\big\} \quad (x, y \in [0,2\pi]), \tag{2.1}$$

and

$$d_{\mathbb{T}^m}(\mathbf{x},\mathbf{y}) = \max_{j=1,\dots,m} d_{\mathbb{T}}(x_j, y_j) \quad (\mathbf{x},\mathbf{y} \in \mathbb{T}^m). \tag{2.2}$$

Note that the metric $d_{\mathbb{T}}(x, y)$ is equivalent to the Euclidean metric $|x - y|$ on the unit circle $\mathbb{T}$ (identified with $[0, 2\pi)$).

Let $K : (0, 2\pi) \to [-\infty, \infty)$ be a concave function which is not identically $-\infty$, and suppose

$$K(0) := \lim_{t\downarrow 0} K(t) = \lim_{t\uparrow 2\pi} K(t) =: K(2\pi).$$

Such a function K will be called a *concave kernel function* and can be regarded as a function on the torus $\mathbb{T}$.

One of the conditions on the kernels that will be considered is the following:

$$K(0) = K(2\pi) = -\infty. \tag{∞}$$

Denote by $D_- f$ and $D_+ f$ the left and right derivatives of a function f defined on an interval, respectively. A *concave* function f, defined on an open interval possesses at each points left and right derivatives, and $D_- f$, $D_+ f$ are non-increasing functions. Then, under condition (∞)

it is obvious that we must also have that

$$\lim_{t\uparrow 2\pi} D_+K(t) = \lim_{t\uparrow 2\pi} D_-K(t) = -\infty, \tag{∞'_-}$$

and

$$\lim_{t\downarrow 0} D_-K(t) = \lim_{t\downarrow 0} D_+K(t) = \infty \tag{∞'_+}$$

(equivalently written in the form $D_\pm K(0) = \pm\infty$ or $K'(\pm 0) = \pm\infty$). The two conditions (∞'_-) and (∞'_+) together constitute

$$D_-K(0) = -\infty \quad \text{and} \quad D_+K(0) = \infty. \tag{$\infty'_\pm$}$$

More often, however, we shall make the following assumption on the kernel K:

$$D_-K(0) = -\infty \quad \text{or} \quad D_+K(0) = \infty. \tag{∞'}$$

For $n \in \mathbb{N}$ fixed let $K_0, \dots, K_n$ be concave kernel functions. We take $n+1$ points $y_0, y_1, y_2, \dots, y_n \in [0, 2\pi)$, called *nodes*. As a matter of fact, for definiteness, we shall always take $y_0 = 0 \equiv 2\pi \mod 2\pi$. Then $\mathbf{y} = (y_1, \dots, y_n)$ is called a *node system*. For notational convenience we also set $y_{n+1} = 2\pi$. For a given node system $\mathbf{y}$ we consider the function

$$F(\mathbf{y}, t) := \sum_{j=0}^{n} K_j(t - y_j) = K_0(t) + \sum_{j=1}^{n} K_j(t - y_j). \tag{2.3}$$

For a permutation σ of $\{1, \dots, n\}$ we introduce the notation $\sigma(0) = 0$ and $\sigma(n+1) = n+1$, and define the simplex

$$S_\sigma := \left\{ \mathbf{y} \in \mathbb{T}^n : 0 = y_{\sigma(0)} < y_{\sigma(1)} < \dots < y_{\sigma(n)} < y_{\sigma(n+1)} = 2\pi \right\}.$$

In this paper the term *simplex* is reserved exclusively for domains of this form. Then S_σ is an open subset of $\mathbb{T}^n$ with

$$\bigcup_\sigma \overline{S}_\sigma = \mathbb{T}^n$$

and the complement $\mathbb{T}^n \setminus X$ of the set $X := \bigcup_\sigma S_\sigma$ is the union of less than n-dimensional simplexes. Given a permutation σ and $\mathbf{y} \in \overline{S}_\sigma = S$ (where $\overline{S}_\sigma$ is the closure of S_σ), for $k = 0, \dots, n$ we define the arc $I_{\sigma(k)}$ (in the counterclockwise direction)

$$I_{\sigma(k)}(\mathbf{y}) := [y_{\sigma(k)}, y_{\sigma(k+1)}].$$

For $j = 0, \dots, n$ we have $I_j = [y_j, y_{\sigma(\sigma^{-1}(j)+1)}]$. Of course, a priori, nothing prevents that some of these arcs I_j reduce to a singleton, but their lengths sum up to 2π

$$\sum_{j=0}^{n} |I_j| = 2\pi.$$

Given $\mathbf{y} \in \mathbb{T}^n$ the arcs $I_j(\mathbf{y})$ are defined uniquely as soon as we specify σ with $\mathbf{y} \in \overline{S}_\sigma$, where $\overline{S}_\sigma$ denotes the closure of S_σ. This is, in particular, the case if $\mathbf{y} \in S_\sigma$, because different (open) simplexes are disjoint. However, for $\sigma \neq \pi$ and for $\mathbf{y} \in \overline{S}_\sigma \cap \overline{S}_\pi$ on the (common) boundary, the system of arcs is still well defined but their numbering does depend on the permutations π and σ.

We set

$$m_j(\mathbf{y}) := \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t).$$

We also introduce the functions

$$\overline{m} : \mathbb{T}^n \to [-\infty, \infty), \quad \overline{m}(\mathbf{y}) := \max_{j=0,\dots,n} m_j(\mathbf{y}) = \sup_{t \in \mathbb{T}} F(\mathbf{y}, t),$$
$$\underline{m} : \mathbb{T}^n \to [-\infty, \infty), \quad \underline{m}(\mathbf{y}) := \min_{j=0,\dots,n} m_j(\mathbf{y}).$$

Of interest are then the following two minimax type expressions:

$$M := \inf_{\mathbf{y}\in\mathbb{T}^n} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y}\in\mathbb{T}^n} \max_{j=0,\dots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y},t), \tag{2.4}$$

$$m := \sup_{\mathbf{y}\in\mathbb{T}^n} \underline{m}(\mathbf{y}) = \sup_{\mathbf{y}\in\mathbb{T}^n} \min_{j=0,\dots,n} m_j(\mathbf{y}) \tag{2.5}$$

Or, more specifically, for any given simplex $S = S_\sigma$ we may consider the problems:

$$M(S) := \inf_{\mathbf{y}\in S} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y}\in S} \max_{j=0,\dots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y}\in S} \sup_{t\in\mathbb{T}} F(\mathbf{y},t), \tag{2.6}$$

$$m(S) := \sup_{\mathbf{y}\in S} \underline{m}(\mathbf{y}) = \sup_{\mathbf{y}\in S} \min_{j=0,\dots,n} m_j(\mathbf{y}). \tag{2.7}$$

For notational convenience for any given set $A \subseteq \mathbb{T}^n$ we also define

$$M(A) : = \inf_{\mathbf{y}\in A} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y}\in A} \max_{j=0,\dots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y}\in A} \sup_{t\in\mathbb{T}} F(\mathbf{y},t),$$

$$m(A) : = \sup_{\mathbf{y}\in A} \underline{m}(\mathbf{y}) = \sup_{\mathbf{y}\in A} \min_{j=0,\dots,n} m_j(\mathbf{y}).$$

It will be proved in Proposition 3.11 below that $m(S) = m(\overline{S})$ and $M(S) = M(\overline{S})$. Observe that then we can also write

$$M = \min_{\sigma} \inf_{\mathbf{y}\in\overline{S}_\sigma} \overline{m}(\mathbf{y}) = \min_{\sigma} M(\overline{S}_\sigma), \tag{2.8}$$

$$m = \max_{\sigma} \sup_{\mathbf{y}\in\overline{S}_\sigma} \underline{m}(\mathbf{y}) = \max_{\sigma} m(\overline{S}_\sigma). \tag{2.9}$$

We are interested in whether the infimum or supremum are always attained, and if so, what can be said about the extremal configurations.

Example 2.1. If the kernels are only concave and not strictly concave, then the minimax problem (2.6) may have many solutions, even on the boundary ∂S of $S = S_\sigma$. Let n be fixed, $K_0 = K_1 = \cdots = K_n = K$ and let K be a symmetric ($K(t) = K(2\pi - t)$) kernel which is constant c_0 on the interval $[\delta, 2\pi - \delta]$, where $\delta < \frac{\pi}{n+1}$. Then for any node system $\mathbf{y}$ we have $\max_{t\in\mathbb{T}^n} F(\mathbf{y},t) = (n+1)c_0$, because the 2δ long intervals around the nodes cannot cover $[0, 2\pi]$.

Proposition 2.2. *For every $\delta > 0$ there is $L = L(K_0,\dots,K_n,\delta) \ge 0$ such that for every $\mathbf{y} \in \mathbb{T}^n$ and for every $j \in \{0,\dots,n\}$ with $|I_j(\mathbf{y})| > \delta$ one has $m_j(\mathbf{y}) \ge -L$.*

Proof. Let $\delta \in (0, 2\pi)$. Each function K_j, $j = 0,\dots,n$ is bounded from below by $-L_j := -L_j(\delta) \le 0$ on $\mathbb{T} \setminus (-\delta/2, \delta/2)$. So that for $\mathbf{y} \in \mathbb{T}^n$ the function $F(\mathbf{y},t)$ is bounded from below by $-L := -(L_0 + \cdots + L_n)$ on $B := \mathbb{T} \setminus \bigcup_{j=0}^n (y_j - \delta/2, y_j + \delta/2)$. Let $\mathbf{y} \in \mathbb{T}^n$ and $j \in \{0,\dots,n\}$ be such that $|I_j(\mathbf{y})| > \delta$, then there is $t \in B \cap I_j(\mathbf{y})$, hence $m_j(\mathbf{y}) \ge -L$. □

Corollary 2.3.
(a) *The mapping $\overline{m}$ is finite valued on $\mathbb{T}^n$.*
(b) *$\overline{m}$ is uniformly bounded.*
(c) *For each simplex $S := S_\sigma$ we have that $m(S), M(S)$ are finite, in particular $m, M \in \mathbb{R}$.*

Proof. Since $K_0,\dots,K_n$ are bounded from above, say by $C \ge 0$, $F(\mathbf{y},t) \le (n+1)C$ for every $t \in \mathbb{T}$ and $\mathbf{y} \in \mathbb{T}^n$. This yields $m(S), M(S) \le (n+1)C$.

Take any $\mathbf{y} \in S$ consisting of distinct nodes, so $m_j(\mathbf{y}) > -\infty$ for each $j = 0, \ldots, n$. Hence $m(S) \geq \min_{j=0,\ldots,n} m_j(\mathbf{y}) > -\infty$. This yields (a) and (b).

For $\delta := \frac{2\pi}{n+2}$ take $L \geq 0$ as in Proposition 2.2. Then for every $\mathbf{y} \in S$ there is $j \in \{0, \ldots, n\}$ with $|I_j(\mathbf{y})| > \delta$, so that for this j we have $m_j(\mathbf{y}) \geq -L$. This implies $M(S) \geq M \geq -L > -\infty$. □

3. *Continuity properties*

In this section we study the continuity properties of the various functions defined in Section 2. As a consequence, we prove that for each of the problems (2.6), (2.7) extremal configurations exist.

To facilitate the argumentation we shall consider $\bar{\mathbb{R}} = [-\infty, \infty]$ endowed with the metric

$$d_{\bar{\mathbb{R}}} : [-\infty, \infty] \to \mathbb{R}, \quad d_{\bar{\mathbb{R}}}(x, y) := |\arctan(x) - \arctan(y)|$$

which makes it a compact metric space, with convergence meaning the usual convergence of real sequences to some finite or infinite limit. In this way, we may speak about uniformly continuous functions with values in $[-\infty, \infty]$. Moreover, $\arctan : [-\infty, \infty] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is an order preserving homeomorphism, and hence $[-\infty, \infty]$ is order complete, and therefore a continuous function defined on a compact set attains maximum and minimum (possibly ∞ and $-\infty$).

By assumption any concave kernel function $K : \mathbb{T} \to [-\infty, \infty)$ is (uniformly) continuous in this extended sense.

Proposition 3.1. *For any concave kernel functions $K_0, \ldots, K_n$ the sum of translates function*

$$F : \mathbb{T}^n \times \mathbb{T} \to [-\infty, \infty)$$

defined in (2.3) *is uniformly continuous (in the above defined extended sense).*

Proof. Continuity of F (in the extended sense) is trivial since the K_j's are continuous in the sense described in the preceding paragraph. Also, they do not take the value ∞. Since $\mathbb{T}^n \times \mathbb{T}$ is compact uniform continuity follows. □

Next, a node system $\mathbf{y}$ determines $n + 1$ arcs on $\mathbb{T}$, and we would like to look at the continuity (in some sense) of the arcs as a function of the nodes. The technical difficulties are that the nodes may coincide and they may jump over $0 \equiv 2\pi$. Note that passing from one simplex to another one may indeed cause jumps in the definitions of the arcs $I_j(\mathbf{y})$, entailing jumps† also in the definition of the corresponding m_j.

These problems can be overcome by the next considerations.

†Indeed, at points $\mathbf{y} \in \mathbb{T}^n \setminus X$, on the (common) boundary of some simplexes, the change of the arcs I_j may be discontinuous. E.g., when y_j and y_k changes place (ordering changes between them, e.g., from $y_\ell < y_j \leq y_k < y_r$ to $y_\ell < y_k < y_j < y_r$), then the three arcs between these points will change from the system $I_\ell = [y_\ell, y_j], I_j = [y_j, y_k], I_k = [y_k, y_r]$ to the system $I_\ell = [y_\ell, y_k], I_k = [y_k, y_j], I_j = [y_j, y_r]$. This also means that the functions m_j may be defined *differently* on a boundary point $\mathbf{y} \in \mathbb{T}^n \setminus X$ depending on the simplex we use: the interpretation of the equality $y_j = y_k$ as part of the simplex with $y_j \leq y_k$ in general furnishes a different value of m_j (which is then $F(\mathbf{y}, z_j) = F(\mathbf{y}, y_j)$) than the interpretation as (boundary) part of the simplex with $y_k \leq y_j$ (when it becomes $\max_{t \in [y_j, y_r]} F(\mathbf{y}, t)$).

REMARK 3.2. Let us fix any node system $\mathbf{y}_0$, together with a small $0<\delta<\pi/(2n+2)$, then there exists an arc $I_j(\mathbf{y}_0)$, together with its center point $c=c_j$ such that $|I_j(\mathbf{y}_0)|>2\delta$, so in a (uniform-) δ-neighborhood $U:=U(\mathbf{y}_0,\delta):=\{\mathbf{x}\in\mathbb{T}^n \ : \ d_{\mathbb{T}^n}(\mathbf{x},\mathbf{y}_0)<\delta\}$ of $\mathbf{y}_0\in\mathbb{T}^n$, no node can reach c. We cut the torus at c and represent the points of the torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ by the interval $[c,c+2\pi)\simeq[0,2\pi)$ and use the ordering of this interval. (Henceforth, such a cut—as well as the cutting point c—will be termed as an *admissible cut*.)

Moreover, for $i=1,\dots,n$ we define

$$\begin{aligned}
\ell_i(\mathbf{y}) &:= \min\{t\in[c,c+2\pi) \ : \ \#\{k : y_k\le t\}\ge i\},\\
r_i(\mathbf{y}) &:= \sup\{t\in[c,c+2\pi) \ : \ \#\{k : y_k\le t\}\le i\},\\
\hat{I}_i(\mathbf{y}) &:= [\ell_i(\mathbf{y}),r_i(\mathbf{y})],
\end{aligned}$$

and we set

$$\hat{I}_0(\mathbf{y}) := [c,\ell_1(\mathbf{y})]\cup[r_n(\mathbf{y}),c+2\pi] =: [\ell_0(\mathbf{y}),r_0(\mathbf{y})]\subseteq\mathbb{T} \quad \text{(as an arc).}$$

Then $\hat{I}_i(\mathbf{y})$ is the i^{th} arc in this *cut* of torus along c corresponding to the node system $\mathbf{y}$. We immediately see the continuity of the mappings

$$\mathbb{T}^n\ni\mathbf{y}\mapsto\ell_i(\mathbf{y})\in\mathbb{T} \quad\text{and}\quad \mathbb{T}^n\ni\mathbf{y}\mapsto r_i(\mathbf{y})\in\mathbb{T}$$

at $\mathbf{y}_0$ for each $i=0,\dots,n$. Obviously, the *system of arcs* $\{I_j \ : \ j=0,\dots,n\}$ is the same as $\{\hat{I}_i \ : \ i=0,\dots,n\}$.

PROPOSITION 3.3. *Let $K_0,\dots,K_n$ be any concave kernel functions, let $\mathbf{y}_0\in\mathbb{T}^n$ be a node system and let c be an admissible cut (as in Remark 3.2). Then for $i=0,\dots,n$ the functions*

$$\mathbf{y}\mapsto\hat{m}_i(\mathbf{y}) := \sup_{t\in\hat{I}_i(\mathbf{y})} F(\mathbf{y},t)\in[-\infty,\infty]$$

are continuous at $\mathbf{y}_0$ (in the extended sense).

Proof. By Proposition 3.1 the function $\arctan\circ F:\mathbb{T}^n\times\mathbb{T}\to[-\frac{\pi}{2},\frac{\pi}{2}]$ is continuous at $\{\mathbf{y}_0\}\times\mathbb{T}$. Hence $f_i(\mathbf{y}):=\max_{t\in\hat{I}_i(\mathbf{y})}\arctan\circ F(\mathbf{y},t)$ (and thus also $\hat{m}_i=\tan\circ f_i$) is continuous, since ℓ_i and r_i are continuous (cf. Remark 3.2). □

The continuity of $\hat{m}_i$ for fixed i involves the cut of the torus at c. However, if we consider the system $\{m_0,\dots,m_n\}=\{\hat{m}_0,\dots,\hat{m}_n\}$ the dependence on the cut of the torus can be cured. For $\mathbf{x}\in\mathbb{T}^{n+1}$ define

$$T_i(\mathbf{x}) := \min\{t\in[c,c+2\pi) \ : \ \exists k_0,\dots,k_i \text{ s.t. } x_{k_0},\dots,x_{k_i}\le t\} \quad (i=0,\dots,n)$$

and

$$T(\mathbf{x}) := (T_0(\mathbf{x}),\dots,T_n(\mathbf{x})).$$

The mapping T arranges the coordinates of $\mathbf{x}$ non-decreasingly and it is easy to see that $T:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ is continuous.

COROLLARY 3.4. *For any concave kernel functions $K_0,\dots,K_n$ the mapping*

$$\mathbb{T}^n\ni\mathbf{y}\mapsto T(m_0(\mathbf{y}),\dots,m_n(\mathbf{y}))$$

is (uniformly) continuous (in the extended sense).

Proof. We have $T(m_0(\mathbf{y}),\dots,m_n(\mathbf{y})) = T(\hat{m}_0(\mathbf{y}),\dots,\hat{m}_n(\mathbf{y}))$ for any $\mathbf{y} \in \mathbb{T}$, while $\mathbf{y} \mapsto (\hat{m}_0(\mathbf{y}),\dots,\hat{m}_n(\mathbf{y}))$ is continuous at any given point $\mathbf{y}_0 \in \mathbb{T}^n$ and for any given admissible cut. But the left-hand term here does not depend on the cut, so the assertion is proved. □

COROLLARY 3.5. *Let $K_0,\dots,K_n$ be any concave kernel functions. The functions $\overline{m} : \mathbb{T}^n \to (-\infty,\infty)$ and $\underline{m} : \mathbb{T}^n \to [-\infty,\infty)$ are continuous (in the extended sense).*

Proof. The assertion immediately follows from Proposition 3.3 and Corollary 2.3 (a) and (b). □

COROLLARY 3.6. *Let $K_0,\dots,K_n$ be any concave kernel functions, and let $S := S_\sigma$ be a simplex. For $j = 0,\dots,n$ the functions*

$$m_j : \overline{S} \to [-\infty,\infty]$$

are (uniformly) continuous (in the extended sense).

Proof. Let $\mathbf{y}_0 \in \overline{S}$, then there is an admissible cut at some c (cf. Remark 3.2) and there is some i, such that we have $m_j(\mathbf{y}) = \hat{m}_i(\mathbf{y})$ for all $\mathbf{y}$ in a small neighborhood U of $\mathbf{y}_0$ in S. So the continuity follows from Proposition 3.3. □

REMARK 3.7. Suppose that the kernel functions are concave and at least one of them is strictly concave. For fixed $\mathbf{y}$ also $F(\mathbf{y},\cdot)$ is strictly concave on the interior of each arc $I_j(\mathbf{y})$ and continuous on $I_j(\mathbf{y})$ (in the extended sense), so there is a *unique* $z_j(\mathbf{y}) \in I_j(\mathbf{y})$ with

$$m_j(\mathbf{y}) = F(\mathbf{y}, z_j(\mathbf{y}))$$

(this being trivially true if $I_j(\mathbf{y})$ is degenerate).

If condition (∞) holds, then it is evident that $z_j(\mathbf{y})$ belongs to the interior of $I_j(\mathbf{y})$ (if this latter is non-empty). However, we obtain the same even under the weaker assumption (∞').

LEMMA 3.8. *Suppose that $K_0,\dots,K_n$ are concave kernel functions, with at least one of them strictly concave.*
(a) *If condition (∞'_+) holds for K_j, then for any $\mathbf{y} \in \mathbb{T}^n$ the sum of translates function $F(\mathbf{y},\cdot)$ is strictly increasing on $(y_j, y_j+\varepsilon)$ for some $\varepsilon > 0$.*
(b) *If condition (∞'_-) holds for K_j, then for any $\mathbf{y} \in \mathbb{T}^n$ the sum of translates function $F(\mathbf{y},\cdot)$ is strictly decreasing on $(y_j-\varepsilon, y_j)$ for some $\varepsilon > 0$.*

Proof. (a) Obviously, in case $K_j(0) = -\infty$, we also have $F(\mathbf{y}, y_j) = -\infty$ and the assertion follows trivially since $F(\mathbf{y},\cdot)$ is concave on an interval $(y_j, y_j+\varepsilon)$, $\varepsilon > 0$. So we may assume $K_j(0) \in \mathbb{R}$, in which case $F(\mathbf{y},\cdot)$ is finite, continuous and concave on $[y_j, y_j+\varepsilon]$ for some $\varepsilon > 0$. Then for the fixed $\mathbf{y}$ and for the function $f = F(\mathbf{y},\cdot)$ we have for any fixed $t \in (y_j, y_j+\varepsilon)$ that

$$D_+ f(y_j) = \lim_{s \downarrow y_j} \sum_{k=0}^{n} D_+ K_k(s-y_k) \geq \sum_{k=0, k\neq j}^{n} D_+ K_k(t-y_k) + \lim_{s\downarrow y_j} D_+ K_j(s-y_j) = \infty,$$

since $D_+ K_k(\cdot - y_k)$ is non-increasing by concavity. Therefore, choosing ε even smaller, we find that $D_+ F(\mathbf{y},\cdot) > 0$ in the interval $(y_j, y_j+\varepsilon)$, which implies that $F(\mathbf{y},\cdot)$ is strictly increasing in this interval.

(b) Under condition (∞'_-) the proof is similar for the interval $(y_j - \varepsilon, y_j)$. □

PROPOSITION 3.9. *Suppose that $K_0, \ldots, K_n$ are concave kernel functions, with at least one of them strictly concave.*
(a) *For each $\mathbf{y} \in \mathbb{T}$ and $j = 0, \ldots, n$ there is a unique maximum point $z_j(\mathbf{y})$ of $m_j(\mathbf{y})$ in $I_j(\mathbf{y})$.*
(b) *If condition (∞'_+) holds for K_j, and $I_j(\mathbf{y}) = [y_j, y_r]$ is non-degenerate, then $z_j(\mathbf{y}) \neq y_j$.*
(c) *If condition (∞'_-) holds for K_j, and $I_\ell(\mathbf{y}) = [y_\ell, y_j]$ is non-degenerate, then $z_\ell(\mathbf{y}) \neq y_j$.*
(d) *If condition $(\infty'_\pm)$ holds for each K_j, $j = 0, \ldots, n$, then $z_j(\mathbf{y})$ belongs to the interior of $I_j(\mathbf{y})$ whenever it is non-degenerate.*

Proof. (a) Uniqueness of a maximum point, i.e., the definition of $z_j(\mathbf{y})$ has been already discussed in Remark 3.7.

The assertions (b) and (c) follow from Lemma 3.8, and they imply (d). □

For the next lemma we need that the function z_j is well-defined for each $j = 0, \ldots, n$, so we need $F(\mathbf{y}, \cdot)$ to be strictly concave, in order to which it suffices if at least one of the kernels is strictly concave.

LEMMA 3.10. *Suppose that $K_0, \ldots, K_n$ are concave kernel functions, with at least one of them strictly concave (hence the maximum point $z_j(\mathbf{y})$ of $F(\mathbf{y}, \cdot)$ in $I_j(\mathbf{y})$ is unique for every $j = 0, \ldots, n$). For each $j = 0, \ldots, n$ and for each simplex $S = S_\sigma$ the mapping*

$$z_j : \overline{S} \to \mathbb{T}, \quad \mathbf{y} \mapsto z_j(\mathbf{y})$$

is continuous. Moreover, for a given $\mathbf{y}_0 \in \mathbb{T}^n$ consider an admissible cut of the torus (cf. Remark 3.2). Then the mapping

$$\mathbf{y} \mapsto \hat{z}_i(\mathbf{y})$$

is continuous at $\mathbf{y}_0$.

Proof. Let $(\overline{S} \ni)\mathbf{y}_n \to \mathbf{y} \in \overline{S}$. Then, by Proposition 3.3, $m_j(\mathbf{y}_n) \to m_j(\mathbf{y}) \in [-\infty, \infty)$. Let $x \in \mathbb{T}$ be any accumulation point of the sequence $z_j(\mathbf{y}_n)$, and by passing to a subsequence assume $z_j(\mathbf{y}_n) \to x$.
By definition of z_j, we have $F(\mathbf{y}_n, z_j(\mathbf{y}_n)) = m_j(\mathbf{y}_n) \to m_j(\mathbf{y})$, and by continuity of F also $F(\mathbf{y}_n, z_j(\mathbf{y}_n)) \to F(\mathbf{y}, x)$, so $F(\mathbf{y}, x) = m_j(\mathbf{y})$. But we have already remarked that by strict concavity there is a *unique* point, where $F(\mathbf{y}, \cdot)$ can attain its maximum on I_j (this provided us the definition of $z_j(\mathbf{y})$ as a uniquely defined point in I_j). Thus we conclude $z_j(\mathbf{y}) = x$. The second assertion follows from this in an obvious way. □

PROPOSITION 3.11. *For a simplex S we always have $M(S) = M(\overline{S})$ and $m(S) = m(\overline{S})$. Furthermore, both minimax problems* (2.6) *and* (2.7) *have finite extremal values, and both have an extremal node system, i.e., there are $\mathbf{w}^*, \mathbf{w}_* \in \overline{S}$ such that*

$$\overline{m}(\mathbf{w}^*) = M(S) := \inf_{\mathbf{y} \in S} \overline{m}(\mathbf{y}) = M(\overline{S}) = \min_{\mathbf{y} \in \overline{S}} \overline{m}(\mathbf{y}) \in \mathbb{R},$$
$$\underline{m}(\mathbf{w}_*) = m(S) := \sup_{\mathbf{y} \in S} \underline{m}(\mathbf{y}) = m(\overline{S}) = \max_{\mathbf{y} \in \overline{S}} \underline{m}(\mathbf{y}) \in \mathbb{R}.$$

Proof. By Proposition 3.3 the functions $\underline{m}$ and $\overline{m}$ are continuous (in the extended sense), whence we conclude $m(S) = m(\overline{S})$ and $M(S) = M(\overline{S})$. Since $\overline{S}$ is compact, the function $\underline{m}$ has a maximum on $\overline{S}$, i.e., (2.6) has an extremal node system $\mathbf{w}_*$. Similarly, $\overline{m}$ has a minimum, meaning that (2.7) has an extremal node system $\mathbf{w}^*$.

Both of these extremal values, however, must be *finite*, according to Corollary 2.3. □

As a consequence, we obtain the following.

COROLLARY 3.12. *Both minimax problems* (2.4) *and* (2.5) *have an extremal node system.*

To decide whether the extremal node systems belong to S or to the boundary ∂S is the subject of the next sections.

4. *Approximation of kernels*

In what follows we shall consider a sequence $K_j^{(k)}$ of kernel functions converging to K_j as $k \to \infty$ for $j = 0, \dots, n$ (in some sense or another). The corresponding values of local maxima and related quantities will be denoted by $m_j^{(k)}(\mathbf{x})$, $\underline{m}^{(k)}(\mathbf{x})$, $\overline{m}^{(k)}(\mathbf{x})$, $m^{(k)}(S)$, $M^{(k)}(S)$, and we study the limit behavior of these as $k \to \infty$. Of course, one has here a number of notions of convergence for the kernels, and we start with the easiest ones.

Let Ω be a compact space and let $f_n, f \in \mathrm{C}(\Omega; \bar{\mathbb{R}})$ (the set of continuous functions with values in $\bar{\mathbb{R}}$). We say that $f_n \to f$ *uniformly* (in the extended sense, e.s. for short) if $\arctan f_n \to \arctan f$ uniformly in the ordinary sense (as real valued functions). We say that $f_n \to f$ *strongly uniformly* if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$f(x) - \varepsilon \le f_n(x) \le f(x) + \varepsilon \quad \text{for every } x \in K \text{ and } n \ge n_0.$$

LEMMA 4.1. *Let $f, f_n \in \mathrm{C}(\Omega; \bar{\mathbb{R}})$ be uniformly bounded by $C \in \mathbb{R}$ from above. We then have $f_n \to f$ uniformly (e.s.) if and only if for each $R > 0, \eta > 0$ there is $n_0 \in \mathbb{N}$ such that for all $x \in \Omega$ and all $n \ge n_0$*

$$\begin{aligned} f_n(x) &< -R + \eta \quad \textit{whenever } f(x) < -R \textit{ and} \\ f(x) - \eta \le f_n(x) &\le f(x) + \eta \quad \textit{whenever } f(x) \ge -R. \end{aligned} \tag{4.1}$$

Proof. Suppose first that $f_n \to f$ uniformly (e.s.), and let $\eta > 0$, $R > 0$ be given. The set $L := \arctan[-R-1, C+1]$ is compact in $(-\frac{\pi}{2}, \frac{\pi}{2})$, and tan is uniformly continuous thereon. Therefore there is $\varepsilon \in (0, 1]$ sufficiently small such that

$$\tan(s) - \eta \le \tan(t) \le \tan(s) + \eta$$

whenever $|s - t| \le \varepsilon$, $s \in \arctan[-R, C]$, and such that $\tan(\arctan(-R) + \varepsilon) \le -R + \eta$. Let $n_0 \in \mathbb{N}$ be so large that $\arctan f(x) - \varepsilon \le \arctan f_n(x) \le \arctan f(x) + \varepsilon$ holds for every $n \ge n_0$. Apply the tan function to this inequality to obtain that $f(x) - \eta \le f_n(x) \le f(x) + \eta$ for $x \in \Omega$ with $f(x) \in [-R, C]$, and

$$f_n(x) \le \tan(\arctan f(x) + \varepsilon) < \tan(\arctan(-R) + \varepsilon) < -R + \eta$$

for $x \in \Omega$ with $f(x) < -R$.

Suppose now that condition (4.1) involving η and R is satisfied, and let $\varepsilon > 0$ be arbitrary. Take $R > 0$ so large that $\arctan(t) < -\frac{\pi}{2} + \varepsilon$ whenever $t < -R + 1$. For $\varepsilon > 0$ take $1 > \eta > 0$ according to the uniform continuity of arctan. By assumption there is $n_0 \in \mathbb{N}$ such that for all

$n \geq n_0$ we have (4.1). Let $x \in \Omega$ be arbitrary. If $f(x) < -R$, then

$$\begin{aligned}\arctan f(x) - \varepsilon < -\frac{\pi}{2} &\leq \arctan f_n(x) \\ &\leq \arctan(-R+\eta) < -\frac{\pi}{2} + \varepsilon < \arctan f(x) + \varepsilon.\end{aligned}$$

On the other hand, if $f(x) \geq -R$, then by the choice of η and by the second part of (4.1) we immediately obtain

$$\arctan f(x) - \varepsilon < \arctan f_n(x) \leq \arctan f(x) + \varepsilon.$$

□

The previous lemma has an obvious version for sequences that are not uniformly bounded from above. This is, however a bit more technical and will not be needed. It is now also clear that strong uniform convergence implies uniform convergence. Furthermore, the next assertions follow immediately from the corresponding classical results about real-valued functions.

LEMMA 4.2. *For $n \in \mathbb{N}$ let $f_n, g_n, f, g \in \mathrm{C}(\Omega; \bar{\mathbb{R}})$.*

(a) *If $f_n, g_n \leq C < \infty$ and $f_n \to f$ and $g_n \to g$ uniformly (e.s.), then $f_n + g_n \to f + g$ uniformly (e.s.).*
(b) *If $f_n \downarrow f$ pointwise, i.e., if $f_n(x) \to f(x)$ non-increasingly for each $x \in \Omega$, then $f_n \to f$ uniformly (e.s.).*
(c) *If $f_n \to f$ uniformly (e.s.), then $\sup f_n \to \sup f$ in $[-\infty, \infty]$.*

Proof. (a) The proof can be based on Lemma 4.1.

(b) This is a consequence of Dini's theorem.

(c) Follows from standard properties of arctan and tan, and from the corresponding result for real-valued functions. □

PROPOSITION 4.3. *Suppose the sequence of kernel functions $K_j^{(k)} \to K_j$ uniformly (e.s.) for $k \to \infty$ and $j = 0, 1, \ldots, n$. Then for each simplex $S := S_\sigma$ we have that $m_j^{(k)} \to m_j$ uniformly (e.s.) on $\bar{S}$ $(j = 0, 1, \ldots, n)$. As a consequence, $m^{(k)}(S) \to m(S)$ and $M^{(k)}(S) \to M(S)$ as $k \to \infty$.*

Proof. The functions $F^{(k)}(\mathbf{x}, t) = \sum_{j=0}^{n} K_j^{(k)}(t - x_j)$ are continuous on $\mathbb{T}^{n+1}$ and converge uniformly (e.s.) to $F(\mathbf{x}, t) = \sum_{j=0}^{n} K_j(t - x_j)$ by (a) of Lemma 4.2. So that we can apply part (c) of the same lemma, to obtain the assertion. □

We now relax the notion of convergence of the kernel functions, but, contrary to the above, we shall make essential use of the concavity of kernel functions. We say that a sequence of functions over a set Ω converges *locally uniformly*, if this sequence of functions converges uniformly on each compact subset of Ω.

REMARK 4.4. By using the facts that pointwise convergence of continuous monotonic functions, and pointwise convergence of concave functions, with a continuous limit function, is actually uniform (on compact intervals, see, e.g., [**26**, Problems 9.4.6, 9.9.1] and [**13**]), it is not hard to see that if the kernel functions K_n converge to K pointwise on $[0, 2\pi]$, then they even converge uniformly in the extended sense.

Recall the definitions of $d_{\mathbb{T}}(x,y)$ and $d_{\mathbb{T}^m}(\mathbf{x},\mathbf{y})$ from (2.1) and (2.2). Define the compact set

$$D := \big\{(\mathbf{x},t) \ : \ \exists i \in \{0,1,\dots,n\}, \text{ such that } t = x_i\big\} = \bigcup_{i=0}^{n} \big\{(\mathbf{x},t) \ : \ t = x_i\big\} \subseteq \mathbb{T}^{n+1}.$$

LEMMA 4.5. *Suppose the sequence of kernel functions $K_j^{(k)}$ converges to the kernel function K_j locally uniformly on $(0,2\pi)$. Then $F^{(k)}(\mathbf{x},t) \to F(\mathbf{x},t)$ locally uniformly on $\mathbb{T}^{n+1} \setminus D$, i.e., for every compact subset $H \subseteq \mathbb{T}^{n+1} \setminus D$ one has $F^{(k)}(\mathbf{x},t) \to F(\mathbf{x},t)$ uniformly on H as $k \to \infty$.*

Note that in general F can attain $-\infty$, and that convergence in 0 of the kernels is not postulated.

Proof. Because of compactness of H and D we have $0 < \rho := d_{\mathbb{T}^{n+1}}(H,D)$.

Take $0 < \delta < \rho$ arbitrarily and consider for any $(\mathbf{x},t) \in H$ the defining expression $F^{(k)}(\mathbf{x},t) := \sum_{i=0}^n K_i^{(k)}(t - x_i)$. For points of H we have $|t - x_i| \geq \min\left(|t-x_i|, 2\pi - |t - x_i|\right) = d_{\mathbb{T}}(t,x_i) = d_{\mathbb{T}^{n+1}}\left((\mathbf{x},t),(\mathbf{x},x_i)\right) \geq \rho > \delta$. In other words, $\Phi_i(H) \subset [\delta, 2\pi - \delta]$ for $i = 0,1,\dots,n$, where $\Phi_i(\mathbf{x},t) := t - x_i$ is continuous—hence also uniformly continuous—on the whole $\mathbb{T}^{n+1}$.

As the locally uniform convergence of $K_i^{(k)}$ (to K_i) on $(0,2\pi)$ entails uniform convergence on $[\delta, 2\pi - \delta]$, we have uniform convergence of $f_i^{(k)} := K_i^{(k)} \circ \Phi_i$ on the compact set H (to the function $K_i \circ \Phi_i$). It follows that $F^{(k)} = \sum_{i=0}^n f_i^{(k)}$ converges uniformly (to $F = \sum_{i=0}^n f_i$) on H, whence the assertion follows. □

LEMMA 4.6. *Let $K : (0,2\pi) \to \mathbb{R}$ be a concave function (so K has limits, possibly $-\infty$, at 0 and 2π). For each $u,v \in [0,1]$ we have*

$$K(u) \leq K(u+v) - v\big(K(\pi + 1/2) - K(\pi - 1/2)\big),$$
$$K(2\pi - u) \leq K(2\pi - u - v) + v\big(K(\pi + 1/2) - K(\pi - 1/2)\big).$$

Proof. It is sufficient to prove the statement for $u > 0$ only, as the case $u = 0$ follows from that by passing to the limit.

Also we may suppose $v > 0$ otherwise the inequalities are trivial. By concavity of K for any system of four points $0 < a < b < c < d < 2\pi$ we clearly have the inequality

$$\frac{K(b) - K(a)}{b - a} \geq \frac{K(d) - K(c)}{d - c}$$

see e.g. [**21**], p. 2, formula (2). Specifying $a := u$, $b := u + v \leq 2 < c := \pi - 1/2$ and $d := \pi + 1/2$ yields the first inequality, while for $a := \pi - 1/2$, $b := \pi + 1/2 < 4 < c := 2\pi - u - v$ and $d := 2\pi - u$, we obtain the second one. □

THEOREM 4.7. *Suppose that the kernels are such that for all $\mathbf{x} \in \mathbb{T}^n$ and $z \in \mathbb{T}$ with $F(\mathbf{x},z) = \overline{m}(\mathbf{x})$ one has $z \neq x_j$, $j = 0,\dots,n$. If the sequence of kernel functions $K_j^{(k)} \to K_j$ locally uniformly on $(0,2\pi)$, then $\overline{m}^{(k)}(\mathbf{x}) \to \overline{m}(\mathbf{x})$ uniformly on $\mathbb{T}^n$.*

Proof. Let us define the set $H := \{(\mathbf{x},z) \ : \ F(\mathbf{x},z) = \overline{m}(\mathbf{x})\} \subset \mathbb{T}^{n+1}$, which is obviously closed by virtue of the continuity of the occurring functions. By assumption $H \subseteq \mathbb{T}^{n+1} \setminus D$, so the condition of Lemma 4.5 is satisfied, hence $F^{(k)} \to F$ uniformly on H.

Let now $\mathbf{x} \in \mathbb{T}^n$ be arbitrary, and take any $z \in \mathbb{T}$ such that $F(\mathbf{x},z) = \overline{m}(\mathbf{x})$ (such a z exists by compactness and continuity). Now, $\overline{m}^{(k)}(\mathbf{x}) \geq F^{(k)}(\mathbf{x},z) > F(\mathbf{x},z) - \varepsilon = \overline{m}(\mathbf{x}) - \varepsilon$ whenever

$k > k_0(\varepsilon)$, hence $\liminf_{k\to\infty} \overline{m}^{(k)}(\mathbf{x}) \geq \overline{m}(\mathbf{x})$ is clear, moreover, according to the above, this holds uniformly on $\mathbb{T}^n$, as $\overline{m}^{(k)}(\mathbf{x}) > \overline{m}(\mathbf{x}) - \varepsilon$ for each $\mathbf{x} \in \mathbb{T}^n$ whenever $k > k_0(\varepsilon)$.

It remains to see that, given $\mathbf{x} \in \mathbb{T}^n$ and $\varepsilon > 0$, there exists $k_1(\varepsilon)$ such that $\overline{m}^{(k)}(\mathbf{x}) < \overline{m}(\mathbf{x}) + \varepsilon$ for all $k > k_1(\varepsilon)$. Let us define the constant

$$C := \max_{j=0,1,\dots,n} \max_{k\in\mathbb{N}} |K_j^{(k)}(\pi + 1/2) - K_j^{(k)}(\pi - 1/2)|.$$

The inner expression is indeed a finite maximum, as $K_j^{(k)}(\pi \pm 1/2) \to K_j(\pi \pm 1/2)$ for $k \to \infty$. By Lemma 4.6 for all $u, v \in [0,1]$

$$K_j^{(k)}(u) \leq K_j^{(k)}(u+v) + Cv, \quad K_j^{(k)}(2\pi - u) \leq K_j^{(k)}(2\pi - u - v) + Cv. \tag{4.2}$$

For the given $\varepsilon > 0$ choose $\delta \in (0, 1/2)$ such that $\overline{m}(\mathbf{y}) \leq \overline{m}(\mathbf{x}) + \frac{\varepsilon}{3}$ holds for all $\mathbf{y}$ with $d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) < \delta$ (use Corollary 3.5, the uniform continuity of $\overline{m} : \mathbb{T}^n \to \mathbb{R}$). Fix moreover $0 < h < \min\{\delta/2, \varepsilon/(3C(n+1))\}$ and define

$$H := \big\{(\mathbf{y}, w) \in \mathbb{T}^{n+1} \ : \ d_{\mathbb{T}}(y_i, w) \geq h \ (i = 0, 1, \dots, n)\big\}.$$

For an arbitrarily given point $(\mathbf{x}, z) \in \mathbb{T}^{n+1}$ we construct another one $(\mathbf{y}, w) \in \mathbb{T}^{n+1}$, which we will call "approximating point", in two steps as follows. First, we shift them (even x_0 which was assumed to be 0 all the time), and then correct them. So we set for $i = 0, 1, \dots, n$

$$x_i' := \begin{cases} x_i & \text{if} \quad d_{\mathbb{T}}(x_i, z) \geq h, \\ x_i \pm h & \text{if} \quad d_{\mathbb{T}}(x_i, z) \leq h, \end{cases}$$

where we add h or $-h$ such that $d_{\mathbb{T}}(x_i \pm h, z) \geq h$. Then we set $y_i := x_i' - x_0'$ $(i = 0, 1, \dots, n)$ and $w := z - x_0'$. This new approximating point $(\mathbf{y}, w)$ has the following properties:

$$d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) = \max_{i=1,\dots,n} d_{\mathbb{T}}(x_i, y_i) \leq 2h < \delta, \qquad d_{\mathbb{T}}(z, w) \leq h < \delta. \tag{4.3}$$

Moreover, we have $(\mathbf{y}, w) \in H$, since $d_{\mathbb{T}}(y_i, w) = d_{\mathbb{T}}(x_i', z_i) \geq h$ for $i = 0, 1, \dots, n$.

By construction of $(\mathbf{y}, w)$ we have

$$\begin{aligned} y_i - w &= x_i - z & &\text{if} \quad d_{\mathbb{T}}(x_i, z) \geq h, \\ y_i - w &= x_i - z \pm h & &\text{if} \quad d_{\mathbb{T}}(x_i, z) \leq h. \end{aligned} \tag{4.4}$$

So by using both inequalities in (4.2) we conclude

$$K_j^{(k)}(x_j - z) \leq K_j^{(k)}(y_j - w) + Ch \qquad (j = 0, 1, \dots, n),$$

providing us

$$F^{(k)}(\mathbf{x}, z) = \sum_{j=0}^{n} K_j^{(k)}(x_j - z) \leq \sum_{j=0}^{n} (K_j^{(k)}(y_j - w) + Ch) = F^{(k)}(\mathbf{y}, w) + (n+1)Ch.$$

Now, for given $\mathbf{x} \in \mathbb{T}^n$ let $z_k \in \mathbb{T}$ be any point with $F^{(k)}(\mathbf{x}, z_k) = \overline{m}^{(k)}(\mathbf{x})$, and let $(\mathbf{y}^{(k)}, w_k) \in H$ be the corresponding approximating point. So that we have

$$\overline{m}^{(k)}(\mathbf{x}) = F^{(k)}(\mathbf{x}, z_k) \leq F^{(k)}(\mathbf{y}^{(k)}, w_k) + (n+1)Ch. \tag{4.5}$$

Since $(\mathbf{y}^{(k)}, w_k) \in H \subseteq \mathbb{T}^n \setminus D$ we can invoke Lemma 4.5 to get $F^{(k)} \to F$ uniformly on H. Therefore, for the given $\varepsilon > 0$ there exists $k_1(\varepsilon)$ with

$$F^{(k)}(\mathbf{y}^{(k)}, w_k) \leq \max\big\{F(\mathbf{y}, w) \ : \ (\mathbf{y}, w) \in H, d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) \leq \delta, d_{\mathbb{T}}(z, w) \leq \delta\big\} + \tfrac{\varepsilon}{3}$$

for all $k \geq k_1(\varepsilon)$. Extending further the maximum on the right-hand side to arbitrary $w \in \mathbb{T}$ we are led to

$$F^{(k)}(\mathbf{y}^{(k)}, w_k) \leq \max\big\{\overline{m}(\mathbf{y}) \ : \ d_{\mathbb{T}^n}(\mathbf{x}, \mathbf{y}) \leq \delta\big\} + \tfrac{\varepsilon}{3} \qquad (k > k_1(\varepsilon)). \tag{4.6}$$

From (4.5), (4.6) and by the choices of $h, \delta > 0$ we conclude

$$\overline{m}^{(k)}(\mathbf{x}) \leq F^{(k)}(\mathbf{y}^{(k)}, w_k) + C(n+1)h \leq (\overline{m}(\mathbf{x}) + \tfrac{\varepsilon}{3}) + \tfrac{\varepsilon}{3} + C(n+1)h < \overline{m}(\mathbf{x}) + \varepsilon$$

for all $k > k_1(\varepsilon)$. So that we get that uniformly on $\mathbb{T}^n$ $\limsup_{k\to\infty} \overline{m}^{(k)}(\mathbf{x}) \leq \overline{m}(\mathbf{x})$ holds.

Since $k_1(\varepsilon)$ does not depend on $\mathbf{x}$, by using also the first part we obtain $\lim_{k\to\infty} \overline{m}^{(k)}(\mathbf{x}) = \overline{m}(\mathbf{x})$ uniformly on $\mathbb{T}^n$. □

5. *Elementary properties*

In this section we record some elementary properties of the function m_j that are useful in the study of minimax and maximin problems and constitute also a substantial part of the abstract framework of [**23**]. Moreover, our aim is to reveal the structural connections between these notions.

PROPOSITION 5.1. *Suppose that the kernels $K_0, \dots, K_n$ satisfy (∞). Let $S = S_\sigma$ be a simplex. Then*

$$\lim_{\mathbf{y}\to\partial S} \max_{j=0,\dots,n-1} \left| m_{\sigma(j)}(\mathbf{y}) - m_{\sigma(j+1)}(\mathbf{y}) \right| = \infty. \tag{5.1}$$

Proof. Without loss of generality we may suppose that $\sigma = \mathrm{id}$, i.e., $\sigma(j) = j$. Let $\mathbf{y}^{(i)} \in S$ be convergent to some $\mathbf{y}^{(0)} \in \partial S$ as $i \to \infty$. This means that some arcs determined by the nodes $\mathbf{y}^{(i)}$ and $y_0 = 0 \equiv 2\pi$ shrink to a singleton. On any such arc $I_j(\mathbf{y}^{(i)})$ we obviously have, with the help of (∞),

$$m_j\left(\mathbf{y}^{(i)}\right) \to -\infty \quad \text{as } i \to \infty.$$

Of course, there is at least one such arc, say with index j_0, that has a neighboring arc with index $j_0 \pm 1$ which is not shrinking to a singleton as $i \to \infty$. This means

$$\left| m_{j_0}\left(\mathbf{y}^{(i)}\right) - m_{j_0 \pm 1}\left(\mathbf{y}^{(i)}\right) \right| \to \infty \quad \text{as } i \to \infty,$$

and the proof is complete. □

The properties introduced below have nothing to do with the conditions we pose on the kernel functions $K_0, \dots, K_n$ (concavity and some type of singularity at 0 and 2π), so we can formulate them in whole generality. (Note that m_j, in contrast to z_j, is well-defined even if the kernels are not strictly concave).

DEFINITION 5.2. Let $S = S_\sigma$ be a simplex.

(a) **Jacobi Property:**
The functions $m_0, \dots, m_n$ are in $\mathrm{C}^1(S)$ and

$$\det\Big(\partial_i m_{\sigma(j)}\Big)_{i=1, j=0, j\neq k}^{n,n} \neq 0 \quad \text{for each } k \in \{0, \dots, n\}.$$

(b) **Difference Jacobi Property:**
The functions $m_0, \dots, m_n$ belong to $\mathrm{C}^1(S)$ and

$$\det\Big(\partial_i (m_{\sigma(j)} - m_{\sigma(j+1)})\Big)_{i=1, j=0}^{n, n-1} \neq 0.$$

REMARK 5.3. Shi [**23**] proved that under the condition (5.1) (which is now a consequence of the assumption (∞)) the Jacobi Property implies the Difference Jacobi Property.

DEFINITION 5.4. Let $S = S_\sigma$ be a simplex.

(a) **Equioscillation Property:**
There exists an **equioscillation point** $\mathbf{y} \in S$, i.e.,

$$\overline{m}(\mathbf{y}) = \underline{m}(\mathbf{y}) = m_0(\mathbf{y}) = m_1(\mathbf{y}) = \cdots = m_n(\mathbf{y}).$$

(b) **(Lower) Weak Equioscillation Property:**
There exists a **weak equioscillation point** $\mathbf{y} \in \overline{S}$, i.e.,

$$m_j(\mathbf{y}) \begin{cases} = \overline{m}(\mathbf{y}) & \text{if } I_j \text{ is non-degenerate,} \\ < \overline{m}(\mathbf{y}) & \text{if } I_j \text{ is degenerate.} \end{cases}$$

REMARK 5.5. For given $S = S_\sigma$ the Equioscillation Property implies the inequality $M(S) \leq m(S)$.

Proof. Let $\mathbf{y} \in S$ be an equioscillation point. Then for this particular point

$$\overline{m}(\mathbf{y}) = \max_{j=0,\dots,n} m_j(\mathbf{y}) = \min_{j=0,\dots,n} m_j(\mathbf{y}) = \underline{m}(\mathbf{y}),$$

hence

$$M(S) \leq \overline{m}(\mathbf{y}) = \underline{m}(\mathbf{y}) \leq m(S).$$

□

PROPOSITION 5.6. *Given a simplex $S = S_\sigma$ the following are equivalent:*

(i) $M(S) \geq m(S)$.
(ii) *For every $\mathbf{x} \in S$ one has $\underline{m}(\mathbf{x}) = \min_{j=0,\dots,n} m_j(\mathbf{x}) \leq M(S)$.*
(iii) *For every $\mathbf{y} \in S$ one has $\overline{m}(\mathbf{y}) = \max_{j=0,\dots,n} m_j(\mathbf{y}) \geq m(S)$.*
(iv) *There exists a value $\mu \in \mathbb{R}$ such that for each $\mathbf{y} \in S$*

$$\overline{m}(\mathbf{y}) = \max_{j=0,\dots,n} m_j(\mathbf{y}) \geq \mu \geq \underline{m}(\mathbf{y}) = \min_{j=0,\dots,n} m_j(\mathbf{y}).$$

Proof. Recalling the inequalities

$$\overline{m}(\mathbf{y}) = \max_{j=0,\dots,n} m_j(\mathbf{y}) \geq M(S) = \inf_S \overline{m}, \quad \sup_S \underline{m} = m(S) \geq \underline{m}(\mathbf{x}) = \min_{j=0,\dots,n} m_j(\mathbf{x})$$

being true for each $\mathbf{x}, \mathbf{y} \in S$, the equivalence of (i), (ii) and (iii) is obvious. Suppose (i) and take $\mu \in [m(S), M(S)]$. Then (iv) is evident. From (iv) assertion (i) follows trivially. □

DEFINITION 5.7. Let $S = S_\sigma$ be a simplex. We say that the **Sandwich Property** is satisfied if any of the equivalent assertions in Proposition 5.6 holds true, i.e., if for each $\mathbf{x}, \mathbf{y} \in S$

$$\max_{j=0,\dots,n} m_j(\mathbf{y}) = \overline{m}(\mathbf{y}) \geq \underline{m}(\mathbf{x}) = \min_{j=0,\dots,n} m_j(\mathbf{x}).$$

REMARK 5.8. For given $S = S_\sigma$ the Equioscillation Property and the Sandwich Property together imply that $M(S) = m(S)$.

REMARK 5.9. The above are fundamental properties in interpolation theory, and thus have been extensively investigated. First, for the Lagrange interpolation on $n + 1$ nodes in $[-1, 1]$ the maximum norm of the Lebesgue function is minimal if and only if all its local maxima are equal. This equioscillation property was conjectured by Bernstein [**5**] and proved by Kilgore

[**16**], using also a lemma (Lemma 10 in the paper [**16**]) whose proof, in some extent, was based on direct input from de Boor and Pinkus [**9**]. Second, the property that the minimum of the local maxima is always below this equioscillation value was conjectured by Erdős in [**11**], and proved in the paper [**9**] of de Boor and Pinkus, which appeared in the same issue as the article of Kilgore [**16**], and which is based very much on the analysis of Kilgore. This latter property is just an equivalent formulation of the Sandwich Property, see Proposition 5.6. For more details on the history of these prominent questions of interpolation theory see in particular [**16**]. The name "Sandwich Property" seems to have appeared first in [**24**], see p. 96.

Definition 5.10. We say that $\mathbf{x}$ *majorizes* (or *strictly majorizes*) $\mathbf{y}$—and $\mathbf{y}$ *minorizes* (or *strictly minorizes*) $\mathbf{x}$—if $m_j(\mathbf{x}) \geq m_j(\mathbf{y})$ (or if $m_j(\mathbf{x}) > m_j(\mathbf{y})$) for all $j = 0, \ldots, n$.

Let $S = S_\sigma$ be a simplex. We define the following properties on S_σ.

(a) **Local (Strict) Comparison Property at z:**
There exists $\delta > 0$ such that if $\mathbf{x}, \mathbf{y} \in B(\mathbf{z}, \delta)$ and $\mathbf{x}$ (strictly) majorizes $\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. In other words, there are no two different $\mathbf{x} \neq \mathbf{y} \in B(\mathbf{z}, \delta)$ with $\mathbf{x}$ (strictly) majorizing $\mathbf{y}$.

(b) **Local (Strict) non-Majorization Property at y:**
There exists $\delta > 0$ such that there is no $\mathbf{x} \in (S \cap B(\mathbf{y}, \delta)) \setminus \{\mathbf{y}\}$ which (strictly) majorizes $\mathbf{y}$.

(c) **Local (Strict) non-Minorization Property at y:**
There exists $\delta > 0$ such that there is no $\mathbf{x} \in (S \cap B(\mathbf{y}, \delta)) \setminus \{\mathbf{y}\}$ which (strictly) minorizes $\mathbf{y}$.

Further, we will pick the following special cases as important.

(A) **(Strict) Comparison Property on S:**
If $\mathbf{x}$, $\mathbf{y} \in S$ and $\mathbf{x}$ (strictly) majorizes $\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. In other words, there exists no two different $\mathbf{x} \neq \mathbf{y} \in S$ with $\mathbf{x}$ (strictly) majorizing $\mathbf{y}$.

(B) **Local (Strict) Comparison Property on S:**
At each point $\mathbf{z} \in \overline{S}$, the Local (Strict) Comparison Property holds.

(C) **Local (Strict) non-Majorization Property on S:**
At each point $\mathbf{y} \in \overline{S}$, the Local (Strict) non-Majorization Property holds.

(D) **Local (Strict) non-Minorization Property on S:**
At each point $\mathbf{y} \in \overline{S}$, the Local (Strict) non-Minorization Property holds.

(E) **Singular (Strict) Comparison Property on S:**
At each *equioscillation point* $\mathbf{z} \in S$ the Local (Strict) Comparison Property holds.

(F) **Singular (Strict) non-Majorization Property:**
At each *equioscillation point* $\mathbf{y} \in S$ the Local (Strict) non-Majorization Property holds.

(G) **Singular (Strict) non-Minorization Property:**
At each *equioscillation point* $\mathbf{y} \in S$ the Local (Strict) non-Minorization Property holds.

Remark 5.11. The comparison properties are symmetric in $\mathbf{x}$ and $\mathbf{y}$, while the non-majorization and non-minorization properties are not. One has the following relations between the previously defined properties: (a)⇒(b) and (c), (A)⇒(B)⇒(E), (B)⇒(C) and (D), (E)⇒(F) and (G), (C)⇒(F), (D)⇒(G). It will be proved in Corollary 8.1 that for *strictly concave kernels* all comparison, non-majorization and non-minorization properties (A), (B), (C), (D) (with their strict version as well) are equivalent to each other.

Remark 5.12. Shi [**23**] proved that (under condition (5.1)) the Jacobi Property implies the Comparison Property, the Sandwich Property, and that the Difference Jacobi Property implies the Equioscillation Property. Example 5.13 below shows that the Comparison Property (even the Local Strict non-Majorization Property) fails in general, even though one has the

Difference Jacobi Property. In Proposition 9.2 we will show that in our setting we always have the Difference Jacobi Property provided the kernels are at least twice continuously differentiable and, moreover we have the Equioscillation Property.

EXAMPLE 5.13. Let $n = 1$ and $K_0 : (0, 2\pi) \to \mathbb{R}$ be a strictly concave kernel function in $\mathrm{C}^\infty(0, 2\pi)$ satisfying (∞) and such that the maximum of K_0 is 0, while with some fixed $0 < \alpha < \pi$ the function K_0 is increasing in $(0, \alpha)$ and is decreasing in $(\alpha, 2\pi)$, and let $K_1(t) := K_0(2\pi - t)$. For $\mathbf{y} := y \in (0, 2\pi)$ we have $F(\mathbf{y}, t) = K_0(t) + K_1(t - y) = K_0(t) + K_0(2\pi + y - t)$, so by symmetry and concavity we obtain $z_0(\mathbf{y}) = \frac{y}{2}$ and $z_1(\mathbf{y}) = \frac{2\pi+y}{2}$. So that

$$\begin{aligned} m_0(\mathbf{y}) &= F(\mathbf{y}, z_0(\mathbf{y})) = K_0(\tfrac{y}{2}) + K_0(2\pi + y - \tfrac{y}{2}) = 2K_0(\tfrac{y}{2}), \\ m_1(\mathbf{y}) &= F(\mathbf{y}, z_1(\mathbf{y})) = K_0(\tfrac{2\pi+y}{2}) + K_0(2\pi + y - \tfrac{2\pi+y}{2}) = 2K_0(\tfrac{2\pi+y}{2}). \end{aligned}$$

Whence we conclude that

$$m_0(\mathbf{y} + h) < m_0(\mathbf{y}) \quad \text{and} \quad m_1(\mathbf{y} + h) < m_1(\mathbf{y}),$$

whenever $y \in (2\alpha, 2\pi)$ and $h > 0$ with $y + h \in (2\alpha, 2\pi)$. This shows that the non-Majorization Property does not hold in general. Since $m_0'(2\alpha) = 0$, the Jacobi Property fails for this example (which anyway follows from Remark 5.3). Notice also that

$$m_0'(\mathbf{y}) - m_1'(\mathbf{y}) = K_0'(\tfrac{y}{2}) - K_0'(\tfrac{2\pi+y}{2}) > 0,$$

since K_0' is strictly decreasing, meaning that we have the Difference Jacobi Property (this holds in general, see Proposition 9.2). Finally, we remark that we have the Singular non-Majorization Property. Indeed, $\mathbf{y}$ is an equioscillation point if and only if

$$2K_0(\tfrac{y}{2}) = m_0(\mathbf{y}) = m_1(\mathbf{y}) = 2K_0(\tfrac{2\pi+y}{2}),$$

i.e., at the corresponding points in the graph of K_0 there is a horizontal chord of length π. This implies that $y/2$ falls in the interval where K_0 is strictly increasing, whereas $\pi + y/2$ belongs to the interval where K_0 is strictly decreasing. Hence if we move $\mathbf{y} = y$ slightly, m_0 and m_1 will change in different directions.

This example shows that Shi's results are not applicable in this general setting, even if we supposed the kernels to be in $\mathrm{C}^\infty(0, 2\pi)$.

6. *Distribution of local minima of* $\overline{m}$

REMARK 6.1. Suppose f_j are (strictly) concave functions for $j = 0, \ldots, n$ and let $f = \sum_{j=0}^n f_j$. Let μ_j be the slope of a supporting line of f_j at some point t. Then $\mu := \sum_{j=0}^n \mu_j$ is the slope of a supporting line of f at the same point t. Conversely, if μ is given as the slope of a supporting line at some point t, then it is not hard to find some μ_j, $j = 0, \ldots, n$ being the slope of some supporting line of f_j at t with $\mu = \sum_{j=0}^n \mu_j$.

LEMMA 6.2 (Perturbation lemma). *Suppose that $K_0, \ldots, K_n$ are strictly concave. Let $\mathbf{y} \in \mathbb{T}^n$ be a node system, and for $k \in \mathbb{N}$, $1 \le k \le n$ let $t_1, \ldots, t_k \in (0, 2\pi)$ be all different from the nodes in $\mathbf{y}$. Let*

$$\delta := \tfrac{1}{2} \min\bigl\{|t_i - y_j| : i = 1, \ldots, k,\ j = 0, \ldots, n\bigr\}.$$

For $i = 1, \ldots, k$ let $\mu^{(i)}$ be the slope of a supporting line to the graph of $F(\mathbf{y}, \cdot)$ at the point t_i. Finally, let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-k} \in \mathbb{R}^n$ be fixed arbitrarily.

(a) *Then there is* $\mathbf{a} \in [-1,1]^n \setminus \{\mathbf{0}\}$ *such that* $\mathbf{x}_\ell^\top \mathbf{a} = 0$ *for* $\ell = 1, \dots, n-k$ *and for all* $0 < h < \delta$ *we have*

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, t_i) + \mu^{(i)}(s_i - t_i)$$

for all s_i *with* $|s_i - t_i| < \delta$, $i = 1, \dots, k$.

(b) *If* $F(\mathbf{y}, \cdot)$ *has local maximum in* t_i *for some* $i \in \{1, \dots, k\}$, *i.e., if* $t_i = z_j(\mathbf{y}) \in \operatorname{int} I_j(\mathbf{y})$ *for some* $j \in \{0, \dots, n\}$, *then*

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, z_j(\mathbf{y})) = m_j(\mathbf{y}) \quad \textit{for all } s_i \textit{ with } |s_i - z_j(\mathbf{y})| < \delta.$$

(c) *For the fixed node system* $\mathbf{y}$ *consider an admissible cut of the torus (cf. Remark 3.2). Let* $i_1, \dots, i_k \in \{0, \dots, n\}$ *be pairwise different, and suppose that* $\hat{I}_{i_1}(\mathbf{y}), \dots, \hat{I}_{i_k}(\mathbf{y})$ *are non-degenerate and* $\hat{z}_{i_j} \in \operatorname{int} \hat{I}_{i_j}$ *for each* $j = 1, \dots, k$. *Then there is* $\eta > 0$ *such that for all* $0 < h < \eta$

$$\hat{m}_{i_j}(\mathbf{y} + h\mathbf{a}) < \hat{m}_{i_j}(\mathbf{y}) \quad j = 1, \dots, k.$$

Proof. By Remark 6.1 for $i = 1, \dots, k$ and $j = 0, \dots, n$ there are μ_{ij} each of them being the slope of a supporting line to the graph of K_j at $t_i - y_j$, i.e., with

$$\mu^{(i)} = \sum_{j=0}^{n} \mu_{ij}.$$

Take a vector $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ with $a_j \in [-1,1]$ $(j = 1, \dots, n)$ and such that

$$\sum_{j=1}^{n} a_j \mu_{ij} \geq 0 \quad \text{for } i = 1, \dots, k$$

and

$$\mathbf{x}_\ell^\top \mathbf{a} = \sum_{j=1}^{n} a_j x_{\ell j} = 0 \quad \text{for } \ell = 1, \dots, n-k.$$

Such a vector does exist by standard linear algebra. We set $a_0 := 0$.

(a) Since K_j is concave, it follows

$$K_j(s_i - (y_j + ha_j)) \leq K_j(t_i - y_j) + \mu_{ij}(s_i - t_i - ha_j)$$

for s_i with $|s_i - t_i| < \delta$ and $0 \leq h < \delta$, because then $|s_i - t_i - ha_j| < \delta + |a_j|h < 2\delta$ and $|t_i - y_j| \geq 2\delta$ guarantees that the full interval between the points $t_i - y_j$ and $s_i - (y_j + ha_j)$ stays in $(0, 2\pi)$, i.e., the continuous change of $t_i - y_j$ to $s_i - (y_j + ha_j)$ happens within the concavity interval of K_j.

Observe that here in view of strict concavity equality holds for some i, j if and only if $s_i - t_i - ha_j = 0$. However, for any given value of i, this cannot occur for all $j = 0, \dots, n$. Indeed, if this were so, then $a_0 = 0$ would imply $s_i = t_i$ and, by $h > 0$, it would follow that $\mathbf{a} = 0$, which was excluded.

Summing for all j, with at least one of the inequalities being strict, we obtain

$$\sum_{j=0}^{n} K_j(s_i - (y_j + ha_j)) < \sum_{j=0}^{n} K_j(t_i - y_j) + \sum_{j=0}^{n} \mu_{ij}(s_i - t_i - ha_j)$$

for $|s_i - t_i| < \delta$, $i = 1, \dots, k$, i.e., dropping also $a_0 = 0$

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, t_i) + \mu^{(i)}(s_i - t_i) - h \sum_{j=1}^{n} \mu_{ij} a_j.$$

Now, by the choice of $\mathbf{a}$, the last sum is non-negative, and since $h > 0$ the last term can be estimated from above by 0, and we obtain the first statement.

(b) In the case when $t_i = z_j(\mathbf{y})$ for some j (and only then) the supporting line can be chosen horizontal, i.e., $\mu^{(i)} = 0$. Therefore, with this choice the already proven result directly implies the second statement.

(c) Take a fixed $\mathbf{y}$ and an admissible cut of the torus at some c (cf. Remark 3.2). For sufficiently small η we have $\hat{z}_{i_j} \in \hat{I}_{i_j}(\mathbf{y} + h\mathbf{a})$ for all $0 < h < \eta$ and $j = 1, \dots, k$. Since $\mathbf{x} \mapsto \hat{z}_{i_j}(\mathbf{x})$ is continuous at $\mathbf{y}$ (see Lemma 3.10), for some possibly even smaller $\eta > 0$ we have $|\hat{z}_{i_j}(\mathbf{y}) - \hat{z}_{i_j}(\mathbf{y} + h\mathbf{a})| < \delta$, whenever $0 < h < \eta$. From this we conclude, by the already proven part (b), that for all $j = 1, \dots, k$

$$\hat{m}_{i_j}(\mathbf{y} + h\mathbf{a}) = F(\mathbf{y} + h\mathbf{a}, \hat{z}_{i_j}(\mathbf{y} + h\mathbf{a})) < \hat{m}_{i_j}(\mathbf{y}).$$

□

The next lemma is an analogue of Lemma 3.8 for kernels in $\mathrm{C}^1(0, 2\pi)$.

LEMMA 6.3. *Suppose the kernels $K_0, \dots, K_n$ are in $\mathrm{C}^1(0, 2\pi)$ and are non-constant. Let $S = S_\sigma$ be a simplex, let $\mathbf{y} \in \overline{S}$ and let $j \in \{0, \dots, n\}$. Then there exists $\varepsilon > 0$ such that either for all $t \in (y_j - \varepsilon, y_j)$ or for all $t \in (y_j, y_j + \varepsilon)$ we have $F(\mathbf{y}, t) > F(\mathbf{y}, y_j)$.*

Proof. Let the left and right neighboring non-degenerate arcs to y_j be $[y_\ell, y_j]$ and $[y_j, y_r]$, respectively.[†] Let us write $y_\ell < y_{j_1} = \dots = y_{j_\nu} < y_r$ with $j_1 = j$. We can assume $K_{j_\lambda} > -\infty$ for all $\lambda = 1, \dots, \nu$, otherwise $F(\mathbf{y}, y_j) = -\infty$, and $F(\mathbf{y}, \cdot)$ is finite valued on $(y_\ell, y_j) \cup (y_j, y_r)$, and the statement is trivial. So summing up, $F(\mathbf{y}, \cdot)$ is concave and continuously differentiable both on (y_ℓ, y_j) and (y_j, y_r), and continuous on $[y_\ell, y_r]$. Now, assume for a contradiction that there exists no $\varepsilon > 0$ with the required property. Therefore, e.g., on the left-hand side of y_j it is possible to converge to y_j by some sequence $x_m \uparrow y_j$ $(m \to \infty)$ satisfying $F(\mathbf{y}, x_m) \le F(\mathbf{y}, y_j)$.

Since $F(\mathbf{y}, \cdot)$ is concave, there is some maximum point $z_\ell \in [y_\ell, y_j]$ (which, however, need not be unique if F is not strictly concave), and by concavity $F(\mathbf{y}, \cdot)$ is non-decreasing on $[y_\ell, z_\ell]$ and non-increasing on $[z_\ell, y_j]$. If $z_\ell \ne y_j$, then by the indirect assumption $F(\mathbf{y}, x_m) = F(\mathbf{y}, y_j)$ for sufficiently large m. But then by concavity the function $F(\mathbf{y}, \cdot)$ is constant $F(\mathbf{y}, y_j)$ on $[z_\ell, y_j]$. So that $F(\mathbf{y}, y_j) = \max_{t \in [y_\ell, y_j]} F(\mathbf{y}, t)$. On the other hand, if $z_\ell = y_j$, then evidently $F(\mathbf{y}, y_j) = \max_{t \in [y_\ell, y_j]} F(\mathbf{y}, t)$. By the same reasoning we obtain the same for $[y_j, y_r]$. So altogether $F(\mathbf{y}, y_j) = \max_{t \in [y_\ell, y_r]} F(\mathbf{y}, t)$.

It follows that $F(\mathbf{y}, \cdot)$ stays below $F(\mathbf{y}, y_j)$ on $[y_\ell, y_r]$, and hence we find

$$D_- F(\mathbf{y}, y_j) \ge 0 \ge D_+ F(\mathbf{y}, y_j).$$

Using the non-constancy of the kernel functions K_i in the form that $D_- K_i(0) < D_+ K_i(0)$, we find

$$\begin{aligned} D_- F(\mathbf{y}, y_j) &= \lim_{t \uparrow y_j} \sum_{i=0}^{n} K_i'(t - y_i) = \sum_{\substack{\lambda=0 \\ \lambda \ne j_1, \dots, j_\nu}}^{n} K_\lambda'(y_j - y_\lambda) + \sum_{\lambda=1}^{\nu} D_- K_{j_\lambda}(0) \\ &< \sum_{\substack{\lambda=0 \\ \lambda \ne j_1, \dots, j_\nu}}^{n} K_\lambda'(y_j - y_\lambda) + \sum_{\lambda=1}^{\nu} D_+ K_{j_\lambda}(0) = \lim_{t \downarrow y_j} \sum_{i=0}^{n} K_i'(t - y_i) = D_+ F(\mathbf{y}, y_j), \end{aligned}$$

which furnishes the required contradiction. Whence the statement follows. □

[†] If all nodes are positioned at $y_0 = 0$, these arcs can be the same.

LEMMA 6.4. *Let the kernel functions $K_0, \dots, K_n$ be concave. Suppose that the interval $I_j(\mathbf{y}) = [y_j, y_{j'}]$ is degenerate, i.e., a singleton.*

(a) *Suppose that the kernel K_j satisfies condition (∞'_-). Then there exists $\varepsilon > 0$ such that for all $t \in (y_j - \varepsilon, y_j)$ we have $F(\mathbf{y}, t) > m_j(\mathbf{y})$.*

(b) *Suppose that the kernel K_j satisfies condition (∞'_+). Then there exists $\varepsilon > 0$ such that for all $t \in (y_j, y_j + \varepsilon)$ we have $F(\mathbf{y}, t) > m_j(\mathbf{y})$.*

(c) *Suppose the kernels $K_0, \dots, K_n$ are in $\mathrm{C}^1(0, 2\pi)$ and are non-constant. Then there exists $\varepsilon > 0$ such that either for all $t \in (y_j - \varepsilon, y_j)$ or for all $t \in (y_j, y_j + \varepsilon)$ we have $F(\mathbf{y}, t) > m_j(\mathbf{y})$.*

Proof. Let $I_j(\mathbf{y}) = \{y_j\} = \{y_{j'}\} = \{z_j(\mathbf{y})\}$ and let $\varepsilon > 0$ be so small that the functions $K_k(\cdot - y_k)$ are all finite and concave on $(y_j - \varepsilon, y_j)$ and $(y_j, y_j + \varepsilon)$. In particular, for a point t in one of these intervals $F(\mathbf{y}, t) \in \mathbb{R}$, so in case of $K_j(0) = -\infty$, we also have $F(\mathbf{y}, z_j(\mathbf{y})) = -\infty < F(\mathbf{y}, t)$ and there is nothing to prove.

(a) and (b) follow from Lemma 3.8 and from the fact that $F(\mathbf{y}, y_j) = m_j(\mathbf{y})$.

(c) follows from Lemma 6.3 by also taking into account that $F(\mathbf{y}, y_j) = m_j(\mathbf{y})$. □

COROLLARY 6.5. *Let the kernel functions $K_0, \dots, K_n$ be concave. Suppose that $I_j(\mathbf{y})$ is degenerate.*

(a) *Suppose that at least n of the kernels $K_0, \dots, K_n$ satisfy condition (∞'). Then for at least one neighboring, non-degenerate arc $I_\ell(\mathbf{y})$ we have $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.*

(b) *Suppose the kernels are in $\mathrm{C}^1(0, 2\pi)$ and are non-constant. Then for at least one neighboring, non-degenerate arc $I_\ell(\mathbf{y})$ we have $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.*

COROLLARY 6.6. *If $K_0, \dots, K_n$ are non-constant, concave kernel functions and either n of them satisfy (∞'), or all belong to $\mathrm{C}^1(0, 2\pi)$, then an equioscillation point $\mathbf{e} \in \mathbb{T}^n$ must belong to the interior of some simplex $S = S_\sigma$, i.e., we have $\mathbf{e} \in X = \bigcup_\sigma S_\sigma$.*

Proof. Let $\mathbf{y} \in \mathbb{T} \setminus X$ be arbitrary. Then there exists some j with $I_j(\mathbf{y})$ being degenerate. According to the above, there exists some $\ell \neq j$ with $m_j(\mathbf{y}) < m_\ell(\mathbf{y})$, so there is no equioscillation at $\mathbf{y}$. □

EXAMPLE 6.7. It can happen that an equioscillation point falls on the boundary of a simplex S, and that maximum points of non-degenerate arcs lie on the endpoints. Indeed, let $K_0 := -4\pi^3/|x|$ on $[-\pi, \pi)$, extended periodically, and let $K_1(x) := K_2(x) := -(x-\pi)^2$ on $(0, 2\pi)$, again extended periodically. Observe that K_0 satisfies $(\infty'_\pm)$ (and belongs to $\mathrm{C}^1((0,\pi) \cup (\pi, 2\pi))$, and $K_1, K_2 \in \mathrm{C}^1(0, 2\pi)$. Still, for the node system $y_1 = y_2 = \pi$, we have $\mathbf{y} \in \partial S = \partial S_{\mathrm{Id}}$, $F(\mathbf{y}, x) = F(\mathbf{y}, 2\pi - x) = -4\pi^3/x - 2x^2$ $(0 \le x \le \pi)$, hence $z_0 = z_1 = z_2 = \pi$ and $m_0 = m_1 = m_2 = F(\mathbf{y}, \pi) = -6\pi^2$, showing that $\mathbf{y}$ is in fact an equioscillation point.

LEMMA 6.8. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave and either all satisfy (∞'), or all belong to $\mathrm{C}^1(0, 2\pi)$. Let $\mathbf{w} \in \mathbb{T}^n$ and $j \in \{0, \dots, n\}$ be such that $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$. Then $I_j(\mathbf{w})$ is non-degenerate and $z_j(\mathbf{w})$ belongs to the interior of $I_j(\mathbf{w})$.*

Proof. By Corollary 6.5 it follows that the arc $I_j(\mathbf{w}) = [w_j, w_r]$ is non-degenerate.

Suppose first that all kernels satisfy (∞'). Here, F cannot attain global maximum neither at w_j nor at w_r because F is strictly increasing on a left or a right neighborhood of these nodes due to condition (∞') (use Lemma 3.8). Therefore, in this case $z_j(\mathbf{w})$ belongs to the interior of $I_j(\mathbf{w})$.

Next, let us suppose that the kernels are in $\mathrm{C}^1(0,2\pi)$. By an application of Lemma 6.3 we obtain $\overline{m}(\mathbf{w}) > F(\mathbf{w}, w_i)$ for all $i = 0, 1 \ldots, n$. Hence, in the case $\overline{m}(\mathbf{w}) = m_j(\mathbf{w}) = F(\mathbf{w}, z_j)$, we cannot have $z_j = w_j$ or $z_j = w_r$. $\square$

As usual, we say that a point $\mathbf{w} \in \mathbb{T}^n$ is a local minimum point of $\overline{m}$ if there exists $\eta > 0$ such that

$$\overline{m}(\mathbf{w}^*) = \min\{\overline{m}(\mathbf{y}) : d_{\mathbb{T}^n}(\mathbf{y}, \mathbf{w}^*) < \eta\}. \tag{6.1}$$

Note that the η-neighborhood here may intersect several different simplexes.

Proposition 6.9. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave and either all satisfy (∞'), or all belong to $\mathrm{C}^1(0,2\pi)$. Let $\mathbf{w}^* \in \mathbb{T}^n$ be a local minimum point of $\overline{m}$, see (6.1). Then $\mathbf{w}^*$ is an equioscillation point, i.e.,*

$$m_j(\mathbf{w}^*) = \overline{m}(\mathbf{w}^*) \quad \text{for all } j = 0, \ldots, n.$$

As a consequence, such a local minimum point belongs to $X = \bigcup_\sigma S_\sigma$.

Proof. Consider an admissible cut of the torus (cf. Remark 3.2). Suppose for a contradiction that $i_1, \ldots, i_k \in \{0, \ldots, n\}$ with $k \le n$ are precisely the indices i with

$$\hat{m}_i(\mathbf{w}^*) = \overline{m}(\mathbf{w}^*) =: M_0.$$

By Lemma 6.8 $t_j := z_{i_j}(\mathbf{w}^*)$ (for $j = 1, \ldots, k$) belong to the interior of non-degenerate arcs. With this choice we can use the Perturbation Lemma 6.2 to slightly move $\mathbf{w}^* = (w_1, \ldots, w_n)$ to $\mathbf{w}' = (w_1', \ldots, w_n')$, $|\mathbf{w}' - \mathbf{w}^*| < \eta$ and achieve

$$\max_{j=1,\ldots,k} \hat{m}_{i_j}(\mathbf{w}') < M_0,$$

while at the same time $\hat{m}_q(\mathbf{w}')$ for $q \ne i_j$, $j = 1, \ldots, k$ do not increase too much (because by Proposition 3.3 the functions $\hat{m}_q$ are continuous), i.e.,

$$\max_{p=0,\ldots,n} m_p(\mathbf{w}') = \max_{j=1,\ldots,k} \hat{m}_{i_j}(\mathbf{w}') < M_0,$$

a contradiction.

The last assertion follows now immediately from Corollary 6.6. $\square$

Corollary 6.10. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave, and either all satisfy (∞'), or all belong to $\mathrm{C}^1(0,2\pi)$. Let $S = S_\sigma$ be a simplex, and let $\mathbf{w}^* \in \overline{S}$ be an extremal node system for (2.6). Then the following assertions hold.*

(a) *If $\mathbf{w}^* \in S$, then $\mathbf{w}^*$ is an equioscillation point.*
(b) *Even in case $\mathbf{w}^* \in \partial S$ we have that $\mathbf{w}^*$ is a weak equioscillation point.*
(c) *Furthermore, if also (∞) holds, then we have $\{m_0(\mathbf{w}^*), \ldots, m_n(\mathbf{w}^*)\} \subseteq \{-\infty, M(S)\}$, with $m_j(\mathbf{w}^*) = -\infty$ iff $I_j(\mathbf{w}^*)$ is degenerate.*
(d) *If $\mathbf{w}^* \in \partial S$, then there exists another simplex $S' = S_\pi$ with $\mathbf{w}^* \in \overline{S} \cap \overline{S'}$ and $M(S') < M(S)$, moreover $\mathbf{w}^*$ is not even a local (conditional) minimum within $\overline{S'}$.*

Proof. (a) When the extremal node system $\mathbf{w}^*$ lies in the interior of the simplex S, it is necessarily a local minimum point, hence the previous Proposition 6.9 applies.

(b) For notational convenience we assume without loss of generality that $\sigma = \mathrm{id}$, the identical pertmutation. Let $\mathbf{w}^* = (w_1, \dots, w_n) \in \partial S$ and assume that

$$0 = w_0 = \cdots = w_{i_0} < w_{i_0+1} = \cdots = w_{i_0+i_1} < w_{i_0+i_1+1} = \cdots = w_{i_0+i_1+i_2}$$
$$< \cdots < w_{i_0+\cdots+i_{s-1}+1} = \cdots = w_{i_0+\cdots+i_s} < w_{i_0+\cdots+i_s+1} := 2\pi$$

is the listing of nodes with the number of equal ones being exactly $i_0, i_1, \dots, i_s$. Thus we have $i_0 + \cdots + i_s = n$ with i_0 possibly 0 but the other i_j's are at least 1, and the number of distinct nodes strictly in $(0, 2\pi)$ is s.

In between the equal nodes there are degenerate arcs I_k, where—in view of Corollary 6.5—the respective maximum $m_k(\mathbf{w}^*)$ of the function $F(\mathbf{w}^*, \cdot)$ is strictly smaller, than one of the maximums on the neighboring non-degenerate arcs, hence $m_k(\mathbf{w}^*)$ is also smaller than $\overline{m}(\mathbf{w}^*)$.

So in particular if $s = 0$ and there is only one non-degenerate arc $I_{i_0} = [0, 2\pi]$, with all the nodes merging to 0, then weak equioscillation (of this one value m_{i_0}) trivially holds.

Next, assume that there exists at least one node $0 < w_k < 2\pi$, and let us now define a new system of s $(1 \le s < n)$ nodes $\mathbf{y} = (y_1, \dots, y_s)$ with $y_j = w_{i_0+\cdots+i_j}$ $(j = 1, \dots, s)$ extended the usual way by $y_0 = 0$. Note that we will thus have $0 = y_0 < y_1 < \cdots < y_s < 2\pi$, and the arising s non-degenerate arcs between these nodes are exactly the same as the non-degenerate arcs determined by the node system $\mathbf{w}^*$.

Further, let us define new kernel functions $L_j := K_{i_0+\cdots+i_{j-1}+1} + \cdots + K_{i_0+\cdots+i_j}$ for $j = 1, \dots, s$, and $L_0 = K_0 + K_1 + \cdots + K_{i_0}$. Obviously, the new $s+1$-element system $L_0, L_1, \dots, L_s$ consists of strictly concave kernels, either all satisfying (∞'), or all belonging to $\mathrm{C}^1(0, 2\pi)$, and now the node system $\mathbf{y}$ belongs to the interior of the respective s-dimensional simplex $\tilde{S}$.

Observe that by construction we now have

$$\tilde{F}(\mathbf{y}, t) = \sum_{j=0}^{s} L_j(t - y_j) = \sum_{i=0}^{n} K_i(t - w_i) = F(\mathbf{w}^*, t),$$

and so from the assumption that $\overline{m}(\mathbf{w}^*)$ is minimal within the simplex S, it also follows that $\sup_{t \in \mathbb{T}} \tilde{F}(\mathbf{y}, t)$ is minimal within $\tilde{S}$. Therefore, by part (a) the maximum values $\tilde{m}_j$ of the function $\tilde{F}$ on these non-degenerate arcs are all equal, and this was to be proven.

(c) is obvious once we have the weak equioscillation in view of (b).

(d) If we had $\mathbf{w}^*$ being a local conditional minimum point in each of the simplexes to the boundary of which it belongs, then altogether it would even be a local minimum point on $\mathbb{T}^n$. Then Proposition 6.9 would yield $\mathbf{w}^* \in X$, contradicting the assumption. So there has to be some simplex S', containing $\mathbf{w}^*$ in $\partial S'$, where $\mathbf{w}^*$ is not a local conditional minimum point. Consequently, $M(S') < \overline{m}(\mathbf{w}^*) = M(S)$, whence the assertion follows. □

Corollary 6.11. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave and either all satisfy (∞'), or all belong to $\mathrm{C}^1(0, 2\pi)$. If $\mathbf{w}$ is an extremal node system for (2.4), i.e.,*

$$\overline{m}(\mathbf{w}) = \min_{\mathbf{y} \in \mathbb{T}^n} \overline{m}(\mathbf{y}) = M,$$

then the nodes w_j $(j = 0, \dots, n)$ are pairwise different (i.e., $\mathbf{w} \in X$) and, moreover, $\mathbf{w}$ is an equioscillation point, i.e., we have

$$m_j(\mathbf{w}) = M \quad \textit{for } j = 0, \dots, n.$$

7. *Distribution of local maxima of $\underline{m}$*

Lemma 7.1. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave. Let $S = S_\sigma$ be a simplex. Then $F(\mathbf{y}, s) : \mathbb{T}^n \times \mathbb{T} \to [-\infty, \infty)$ restricted to the convex open set*

$$\mathcal{D} := \mathcal{D}_{\sigma,i} := \big\{(\mathbf{y}, s) : \mathbf{y} \in S \text{ and } s \in \operatorname{int} I_i(\mathbf{y})\big\}$$

is strictly concave.

Proof. First, note that the set $\mathcal{D} := \mathcal{D}_{\sigma,i}$ is a convex subset of $\mathbb{T}^{n+1}$. Indeed, let $(\mathbf{x}, r), (\mathbf{y}, s) \in \mathcal{D}$ and $t \in [0, 1]$. Then $x_i < x_\ell$ and $y_i < y_\ell$ imply $tx_i + (1-t)y_i < tx_\ell + (1-t)y_\ell$, and $x_i < r < x_\ell$, $y_i < s < y_\ell$ entails also $tx_i + (1-t)y_i < tr + (1-t)s < tx_\ell + (1-t)y_\ell$.

Now, consider the sum representation of F and concavity of each K_ℓ to conclude

$$\begin{aligned} F(t(\mathbf{x}, r) + (1-t)(\mathbf{y}, s)) &= \sum_{\ell=0}^{n} K_\ell(tr + (1-t)s - (tx_\ell + (1-t)y_\ell)) \\ &\ge \sum_{\ell=0}^{n} tK_\ell(r - x_\ell) + (1-t)K_\ell(s - y_\ell) \\ &= tF(\mathbf{x}, r) + (1-t)F(\mathbf{y}, s). \end{aligned} \tag{7.1}$$

This shows concavity of F. To see strict concavity suppose $t \neq 0, 1$ and that $(\mathbf{x}, r), (\mathbf{y}, s) \in \mathcal{D}$ are different points. If $r \neq s$, then using the strict concavity of K_0 we must have

$$K_0(tr + (1-t)s) > tK_0(r) + (1-t)K_0(s),$$

and if $r = s$, but $x_\ell \neq y_\ell$ for some $1 \le \ell \le n$, then using strict concavity of K_ℓ (and also that $r = s$) it follows that

$$K_\ell(tr + (1-t)s - (tx_\ell + (1-t)y_\ell)) = K_\ell(s - (tx_\ell + (1-t)y_\ell)) > tK_\ell(s - x_\ell) + (1-t)K_\ell(s - y_\ell).$$

Altogether we obtain strict inequality in (7.1). □

Proposition 7.2. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave. Then for all $i = 0, 1, \ldots, n$, the functions $m_i(\mathbf{y}) : S \to \mathbb{R}$ are also strictly concave. As a consequence,*

$$\underline{m} : S \to [-\infty, \infty), \quad \underline{m}(\mathbf{y}) := \min_{j=0,\ldots,n} m_j(\mathbf{y})$$

is a strictly concave function.

Proof. Take $i \in \{0, 1, \ldots, n\}$, $\mathbf{x}, \mathbf{y} \in S$ and abbreviate $w := z_i(\mathbf{x})$, $v := z_i(\mathbf{y})$ (the unique maximum points of $F(\mathbf{x}, \cdot)$ and $F(\mathbf{y}, \cdot)$ in $I_i(\mathbf{x})$ and $I_i(\mathbf{y})$, respectively, i.e., $m_i(\mathbf{x}) = F(\mathbf{x}, w)$, $m_i(\mathbf{y}) = F(\mathbf{y}, v)$). Let $\zeta(t) := z_i(t\mathbf{x} + (1-t)\mathbf{y})$, $\zeta(0) = v$, $\zeta(1) = w$. According to the previous Lemma 7.1 the function F is strictly concave on $\mathcal{D}_{\sigma,i}$, hence for different $\mathbf{x} \neq \mathbf{y}$ we necessarily have

$$F(t(\mathbf{x}, w) + (1-t)(\mathbf{y}, v)) > tF(\mathbf{x}, w) + (1-t)F(\mathbf{y}, v) = tm_i(\mathbf{x}) + (1-t)m_i(\mathbf{y}).$$

Here the left-hand side can be written as $F(t\mathbf{x} + (1-t)\mathbf{y}, \omega(t))$ with

$$\omega(t) = tw + (1-t)v \in I_i(t\mathbf{x} + (1-t)\mathbf{y}).$$

Thus by the definition of m_i we have

$$m_i(t\mathbf{x} + (1-t)\mathbf{y}) = \max_{s \in I_i(t\mathbf{x} + (1-t)\mathbf{y})} F(t\mathbf{x} + (1-t)\mathbf{y}, s) \ge F(t(\mathbf{x}, w) + (1-t)(\mathbf{y}, v)).$$

Hence, the previous considerations yield even $m_i(t\mathbf{x}+(1-t)\mathbf{y}) > tm_i(\mathbf{x})+(1-t)m_i(\mathbf{y})$, whence the first assertion follows. Since minimum of strictly concave functions is strictly concave, the last assertion follows, too. $\square$

COROLLARY 7.3. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave, and let $S := S_\sigma$ be a simplex.*
(a) *In $\overline{S}$ the function $\underline{m}$ has a unique global maximum point $\mathbf{y}_*$, and no local minimum point in S.*
(b) *If the kernels satisfy (∞), then $\mathbf{y}_* \in S$.*
(c) *There is no other point in $\overline{S}$ majorizing $\mathbf{y}_*$ than $\mathbf{y}_*$ itself.*

Proof. (a) Since $\underline{m}$ is strictly concave on S and continuous on $\overline{S}$ the assertion is evident.

(b) Under condition (∞) we have $\underline{m}|_{\partial S} = -\infty$, whence the assertion is immediate.

(c) If $\mathbf{x} \in \overline{S}$ with $m_j(\mathbf{x}) \geq m_j(\mathbf{y}_*)$ for all $j = 0, 1, \ldots, n$, then for $\underline{m} = \min_{j=0,\ldots,n} m_j$ we also have $\underline{m}(\mathbf{x}) \geq \underline{m}(\mathbf{y}_*)$, hence $\mathbf{x}$ is also a maximum point, and by uniqueness (part (a)) this entails $\mathbf{x} = \mathbf{y}_*$. $\square$

8. *Local properties of sums of translates*

COROLLARY 8.1. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave. Let $S := S_\sigma$ be a simplex.*
(a) *Let $\mathbf{y} \in S$, $\mathbf{x} \in \overline{S}$, $\mathbf{x} \neq \mathbf{y}$ be such that $\mathbf{x}$ majorizes $\mathbf{y}$, i.e., $m_j(\mathbf{x}) \geq m_j(\mathbf{y})$ for each $j = 0, \ldots, n$. Then there are $\mathbf{a} \in \mathbb{R}^n$ and $\delta > 0$ such that for every $j = 0, \ldots, n$*

$$\begin{aligned} m_j(\mathbf{y}+t\mathbf{a}) &> m_j(\mathbf{y}) \qquad \big(t \in (0,\delta)\big), \\ m_j(\mathbf{y}-t\mathbf{a}) &< m_j(\mathbf{y}) \qquad \big(t \in (0,\delta)\big). \end{aligned}$$

In particular, the Local Strict non-Majorization Property (b) and non-Minorization Property (c) fail at $\mathbf{y}$.
(b) *On S the Local non-Majorization Property (C), the Local non-Minorization Property (D), the Local Comparison Property (B) and the Comparison Property (A) are all equivalent, also together with their strict versions.*

Proof. (a) Take $\mathbf{a} := \mathbf{x} - \mathbf{y}$ and let

$$\mathbf{y}_t := \mathbf{y} + t\mathbf{a} = (1-t)\mathbf{y} + t\mathbf{x}.$$

For sufficiently small $\delta > 0$ we have $\mathbf{y}_t \in S$ for every $(-\delta, 1]$ (since S is convex and open). By the strict concavity of m_j we obtain for $t \in (0,1)$ that

$$m_j(\mathbf{y}_t) > (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{x}) \geq (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{y}) = m_j(\mathbf{y})$$

and for $t \in (-\delta, 0)$

$$m_j(\mathbf{y}_t) < (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{x}) \leq (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{y}) = m_j(\mathbf{y}).$$

This proves the first assertion.

(b) The Comparison Property evidently implies the Local Comparison Property and that implies further the Local non-Minorization and non-Majorization Properties. The already established first assertion (a) provides the converse implications if we start with the even weaker Local Strict non-Minorization or non-Majorization Properties. $\square$

Proposition 8.2. *Suppose that the kernel functions $K_0, \ldots, K_n$ are strictly concave. Let $S = S_\sigma$ be a fixed simplex and let $\mathbf{e}, \mathbf{f} \in \overline{S}$ be two different equioscillation points.*
(a) *Then we have $M(S) < m(S)$, and the Sandwich Property (see Definition 5.7 and Remark 5.6) fails.*
(b) *If $\overline{m}(\mathbf{e}) \le \overline{m}(\mathbf{f})$ and $\mathbf{e} \in S$, then the Local Strict non-Majorization (b) and all the non-Minorization Properties fail to hold at $\mathbf{e}$.*
(c) *If the kernels either all satisfy (∞'), or are all in $\mathrm{C}^1(0, 2\pi)$, then the Comparison Property (A) fails (see Definition 5.10).*

Proof. For definiteness assume, as we may, that $\overline{m}(\mathbf{e}) \le \overline{m}(\mathbf{f})$.

(a) If $\overline{m}(\mathbf{e}) < \overline{m}(\mathbf{f})$, then we obviously have $M(S) \le \overline{m}(\mathbf{e}) < \overline{m}(\mathbf{f}) = \underline{m}(\mathbf{f}) \le m(S)$. If, on the other hand, $\overline{m}(\mathbf{e}) = \overline{m}(\mathbf{f})$, then for the point $\mathbf{g} := \frac{1}{2}(\mathbf{e} + \mathbf{f}) \in \overline{S}$ by the strict concavity we find $m_j(\mathbf{g}) > \frac{1}{2}(m_j(\mathbf{e}) + m_j(\mathbf{f})) = \overline{m}(\mathbf{e})$ for all $j = 0, \ldots, n$, hence also $\underline{m}(\mathbf{g}) > \overline{m}(\mathbf{e})$ and thus also $m(S) \ge \underline{m}(\mathbf{g}) > \overline{m}(\mathbf{e}) \ge M(S)$. In both cases the Sandwich Property must fail, because by Remark 5.6 this property is equivalent to $M(S) \ge m(S)$.

(b) If $\overline{m}(\mathbf{e}) \le \overline{m}(\mathbf{f})$, then $\mathbf{f}$ majorizes $\mathbf{e}$, so Corollary 8.1 (a) finishes the proof.

(c) Under the conditions we have $\mathbf{e}, \mathbf{f} \in S$ in view of Corollary 6.6. According to the previous part (b), we find that the the Local Strict non-Majorization (b) and non-Minorization Properties (c), (D) and (G) fail to hold at $\mathbf{e}$. However, it has already been noted in Remark 5.11 that in this case the Comparison Property (A) must fail as well. □

Corollary 8.3. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave. Let $S := S_\sigma$ be a simplex and let $\mathbf{y}^* \in S$ be a local minimum point of $\overline{m}$, see (6.1).*
(a) *Then there exists no other point different from $\mathbf{y}^*$ in $\overline{S}$ majorizing $\mathbf{y}^*$.*
(b) *Suppose the kernels either all satisfy (∞'), or all are in $\mathrm{C}^1(0, 2\pi)$. Then there exists no other local minimum point of $\overline{m}$ in the sense (6.1) in the closure $\overline{S}$ of S.*

Proof. (a) Suppose $\mathbf{x} \in \overline{S}$ majorizes $\mathbf{y}^*$ and $\mathbf{x} \neq \mathbf{y}^*$. Then by Corollary 8.1 (a) there are $\mathbf{a} \in \mathbb{R}^n$ and $\delta > 0$ with $m_j(\mathbf{y}^* - t\mathbf{a}) < m_j(\mathbf{y}^*)$ for every $t \in (0, \delta)$ and $j = 0, \ldots, n$. Hence $\mathbf{y}^*$ cannot be a local minimum point for $\overline{m}$.

(b) By Proposition 6.9, under the conditions on the kernels the local minimum points of $\overline{m}$ are also equioscillation points. Therefore, if $\mathbf{y} \in \overline{S}$, $\mathbf{y} \neq \mathbf{y}^*$ is another local minimum point of $\overline{m}$, then one of $\mathbf{y}$ and $\mathbf{y}^*$ majorizes the other. But then by part (a) the two points must be equal. □

To sum up our findings we can state:

Proposition 8.4. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave and either all satisfy (∞'), or all belong to $\mathrm{C}^1(0, 2\pi)$. Let $S := S_\sigma$ be a simplex. If $\overline{m}$ has a local minimum point $\mathbf{y}^* \in S$, then $\mathbf{y}^*$ is a unique point of equioscillation in $\overline{S}$, and $\underline{m}$ has there its (unique, global) maximum. In particular, then $M(S) = m(S)$. Moreover, the Sandwich Property holds true in S. Furthermore, the Singular non-Majorization and non-Minorization Properties hold on S.*

Proof. Let $\mathbf{y}_* \in \overline{S}$ be the (unique, global) maximum point of $\underline{m}$, see Corollary 7.3 (a). Obviously,

$$\min_{j=0,\ldots,n} m_j(\mathbf{y}_*) = \underline{m}(\mathbf{y}_*) \ge \underline{m}(\mathbf{y}^*).$$

By assumption we can apply Proposition 6.9 to conclude that $\mathbf{y}^*$ is an equioscillation point, i.e., $\underline{m}(\mathbf{y}^*) = \overline{m}(\mathbf{y}^*) = m_j(\mathbf{y}^*)$ for $j = 0, \dots, n$. Thus we find that $\mathbf{y}_*$ majorizes the point $\mathbf{y}^*$. According to Corollary 8.3 (a) this is not possible unless $\mathbf{y}_* = \mathbf{y}^*$. Therefore we obtain $M(S) = m(S)$, and Remark 5.6 yields the Sandwich Property. If $\mathbf{e} \in \overline{S}$ is another equioscillation point, then $\overline{m}(\mathbf{e}) \geq \overline{m}(\mathbf{y}^*)$ (since $\mathbf{y}^*$ is a minimum point). By Proposition 8.2 (a) this would imply $M(S) < m(S)$, which would be a contradiction. Therefore, there exists no other equioscillation point in $\overline{S}$ than $\mathbf{y}^*$ itself. Since $\mathbf{y}^* \in S$ is a local minimum point of $\overline{m}$, by Corollary 8.3 (a) there is no point majorizing it. But also $\mathbf{y}^*$ is the unique global minimum point of $\overline{m}$, so there is no point in $\overline{S}$ minorizing it. □

9. *The Difference Jacobi Property*

PROPOSITION 9.1. *Suppose that $K_0, \dots, K_n$ are in $\mathrm{C}^2(0, 2\pi)$ with $K_j'' < 0$ ($j = 0, \dots, n$), and let $S = S_\sigma$ be a simplex. For $j = 0, \dots, n$ the functions $m_j(\mathbf{y})$ are continuously differentiable in S and*

$$\frac{\partial m_j}{\partial y_r}(\mathbf{y}) = -K_r'\big(z_j(\mathbf{y}) - y_r\big) \quad \text{for } r = 1, \dots, n. \tag{9.1}$$

Proof. Let $\mathbf{y} \in S$ be fixed. Recall that $t = z_j(\mathbf{y})$ is the unique maximum point in $I_j(\mathbf{y})$, i.e., with $F'(\mathbf{y}, t) = 0$. Since

$$F''(\mathbf{y}, t) = K_0''(t) + \sum_{j=1}^{n} K_j''(t - y_j) < 0$$

by the implicit function theorem, for a suitable neighborhood $U \times V \subseteq S \times I_j(\mathbf{y})$ we have that $z_j : U \to V$ is continuously differentiable. Since $m_j(\mathbf{y}) = F(\mathbf{y}, z_j(\mathbf{y}))$ we obtain that m_j, too is continuously differentiable and

$$\begin{aligned}\frac{\partial m_j}{\partial y_r}(\mathbf{y}) &= \frac{\partial}{\partial y_r}\Big(F\big(\mathbf{y}, z_j(\mathbf{y})\big)\Big) = \frac{\partial F}{\partial y_r}\big(\mathbf{y}, z_j(\mathbf{y})\big) + \frac{\partial}{\partial t}F\big(\mathbf{y}, t\big)|_{t=z_j(\mathbf{y})}\frac{\partial}{\partial y_r}z_j(\mathbf{y}) \\ &= -K_r'\big(z_j(\mathbf{y}) - y_r\big).\end{aligned}$$

□

As a consequence, the Jacobian matrix $D\mathbf{m}$ of $\mathbf{m} = (m_0, \dots, m_n)^\top$ is

$$D\mathbf{m} = \begin{array}{c c} & r \\ j & \begin{pmatrix} & \vdots & \\ \cdots & -K_r'\big(z_j(\mathbf{y}) - y_r\big) & \cdots \\ & \vdots & \end{pmatrix}\end{array} \tag{9.2}$$

where $r = 1, \dots, n$ and $j = 0, \dots, n$.

For a given permutation σ of $\{1, \dots, n\}$ let us consider the mapping Δ_σ defined by

$$\Delta_\sigma(\mathbf{y}) := \big(m_{\sigma(1)}(\mathbf{y}) - m_{\sigma(0)}(\mathbf{y}), \dots, m_{\sigma(n)}(\mathbf{y}) - m_{\sigma(n-1)}(\mathbf{y})\big)^\top. \tag{9.3}$$

Its Jacobian matrix $D\Delta_\sigma$ is

$$D\Delta_\sigma(\mathbf{y}) = \begin{array}{c c} & r \\ j & \begin{pmatrix} & \vdots & \\ \cdots & -K_r'\big(z_{\sigma(j)}(\mathbf{y}) - y_r\big) + K_r'\big(z_{\sigma(j-1)}(\mathbf{y}) - y_r\big) & \cdots \\ & \vdots & \end{pmatrix}\end{array} \tag{9.4}$$

where $r = 1, \ldots, n$ and $j = 1, \ldots, n$.

Proposition 9.2. *Suppose that for each $j = 0, \ldots, n$ the kernel K_j belongs to $\mathrm{C}^2(0, 2\pi)$ with $K_j'' < 0$. Let $S = S_\sigma$ be a simplex and let $\mathbf{y} \in S$ be such that for each $j = 0, 1, \ldots, n$ we have $z_j(\mathbf{y}) \in \operatorname{int} I_j(\mathbf{y})$. Then, the Jacobian matrix of $\Delta_\sigma(\mathbf{y})$ is non-singular. That is, on S, we have the Difference Jacobi Property.*

Proof. For the sake of brevity we may suppose $\sigma = \mathrm{id}$, i.e., $\sigma(j) = j$, otherwise we can relabel the kernels K_j accordingly. We abbreviate $z_j := z_j(\mathbf{y})$ and have according to the assumption

$$z_{j-1} < y_j < z_j \quad \text{for } j = 1, \ldots, n.$$

Write $A := -D\Delta_\sigma(\mathbf{y})$. First, we show that A is a so-called Z-matrix, that is, the entries are non-negative on the diagonal and are non-positive off the diagonal.

On the diagonal the entries are $-K_r'(z_r - y_r) + K_r'(z_{r-1} - y_r)$, $r = 1, \ldots, n$. Since $z_{r-1} < y_r < z_r$, $z_r < 2\pi$ and $z_{r-1} > 0$, we obtain $z_{r-1} - y_r < 0 < z_r - y_r$ and $z_r - y_r < 2\pi + z_{r-1} - y_r$. Now, using the 2π periodicity of K_r' and that K_r' is strictly monotone decreasing, we obtain $K_r'(z_{r-1} - y_r) < K_r'(z_r - y_r)$, that is, $K_r'(z_r - y_r) - K_r'(z_{r-1} - y_r) > 0$.

For $j < r$ we have $z_{j-1} < z_j \le z_{r-1} < y_r$. Therefore, $-2\pi < z_{j-1} - y_r < z_j - y_r < 0$ and using that K_r' is strictly monotone decreasing and 2π periodic, we can write

$$K_r'(z_j - y_r) - K_r'(z_{j-1} - y_r) < 0.$$

Therefore the elements above the diagonal of A are strictly negative.

If $j > r$, then $y_r < z_r \le z_{j-1} < z_j$. As above, $0 < z_{j-1} - y_r < z_j - y_r < 2\pi$ and using that K_r' is strictly monotone decreasing, we can write

$$K_r'(z_j - y_r) - K_r'(z_{j-1} - y_r) < 0,$$

meaning that the entries below the diagonal of A are strictly negative, too. So we have seen that A is a Z-matrix.

We now show that the column sums of A are strictly positive. Indeed, the sum of the r^{th} column of A is telescopic

$$\sum_{i=1}^{n} \Big(K_r'(z_i - y_r) - K_r'(z_{i-1} - y_r) \Big) = K_r'(z_n - y_r) - K_r'(z_0 - y_r).$$

Since $0 < z_0 < y_r < z_n < 2\pi$, we have $0 < z_n - y_r < 2\pi + z_0 - y_r < 2\pi$. Since K_r' is strictly decreasing and 2π periodic, it follows $K_r'(z_n - y_r) - K_r'(z_0 - y_r) > 0$.

Therefore, with $\mathbf{x} = (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$ we have $A^\top \mathbf{x}$ is a strictly positive vector. This means that $A^\top$ satisfies condition I27 in [**4**] (see page 136). Hence by Theorem 2.3 on pp. 134–138 in [**4**] it follows that $A^\top$ is an M-matrix and is non-singular, this yielding also the non-singularity of $-A$. The proof is hence complete. □

Corollary 9.3. *Suppose that for each $j = 0, \ldots, n$ the kernel K_j belongs to $\mathrm{C}^2(0, 2\pi)$ with $K_j'' < 0$ and satisfies (∞). Let $S = S_\sigma$ be a simplex. The mapping $\Delta_\sigma : S \to \mathbb{R}^n$ is then a homeomorphism.*

Proof. By Proposition 9.2 the mapping Δ_σ is locally a homeomorphism (onto its image), and by Proposition 5.1 it carries the boundary ∂S onto the boundary of the one-point compactified $\mathbb{R}^n$. By a well-known result— see e.g. [**24**, p. 105, Lemma 3.24], [**20**, Corollary 4.3], or [**19**, pp. 136–137, Theorem 5.3.8]—Δ_σ is a homeomorphism. $\square$

Here is a proof of existence (and even uniqueness) of equioscillation points in a given simplex under the special conditions of this section.

Corollary 9.4. *Suppose that for each $j = 0, \dots, n$ the kernel K_j belongs to $\mathrm{C}^2(0, 2\pi)$ with $K_j'' < 0$ and satisfies (∞). Then all equioscillation points belong to some (open) simplex, and in each simplex $S = S_\sigma$ there is a unique equioscillation point.*

Proof. An equioscillation point must belong to X according to Corollary 6.6. In a fixed simplex S_σ, an equioscillation point is the inverse image of $\mathbf{0} \in \mathbb{R}^n$ under the homeomorphism Δ_σ from Corollary 9.3. $\square$

10. *Equioscillation points*

In this section we prove the existence of equioscillation points in each simplex $S = S_\sigma$, and discuss the uniqueness of such points. The main tool will be the approximation of kernels by a sequence of kernel functions having special properties, so the arguments rely on the results of Section 4.

Lemma 10.1. *Suppose that $K_0, \dots, K_n$ are strictly concave kernel functions and that a sequence of strictly concave kernel functions $(K_j^{(k)})_{k\in\mathbb{N}}$ converges uniformly (e.s.) to K_j as $k \to \infty$, $j = 1, \dots, n$. For each $k \in \mathbb{N}$ let $\mathbf{e}^{(k)} \in \overline{S}$ be an equioscillation point for the system of kernels $K_j^{(k)}$, $j = 0, \dots, n$. Then any accumulation point $\mathbf{e} \in \overline{S}$ of the sequence $(\mathbf{e}^{(k)})_{k\in\mathbb{N}}$ is an equioscillation point of the system K_j, $j = 0, \dots, n$.*

Proof. By passing to a subsequence we may assume that $\mathbf{e}^{(k)} \to \mathbf{e} \in \overline{S}$. By assumption and by Proposition 4.3 $m_j^{(k)} \to m_j$ uniformly (e.s.) on $\overline{S}$ as $k \to \infty$. It follows that $m_j^{(k)}(\mathbf{e}_k) \to m_j(\mathbf{e})$ as $k \to \infty$, so $\mathbf{e} \in \overline{S}$ is an equioscillation point. $\square$

We need another lemma, similar to [**3**, Theorem 1], in order to be able to apply the previous result.

Lemma 10.2. *Let $f : [0, 1) \to \mathbb{R}$ be a strictly concave, non-increasing function. Then for each $\varepsilon > 0$ there exists another strictly concave decreasing function $g : [0, 1) \to \mathbb{R}$ such that $g \in \mathrm{C}^\infty[0, 1)$, $g'' < 0$ on $[0, 1)$, and $f(x) - \varepsilon \le g(x) \le f(x)$ for each $x \in [0, 1)$.*

Proof. This lemma is fairly standard, but for sake of completeness, we include a proof.

Assume, without loss of generality, that $f(0) = 0$. Let us consider the right (hence right continuous) derivative f_+' of f for our construction: We can write $f(x) = \int_0^x \phi(t)dt$, where $\phi(t) := f_+'(t)$ and $\phi : [0, 1) \to (-\infty, 0]$, defining the value of $\phi(0)$ by the limit at the endpoint.

It suffices to construct a C^∞-approximation $\gamma : [0, 1) \to (-\infty, 0]$ to the non-increasing function ϕ, which has non-positive, continuous derivative $\gamma' \in \mathrm{C}^\infty[0, 1)$, and which satisfies $\gamma(x) \le \phi(x)$ on $[0, 1)$ and $\int_0^1 (\phi(x) - \gamma(x))dx < \varepsilon$. Indeed, then $g(x) := \int_0^x \gamma(t)dt$ is a suitable approximant to f. (If needed, we can easily achieve $g'' < 0$ by adding $-\eta \cdot (x + 1)^2$ to g where $\eta > 0$ is small enough, still satisfying $f(x) - \varepsilon - 4\eta \le g(x) - \eta \cdot (x + 1)^2 \le f(x)$).

Write $\phi(x) = \alpha(x) + \beta(x)$, where $\alpha(x)$ is a pure jump function and $\beta(x)$ is continuous. Both α and β are non-increasing.

Approximate β with a pure jump function β_1 such that β_1 is non-increasing and $\beta(x) - \varepsilon/2 \le \beta_1(x) \le \beta(x)$ for all $x \in [0,1)$.

Consider $\alpha(x) + \beta_1(x) = \sum_{j=1}^{\infty} s_j H(x - r_j)$, where $H(x)$ is the usual Heaviside function, $H(x) = 1$ for $x \ge 0$ and otherwise zero. Here $s_j < 0$, $r_j \in [0,1)$, $\sum_{i\ :r_i<x} |s_i| < \infty$ (for all $x < 1$). By construction, $\phi(x) - \varepsilon/2 \le \sum_{j=1}^{\infty} s_j H(x - r_j) \le \phi(x)$.

Take $\psi \in \mathrm{C}^{\infty}(\mathbb{R})$ with $\psi \ge 0$, $\operatorname{supp}\psi = [-1,0]$, $\int_{\mathbb{R}} \psi(t)dt = 1$ and define $\theta(x) := \int_{-\infty}^{x} \psi(t)dt$. Consider the translated and dilated versions $\tau_{r,h}(x) := \theta((x-r)/h)$ of θ. Then $\tau_{r,h} \in \mathrm{C}^{\infty}[0,1)$ for any $h > 0$, and these functions are non-decreasing, and $H(x-r) \le \tau_{r,h}(x)$ with strict inequality holding precisely for $x \in (r-h, r)$. As a result, we have $\int_0^1 |\tau_{r,h}(x) - H(x-r)|dx \le h$. Approximate now the constructed pure jump function from below as follows:

$$\sum_{i=1}^{\infty} s_i H(t - r_i) \ge \sum_{i=1}^{\infty} s_i \tau_{r_i,h_i}(t),$$

where both sums are absolutely and uniformly convergent for all $t \in [0,x]$ for any fixed $x < 1$, if only we assume $h_i \le \frac{1}{2}(1 - r_i)$. (Indeed, this follows for the first sum by $\sum_{i\ :r_i<x} |s_i| < \infty$, while the assumption entails that $r_i - h_i < x \Rightarrow r_i < x + \frac{1}{2}(1-r_i) \Leftrightarrow r_i < \frac{2x+1}{3} (< 1)$, whence the sum $\sum_{i\ :r_i - h_i < x} |s_i| \le \sum_{i\ :r_i<(2x+1)/3} |s_i|$ also converges.) Furthermore, we also have

$$\begin{aligned} 0 &\le \int_0^x \sum_{i=1}^{\infty} s_i H(t-r_i) - \sum_{i=1}^{\infty} s_i \tau_{r_i,h_i}(t) dt = \sum_{i=1}^{\infty} s_i \int_0^x H(t-r_i) - \tau_{r_i,h_i}(t)\ dt \\ &\le \sum_{i:\ r_i - h_i < x} |s_i|\, h_i < \frac{\varepsilon}{2}, \end{aligned}$$

if we also know that h_i are so small that $\sum_{i=1}^{\infty} |s_i|\, h_i < \varepsilon/2$. Here we can choose $h_i := \min\left(\frac{1}{2} - \frac{r_i}{2}, 2^{-i}\frac{\varepsilon}{4|s_i|}\right)$.

Finally, let $\gamma(x) := \sum_{i=1}^{\infty} s_i \tau_{r_i,h_i}(t)$. Then $\phi(x) \ge \gamma(x)$ and $\int_0^1 \gamma(x) - \phi(x)dx < \varepsilon$. This finishes the proof of this lemma. ☐

Lemma 10.3. *Let K be a strictly concave kernel function. Then for each $\varepsilon > 0$ there exists another strictly concave decreasing function $k \in \mathrm{C}^2(0,2\pi)$, $k'' < 0$ on $(0,2\pi)$, and $K(x) - \varepsilon \le k(x) \le K(x)$ for each $x \in (0,2\pi)$.*

Proof. This approximation is indeed possible, for given $\varepsilon > 0$ and a given (strictly) concave function $K : (0,2\pi) \to \mathbb{R}$ satisfying (∞), we can choose the maximum point $c \in (0,2\pi)$, and consider the intervals ($[c,2\pi)$ and $(0,c]$ separately: applying Lemma 10.2 for $-K((x-c)/(2\pi - c))$ and $-K((c-x)/c)$ separately provides an approximating strictly concave kernel function $k \in \mathrm{C}^2((0,2\pi)\setminus\{c\})$ with $k'' < 0$ and $K - \varepsilon < k < K$. By a modification of this kernel function even a smooth approximating kernel function, as in the assertion, can be easily found. ☐

Theorem 10.4. *Suppose that for each $j = 0,\dots,n$ the kernels K_j are strictly concave. Then for each simplex $S = S_\sigma$ there exists an equioscillation point in $\overline{S}$.*

Moreover, if the kernels are either all in $\mathrm{C}^1(0,2\pi)$ or at least n of them satisfy (∞'), then any equioscillation point is in the open simplex S.

Proof. We split the proof into several steps.

Step 1. First, let us suppose that all the kernel functions $K_0, \dots, K_n$ satisfy (∞). By Lemma 10.3 we can take a sequence $(K_i^{(k)})_{k\in\mathbb{N}}$ of strictly concave functions in $\mathrm{C}^2(0,2\pi)$ satisfying $\frac{d^2}{dt^2}K_i^{(k)}(t) < 0$ and converging strongly uniformly (and therefore locally uniformly, too) to the functions K_i. Note that hence we also require that $K_j^{(k)}$ satisfy (∞).

According to Corollary 9.4 each system $K_j^{(k)}$, $j = 0, \dots, n$, has a unique equioscillation point $\mathbf{e}^{(k)}$. By Lemma 10.1 any accumulation point $\mathbf{e}$ of this sequence (and, by compactness, there is one) is an equioscillation point. Finally, by Corollary 6.6 an equioscillation point is necessarily inside S. This concludes the proof for the special case when all the kernels satisfy (∞).

Step 2. Now, let us consider the case when the kernels are strictly concave but satisfy $(\infty'_\pm)$ only. Let us fix the auxiliary functions $L_k(x) := \log_-(k|x|)$, which are concave, even, non-positive functions on $(-\pi, 0) \cup (0, \pi)$ with singularity at 0. We extend these functions to $\mathbb{R}$ periodically. For $k \in \mathbb{N}$ and $j = 0, \dots, n$ define $K_j^{(k)} := L_k + K_j$. Then $K_j^{(k)} \uparrow K_j$ on $\mathbb{T} \setminus \{0\}$. By Step 1, for each $k \in \mathbb{N}$ there is an equioscillation point $\mathbf{e}^{(k)}$ for the system $K_j^{(k)}$, $j = 0, \dots, n$. By passing to a subsequence we can assume $\mathbf{e}^{(k)} \to \mathbf{e} \in \overline{S}$. For $j \in \{0, \dots, n\}$ we have

$$m_j^{(k)}(\mathbf{e}^{(k)}) = \max_{t\in I_j(\mathbf{e}^{(k)})} F^{(k)}(\mathbf{e}^{(k)}, t) \le \max_{t\in I_j(\mathbf{e}^{(k)})} F(\mathbf{e}^{(k)}, t) = m_j(\mathbf{e}^{(k)}).$$

Since m_j is continuous on $\overline{S}$, we obtain

$$\limsup_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) \le m_j(\mathbf{e}). \tag{10.1}$$

Suppose first that the arc $I_j(\mathbf{e})$ is non-degenerate for all $j = 0, 1, \dots, n$, i.e., assume $\mathbf{e} \in S$. Then Proposition 3.9 (d) yields $z_j(\mathbf{e}) \in \operatorname{int} I_j(\mathbf{e}) = (e_j, e_r)$, so for sufficiently large k we have $z_j(\mathbf{e}) \in \operatorname{int} I_j(\mathbf{e}^{(k)})$, too; furthermore, since by construction $K_j(t) = K_j^{(k)}(t)$ for $t \notin [-\frac{1}{k}, \frac{1}{k}]$, for sufficiently large k we even have $e_j^{(k)} + 1/k < z_j(\mathbf{e}) < e_r^{(k)} - 1/k$, whence $F^{(k)}(\mathbf{e}^{(k)}, z_j(\mathbf{e})) = F(\mathbf{e}^{(k)}, z_j(\mathbf{e}))$, too. Therefore we obtain

$$m_j^{(k)}(\mathbf{e}^{(k)}) = \max_{t\in I_j(\mathbf{e}^{(k)})} F^{(k)}(\mathbf{e}^{(k)}, t) \ge F^{(k)}(\mathbf{e}^{(k)}, z_j(\mathbf{e})) = F(\mathbf{e}^{(k)}, z_j(\mathbf{e})).$$

This implies

$$\liminf_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) \ge \liminf_{k\to\infty} F(\mathbf{e}^{(k)}, z_j(\mathbf{e})) = F(\mathbf{e}, z_j(\mathbf{e})) = m_j(\mathbf{e}). \tag{10.2}$$

So the proof of Step 2 is complete if $\mathbf{e} \in S$.

Finally, we show that $\mathbf{e} \in \partial S$ is impossible. Indeed, if there is a degenerate arc $I_j(\mathbf{e})$, then by Corollary 6.5 there is a neighboring non-degenerate arc $I_i(\mathbf{e})$ such that $m_i(\mathbf{e}) > m_j(\mathbf{e})$. But then we are led to a contradiction, because using (10.1) and (10.2) we also have

$$m_j(\mathbf{e}) \ge \limsup_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) \ge \liminf_{k\to\infty} m_j^{(k)}(\mathbf{e}^{(k)}) = \liminf_{k\to\infty} m_i^{(k)}(\mathbf{e}^{(k)}) \ge m_i(\mathbf{e}).$$

Step 3. Finally, we suppose only that $K_0, \dots, K_n$ are strictly concave kernel functions. We now take the functions $L_k(x) := (\sqrt{|x|} - 1/k)_-$, which are negative only for $-1/k^2 < x < 1/k^2$ and zero otherwise, and converge uniformly to zero. Restricting L_k to $[-\pi, \pi)$ and then extending it periodically we thus obtain a function on $\mathbb{T}$ which is concave on $(0, 2\pi)$ and converges to 0 uniformly on $[0, 2\pi]$. Note that $\lim_{x\to 0\pm 0} L'_k(x) = \pm\infty$, hence the perturbed kernels $K_j^{(k)} := K_j + L_k$, $j = 0, \dots, n$, satisfy $(\infty'_\pm)$. Again, in view of the already proven case in Step 2, there exist some equioscillation points $\mathbf{e}^{(k)}$ for the system $K_j^{(k)}$, $j = 0, \dots, n$, and by compactness, there exists an accumulation point $\mathbf{e} \in \overline{S}$ of the sequence $(\mathbf{e}^{(k)})_{k\in\mathbb{N}}$. By uniform convergence of the kernels we can apply Lemma 10.1 to conclude that $\mathbf{e}$ is an equioscillation point of the system K_j, $j = 0, \dots, n$.

It remains to prove that $\mathbf{e} \in S$ if the additional assumptions are fulfilled, but this has already been done in Corollary 6.6. $\square$

Corollary 10.5. *Let the kernel functions $K_0, \ldots, K_n$ be strictly concave. Then in any simplex $S = S_\sigma$ the Equioscillation Property holds, and we have $M(S) \leq m(S)$.*

Corollary 10.6. *Let the kernel functions $K_0, \ldots, K_n$ be strictly concave and let $S = S_\sigma$ be a simplex. Suppose that $M(S) = m(S)$. Then there is $\mathbf{w}_* \in \overline{S}$ with $m(S) = \underline{m}(\mathbf{w}_*)$ and $\mathbf{w}_*$ is the unique equioscillation point in $\overline{S}$.*

Proof. Let $\mathbf{e} \in \overline{S}$ be an equioscillation point (see Corollary 10.5), and let $\mathbf{w}_* \in \overline{S}$ be such that $\underline{m}(\mathbf{w}_*) = m(S)$ (see Proposition 3.11). Because $\underline{m}(\mathbf{e}) = \overline{m}(\mathbf{e}) \geq M(S) = m(S) = \underline{m}(\mathbf{w}_*)$, we find that $\mathbf{e}$ is also a maximum point of $\underline{m}$, and that $\underline{m}(\mathbf{e}) = M(S)$. By Corollary 7.3 (a), $\mathbf{e} = \mathbf{w}_*$, and by $M(S) = m(S)$ and in view of Proposition 8.2 (a), the equioscillation point is unique. $\square$

11. *Summary and conclusions*

Theorem 11.1. *Suppose the kernel functions $K_0, K_1, \ldots, K_n$ are strictly concave and either all satisfy (∞'), or all belong to $\mathrm{C}^1(0, 2\pi)$. Then there is $\mathbf{w}^* \in \mathbb{T}^n$, $\mathbf{w}^* = (w_1, \ldots, w_n)$ with*

$$M := \inf_{\mathbf{y} \in \mathbb{T}^n} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t) = \sup_{t \in \mathbb{T}} F(\mathbf{w}^*, t).$$

Moreover, we have the following:
(a) *$\mathbf{w}^*$ is an equioscillation point, i.e., $m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*)$.*
(b) *$\mathbf{w}^* \in S := S_\sigma$ for some simplex, i.e., the nodes in $\mathbf{w}^*$ are different, and*

$$M(S) = \inf_{\mathbf{y} \in S} \max_{j=0,\ldots,n} \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t) = M = \sup_{\mathbf{y} \in S} \min_{j=0,\ldots,n} \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t) = m(S).$$

(c) *We have the Sandwich Property on S, i.e., for each $\mathbf{x}, \mathbf{y} \in S$*

$$\underline{m}(\mathbf{x}) \leq M \leq \overline{m}(\mathbf{y}).$$

Proof. In view of Corollary 3.12, a global minimum point $\mathbf{w}^*$ of $\overline{m}$ must exist. Next, Corollary 6.11 furnishes part (a) and $\mathbf{w}^* \in X$, i.e. the first half of (b). Finally, Proposition 8.4 implies the second half of (b) and the assertion in (c). $\square$

Example 11.2. We present an example showing that on different simplexes we may have different values of $M(S)$. Consider the functions

$$\begin{aligned} K(x) &:= \pi - |x - \pi| && \text{for } x \in [0, 2\pi], \\ Q(x) &:= x(2\pi - x) && \text{for } x \in [0, 2\pi], \end{aligned}$$

and extend them periodically to $\mathbb{R}$. We take $K_0 = K_1 = K$ and $K_2 = K_3 = \varepsilon Q$ where $\varepsilon \in (0, \frac{1}{4})$ is fixed arbitrarily.

Note that this system of kernels almost satisfy the conditions of Theorem 11.1: two kernels satisfy $(\infty'_\pm)$ and all the kernels are in $\mathrm{C}^1((0, 2\pi) \setminus \{\pi\})$, and the two not satisfying $(\infty'_\pm)$ are even in $\mathrm{C}^1(0, 2\pi)$ (which, again, could have been enough if satisfied by all).

We consider two simplexes $S = S_\sigma$ for $\sigma = (2,1,3)$ and $S' = S_{\sigma'}$ with $\sigma' = (3,2,1)$. We prove that there is an equioscillation point $\mathbf{e} \in S$ and for this equioscillation point we have $\overline{m}(\mathbf{e}) > \overline{m}(S')$. This will be done first in two steps below, then in Step 3 we shall take an appropriate sequence of kernel functions $K_j^{(k)}$ converging to K_j $(j = 0, 1, \ldots, n)$ and obtain

$$M^{(k)}(S) > M^{(k)}(S').$$

Step 1. We take the node system $\mathbf{e} : e_0 = 0$, $e_1 = \pi$, $e_2 = \frac{\pi}{2}$, $e_3 = \frac{3\pi}{2}$. Then we have $\mathbf{e} \in S$ and

$$F(\mathbf{e}, t) = K_0(t) + K_1(t - e_1) + K_2(t - e_2) + K_3(t - e_3) = \pi + \varepsilon Q(t - \tfrac{\pi}{2}) + \varepsilon Q(t - \tfrac{3\pi}{2}).$$

It is easy to see that

$$m_0(\mathbf{e}) = F(\mathbf{e}, 0) = \max_{t \in [0, \frac{\pi}{2}]} F(\mathbf{e}, t) = \pi + 3\varepsilon \tfrac{\pi^2}{2},$$

and by symmetry $m_0(\mathbf{e}) = m_1(\mathbf{e}) = m_2(\mathbf{e}) = m_3(\mathbf{e})$, i.e., $\mathbf{e}$ is an equioscillation point.

Step 2. Consider the node system $x_0 = 0$, $x_1 = \pi + (3 - 2\sqrt{2})\varepsilon\pi^2$, $x_2 = (2\sqrt{2} - 2)\pi$, $x_3 = 0$. Then of course $\mathbf{x} \in \overline{S'} \cap \overline{S}$. It is easy to see that

$$F(\mathbf{x}, t) = \begin{cases} -2\varepsilon t^2 + 2(1 + \varepsilon x_2)t - \varepsilon x_2^2 + 2\pi(\varepsilon x_2 + 1) - x_1, & \text{if } 0 \le t \le x_1 - \pi, \\ -2\varepsilon t^2 + 2\varepsilon x_2 t - \varepsilon x_2\left(-2\pi + x_2\right) + x_1, & \text{if } x_1 - \pi \le t \le x_2, \\ -2\varepsilon t^2 + 2\varepsilon\left(2\pi + x_2\right)t - \varepsilon x_2\left(2\pi + x_2\right) + x_1, & \text{if } x_2 \le t \le \pi, \\ -2\varepsilon t^2 + 2(\varepsilon x_2 + 2\varepsilon\pi - 1)t - \varepsilon x_2(2\pi + x_2) + x_1 + 2\pi, & \text{if } \pi \le t \le x_1, \\ -2\varepsilon t^2 + 2\varepsilon\left(2\pi + x_2\right)t - \varepsilon x_2(2\pi + x_2) - x_1 + 2\pi, & \text{if } x_1 \le t \le 2\pi. \end{cases}$$

For definiteness of indexing, let us consider the node system $\mathbf{x}$ as an element of the simplex S' where $\sigma' = (3,2,1)$.

Now, an easy but tedious computation leads to the following. The maximum of $F(\mathbf{x}, \cdot)$ on $I_0(\mathbf{x}) = [x_0, x_3] = [0, 0]$ is

$$m_0(\mathbf{x}) = F(\mathbf{x}, 0) = \pi + \varepsilon\pi^2(14\sqrt{2} - 19),$$

the maximum of $F(\mathbf{x}, \cdot)$ on $I_1(\mathbf{x}) = [x_1, 2\pi]$ is attained at $z_1(\mathbf{x}) = \pi + \frac{x_2}{2}$ and

$$m_1(\mathbf{x}) = F(\mathbf{x}, \pi + \tfrac{w_2}{2}) = \pi + \varepsilon\pi^2(6\sqrt{2} - 7),$$

the maximum of $F(\mathbf{x}, \cdot)$ on $I_2(\mathbf{x}) = [x_2, x_1]$ is attained at $z_2(\mathbf{x}) = \pi$ and

$$m_2(\mathbf{x}) = F(\mathbf{x}, \pi) = \pi + \varepsilon\pi^2(6\sqrt{2} - 7),$$

the maximum of $F(\mathbf{x}, \cdot)$ on $I_3(\mathbf{x}) = [x_3, x_2] = [0, x_2]$ is attained at $z_3(\mathbf{x}) = \frac{x_2}{2}$ and

$$m_3(\mathbf{x}) = F(\mathbf{x}, \tfrac{x_2}{2}) = \pi + \varepsilon\pi^2(6\sqrt{2} - 7).$$

From this we conclude

$$\overline{m}(\mathbf{x}) = \pi + \varepsilon\pi^2(6\sqrt{2} - 7) < \pi + 3\varepsilon\tfrac{\pi^2}{2} = \overline{m}(\mathbf{e}),$$

and hence

$$M(S),\ \ M(S') \le \overline{m}(\mathbf{x}) < \overline{m}(\mathbf{e}).$$

Note that the equioscillation point $\mathbf{e} \in S$ thus cannot be a minimum point of $\overline{m}$ on the simplex S, while $\mathbf{x} \in \overline{S} \cap \overline{S'}$ is a weak equioscillation point on the boundary of both simplexes.

Step 3. Now, let

$$K_j^{(k)}(x) := K_j(x) + \frac{1}{k}\sqrt{\pi^2 - (x-\pi)^2},$$

for $j = 0, 1, 2, 3$. Then $K_0^{(k)}$, $K_1^{(k)}$, $K_2^{(k)}$, $K_3^{(k)}$ are strictly concave, symmetric, satisfying the condition $(\infty'_\pm)$ and

$$K_j^{(k)} \to K_j \quad \text{uniformly as } k \to \infty \text{ for } j = 0, 1, 2, 3.$$

Since the configuration of the kernel functions for the simplex S is symmetric and the node system $\mathbf{e}$ is symmetric, it is easy to see that $\mathbf{e}$ is an equioscillation point in S also in the case of the kernels $K_j^{(k)}$. By Proposition 4.3 we have $M^{(k)}(S) \to M(S)$, $m^{(k)}(S) \to m(S)$ and $m_j^{(k)}(\mathbf{e}) \to m_j(\mathbf{e})$ as $k \to \infty$. Let $\mathbf{w}^{*(k)} \in \overline{S}$ be such that $M^{(k)}(S) = \overline{m}(\mathbf{w}^{*(k)})$.

Now if for some $k \in \mathbb{N}$ we have $\overline{m}^{(k)}(\mathbf{e}) \neq M^{(k)}(S)$, then $\mathbf{w}^{*(k)} \in \partial S$ (by Proposition 8.4) and $m^{(k)}(S) \geq \underline{m}^{(k)}(\mathbf{e}) > M^{(k)}(S)$. By Corollary 6.10 (d) we have then $M^{(k)}(S'') < M^{(k)}(S)$ for some neighboring simplex S''. Since by symmetry there are basically two simplexes, we must have $M^{(k)}(S'') = M^{(k)}(S')$ (recall $S' = S_{\sigma'}$ for $\sigma' = (3, 2, 1)$). Therefore

$$M^{(k)}(S) > M^{(k)}(S').$$

On the other hand, had we† for all $k \in \mathbb{N}$ $\overline{m}^{(k)}(\mathbf{e}) = M^{(k)}(S)$, then for large enough k we would find

$$M^{(k)}(S) = \overline{m}^{(k)}(\mathbf{e}) > \overline{m}^{(k)}(\mathbf{x}) \geq M^{(k)}(S').$$

We sum up what has been found in this example: There are strictly concave kernel functions $K_j^{(k)}$, $j = 0, 1, 2, 3$ satisfying $(\infty'_\pm)$, and there are two simplexes S and S' such that $M^{(k)}(S) > M^{(k)}(S')$.

The phenomenon observed in the previous example can be present also for strictly concave kernels with the (∞) property.

EXAMPLE 11.3. Consider some symmetric kernel functions K_0, K_1, K_2, K_3 satisfying $(\infty'_\pm)$ with $M(S_\sigma) > M(S_{\sigma'})$ (see the previous Example 11.2). Let L be a strictly concave, symmetric kernel function with (∞), and consider $K_j^{(k)} := \frac{1}{k}L + K_j$, $j = 0, \ldots, 3$. Then, as in Example 11.2, by means of Proposition 4.3 we obtain

$$M^{(k)}(S_\sigma) > M^{(k)}(S_{\sigma'})$$

for large k.

EXAMPLE 11.4. It can happen that $M(\mathbb{T}^3) < m(\mathbb{T}^3)$.

Indeed, Let K_0, K_1, K_2, $K_3 \in \mathrm{C}^2(0, 2\pi)$ be strictly concave symmetric kernel functions satisfying (∞) with

$$M(S_\sigma) > M(S_{\sigma'})$$

for different simplexes S_σ and $S_{\sigma'}$. Consider, for example, the situation of the preceding Example 11.3.

†In fact, this case cannot occur at all. Indeed, the inequality $M^{(k)}(S) > \underline{m}^{(k)}(\mathbf{x})$ and $\mathbf{x} \in \overline{S}$ already contradict each other, whence this is impossible.

Let $\mathbf{w}^* \in \mathbb{T}^3$ be a global minimum point of $\overline{m}$ on $\mathbb{T}^3$. Let $S_{\sigma''}$ denote the simplex in which $\mathbf{w}^*$ lies. We then have

$$M(\mathbb{T}^3) = m(S_{\sigma''}) = M(S_{\sigma''}) \leq M(S_{\sigma'}) < M(S_\sigma) \leq m(S_\sigma)$$

by Theorem 11.1 (b) and by Corollary 10.5. This implies $M(\mathbb{T}^3) < m(\mathbb{T}^3)$.

Next, let us discuss the case when all but one kernel functions are the same. This is analogous to the setting of Fenton [**12**] in the interval case. Under these circumstances the phenomenon in the previous example is not present anymore. We first need the next lemma, whose similar versions have appeared already in [**12**] and [**14**].

LEMMA 11.5. *Let K be strictly concave and let $a, b > 0$, $x, y \in (0, 2\pi)$ with $x \leq y$ be given. Then for sufficiently small $\delta > 0$ we have that*

$$\frac{1}{a}K(t-(y+ah)) + \frac{1}{b}K(t-(x-bh)) < \frac{1}{a}K(t-y) + \frac{1}{b}K(t-x)$$

for each $t \in (0, x-b\delta) \cup (y+a\delta, 2\pi)$ and each $0 < h < \delta$.

Proof. Let $\delta > 0$ be so small that for $h \in (0, \delta)$ we have $x - bh > 0$ and $y + ah < 2\pi$. By strict concavity the difference quotients of K are strictly decreasing in both variables, so that for all $h \in (0, \delta)$ and $t \in (0, x - b\delta)$ or $t \in (y + a\delta, 2\pi)$

$$\frac{K(t-x+bh) - K(t-x)}{bh} < \frac{K(t-y) - K(t-y-ah)}{ah}.$$

But this inequality is equivalent to the assertion. □

THEOREM 11.6. *Suppose the kernel functions L, K are strictly concave and either K satisfies (∞') or both K and L belong to $\mathrm{C}^1(0, 2\pi)$. Set*

$$F(\mathbf{y}, t) := L(t) + \sum_{j=1}^{n} K(t - y_j).$$

Then there is an up to permutation unique $\mathbf{w}^* \in \mathbb{T}^n$, $\mathbf{w}^* = (w_1, \ldots, w_n)$ with

$$M := \inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y}, t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*, t).$$

Moreover, we have the following:

(a) *The nodes $w_0, \ldots, w_n$ are different and $\mathbf{w}^*$ is an equioscillation point, i.e.,*

$$m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*).$$

(b) *We have*

$$M = \inf_{\mathbf{y}\in\mathbb{T}^n} \max_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t) = \sup_{\mathbf{y}\in\mathbb{T}^n} \min_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t) = m.$$

(c) *We have the Sandwich Property on $\mathbb{T}^n$, i.e., for each $\mathbf{x}, \mathbf{y} \in \mathbb{T}^n$*

$$\underline{m}(\mathbf{x}) \leq m = M \leq \overline{m}(\mathbf{y}).$$

(d) *If K is as in the above and $L = K$, then a permutation of the points $w_0 = 0, w_1, \ldots, w_n$ lies equidistantly in $\mathbb{T}$.*

Proof. First of all, notice that assertion (d) is obvious by the complete symmetry of the setup. Furthermore, again by the cyclic symmetry of the situation, even if $K \neq L$, we still have for any two simplexes S_σ and $S_{\sigma'}$ that $M(S_\sigma) = M(S_{\sigma'}) = M$ and $m(S_\sigma) = m(S_{\sigma'}) = m$. Thus, if L and K satisfies (∞'), or if both belong to $\mathrm{C}^1(0, 2\pi)$, existence, uniqueness, and the assertions (a), (b) and (c) are contained in Theorem 11.1.

It remains to prove parts (a), (b) and (c) in the case when K satisfies (∞') while L does not, so that L is a real-valued continuous function on $\mathbb{T}$. Without loss of generality we may assume that K satisfies (∞'_-).

Let $\mathbf{w}^* = (w_1, \ldots, w_n)$ be a global minimum point of $\overline{m}$ in $\mathbb{T}^n$ (Corollary 3.12). We first show that $\mathbf{w}^* \in X$, i.e., $\mathbf{w}^* \in S$ for some simplex S. We argue by contradiction and assume that $\mathbf{w}^* \in \mathbb{T}^n \setminus X$, i.e. $\mathbf{w}^* \in \partial S_\sigma$ for some permutation σ.

As the kernels $K_i = K$ satisfy (∞') for $i = 1, \ldots, n$, Lemma 3.8 (b) immediately provides $M > F(\mathbf{w}^*, w_i)$ for each w_i, $i = 1, \ldots, n$. Now if $\mathbf{w}^* \in \partial S_\sigma$ is such that $w_i = w_0 = 0$ for some $i \in \{1, 2, \ldots, n\}$, then we also have $M > F(\mathbf{w}^*, w_0)$, and so for any maximum point z of $F(\mathbf{w}^*, \cdot)$ we necessarily have $z \in \mathbb{T} \setminus \{w_0, w_1, w_2, \ldots, w_n\}$. That is, for the unique local maximum points $z_{j_i} \in I_{j_i}$ with $M = m_{j_i}(\mathbf{w}^*) = F(\mathbf{w}^*, z_{j_i})$, where $i = 1, \ldots, k$, neither of these points can be endpoints of the respective I_{j_i}, and so they are all located in the interior of the respective arcs. Note that by assumption $\mathbf{w}^* \in \partial S_\sigma$, hence there are at most n non-degenerate arcs, so $k \leq n$ and the Perturbation Lemma 6.2 (c) applies. This provides us some perturbation of the node system $\mathbf{w}^*$ to a new node system $\mathbf{w}'$ with all the maxima $m_{j_i}(\mathbf{w}') < M$ $(i = 1, \ldots, k)$. As the other arcs had maxima strictly below M, and in view of continuity (Proposition 3.3), altogether we would get $\overline{m}(\mathbf{w}') < M$, a contradiction.

Therefore, it remains to settle the case when there is no $i \geq 1$ with $w_i = w_0$ (but we still have $\mathbf{w}^* \in \partial S_\sigma$). So assume that $(0 = w_0 <) w_{\sigma(j)} = \cdots = w_{\sigma(j+k)} (< 2\pi)$ is a complete list of $k+1$ coinciding nodes within $(0, 2\pi)$. As before, by condition (∞') Lemma 3.8 (b) applies providing $M > F(\mathbf{w}^*, w_{\sigma(j)})$. Consider now the perturbed system $\mathbf{w}'$ obtained from $\mathbf{w}^*$ by means of slightly pulling apart $w_{\sigma(j)}$ and $w_{\sigma(j+k)}$, i.e. taking $w'_{\sigma(j)} := w_{\sigma(j)} - h$ and $w'_{\sigma(j+k)} = w_{\sigma(j+k)} + h$ (and leaving the other nodes unchanged). Referring to Lemma 11.5 with $a = b = 1$, we obtain for small enough $h > 0$ that F is strictly decreased in $\mathbb{T} \setminus (w_{\sigma(j)} - h, w_{\sigma(j)} + h))$, whence even $\max_{\mathbb{T} \setminus (w_{\sigma(j)} - h, w_{\sigma(j)} + h)} F(\mathbf{w}', t) < M$, while in the missing interval of length $2h$ continuity of F and $M > F(\mathbf{w}^*, w_{\sigma(j)})$ entails $\max_{[w_{\sigma(j)} - h, w_{\sigma(j)} + h]} F(\mathbf{w}', t) < M$. Altogether, we are led to $\overline{m}(\mathbf{w}') < M$, a contradiction again. This proves that $\mathbf{w}^* \in S$ for some simplex.

Now, by Theorem 10.4 there is an equioscillation point $\mathbf{e} \in S$, which certainly majorizes $\mathbf{w}^*$. By Corollary 8.3 we obtain $\mathbf{w}^* = \mathbf{e}$. This proves (a). Let $\mathbf{w}_*$ be a maximum point of $\underline{m}$ in $\overline{S}$. Then, $\mathbf{w}_*$ majorizes the equioscillation point $\mathbf{w}^*$, so again Corollary 8.3 yields $\mathbf{w}^* = \mathbf{w}_*$. This proves (b) and (c). □

12. *An application: A minimax problem on the torus*

The aim of this section is to prove the next result, which generalizes Theorem 1.1 of Hardin, Kendall and Saff from [**14**] in the extent that we do not assume the kernels to be even. We also add some extra information about the extremal node system: It is the unique solution of the dual maximin problem.

COROLLARY 12.1. *Let K be any concave kernel function, and let $0 = e_0 < e_1 < \cdots < e_n$ be the equidistant node system in $\mathbb{T}$. Consider $F(\mathbf{y}, t) = K(t) + \sum_{j=1}^n K(t - y_j)$.*
(a) *For $\mathbf{e} = (e_1, \ldots, e_n)$ we have*

$$\max_{t \in \mathbb{T}} F(\mathbf{e}, t) = M = \inf_{\mathbf{y} \in \mathbb{T}^n} \max_{t \in \mathbb{T}} F(\mathbf{y}, t),$$

i.e., $\mathbf{e}$ *is a minimum point of* $\overline{m}$. *Moreover,*

$$\inf_{\mathbf{y}\in\mathbb{T}^n}\max_{j=0,\dots,n} m_j(\mathbf{y}) = M = m = \sup_{\mathbf{y}\in\mathbb{T}^n}\min_{j=0,\dots,n} m_j(\mathbf{y}).$$

(b) *If* K *is strictly concave, then* $\mathbf{e}$ *is the unique (up to permutation of the nodes) maximum point of* $\underline{m}$ *and the unique minimum point of* $\overline{m}$.

Proof. Since the permutation of the nodes is irrelevant we may restrict the consideration to the simplex $S := S_{\mathrm{id}}$, where id is the identical permutation. We have $M = M(S)$ and $m = m(S)$.

(a) Approximate K uniformly by strictly concave kernel functions $K^{(k)}$ satisfying $(\infty'_{\pm})$ (cf. Example 11.3). By Theorem 11.6, $M^{(k)} = \overline{m}^{(k)}(\mathbf{e})$ and $M^{(k)} = m^{(k)}$ and obviously we have $M^{(k)} = M^{(k)}(S)$, $m^{(k)} = m^{(k)}(S)$. By Proposition 4.3 we have $M^{(k)}(S) \to M(S) = M$, $m^{(k)}(S) \to m(S) = m$, $\underline{m}^{(k)}(\mathbf{e}) \to \underline{m}(\mathbf{e})$ and $\overline{m}^{(k)}(\mathbf{e}) \to \overline{m}(\mathbf{e})$. So $\overline{m}(\mathbf{e}) = M = M(S) = m(S) = m$.

(b) Let $\mathbf{w}^* \in \overline{S}$ be a minimum point of $\overline{m}$. If $m_j(\mathbf{w}^*) < \overline{m}(\mathbf{w}^*) = M(S)$ held for some $j \in \{0, 1, \dots, n\}$, then by an application of Lemma 11.5 (with $a = b = 1$ there) and Corollary 3.6 we could arrive at a new node system $\mathbf{w}'$ with $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w}^*)$, which is impossible. We conclude therefore that $\mathbf{w}^*$ is an equioscillation point. Since by part (a) we have $m(S) = M(S)$, the equioscillation point is unique by Proposition 8.2 (a). Hence $\mathbf{w}^* = \mathbf{e}$, and uniqueness follows. □

13. *An application: Generalized polynomials and Bojanov's result*

In this section we present an application of the previously developed theory to Chebyshev type problems for generalized polynomials and generalized trigonometric polynomials, thereby refining some results of Bojanov [**6**], see Theorem 13.2 below.

We shall use the following form of our main theorem.

THEOREM 13.1. *Suppose the kernel function* K *is strictly concave and either satisfies* (∞'), *or is in* $\mathrm{C}^1(0, 2\pi)$. *Let* $r_0, r_1, \dots, r_n > 0$, *set* $K_j := r_j K$ *and*

$$F(\mathbf{y}, t) := K_0(t) + \sum_{j=1}^{n} K_j(t - y_j) = r_0 K(t) + \sum_{j=1}^{n} r_j K(t - y_j).$$

Let $S = S_\sigma$ *be a simplex. Then there is a unique* $\mathbf{w}^* \in S$, $\mathbf{w}^* = (w_1, \dots, w_n)$ *with*

$$M(S) := \inf_{\mathbf{y}\in S}\sup_{t\in\mathbb{T}} F(\mathbf{y}, t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*, t).$$

Moreover, we have the following:

(a) *The nodes* $w_0, \dots, w_n$ *are different and* $\mathbf{w}^*$ *is an equioscillation point, i.e.,*

$$m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*).$$

(b) *We have*

$$\inf_{\mathbf{y}\in S}\max_{j=0,\dots,n}\sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t) = M(S) = m(S) = \sup_{\mathbf{y}\in S}\min_{j=0,\dots,n}\sup_{t\in I_j(\mathbf{y})} F(\mathbf{y}, t).$$

(c) *We have the Sandwich Property in* $\overline{S}$, *i.e., for each* $\mathbf{x}, \mathbf{y} \in \overline{S}$

$$\underline{m}(\mathbf{x}) \le M(S) \le \overline{m}(\mathbf{y}).$$

Proof. There is $\mathbf{w} \in \overline{S}$ with $M(S) = \sup_{t\in\mathbb{T}} F(\mathbf{w}, t)$. By Proposition 8.4 we only need to prove that $\mathbf{w}$ belongs to the interior of the simplex, i.e., $\mathbf{w} \in S$. Suppose by contradiction that $w_{\sigma(k-1)} \le w_{\sigma(k)} = w_{\sigma(k+1)} = \cdots = w_{\sigma(\ell)} < 2\pi = w_{\sigma(n+1)}$ with $k \ne \ell$, $k \in \{1, \ldots, n\}$ (the case $k = 0$ will be considered below separately). Then we can apply Lemma 11.5 with $a = \frac{1}{r_{\sigma(\ell)}}$, $b = \frac{1}{r_{\sigma(k)}}$ and $x = w_{\sigma(k)}$, $y = w_{\sigma(\ell)}$, and move the two nodes $w_{\sigma(k)}$ and $w_{\sigma(\ell)}$ away from each other, such that the new node system $\mathbf{w}'$ still belongs to S. We conclude

$$\begin{aligned}&F(\mathbf{w}', t) - F(\mathbf{w}, t)\\ &= K_{\sigma(k)}(t - w'_{\sigma(k)}) + K_{\sigma(\ell)}(t - w'_{\sigma(\ell)}) - K_{\sigma(k)}(t - w_{\sigma(k)}) - K_{\sigma(\ell)}(t - w_{\sigma(\ell)}) < 0\end{aligned}$$

for all $t \in \mathbb{T} \setminus [w'_{\sigma(k)}, w'_{\sigma(\ell)}]$. Hence we obtain

$$m_j(\mathbf{w}') < m_j(\mathbf{w}) \quad \text{for each } j \in \{0, \ldots, n\} \setminus \{\sigma(k), \ldots, \sigma(\ell-1)\}. \tag{13.1}$$

Since by Corollary 6.5 $m_{\sigma(k)}(\mathbf{w}) = m_{\sigma(k+1)}(\mathbf{w}) = \cdots = m_{\sigma(\ell-1)}(\mathbf{w}) < \overline{m}(\mathbf{w})$, if we move the two nodes $w_{\sigma(k)}$ and $w_{\sigma(\ell)}$ by a sufficiently small amount, by Corollary 3.6 we can achieve

$$m_{\sigma(k)}(\mathbf{w}'),\ m_{\sigma(k+1)}(\mathbf{w}'),\ \cdots,\ m_{\sigma(\ell-1)}(\mathbf{w}') < \overline{m}(\mathbf{w}). \tag{13.2}$$

Putting together (13.1) and (13.2), we would obtain $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, which is in contradiction with the choice of $\mathbf{w}$. If finally, w_0 happens to coincide with some $w_{\sigma(\ell)}$, then we can move w_0 and $w_{\sigma(\ell)}$ away from each other as above and obtain a new node system $w'_0 \in \mathbb{T}$, $\mathbf{w}' = (w'_1, \ldots, w'_n)$ with $\overline{m}(\mathbf{w}') < \overline{m}(\mathbf{w})$, and then we need to rotate back by w'_0.

We have seen that $\mathbf{w}^* := \mathbf{w} \in S$, therefore the proof is complete. □

Bojanov proved in [**6**] the following.

THEOREM 13.2 (Bojanov). *Let $\nu_1, \ldots, \nu_n$ be fixed positive integers. Fix $[a, b] \subset \mathbb{R}$. Then, there exists a unique system of points $a < x_1 < \ldots < x_n < b$ such that*

$$\|(x - x_1)^{\nu_1} \ldots (x - x_n)^{\nu_n}\| = \inf_{a \le y_1 < \ldots < y_n \le b} \|(x - y_1)^{\nu_1} \ldots (x - y_n)^{\nu_n}\|$$

where $\|\cdot\|$ denotes the sup-norm over $[a, b]$. The extremal polynomial

$$P^*(x) := (x - x_1)^{\nu_1} \ldots (x - x_n)^{\nu_n}$$

is uniquely characterized by the property that there exist $a = s_0 < s_1 < \ldots < s_{n-1} < s_n = b$ such that $|P^*(s_j)| = \|P^*\|$ for $j = 0, 1, \ldots, n$. Moreover, in this situation

$$P^*(s_{j+1}) = (-1)^{\nu_{j+1}} P^*(s_j) \quad \text{for } j = 0, 1, \ldots, n-1.$$

Now, we are going to establish a similar result for trigonometric polynomials and relate this new result to Bojanov's theorem.

It is well known (see e.g. [**8**] p. 19) that a trigonometric polynomial

$$T(t) = a_0 + \sum_{k=1}^{m} a_k \cos(kt) + b_k \sin(kt)$$

where $|a_m| + |b_m| > 0$, can be written in the form $T(t) = c \prod_{j=1}^{2m} \sin \frac{t - t_j}{2}$ where c, $t_1, \ldots, t_{2m}$ are numbers. More precisely, if $T(t') = 0$, $t' \in \mathbb{C}$, $\Re t' \in [0, 2\pi)$, then t' appears in $t_1, \ldots, t_{2m}$ and if $a_0, a_1, b_1, \ldots, a_m, b_m \in \mathbb{R}$ and $T(t') = 0$, $t' \in \mathbb{C} \setminus \mathbb{R}$, $\Re t' \in [0, 2\pi)$, then the conjugate of t' is also a zero, $T(\overline{t'}) = 0$ and both appear among $t_1, \ldots, t_{2m}$.

Functions of the form

$$a \prod_{j=1}^{m} \left|\sin \frac{t - t_j}{2}\right|^{r_j},$$

where $a, r_j > 0$, $t_j \in \mathbb{C}$ for all $j = 1, \ldots, m$, are called *generalized trigonometric polynomials* (GTP for short), see, e.g. [**8**] Appendix 4. The number $\frac{1}{2}\sum_{j=1}^{m} r_j$ is usually called the degree of this GTP.

In the next theorem, we describe Chebyshev type (having minimal sup norm and fixed leading coefficient) GTPs when the multiplicities of the zeros are fixed and the zeros are real. We thank Lozko Milev for drawing our attention to a result of Kristiansen (see [**17**, Thm. 2], which is also mentioned in [**7**] as Theorem B) concerning trigonometric polynomials with prescribed multiplicities of zeros. However, it is not an extremal type (minimax or maximin), but an existence and uniqueness result, for trigonometric polynomials when the local extrema are also prescribed.

THEOREM 13.3. *Let $r_0, r_1, \ldots, r_n > 0$ be fixed. Then, there exists a unique system of points $0 = w_0 < w_1 < \ldots < w_n < 2\pi$ such that*

$$\left\| \left|\sin\frac{t-w_0}{2}\right|^{r_0} \cdots \left|\sin\frac{t-w_n}{2}\right|^{r_n} \right\| = \inf_{0=y_0\le y_1<\ldots<y_n<2\pi} \left\| \left|\sin\frac{t-y_0}{2}\right|^{r_0} \cdots \left|\sin\frac{t-y_n}{2}\right|^{r_n} \right\|$$

where $\|\cdot\|$ denotes the sup-norm over $[0, 2\pi]$. The extremal GTP

$$T^*(t) := \left|\sin\frac{t-w_0}{2}\right|^{r_0} \cdots \left|\sin\frac{t-w_n}{2}\right|^{r_n}$$

is uniquely determined by properties that there exist $0 < z_0 < z_1 < z_2 < \ldots < z_n < 2\pi$ such that w_j's and z_j's interlace, i.e., $0 = w_0 < z_0 < w_1 < \ldots < w_n < z_n < w_0 + 2\pi = 2\pi$, and $T^*(z_j) = \|T^*\|$ for $j = 0, 1, \ldots, n$.

Proof. Let $K(x) := \log|\sin(x/2)|$ for $-\pi \le x \le \pi$, then extend K 2π-periodically to $\mathbb{R}$. Then K is kernel in $C^2(0, 2\pi)$ with $K'' < 0$. Let $K_j(x) := r_j K(x)$, $j = 0, 1, 2, \ldots, n$ be the kernels and consider the simplex $S := S_{\mathrm{id}}$. Further, let $T(\mathbf{y}, t) := \prod_{j=0}^{n} |\sin\frac{t-y_j}{2}|^{r_j}$ where $\mathbf{y} \in S$ and $F(\mathbf{y}, t) := \log|T(\mathbf{y}, t)|$ is a sum of translates function. Then

$$F(\mathbf{y}, t) = K_0(t) + \sum_{j=1}^{n} K_j(t - y_j) = \sum_{j=0}^{n} r_j K(x - y_j).$$

Applying Theorem 13.1, we obtain $M(S) = \inf_{\mathbf{y}\in S}\sup_{t\in[0,2\pi)} F(\mathbf{y}, t)$ is attained at exactly one point $\mathbf{w}^* = (w_1, \ldots, w_n) \in S$, i.e.,

$$M(S) = \sup_{t\in[0,2\pi)} F(\mathbf{w}^*, t) \quad \text{and} \quad \sup_{t\in[0,2\pi)} F(\mathbf{y}, t) > M(S) \quad \text{when } \mathbf{y} \ne \mathbf{w}^*.$$

Moreover, there exist $0 < z_0 < z_1 < z_2 < \ldots < z_n < 2\pi$ such that $F(\mathbf{w}^*, z_j) = M(S)$, that is, $\mathbf{w}^*$ is an equioscillation point. The interlacing property obviously follows. Rewriting these properties for $T^*(t) := \exp F(\mathbf{w}^*, t)$, we obtain the assertions. □

We turn to the interval case. Suppose the n positive real numbers $r_1, r_2, \ldots, r_n > 0$ are fixed, and consider $P(x) := |x - y_1|^{r_1} \ldots |x - y_n|^{r_n}$. Such functions are sometimes called *generalized algebraic polynomials* (GAP, see, for instance, [**8**] Appendix 4). Now, fix $[a, b] \subset \mathbb{R}$ and consider the following minimization problem

$$\inf_{a\le y_1<\ldots<y_n\le b} \sup_{x\in[a,b]} \left| |x-y_1|^{r_1} \ldots |x-y_n|^{r_n} \right|. \tag{13.3}$$

Based on this, we will investigate the problem

$$\inf \sup_{t\in[0,2\pi]} \left| \left|\sin\frac{t-t_1}{2}\right|^{r_n} \ldots \left|\sin\frac{t-t_n}{2}\right|^{r_1} \left|\sin\frac{t-t_{n+1}}{2}\right|^{r_1} \ldots \left|\sin\frac{t-t_{2n}}{2}\right|^{r_n} \right| \tag{13.4}$$

where the infimum is taken for $0 \le t_1 < \ldots < t_n < t_{n+1} < \ldots < t_{2n} < 2\pi$.

THEOREM 13.4. *With the previous notation, the infimum in* (13.5) *is attained at a unique point* $\mathbf{w}^* = (w_1, w_2, \dots, w_{2n})$ *with* $w_1 + (w_{2n} - 2\pi) = 0$ *and* $0 < w_1 < \dots < w_{2n} < 2\pi$. *Furthermore,* $\mathbf{w}^*$ *is symmetric:* $w_k = 2\pi - w_{2n+1-k}$ *for* $k = 1, 2, \dots, n$.

As a consequence the minimization in (13.4) has the same (unique) solution as

$$\inf \sup_{t\in[0,2\pi]} \left| \left|\sin \frac{t-t_1}{2}\right|^{r_n} \dots \left|\sin \frac{t-t_n}{2}\right|^{r_1} \left|\sin \frac{t-t_{n+1}}{2}\right|^{r_1} \dots \left|\sin \frac{t-t_{2n}}{2}\right|^{r_n} \right|, \tag{13.5}$$

where the infimum is taken for $0 \le t_1 < \dots < t_n < \pi$ satisfying $t_j = 2\pi - t_{2n+1-j}$, for all $j = 1, \dots, n$.

The previous theorem follows from the next, more general, symmetry theorem.

THEOREM 13.5. *Let* $K_1, \dots, K_n$ *be strictly concave kernels such that* K_j *is even for all* $j = 1, \dots, n$. *Assume that the kernels are either all in* $\mathrm{C}^1(0, 2\pi)$ *or all satisfy* (∞'). *Take the simplex* $S := \{0 \le y_1 < y_2 < \dots < y_{2n} < 2\pi\}$. *Define the symmetric sum of translates function*

$$\begin{aligned} F_{\mathrm{symm}}(\mathbf{y}, t) := K_1(t-y_1) + \dots + K_{n-1}(t-y_{n-1}) + K_n(t-y_n) \\ + K_n(t-y_{n+1}) + K_{n-1}(t-y_{n+2}) + \dots + K_1(t-y_{2n}) \end{aligned} \tag{13.6}$$

and consider the "doubled" problem

$$M_{\mathrm{symm}} := \inf_{\mathbf{y}\in S} \sup_{t\in[0,2\pi)} F_{\mathrm{symm}}(\mathbf{y}, t). \tag{13.7}$$

Then there is a unique minimum point $\mathbf{w}^* = (w_1, w_2, \dots, w_{2n}) \in S$ *with* $w_1 + (w_{2n} - 2\pi) = 0$. *Furthermore,* $\mathbf{w}^*$ *is symmetric:* $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \dots, n)$ *and there are exactly* $2n$ *points:* $0 = z_1 < z_2 < \dots < z_{n+1} = \pi < \dots < z_{2n}$ *where* $F_{\mathrm{symm}}(\mathbf{w}^*, \cdot)$ *attains its supremum. Moreover,* z_j*'s and* w_j*'s interlace and* z_j*'s are symmetric too:* $z_k = 2\pi - z_{2n+1-k}$ $(k = 1, 2, \dots, n)$.

Proof. Following the symmetric definition, we denote $K_{n+k}(t) := K_{n+1-k}(-t)$ where $k = 1, 2, \dots, n$, and by symmetry,

$$K_{n+k}(t) = K_{n+1-k}(t) \quad \text{for } k = -n+1, \dots, n. \tag{13.8}$$

Hence $F_{\mathrm{symm}}(\mathbf{y}, t) = \sum_{j=1}^{2n} K_j(t - y_j)$.

The existence and uniqueness follow from Theorem 13.1. That is, there exists a unique $\mathbf{w}^* = (w_1, w_2, \dots, w_{2n}) \in S$ (unique with $w_1 = 0$ such that $M(S) = \overline{m}(\mathbf{w}^*)$). Furthermore, $M(S) = m(S)$ and $F(\mathbf{w}^*, \cdot)$ equioscillates, hence $m(S) = \underline{m}(\mathbf{w}^*)$. Using rotation, we can assume that $w_1 > 0$ is such that $w_1 + (w_{2n} - 2\pi) = 0$.

Now, we establish $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \dots, n)$. By the assumption, it holds for $k = 1$, i.e., $w_1 = 2\pi - w_{2n}$. Reflect the w_k's: $v_k := 2\pi - w_{2n+1-k}$, $k = 1, \dots, 2n$ and write $\mathbf{v} := (v_1, \dots, v_{2n})$. Then $v_1 = w_1$ and $v_{2n} = w_{2n}$. Furthermore, put $L_k(t) := K_{2n+1-k}(-t)$ and consider

$$\tilde{F}(\mathbf{v}, t) := \sum_{k=-n+1}^{n} L_{n+k}(t - v_{n+k})$$

the sum of translates function of the reflected configuration. We obtain, using (13.8) and the even property of the kernels, that

$$\begin{aligned} L_{n+k}(t - v_{n+k}) &= K_{n+1-k}(-t + v_{n+k}) = K_{n+1-k}(t - v_{n+k}) \\ &= K_{n+1-k}(t - 2\pi + w_{n+1-k}) = K_{n+1-k}((2\pi - t) - w_{n+1-k}) \end{aligned}$$

for all $k = -n+1, \dots, n$. Hence

$$\begin{aligned}\tilde{F}(\mathbf{v}, t) &= \sum_{k=-n+1}^{n} L_{n+k}(t - v_{n+k}) = \sum_{k=-n+1}^{n} K_{n+1-k}((2\pi - t) - w_{n+1-k}) \\ &= F_{\mathrm{symm}}(\mathbf{w}^*, 2\pi - t) = F_{\mathrm{symm}}(\mathbf{w}^*, -t).\end{aligned}$$

Obviously $\mathbf{v} \in S$. By definition, $m_0(\mathbf{w}^*) = m_{2n}(\mathbf{w}^*) = \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, t) : w_{2n} - 2\pi \le t \le w_1\}$ and $m_j(\mathbf{w}^*) = \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, t) : w_j \le t \le w_{j+1}\}$, $j = 1, \dots, 2n-1$, and similarly for $\mathbf{v}$, $m_j(\mathbf{v}) = \sup\{\tilde{F}(\mathbf{v}, t) : v_j \le t \le v_{j+1}\}$, $j = 1, \dots, 2n-1$ and

$$m_0(\mathbf{v}) = m_{2n}(\mathbf{v}) = \sup\{\tilde{F}(\mathbf{v}, t) : v_{2n} - 2\pi \le t \le v_1\}.$$

Hence, we also have for $j = 1, \dots, 2n-1$

$$\begin{aligned}m_j(\mathbf{w}^*) &= \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, t) : w_j \le t \le w_{j+1}\} \\ &= \sup\{F_{\mathrm{symm}}(\mathbf{w}^*, -t) : -w_{j+1} \le t \le -w_j\} = \sup\{\tilde{F}(\mathbf{v}, t) : -w_{j+1} \le t \le -w_j\} \\ &= \sup\{\tilde{F}(\mathbf{v}, t) : 2\pi - w_{j+1} \le t \le 2\pi - w_j\} = \sup\{\tilde{F}(\mathbf{v}, t) : v_{2n-j} \le t \le v_{2n+1-j}\} \\ &= m_{2n-j}(\mathbf{v}),\end{aligned}$$

and obviously $m_0(\mathbf{v}) = m_{2n}(\mathbf{v}) = m_0(\mathbf{w}^*) = m_{2n}(\mathbf{w}^*)$. This implies that $\underline{m}(\mathbf{v}) = \underline{m}(\mathbf{w}^*)$. Indirectly, suppose $\mathbf{v} \ne \mathbf{w}^*$. We use Proposition 7.2 hence the strict concavity of $\underline{m}$ implies that there is an $\mathbf{a} = (a_1, \dots, a_{2n}) \in S$ such that $\underline{m}(\mathbf{a}) > \underline{m}(\mathbf{w}^*)$. But this contradicts that $\underline{m}(\mathbf{w}^*) = m(S) = \sup_{\mathbf{y} \in S} \underline{m}(\mathbf{y})$. Therefore $\mathbf{v} = \mathbf{w}^*$, hence $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \dots, n)$.

The symmetry of the w_k's implies the remaining assertions (interlacing and symmetry of the z_j's). □

We connect the "algebraic" problem (13.3) and the "trigonometric" problem (13.5) by using a classical idea of transferring between these situations with $x = \cos t$ (see, e.g., [**25**]).

Lemma 13.6. *Let $L(x) := \frac{b-a}{2}x + \frac{b+a}{2}$. The identities*

$$y_j = L(\cos t_{n+1-j}), \quad t_{n+1-j} = \arccos L^{-1}(y_j), \quad t_{n+j} = 2\pi - \arccos L^{-1}(y_j) \tag{13.9}$$

for $j = 1, \dots, n$ provide a one-to-one correspondence between generalized algebraic polynomials in (13.3) *and generalized trigonometric polynomials in* (13.5)*. Similarly, for the corresponding interlacing points of maxima we have $s_j = L(\cos z_{n+1-j})$, $z_{n+1-j} = \arccos L^{-1}(s_j)$ and $z_{n+j} = 2\pi - \arccos L^{-1}(s_j)$ for $j = 0, \dots, n$.*

Proof. For simplicity, assume that $a = -1$, $b = 1$, hence $L(x) = x$. Recall

$$\sin\frac{t-\alpha}{2}\sin\frac{t+\alpha-2\pi}{2} = \frac{1}{2}(\cos t - \cos\alpha) \tag{13.10}$$

hence

$$\begin{aligned}&\left|\sin\frac{t-t_1}{2}\right|^{r_n} \cdots \left|\sin\frac{t-t_n}{2}\right|^{r_1} \left|\sin\frac{t+t_n-2\pi}{2}\right|^{r_1} \cdots \left|\sin\frac{t+t_1-2\pi}{2}\right|^{r_n} \\ &\qquad = \frac{1}{2^{\sum_{j=1}^n r_j}} |\cos t - \cos t_1|^{r_n} \cdots |\cos t - \cos t_n|^{r_1}.\end{aligned} \tag{13.11}$$

Therefore, for every GAP $P(x) = |x - y_1|^{r_1} \dots |x - y_n|^{r_n}$ there is a GTP $T(t)$ of the form as in (13.5) such that $P(\cos t) = 2^{-\sum_{j=1}^n r_j} T(t)$. Also to every GTP $T(t)$ as appearing in (13.5), there is a corresponding GAP as in (13.3) (modulo a constant factor), where between the zeros t_j, t_{n+1-j} and y_j $(j = 1, \dots, n)$ the asserted relations (13.9) hold and $P(\cos t) = 2^{-\sum_{j=1}^n r_j} T(t)$. The statement about the points of maxima is now obvious. □

From this the following generalization of Bojanov's result can be deduced immediately:

THEOREM 13.7. *Let $\nu_1, \ldots, \nu_n > 0$ be fixed, and let $[a, b] \subset \mathbb{R}$. Then, there exists a unique system of points $a < x_1 < \ldots < x_n < b$ such that*

$$\| |x - x_1|^{\nu_1} \ldots |x - x_n|^{\nu_n} \| = \inf_{a \le y_1 < \ldots < y_n \le b} \| |x - y_1|^{\nu_1} \ldots |x - y_n|^{\nu_n} \|$$

where $\| \cdot \|$ denotes the sup-norm over $[a, b]$. The extremal generalized polynomial

$$P^*(x) := |x - x_1|^{\nu_1} \ldots |x - x_n|^{\nu_n}$$

is uniquely characterized by the existence of $a = s_0 < s_1 < \ldots < s_{n-1} < s_n = b$ with $|P^*(s_j)| = \|P^*\|$ for $j = 0, 1, \ldots, n$.

References

1. G. Ambrus, *Analytic and probabilistic problems in discrete geometry*, Ph.D. thesis, University College London, 2009.
2. Gergely Ambrus, Keith M. Ball, and Tamás Erdélyi, *Chebyshev constants for the unit circle*, Bull. Lond. Math. Soc. **45** (2013), no. 2, 236–248. MR 3064410
3. Daniel Azagra, *Global and fine approximation of convex functions*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 4, 799–824. MR 3108831
4. Abraham Berman and Robert J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Classics in Applied Mathematics, vol. 9, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, Revised reprint of the 1979 original. MR 1298430 (95e:15013)
5. S.N. Bernstein, *Sur la limitation des valeurs d'un polynome $p_n(x)$ de degree n sur tout un segment par ses valeurs en $(n + 1)$-points du segment*, Izv. Akad. Nauk SSSR **7** (1931), 1025–1050.
6. B. D. Bojanov, *A generalization of Chebyshev polynomials*, J. Approx. Theory **26** (1979), no. 4, 293–300. MR 550677 (81a:41011)
7. Borislav Bojanov, *Turán's inequalities for trigonometric polynomials*, J. London Math. Soc. (2) **53** (1996), no. 3, 539–550. MR 1396717
8. Peter Borwein and Tamás Erdélyi, *Polynomials and polynomial inequalities*, Graduate Texts in Mathematics, vol. 161, Springer-Verlag, New York, 1995. MR 1367960 (97e:41001)
9. Carl de Boor and Allan Pinkus, *Proof of the conjectures of Bernstein and Erdős concerning the optimal nodes for polynomial interpolation*, J. Approx. Theory **24** (1978), no. 4, 289–303. MR 523978 (80d:41003)
10. Tamás Erdélyi and Edward B. Saff, *Riesz polarization inequalities in higher dimensions*, J. Approx. Theory **171** (2013), 128–147. MR 3053720
11. P. Erdős, *Some remarks on polynomials*, Bull. Amer. Math. Soc. **53** (1947), 1169–1176. MR 0022942
12. P. C. Fenton, *A min-max theorem for sums of translates of a function*, J. Math. Anal. Appl. **244** (2000), no. 1, 214–222.
13. I. Ja. Guberman, *On the uniform convergence of convex functions in a closed region*, Izv. Vysš. Učebn. Zaved. Matematika **1965** (1965), no. 3 (46), 61–73. MR 0183835
14. Douglas P. Hardin, Amos P. Kendall, and Edward B. Saff, *Polarization Optimality of Equally Spaced Points on the Circle for Discrete Potentials*, Discrete Comput. Geom. **50** (2013), no. 1, 236–243. MR 3070548
15. T. A. Kilgore, *Optimization of the norm of the Lagrange interpolation operator*, Bull. Amer. Math. Soc. **83** (1977), no. 5, 1069–1071. MR 0437985 (55 #10906)
16. Theodore A. Kilgore, *A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm*, J. Approx. Theory **24** (1978), no. 4, 273–288. MR 523977 (80d:41002)
17. G. K. Kristiansen, *Characterization of polynomials by means of their stationary values*, Arch. Math. (Basel) **43** (1984), no. 1, 44–48. MR 758339
18. H. N. Mhaskar and E. B. Saff, *Where does the sup norm of a weighted polynomial live? (A generalization of incomplete polynomials)*, Constr. Approx. **1** (1985), no. 1, 71–91. MR 766096
19. J. M. Ortega and W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, Classics in Applied Mathematics, vol. 30, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000, Reprint of the 1970 original. MR 1744713
20. Richard S. Palais, *Natural operations on differential forms*, Trans. Amer. Math. Soc. **92** (1959), 125–141. MR 0116352
21. A. Wayne Roberts and Dale E. Varberg, *Convex functions*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973, Pure and Applied Mathematics, Vol. 57. MR 0442824
22. Edward B. Saff and Vilmos Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 316, Springer-Verlag, Berlin, 1997, Appendix B by Thomas Bloom. MR 1485778

23. Ying Guang Shi, *A minimax problem admitting the equioscillation characterization of Bernstein and Erdős*, J. Approx. Theory **92** (1998), no. 3, 463–471. MR 1609190 (99a:41003)

24. J. Szabados and P. Vértesi, *Interpolation of functions*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1990. MR 1089431 (92j:41009)

25. Gábor Szegő, *On a problem of the best approximation*, Abh. Math. Sem. Univ. Hamburg **27** (1964), 193–198. MR 0168976 (29 #6231)

26. Brian S. Thomson, Judith B. Bruckner, and Andrew M. Bruckner, *Elementary real analysis*, ClassicalRealAnalysis.com, 2008.

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