Iterated limits for aggregation of randomized INAR(1) processes with Poisson innovations

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Abstract

We discuss joint temporal and contemporaneous aggregation of N independent copies of strictly stationary INteger-valued AutoRegressive processes of order 1 (INAR(1)) with random coefficient $\alpha \in (0,1)$ and with idiosyncratic Poisson innovations. Assuming that α has a density function of the form $\psi(x)(1-x)^\beta$, $x \in (0,1)$, with $\lim_{x\uparrow 1}\psi(x) = \psi_1 \in (0,\infty)$, different limits of appropriately centered and scaled aggregated partial sums are shown to exist for $\beta \in (-1,0)$, $\beta = 0$, $\beta \in (0,1)$ or $\beta \in (1,\infty)$, when taking first the limit as $N \to \infty$ and then the time scale $n \to \infty$, or vice versa. In fact, we give a partial solution to an open problem of Pilipauskaitė and Surgailis [23] by replacing the random-coefficient AR(1) process with a certain randomized INAR(1) process.

1 Introduction

The aggregation problem is concerned with the relationship between individual (micro) behavior and aggregate (macro) statistics. There exist different types of aggregation. The scheme of contemporaneous (also called cross-sectional) aggregation of random-coefficient AR(1) models was firstly proposed by Robinson [28] and Granger [10] in order to obtain the long memory phenomena in aggregated time series. See also Gonçalves and Gouriéroux [9], Zaffaroni [36], Oppenheim and Viano [22], Celov et al. [5] and Beran et al. [4] on the aggregation of more general time-series models with finite variance. Puplinskaitė and Surgailis [26, 27] discussed aggregation of random-coefficient AR(1) processes with infinite variance and innovations in the domain of attraction of a stable law. Related problems for some network traffic models were studied in Willinger et al. [35], Taqqu et al. [33], Gaigalas and Kaj [8] and Dombry and Kaj

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[6], where independent and centered ON/OFF processes are aggregated, in Mikosch et al. [19], where aggregation of $\mathrm{M/G/\infty}$ queues with heavy-tailed activity periods are investigated, in Pipiras et al. [25], where integrated renewal or renewal-reward processes are considered, or in Iglói and Terdik [11], where the limit behavior of the aggregate of certain random-coefficient Ornstein–Uhlenbeck processes is examined. On page 512 in Jirak [13] one can find a lot of references for papers dealing with the aggregation of continuous time stochastic processes.

The present paper extends some of the results in Pilipauskaitė and Surgailis [23], which discusses the limit behavior of sums

$$S_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(j)}, \qquad t \in [0,\infty), \qquad N, n \in \{1,2,\ldots\}, \tag{1.1}$$

where $(X_k^{(j)})_{k\in\{0,1,\ldots\}}$, $j \in \{1,2,\ldots\}$, are independent copies of a stationary random-coefficient AR(1) process

$$X_k = aX_{k-1} + \varepsilon_k, \qquad k \in \{1,2,\ldots\}, \tag{1.2}$$

with standardized independent and identically distributed (i.i.d.) innovations $(\varepsilon_k)_{k\in\{1,2,\ldots\}}$ having $\mathbb{E}(\varepsilon_1) = 0$ and $\mathrm{Var}(\varepsilon_1) = 1$, and a random coefficient a with values in $[0,1)$, being independent of $(\varepsilon_k)_{k\in\{1,2,\ldots\}}$ and admitting a probability density function of the form

$$\psi(x)(1-x)^\beta, \qquad x \in [0,1), \tag{1.3}$$

where $\beta \in (-1,\infty)$ and ψ is an integrable function on $[0,1)$ having a limit $\lim_{x\uparrow 1} \psi(x) = \psi_1 > 0$. Here the distribution of X_0 is chosen as the unique stationary distribution of the model (1.2). Its existence was shown in Puplinskaitė and Surgailis [26, Proposition 1]. We point out that they considered so-called idiosyncratic innovations, i.e., the innovations $(\varepsilon_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, belonging to $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, are independent. In [23] they derived scaling limits of the finite dimensional distributions of $(A_{N,n}^{-1} S_t^{(N,n)})_{t\in[0,\infty)}$, where $A_{N,n}$ are some scaling factors and first $N \to \infty$ and then $n \to \infty$, or vice versa, or both N and n increase to infinity, possibly with different rates. Very recently, Pilipauskaitė and Surgailis [24] extended their results in [23] from the case of idiosyncratic innovations to the case of common innovations, i.e., when $(\varepsilon_k^{(j)})_{k\in\mathbb{Z}_+} = (\varepsilon_k^{(1)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$.

The aim of the present paper is to extend the results of Pilipauskaitė and Surgailis [23, Theorem 2.1] concerning iterated scaling limits to the case of certain randomized first-order Integer-valued AutoRegressive (INAR(1)) processes. The theory and application of integer-valued time series models are rapidly developing and important topics, see, e.g., Steutel and van Harn [31] and Weiß [34]. The INAR(1) process is among the most fertile integer-valued time series models, and it was first introduced by McKenzie [18] and Al-Osh and Alzaid [1]. An INAR(1) time series model is a stochastic process $(X_k)_{k\in\{0,1,\ldots\}}$ satisfying the recursive equation

$$X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \{1,2,\ldots\}, \tag{1.4}$$

where $(\varepsilon_k)_{k\in\{1,2,\dots\}}$ are i.i.d. non-negative integer-valued random variables, $(\xi_{k,j})_{k,j\in\{1,2,\dots\}}$ are i.i.d. Bernoulli random variables with mean $\alpha \in [0,1]$, and X_0 is a non-negative integer-valued random variable such that X_0, $(\xi_{k,j})_{k,j\in\{1,2,\dots\}}$ and $(\varepsilon_k)_{k\in\{1,2,\dots\}}$ are independent. By using the binomial thinning operator $\alpha\circ$ due to Steutel and van Harn [31], the INAR(1) model in (1.4) can be written as

$$X_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \{1,2,\dots\}, \tag{1.5}$$

which form captures the resemblance with the AR model. We note that an INAR(1) process can also be considered as a special branching process with immigration having Bernoulli offspring distribution.

Leonenko et al. [16] introduced the aggregation $\sum_{j=1}^{\infty} X^{(j)}$ of a sequence of independent stationary INAR(1) processes $X^{(j)}$, $j \in \mathbb{N}$, where $X_k^{(j)} = \alpha^{(j)} \circ X_{k-1}^{(j)} + \varepsilon_k^{(j)}$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Under appropriate conditions on $\alpha^{(j)}$, $j \in \mathbb{N}$, and on the distributions of $\varepsilon^{(j)}$, $j \in \mathbb{N}$, they showed that the process $\sum_{j=1}^{\infty} X^{(j)}$ is well-defined in L^2-sense and it has long memory.

We will consider a certain randomized INAR(1) process $(X_k)_{k\in\mathbb{Z}_+}$ with randomized thinning parameter α, given formally by the recursive equation

$$X_k = \alpha \circ X_{k-1} + \varepsilon_k, \qquad k \in \{1,2,\dots\}, \tag{1.6}$$

where α is a random variable with values in $(0,1)$ and X_0 is some appropriate random variable. This means that, conditionally on α, the process $(X_k)_{k\in\mathbb{Z}_+}$ is an INAR(1) process with thinning parameter α. Conditionally on α, the i.i.d. innovations $(\varepsilon_k)_{k\in\{1,2,\dots\}}$ are supposed to have a Poisson distribution with parameter $\lambda \in (0,\infty)$, and the conditional distribution of the initial value X_0 given α is supposed to be the unique stationary distribution, namely, a Poisson distribution with parameter $\lambda/(1-\alpha)$. For a rigorous construction of this process, see Section 4. Here we only note that $(X_k)_{k\in\mathbb{Z}_+}$ is a strictly stationary sequence, but it is not even a Markov chain (so it is not an INAR(1) process) if α is not degenerate, see Appendix A. Let us also remark that the choice of Poisson-distributed innovations serves a technical purpose. It allows us to calculate and use the explicit stationary distribution and the joint generator function given in (2.4). The authors are planning to try releasing this assumption and giving more general results in future research.

Note that there is another way of randomizing the INAR(1) model (1.5), the so-called random-coefficient INAR(1) process (RCINAR(1)), proposed by Zheng et al. [37] and Leonenko et al. [16]. It differs from (1.6), namely, it is a process formally given by the recursive equation

$$X_k = \alpha_k \circ X_{k-1} + \varepsilon_k, \qquad k \in \{1,2,\dots\},$$

where $(\alpha_k)_{k\in\{1,2,\dots\}}$ is an i.i.d. sequence of random variables with values in $[0,1]$. An RCINAR(1) process can be considered as a special kind of branching processes with immigration in a random environment, see Key [15], where a rigorous construction is given on the state space of the so-called genealogical trees.

In the paper first we examine a strictly stationary INAR(1) process (1.5) with deterministic thinning and Poisson innovation, and in Section 2 an explicit formula is given for the joint generator function of $(X_0, X_1, \ldots, X_k)$, $k \in \{0, 1, \ldots\}$. In Section 3 we consider independent copies of this stationary INAR(1) process supposing idiosyncratic Poisson innovations. Applying the natural centering by the expectation, in Propositions 3.1, 3.2 and in Theorem 3.3, we derive scaling limits for the contemporaneously, the temporally and the joint temporally and contemporaneously aggregated processes, respectively. In Section 4 first we give a construction of the stationary randomized INAR(1) process (1.6). Considering independent copies of this randomized INAR(1) process, we discuss the limit behavior of the temporal and contemporaneous aggregation of these processes, both with centering by the expectation and by the conditional expectation, see Propositions 4.1–4.4. Then, assuming that the distribution of α has the form (1.3), we prove iterated limit theorems for the joint temporally and contemporaneously aggregated processes in case of both centralizations, see Theorems 4.7–4.13. As a consequence of our results, we formulate limit theorems with centering by the empirical mean as well, see Corollary 4.14. Note that we have separate results for the different ranges of β (namely, $\beta \in (-1, 0)$, $\beta = 0$, $\beta \in (0, 1)$ and $\beta \in (1, \infty)$), the different orders of the iterations, and the different centralizations. The case $\beta = 1$ is not covered in this paper, nor in Pilipauskaitė and Surgailis [23] for the random coefficient AR(1) processes. We discuss this case for both models in Nedényi and Pap [21]. Section 5 contains the proofs. In the appendices we discuss the non-Markov property of the randomized INAR(1) model, some approximations of the exponential function and some of its integrals, and an integral representation of the fractional Brownian motion due to Pilipauskaitė and Surgailis [23]. We consider three kinds of centralizations (by the conditional and the unconditional expectations and by the empirical mean). In Pilipauskaitė and Surgailis [23] centralization does not appear since they aggregate centered processes. In Jirak [13] the role of centralizations by the conditional and the unconditional expectations is discussed, where an asymptotic theory of aggregated linear processes is developed, and the limit distribution of a large class of linear and nonlinear functionals of such processes are determined.

All in all, we have similar limit theorems for randomized INAR(1) processes that Pilipauskaitė and Surgailis [23, Theorem 2.1] have for random coefficients AR(1) processes. On page 1014, Pilipauskaitė and Surgailis [23] formulated an open problem that concerns possible existence and description of limit distribution of the double sum (1.1) for general i.i.d. processes $(X_t^{(j)})_{t \in \mathbb{R}_+}$, $j \in \mathbb{N}$. We solve this open problem for some randomized INAR(1) processes. Since INAR(1) processes are special branching processes with immigration, based on our results, later on, one may proceed with general branching processes with immigration. The techniques of our proofs differ from those of Pilipauskaitė and Surgailis [23] in many cases, for a somewhat detailed comparison, see the beginning of Section 5.

2 Generator function of finite-dimensional distributions of Galton–Watson branching processes with immigration

Let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{C}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers, and complex numbers, respectively. The Borel σ-field on $\mathbb{R}$ is denoted by $\mathcal{B}(\mathbb{R})$. Every random variable in this section will be defined on a fixed probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

For each $k, j \in \mathbb{Z}_+$, the number of individuals in the k^{th} generation will be denoted by X_k, the number of offsprings produced by the j^{th} individual belonging to the $(k-1)^{\text{th}}$ generation will be denoted by $\xi_{k,j}$, and the number of immigrants in the k^{th} generation will be denoted by ε_k. Then we have

$$X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \mathbb{N},$$

where we define $\sum_{j=1}^{0} := 0$. Here $\big\{X_0,\, \xi_{k,j},\, \varepsilon_k : k, j \in \mathbb{N}\big\}$ are supposed to be independent nonnegative integer-valued random variables. Moreover, $\{\xi_{k,j} : k, j \in \mathbb{N}\}$ and $\{\varepsilon_k : k \in \mathbb{N}\}$ are supposed to consist of identically distributed random variables, respectively.

Let us introduce the generator functions

$$F_k(z) := \mathbb{E}(z^{X_k}), \quad k \in \mathbb{Z}_+, \qquad G(z) := \mathbb{E}(z^{\xi_{1,1}}), \qquad H(z) := \mathbb{E}(z^{\varepsilon_1})$$

for $z \in D := \{z \in \mathbb{C} : |z| \leqslant 1\}$. First we observe that for each $k \in \mathbb{N}$, the conditional generator function $\mathbb{E}(z_k^{X_k} \,|\, X_{k-1})$ of X_k given X_{k-1} takes the form

$$\mathbb{E}(z_k^{X_k} \,|\, X_{k-1}) = \mathbb{E}\Big(z_k^{\sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k} \,\Big|\, X_{k-1}\Big) = \mathbb{E}(z_k^{\varepsilon_k}) \prod_{j=1}^{X_{k-1}} \mathbb{E}(z_k^{\xi_{k,j}}) = H(z_k)\, G(z_k)^{X_{k-1}} \tag{2.1}$$

for $z_k \in D$, where we define $\prod_{j=1}^{0} := 1$. The aim of the following discussion is to calculate the joint generator functions of the finite dimensional distributions of $(X_k)_{k \in \mathbb{Z}_+}$. Using (2.1), we also have the recursion

$$F_k(z) = \mathbb{E}(\mathbb{E}(z^{X_k} \,|\, X_{k-1})) = \mathbb{E}(H(z)\, G(z)^{X_{k-1}}) = H(z)\, \mathbb{E}\big(G(z)^{X_{k-1}}\big) = H(z)\, F_{k-1}(G(z))$$

for $z \in D$ and $k \in \mathbb{N}$. Put $G_{(0)}(z) := z$ and $G_{(1)}(z) := G(z)$ for $z \in D$, and introduce the iterates $G_{(k+1)}(z) := G_{(k)}(G(z))$, $z \in D$, $k \in \mathbb{N}$. The above recursion yields

$$F_k(z) = H(z)\, H(G(z)) \cdots H(G_{(k-1)}(z))\, F_0(G_{(k)}(z)) = F_0(G_{(k)}(z)) \prod_{j=0}^{k-1} H(G_{(j)}(z))$$

for $z \in D$ and $k \in \mathbb{N}$. Supposing that $\mathbb{E}(\xi_{1,1}) = G'(1-) < 1$, $0 < \mathbb{P}(\xi_{1,1} = 0) < 1$, $0 < \mathbb{P}(\xi_{1,1} = 1)$ and $0 < \mathbb{P}(\varepsilon_1 = 0) < 1$, the Markov chain $(X_k)_{k \in \mathbb{Z}_+}$ is irreducible and aperiodic. Further, it is ergodic (positive recurrent) if and only if $\sum_{\ell=1}^\infty \log(\ell)\,\mathbb{P}(\varepsilon_1 = \ell) < \infty$, and in this case the unique stationary distribution has the generator function

$$\widetilde{F}(z) = \prod_{j=0}^\infty H(G_{(j)}(z)), \qquad z \in D,$$

see, e.g., Seneta [29, Chapter 5] and Foster and Williamson [7, Theorem, part (iii)].

Consider the special case with Bernoulli offspring and Poisson immigration distributions, namely,

$$\begin{aligned} &\mathbb{P}(\xi_{1,1} = 1) = \alpha = 1 - \mathbb{P}(\xi_{1,1} = 0), \\ &\mathbb{P}(\varepsilon_1 = \ell) = \frac{\lambda^\ell}{\ell!}\mathrm{e}^{-\lambda}, \qquad \ell \in \mathbb{Z}_+, \end{aligned} \tag{2.2}$$

with $\alpha \in (0,1)$ and $\lambda \in (0,\infty)$. With the special choices (2.2), the Galton–Watson process with immigration $(X_k)_{k\in\mathbb{Z}_+}$ is an INAR(1) process with Poisson innovations. Then

$$G(z) = 1 - \alpha + \alpha z, \qquad H(z) = \sum_{\ell=0}^\infty \frac{z^\ell \lambda^\ell}{\ell!}\mathrm{e}^{-\lambda} = \mathrm{e}^{\lambda(z-1)}, \qquad z \in \mathbb{C},$$

hence

$$G_{(j)}(z) = 1 - \alpha^j + \alpha^j z, \qquad z \in \mathbb{C}, \quad j \in \mathbb{N}.$$

Indeed, by induction, for all $j \in \mathbb{Z}_+$,

$$G_{(j+1)}(z) = G(G_{(j)}(z)) = \alpha G_{(j)}(z) + 1 - \alpha = \alpha(1 - \alpha^j + \alpha^j z) + 1 - \alpha = 1 - \alpha^{j+1} + \alpha^{j+1} z.$$

Since $\mathbb{E}(\xi_{1,1}) = G'(1-) = \alpha \in (0,1)$, $\mathbb{P}(\xi_{1,1} = 0) = 1 - \alpha \in (0,1)$, $\mathbb{P}(\xi_{1,1} = 1) = \alpha > 0$, $\mathbb{P}(\varepsilon_1 = 0) = \mathrm{e}^{-\lambda} \in (0,1)$, and

$$\sum_{\ell=1}^\infty \log(\ell)\frac{\lambda^\ell}{\ell!}\mathrm{e}^{-\lambda} \leqslant \sum_{\ell=1}^\infty \ell\frac{\lambda^\ell}{\ell!}\mathrm{e}^{-\lambda} = \mathbb{E}(\varepsilon_1) = \lambda < \infty,$$

the Markov chain $(X_k)_{k\in\mathbb{Z}_+}$ has a unique stationary distribution admitting a generator function of the form

$$\widetilde{F}(z) = \prod_{j=0}^\infty \mathrm{e}^{\alpha^j\lambda(z-1)} = \mathrm{e}^{(1-\alpha)^{-1}\lambda(z-1)}, \qquad z \in \mathbb{C},$$

thus it is a Poisson distribution with expectation $(1-\alpha)^{-1}\lambda$.

Suppose now that the initial distribution is a Poisson distribution with expectation $(1-\alpha)^{-1}\lambda$, hence the Markov chain $(X_k)_{k\in\mathbb{Z}_+}$ is strictly stationary and

$$F_0(z_0) = \mathbb{E}(z_0^{X_0}) = \mathrm{e}^{(1-\alpha)^{-1}\lambda(z_0-1)}, \qquad z_0 \in \mathbb{C}. \tag{2.3}$$

By induction, one can derive the following result, formulae for the joint generator function of $(X_0, X_1, \ldots, X_k)$, $k \in \mathbb{Z}_+$.

2.1 Proposition. *Under (2.2) and supposing that the distribution of* X_0 *is Poisson distribution with expectation* $(1-\alpha)^{-1}\lambda$, *the joint generator function of* $(X_0, X_1, \dots, X_k)$, $k \in \mathbb{Z}_+$, *takes the form*

$$
\begin{aligned}
F_{0,\dots,k}(z_0,\dots,z_k) &:= \mathbb{E}(z_0^{X_0} z_1^{X_1} \cdots z_k^{X_k}) \\
&= \exp\left\{ \frac{\lambda}{1-\alpha} \sum_{0 \leqslant i \leqslant j \leqslant k} \alpha^{j-i}(z_i - 1) z_{i+1} \cdots z_{j-1}(z_j - 1) \right\}
\end{aligned} \tag{2.4}
$$

for all $k \in \mathbb{N}$ *and* $z_0, \dots, z_k \in \mathbb{C}$, *where, for* $i = j$, *the term in the sum above is* $z_i - 1$. *Alternatively, one can write up the joint generator function as*

$$
F_{0,\dots,k}(z_0,\dots,z_k) = \exp\left\{ \lambda \sum_{0 \leqslant i \leqslant j \leqslant k} (1-\alpha)^{K_{i,j,k}} \alpha^{j-i}(z_i z_{i+1} \cdots z_j - 1) \right\}, \tag{2.5}
$$

where

$$
K_{i,j,k} := \begin{cases} -1 & \text{if } i = 0 \text{ and } j = k, \\ 0 & \text{if } i = 0 \text{ and } 0 \leqslant j \leqslant k-1, \\ 0 & \text{if } 1 \leqslant i \leqslant k \text{ and } j = k, \\ 1 & \text{if } 1 \leqslant i \leqslant j \leqslant k-1. \end{cases}
$$

2.2 Remark. Under the conditions of Proposition 2.1, the distribution of (X_0, X_1) can be represented using independent Poisson distributed random variables. Namely, if U, V and W are independent Poisson distributed random variables with parameters $\lambda(1-\alpha)^{-1}\alpha$, λ and λ, respectively, then $(X_0, X_1) \stackrel{\mathcal{D}}{=} (U + V, U + W)$. Indeed, for all $z_0, z_1 \in \mathbb{C}$,

$$
\begin{aligned}
\mathbb{E}(z_0^{U+V} z_1^{U+W}) &= \mathbb{E}((z_0 z_1)^U z_0^V z_1^W) = \mathbb{E}((z_0 z_1)^U)\, \mathbb{E}(z_0^V)\, \mathbb{E}(z_1^W) \\
&= \mathrm{e}^{\lambda(1-\alpha)^{-1}\alpha(z_0 z_1 - 1)} \mathrm{e}^{\lambda(z_0 - 1)} \mathrm{e}^{\lambda(z_1 - 1)},
\end{aligned}
$$

as desired. Further, note that formula (2.5) shows that $(X_0, \dots, X_k)$ has a $(k+1)$-variate Poisson distribution, see, e.g., Johnson et al. [14, (37.85)]. □

3 Iterated aggregation of INAR(1) processes with Poisson innovations

Let $(X_k)_{k \in \mathbb{Z}_+}$ be an INAR(1) process with offspring and immigration distributions given in (2.2) and with initial distribution given in (2.3), hence the process is strictly stationary. Let $X^{(j)} = (X_k^{(j)})_{k \in \mathbb{Z}_+}$, $j \in \mathbb{N}$, be a sequence of independent copies of the stationary INAR(1) process $(X_k)_{k \in \mathbb{Z}_+}$.

First we consider a simple aggregation procedure. For each $N \in \mathbb{N}$, consider the stochastic process $S^{(N)} = (S_k^{(N)})_{k \in \mathbb{Z}_+}$ given by

$$
S_k^{(N)} := \sum_{j=1}^{N} (X_k^{(j)} - \mathbb{E}(X_k^{(j)})), \qquad k \in \mathbb{Z}_+, \tag{3.1}
$$

where $\mathbb{E}(X_k^{(j)}) = \lambda(1-\alpha)^{-1}$, $k \in \mathbb{Z}_+$, $j \in \mathbb{N}$, since the stationary distribution is Poisson with expectation $(1-\alpha)^{-1}\lambda$. We will use $\stackrel{\mathcal{D}_\mathrm{f}}{\longrightarrow}$ or $\mathcal{D}_\mathrm{f}$-lim for the weak convergence of the finite dimensional distributions, and $\stackrel{\mathcal{D}}{\longrightarrow}$ for the weak convergence of stochastic processes with sample paths in $D(\mathbb{R}_+, \mathbb{R})$, where $D(\mathbb{R}_+, \mathbb{R})$ denotes the space of real-valued càdlàg functions defined on $\mathbb{R}_+$. The almost sure convergence is denoted by $\stackrel{\mathrm{a.s.}}{\longrightarrow}$.

3.1 Proposition. *We have*

$$N^{-\frac{1}{2}} S^{(N)} \stackrel{\mathcal{D}_\mathrm{f}}{\longrightarrow} \mathcal{X} \qquad \text{as } N \to \infty,$$

where $\mathcal{X} = (\mathcal{X}_k)_{k\in\mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\mathcal{X}_0 \mathcal{X}_k) = \operatorname{Cov}(X_0, X_k) = \frac{\lambda\alpha^k}{1-\alpha}, \qquad k \in \mathbb{Z}_+. \tag{3.2}$$

3.2 Proposition. *We have*

$$\left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} S_k^{(1)} \right)_{t\in\mathbb{R}_+} = \left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} (X_k^{(1)} - \mathbb{E}(X_k^{(1)})) \right)_{t\in\mathbb{R}_+} \stackrel{\mathcal{D}}{\longrightarrow} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B$$

as $n \to \infty$, *where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion.*

Note that Propositions 3.1 and 3.2 are about the scaling of the space-aggregated process $S^{(N)}$ and the time-aggregated process $\left(\sum_{k=1}^{\lfloor nt \rfloor} S_k^{(1)}\right)_{t\in\mathbb{R}_+}$, respectively.

For each $N, n \in \mathbb{N}$, consider the stochastic process $S^{(N,n)} = (S_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$S_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} (X_k^{(j)} - \mathbb{E}(X_k^{(j)})), \qquad t \in \mathbb{R}_+. \tag{3.3}$$

3.3 Theorem. *We have*

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} S^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} S^{(N,n)} = \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B,$$

where $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion.*

4 Iterated aggregation of randomized INAR(1) processes with Poisson innovations

Let $\lambda \in (0, \infty)$, and let $\mathbb{P}_\alpha$ be a probability measure on $(0, 1)$. Then there exist a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a random variable α with distribution $\mathbb{P}_\alpha$ and random variables

$\{X_0, \xi_{k,j}, \varepsilon_k : k, j \in \mathbb{N}\}$, conditionally independent given α on $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$\mathbb{P}(\xi_{k,j} = 1 \,|\, \alpha) = \alpha = 1 - \mathbb{P}(\xi_{k,j} = 0 \,|\, \alpha), \qquad k, j \in \mathbb{N}, \tag{4.1}$$

$$\mathbb{P}(\varepsilon_k = \ell \,|\, \alpha) = \frac{\lambda^\ell}{\ell!}\mathrm{e}^{-\lambda}, \qquad \ell \in \mathbb{Z}_+, \qquad k \in \mathbb{N}, \tag{4.2}$$

$$\mathbb{P}(X_0 = \ell \,|\, \alpha) = \frac{\lambda^\ell}{\ell!(1-\alpha)^\ell}\mathrm{e}^{-(1-\alpha)^{-1}\lambda}, \qquad \ell \in \mathbb{Z}_+. \tag{4.3}$$

(Note that the conditional distribution of ε_k does not depend on α.) Indeed, for each $n \in \mathbb{N}$, by Ionescu Tulcea's theorem (see, e.g., Shiryaev [30, II. § 9, Theorem 2]), there exist a probability space $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ and random variables $\alpha^{(n)}$, $X_0^{(n)}$, $\varepsilon_k^{(n)}$ and $\xi_{k,j}^{(n)}$ for $k, j \in \{1, \ldots, n\}$ on $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ such that

$$\begin{aligned}
&\mathbb{P}_n(\alpha^{(n)} \in B,\ X_0^{(n)} = x_0,\ \varepsilon_k^{(n)} = \ell_k,\ \xi_{k,j}^{(n)} = x_{k,j} \ \text{ for all } \ k, j \in \{1, \ldots, n\}) \\
&= \int_B p_n\left(a, x_0, (\ell_k)_{k=1}^n, (x_{k,j})_{k,j=1}^n\right) \, \mathbb{P}_\alpha(\mathrm{d}a)
\end{aligned}$$

for all $B \in \mathcal{B}(\mathbb{R})$, $x_0 \in \mathbb{Z}_+$, $(\ell_k)_{k=1}^n \in \mathbb{Z}_+^n$, $(x_{k,j})_{k,j=1}^n \in \{0,1\}^{n\times n}$, with

$$p_n\left(a, x_0, (\ell_k)_{k=1}^n, (x_{k,j})_{k,j=1}^n\right) := \frac{\lambda^{x_0}}{x_0!(1-a)^{x_0}}\mathrm{e}^{-(1-a)^{-1}\lambda} \prod_{k=1}^n \frac{\lambda^{\ell_k}}{\ell_k!}\mathrm{e}^{-\lambda} \prod_{k,j=1}^n a^{x_{k,j}}(1-a)^{1-x_{k,j}},$$

since the mapping $(0,1) \ni a \mapsto p_n\left(a, x_0, (\ell_k)_{k=1}^n, (x_{k,j})_{k,j=1}^n\right)$ is Borel measurable for all $x_0 \in \mathbb{Z}_+$, $(\ell_k)_{k=1}^n \in \mathbb{Z}_+^n$, $(x_{k,j})_{k,j=1}^n \in \{0,1\}^{n\times n}$, and

$$\sum\left\{p_n\left(a, x_0, (\ell_k)_{k=1}^n, (x_{k,j})_{k,j=1}^n\right) : x_0 \in \mathbb{Z}_+,\ (\ell_k)_{k=1}^n \in \mathbb{Z}_+^n,\ (x_{k,j})_{k,j=1}^n \in \{0,1\}^{n\times n}\right\} = 1$$

for all $a \in (0,1)$. Then the Kolmogorov consistency theorem implies the existence of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and random variables α, X_0, ε_k and $\xi_{k,j}$ for $k, j \in \mathbb{N}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ with the desired properties (4.1), (4.2) and (4.3), since for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum\big\{p_{n+1}\big(a, x_0, (\ell_k)_{k=1}^{n+1}, (x_{k,j})_{k,j=1}^{n+1}\big) \\
&\qquad : \ell_{n+1} \in \mathbb{Z}_+,\ (x_{n+1,j})_{j=1}^n, (x_{k,n+1})_{k=1}^n \in \{0,1\}^n,\ x_{n+1,n+1} \in \{0,1\}\big\} \\
&= p_n\big(a, x_0, (\ell_k)_{k=1}^n, (x_{k,j})_{k,j=1}^n\big).
\end{aligned}$$

Define a process $(X_k)_{k\in\mathbb{Z}_+}$ by

$$X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \mathbb{N}.$$

By Section 2, conditionally on α, the process $(X_k)_{k\in\mathbb{Z}_+}$ is a strictly stationary INAR(1) process with thinning parameter α and with Poisson innovations. Moreover, by the law of total probability, it is also (unconditionally) strictly stationary but it is not a Markov chain (so it is not an INAR(1) process) if α is not degenerate, see Appendix A. The process $(X_k)_{k\in\mathbb{Z}_+}$ can be called a randomized INAR(1) process with Poisson innovations, and the distribution of α is the so-called mixing distribution of the model. The conditional generator function of X_0 given $\alpha \in (0,1)$ has the form

$$F_0(z_0 \,|\, \alpha) := \mathbb{E}(z_0^{X_0} \,|\, \alpha) = \mathrm{e}^{(1-\alpha)^{-1}\lambda(z_0-1)}, \qquad z_0 \in \mathbb{C},$$

and the conditional expectation of X_0 given α is $\mathbb{E}(X_0 \,|\, \alpha) = (1-\alpha)^{-1}\lambda$. Here and in the sequel conditional expectations like $\mathbb{E}(z_0^{X_0} \,|\, \alpha)$ or $\mathbb{E}(X_0 \,|\, \alpha)$ are meant in the generalized sense, see, e.g., in Stroock [32, § 5.1.1]. The joint conditional generator function of $X_0, X_1, \ldots, X_k$ given α will be denoted by $F_{0,\ldots,k}(z_0, \ldots, z_k \,|\, \alpha)$, $z_0, \ldots, z_k \in \mathbb{C}$.

Let $\alpha^{(j)}$, $j \in \mathbb{N}$, be a sequence of independent copies of the random variable α, and let $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, be a sequence of independent copies of the process $(X_k)_{k\in\mathbb{Z}_+}$ with idiosyncratic innovations (i.e., the innovations $(\varepsilon_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, belonging to $(X_k^{(j)})_{k\in\mathbb{Z}_+}$, $j \in \mathbb{N}$, are independent) such that $(X_k^{(j)})_{k\in\mathbb{Z}_+}$ conditionally on $\alpha^{(j)}$ is a strictly stationary INAR(1) process with thinning parameter $\alpha^{(j)}$ and with Poisson innovations for all $j \in \mathbb{N}$.

First we consider a simple aggregation procedure. For each $N \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N)} = (\widetilde{S}_k^{(N)})_{k\in\mathbb{Z}_+}$ given by

$$\widetilde{S}_k^{(N)} := \sum_{j=1}^{N} (X_k^{(j)} - \mathbb{E}(X_k^{(j)} \,|\, \alpha^{(j)})) = \sum_{j=1}^{N} \Big(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}} \Big), \qquad k \in \mathbb{Z}_+.$$

4.1 Proposition. *If* $\mathbb{E}\big(\frac{1}{1-\alpha}\big) < \infty$, *then*

$$N^{-\frac{1}{2}} \widetilde{S}^{(N)} \xrightarrow{\mathcal{D}_\mathrm{f}} \widetilde{\mathcal{Y}} \qquad \text{as } N \to \infty,$$

where $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\widetilde{\mathcal{Y}}_0 \widetilde{\mathcal{Y}}_k) = \operatorname{Cov}\left(X_0 - \frac{\lambda}{1-\alpha}, X_k - \frac{\lambda}{1-\alpha} \right) = \lambda \, \mathbb{E}\Big(\frac{\alpha^k}{1-\alpha} \Big), \qquad k \in \mathbb{Z}_+. \tag{4.4}$$

4.2 Proposition. *We have*

$$\left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{S}_k^{(1)} \right)_{t\in\mathbb{R}_+} = \left(n^{-\frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} (X_k^{(1)} - \mathbb{E}(X_k^{(1)} \,|\, \alpha^{(1)})) \right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}_\mathrm{f}} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B$$

as $n \to \infty$, *where* $B = (B_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion, independent of* α.

In the next two propositions, which are counterparts of Propositions 3.1 and 3.2, we point out that the usual centralization leads to limit theorems similar to Propositions 4.1 and 4.2, but with an occasionally different scaling and with a different limit process. We use again the notation $S^{(N)} = (S_k^{(N)})_{k\in\mathbb{Z}_+}$ given in (3.1) for the simple aggregation (with the usual centralization) of the randomized process.

4.3 Proposition. *If* $\mathbb{E}\big(\frac{1}{(1-\alpha)^2}\big) < \infty$, *then*

$$N^{-\frac{1}{2}} S^{(N)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \mathcal{Y} \qquad \text{as} \quad N \to \infty,$$

where $\mathcal{Y} = (\mathcal{Y}_k)_{k\in\mathbb{Z}_+}$ *is a stationary Gaussian process with zero mean and covariances*

$$\mathbb{E}(\mathcal{Y}_0\mathcal{Y}_k) = \operatorname{Cov}(X_0, X_k) = \lambda\,\mathbb{E}\Big(\frac{\alpha^k}{1-\alpha}\Big) + \lambda^2 \operatorname{Var}\left(\frac{1}{1-\alpha}\right), \qquad k \in \mathbb{Z}_+.$$

4.4 Proposition. *If* $\mathbb{E}\big(\frac{1}{1-\alpha}\big) < \infty$, *then*

$$\left(n^{-1}\sum_{k=1}^{\lfloor nt\rfloor} S_k^{(1)}\right)_{t\in\mathbb{R}_+} = \left(n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(1)} - \mathbb{E}(X_k^{(1)}))\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \left(\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\Big(\frac{\lambda}{1-\alpha}\Big)\right)t\right)_{t\in\mathbb{R}_+}$$

as $n \to \infty$.

In Proposition 4.4 the limit process is simply a line with a random slope.

In the forthcoming Theorems 4.7–4.13, we assume that the distribution of the random variable α, i.e., the mixing distribution, has a probability density of the form

$$\psi(x)(1-x)^\beta, \qquad x \in (0,1), \tag{4.5}$$

where ψ is a function on $(0,1)$ having a limit $\lim_{x\uparrow 1}\psi(x) = \psi_1 \in (0,\infty)$. Note that necessarily $\beta \in (-1,\infty)$ (otherwise $\int_0^1 \psi(x)(1-x)^\beta\,\mathrm{d}x = \infty$), the function $(0,1) \ni x \mapsto \psi(x)$ is integrable on $(0,1)$, and the function $(0,1) \ni x \mapsto \psi(x)(1-x)^\beta$ is regularly varying at the point 1 (i.e., $(0,\infty) \ni x \mapsto \psi(1-\frac{1}{x})\,x^{-\beta}$ is regularly varying at infinity). Further, in case of $\psi(x) = \frac{\Gamma(a+\beta+2)}{\Gamma(a+1)\Gamma(\beta+1)}x^a$, $x \in (0,1)$, with some $a \in (-1,\infty)$, the random variable α is Beta distributed with parameters $a+1$ and $\beta+1$. The special case of Beta mixing distribution is an important one from the historical point of view, since the Nobel prize winner Clive W. J. Granger used Beta distribution as a mixing distribution for random coefficient AR(1) processes, see Granger [10].

4.5 Remark. Under the condition (4.5), for each $\ell \in \mathbb{N}$, the expectation $\mathbb{E}\big(\frac{1}{(1-\alpha)^\ell}\big)$ is finite if and only if $\beta > \ell - 1$. Indeed, if $\beta > \ell - 1$, then, by choosing $\varepsilon \in (0,1)$ with $\sup_{a\in(1-\varepsilon,1)}\psi(a) \leqslant 2\psi_1$, we have $\mathbb{E}\big(\frac{1}{(1-\alpha)^\ell}\big) = I_1(\varepsilon) + I_2(\varepsilon)$, where

$$I_1(\varepsilon) := \int_0^{1-\varepsilon} \psi(a)(1-a)^{\beta-\ell}\,\mathrm{d}a \leqslant \varepsilon^{\beta-\ell}\int_0^{1-\varepsilon}\psi(a)\,\mathrm{d}a < \infty,$$

$$I_2(\varepsilon) := \int_{1-\varepsilon}^1 \psi(a)(1-a)^{\beta-\ell}\,\mathrm{d}a \leqslant 2\psi_1\int_{1-\varepsilon}^1 (1-a)^{\beta-\ell}\,\mathrm{d}a = \frac{2\psi_1\varepsilon^{\beta-\ell+1}}{\beta-\ell+1} < \infty.$$

Conversely, if $\beta \leqslant \ell - 1$, then, by choosing $\varepsilon \in (0,1)$ with $\sup_{a\in(1-\varepsilon,1)}\psi(a) \geqslant \psi_1/2$, we have

$$\mathbb{E}\left(\frac{1}{(1-\alpha)^\ell}\right) \geqslant \int_{1-\varepsilon}^1 \psi(a)(1-a)^{\beta-\ell}\,\mathrm{d}a \geqslant \frac{\psi_1}{2}\int_{1-\varepsilon}^1 (1-a)^{\beta-\ell}\,\mathrm{d}a = \infty.$$

This means that in case of $\beta \in (-1, 0]$, the processes $S^{(N,n)} = (S_t^{(N,n)})_{t\in\mathbb{R}_+}$, $N, n \in \mathbb{N}$, given in (3.3) are not defined for the randomized INAR(1) process introduced in this section with mixing distribution given in (4.5). Moreover, the Propositions 4.1, 4.2, 4.3 and 4.4 are valid in case of $\beta > 0$, $\beta > -1$, $\beta > 1$ and $\beta > 0$, respectively. $\Box$

For each $N, n \in \mathbb{N}$, consider the stochastic process $\widetilde{S}^{(N,n)} = (\widetilde{S}_t^{(N,n)})_{t\in\mathbb{R}_+}$ given by

$$\widetilde{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} (X_k^{(j)} - \mathbb{E}(X_k^{(j)} \,|\, \alpha^{(j)})), \qquad t \in \mathbb{R}_+.$$

4.6 Remark. If $\beta > 0$, then the covariances of the strictly stationary process $(X_k - \mathbb{E}(X_k \,|\, \alpha))_{k\in\mathbb{Z}_+} = (X_k - \frac{\lambda}{1-\alpha})_{k\in\mathbb{Z}_+}$ exist and take the form

$$\operatorname{Cov}\big(X_0 - \mathbb{E}(X_0 \,|\, \alpha), X_k - \mathbb{E}(X_k \,|\, \alpha)\big) = \mathbb{E}\left(\frac{\lambda\alpha^k}{1-\alpha}\right), \qquad k \in \mathbb{Z}_+,$$

see (5.3). Further,

$$\begin{aligned} \sum_{k=0}^{\infty} \Big| \operatorname{Cov}(X_0 - \mathbb{E}(X_0 \,|\, \alpha), X_k - \mathbb{E}(X_k \,|\, \alpha)) \Big| &= \sum_{k=0}^{\infty} \mathbb{E}\left(\frac{\lambda\alpha^k}{1-\alpha}\right) = \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\sum_{k=0}^{\infty}\alpha^k\right) \\ &= \lambda\,\mathbb{E}\left(\frac{1}{(1-\alpha)^2}\right), \end{aligned}$$

which is finite if and only if $\beta > 1$, see Remark 4.5. This means that the strictly stationary process $(X_k - \mathbb{E}(X_k \,|\, \alpha))_{k\in\mathbb{Z}_+}$ has short memory (i.e., it has summable covariances) if $\beta > 1$, and long memory if $\beta \in (0, 1]$ (i.e., it has non-summable covariances). $\Box$

For $\beta \in (0, 2)$, let $(\mathcal{B}_{1-\frac{\beta}{2}}(t))_{t\in\mathbb{R}_+}$ denote a fractional Brownian motion with parameter $1 - \beta/2$, that is a Gaussian process with zero mean and covariance function

$$\operatorname{Cov}(\mathcal{B}_{1-\frac{\beta}{2}}(t_1), \mathcal{B}_{1-\frac{\beta}{2}}(t_2)) = \frac{t_1^{2-\beta} + t_2^{2-\beta} - |t_2 - t_1|^{2-\beta}}{2}, \qquad t_1, t_2 \in \mathbb{R}_+. \tag{4.6}$$

In Appendix C we recall an integral representation of the fractional Brownian motion $(\mathcal{B}_{1-\frac{\beta}{2}}(t))_{t\in\mathbb{R}_+}$ due to Pilipauskaitė and Surgailis [23] in order to connect our forthcoming results with the ones in Pilipauskaitė and Surgailis [23] and in Puplinskaitė and Surgailis [26], [27].

The next three results are limit theorems for appropriately scaled versions of $\widetilde{S}^{(N,n)}$, first taking the limit $N \to \infty$ and then $n \to \infty$ in the case $\beta \in (-1, 1)$, which are counterparts of (2.7), (2.8) and (2.9) of Theorem 2.1 in Pilipauskaitė and Surgailis [23], respectively.

4.7 Theorem. *If* $\beta \in (0, 1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1+\frac{\beta}{2}} N^{-\frac{1}{2}}\, \widetilde{S}^{(N,n)} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}}\,\mathcal{B}_{1-\frac{\beta}{2}}.$$

4.8 Theorem. *If* $\beta \in (-1,0)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{2(1+\beta)}} \widetilde{S}^{(N,n)} = (V_{2(1+\beta)} t)_{t\in\mathbb{R}_+},$$

where $V_{2(1+\beta)}$ *is a symmetric* $2(1+\beta)$*-stable random variable (not depending on* t*) with characteristic function*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta V_{2(1+\beta)}}) = \mathrm{e}^{-K_\beta |\theta|^{2(1+\beta)}}, \qquad \theta \in \mathbb{R},$$

where

$$K_\beta := \psi_1 \left(\frac{\lambda}{2}\right)^{1+\beta} \frac{\Gamma(-\beta)}{1+\beta}.$$

4.9 Theorem. *If* $\beta = 0$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1} (N \log N)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = (W_{\lambda\psi_1} t)_{t\in\mathbb{R}_+},$$

where $W_{\lambda\psi_1}$ *is a normally distributed random variable with mean zero and with variance* $\lambda\psi_1$.

The next result is a limit theorem for an appropriately scaled version of $\widetilde{S}^{(N,n)}$, first taking the limit $n \to \infty$ and then $N \to \infty$ in the case $\beta \in (-1,1)$, which is a counterpart of (2.10) of Theorem 2.1 in Pilipauskaitė and Surgailis [23].

4.10 Theorem. *If* $\beta \in (-1,1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} N^{-\frac{1}{1+\beta}} n^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \mathcal{Y}_{1+\beta},$$

where $\mathcal{Y}_{1+\beta} = \big(\mathcal{Y}_{1+\beta}(t) := \sqrt{Y_{(1+\beta)/2}}\, B_t\big)_{t\in\mathbb{R}_+}$ *is a* $(1+\beta)$*-stable Lévy process. Here* $Y_{(1+\beta)/2}$ *is a positive* $\frac{1+\beta}{2}$*-stable random variable with Laplace transform* $\mathbb{E}(\mathrm{e}^{-\theta Y_{(1+\beta)/2}}) = \mathrm{e}^{-k_\beta \theta^{\frac{1+\beta}{2}}}$, $\theta \in \mathbb{R}_+$, *and with characteristic function*

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Y_{(1+\beta)/2}}) = \exp\left\{-k_\beta |\theta|^{\frac{1+\beta}{2}} \mathrm{e}^{-\mathrm{i}\,\mathrm{sign}(\theta)\frac{\pi(1+\beta)}{4}}\right\}, \qquad \theta \in \mathbb{R},$$

where

$$k_\beta := \frac{(2\lambda)^{\frac{1+\beta}{2}} \psi_1}{1+\beta} \Gamma\left(\frac{1-\beta}{2}\right),$$

and $(B_t)_{t\in\mathbb{R}_+}$ *is an independent standard Wiener process.*

Next we show an iterated scaling limit theorem where the order of the iteration can be arbitrary in the case $\beta \in (1,\infty)$, which is a counterpart of Theorem 2.3 in Pilipauskaitė and Surgailis [23].

4.11 Theorem. *If* $\beta \in (1,\infty)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \sigma B,$$

where $\sigma^2 := \lambda\, \mathbb{E}((1+\alpha)(1-\alpha)^{-2})$ *and* $(B_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process.*

By Remark 4.5, if $\beta > 1$, then $\mathbb{E}\big(\frac{1}{(1-\alpha)^2}\big) < \infty$, and hence $\sigma^2 < \infty$, where σ^2 is given in Theorem 4.11.

In the next theorems we consider the usual centralization with $\mathbb{E}(X_k^{(j)})$ in the cases $\beta \in (0,1)$ and $\beta > 1$. These are the counterparts of Theorems 4.7, 4.10 and 4.11. Recall that, due to Remark 4.5, the expectation $\mathbb{E}(X_0) = \mathbb{E}\big(\frac{\lambda}{1-\alpha}\big)$ is finite if and only if $\beta > 0$, so Theorems 4.8 and 4.9 can not have counterparts in this sense.

4.12 Theorem. *If* $\beta \in (0,1)$, *then*

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{1+\beta}} S^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{1+\beta}} S^{(N,n)} = \big(Z_{1+\beta}\, t\big)_{t\in\mathbb{R}_+},$$

where $Z_{1+\beta}$ *is a* $(1+\beta)$*-stable random variable with characteristic function* $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Z_{1+\beta}}) = \mathrm{e}^{-|\theta|^{1+\beta}\omega_\beta(\theta)}$, $\theta \in \mathbb{R}$, *where*

$$\omega_\beta(\theta) := \frac{\psi_1 \Gamma(1-\beta)\lambda^{1+\beta}}{-\beta(1+\beta)} \mathrm{e}^{-\mathrm{i}\pi\,\mathrm{sign}(\theta)(1+\beta)/2}, \qquad \theta \in \mathbb{R}.$$

4.13 Theorem. *If* $\beta \in (1,\infty)$, *then*

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} n^{-1} N^{-\frac{1}{2}} S^{(N,n)} = \mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} \mathcal{D}_\mathrm{f}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{2}} S^{(N,n)} = (W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}\, t)_{t\in\mathbb{R}_+},$$

where $W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})}$ *is a normally distributed random variable with mean zero and with variance* $\lambda^2 \operatorname{Var}((1-\alpha)^{-1})$.

In case of Theorems 4.8, 4.9, 4.12 and 4.13 the limit processes are lines with random slopes.

We point out that the processes of doubly indexed partial sums, $S^{(N,n)}$ and $\widetilde{S}^{(N,n)}$ contain the expected or conditional expected values of the processes $X^{(j)}$, $j \in \mathbb{N}$. Therefore, in a statistical testing, they could not be used directly. So we consider a similar process

$$\widehat{S}_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[X_k^{(j)} - \frac{\sum_{\ell=1}^{n} X_\ell^{(j)}}{n} \right], \qquad t \in \mathbb{R}_+,$$

which does not require the knowledge of the expectation or conditional expectation of the processes $X^{(j)}$, $j \in \mathbb{N}$. Note that the summands in $\widehat{S}_t^{(N,n)}$ have 0 conditional means with respect to α, so we do not need any additional centering. Moreover, $\widehat{S}^{(N,n)}$ is related to the two previously examined processes in the following way: in case of $\beta \in (0,\infty)$ (which ensures the existence of $\mathbb{E}(X_k^{(j)})$, $k \in \mathbb{Z}_+$), we have

$$\widehat{S}_t^{(N,n)} = \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[X_k^{(j)} - \mathbb{E}(X_k^{(j)}) - \frac{\sum_{\ell=1}^{n} (X_\ell^{(j)} - \mathbb{E}(X_\ell^{(j)}))}{n} \right] = S_t^{(N,n)} - \frac{\lfloor nt \rfloor}{n} S_1^{(N,n)},$$

and in case of $\beta \in (-1,\infty)$,

$$\widehat{S}_t^{(N,n)} = \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt \rfloor} \left[X_k^{(j)} - \mathbb{E}(X_k^{(j)} \mid \alpha^{(j)}) - \frac{\sum_{\ell=1}^{n} (X_\ell^{(j)} - \mathbb{E}(X_\ell^{(j)} \mid \alpha^{(j)}))}{n} \right] = \widetilde{S}_t^{(N,n)} - \frac{\lfloor nt \rfloor}{n} \widetilde{S}_1^{(N,n)}$$

for every $t \in \mathbb{R}_+$. Therefore, by Theorem 4.7, Theorem 4.10, and Theorem 4.11, using Slutsky's lemma, the following limit theorems hold.

4.14 Corollary. *If* $\beta \in (0,1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1+\frac{\beta}{2}} N^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}} \Big(\mathcal{B}_{1-\frac{\beta}{2}}(t) - t\mathcal{B}_{1-\frac{\beta}{2}}(1)\Big)_{t\in\mathbb{R}_+},$$

where the process $\mathcal{B}_{1-\frac{\beta}{2}}$ *is given by* (4.6).

If $\beta \in (-1,1)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} N^{-\frac{1}{1+\beta}} n^{-\frac{1}{2}} \widehat{S}^{(N,n)} = (\mathcal{Y}_{1+\beta}(t) - t\mathcal{Y}_{1+\beta}(1))_{t\in\mathbb{R}_+},$$

where the process $\mathcal{Y}_{1+\beta}$ *is given in Theorem 4.10.*

If $\beta \in (1,\infty)$, *then*

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} (nN)^{-\frac{1}{2}} \widehat{S}^{(N,n)} = \sigma(B_t - tB_1)_{t\in\mathbb{R}_+},$$

where σ^2 *and the process* B *are given in Theorem 4.11.*

In Corollary 4.14, the limit processes restricted on the time interval $[0,1]$ are bridges in the sense that they take the same value (namely, 0) at the time points 0 and 1, and especially, in case of $\beta \in (1,\infty)$, it is a Wiener bridge. We note that no counterparts appear for the rest of the theorems because in those cases the limit processes are lines with random slopes, which result the constant zero process in this alternative case. In case of $\beta \in (-1,0]$, by applying some smaller scaling factors, one could try to achieve a non-degenerate weak limit of $\widehat{S}^{(N,n)}$ by first taking the limit $N \to \infty$ and then that of $n \to \infty$.

5 Proofs

Theorem 4.7 is a counterpart of (2.7) of Theorem 2.1 in Pilipauskaitė and Surgailis [23]. We will present two proofs of Theorem 4.7, and we call the attention that both proofs are completely different from the proof of (2.7) in Theorem 2.1 in Pilipauskaitė and Surgailis [23] (suspecting also that their result in question might be proved by our method as well). Theorems 4.8 and 4.9 are counterparts of (2.8) and (2.9) of Theorem 2.1 in Pilipauskaitė and Surgailis [23]. The proofs of these theorems use the same technique, namely, expansions of characteristic functions, and we provide all the technical details. Theorem 4.10 is a counterpart of (2.10) of Theorem 2.1 in Pilipauskaitė and Surgailis [23]. We give two proofs of Theorem 4.10: the first one is based on expansions of characteristic functions (as the proof of (2.10) of Theorem 2.1 in Pilipauskaitė and Surgailis [23]), the second one reduces to show that $\frac{\lambda(1+\alpha)}{(1-\alpha)^2}$ belongs to the domain of normal attraction of the $\frac{1+\beta}{2}$-stable law of $Y_{\frac{1+\beta}{2}}$. Theorem 4.11 is a counterpart of Theorem 2.3 in Pilipauskaitė and Surgailis [23]. The proof of Theorem 4.11 is based on the multidimensional central limit theorem and checking convergence of covariances of some Gaussian processes.

The notations $\mathrm{O}(1)$ and $|\,\mathrm{O}(1)|$ stand for a possibly complex and respectively real sequence $(a_k)_{k\in\mathbb{N}}$ that is bounded and can only depend on the parameters λ, ψ_1, β, and on some fixed

$m \in \mathbb{N}$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$. Further, we call the attention that several $\mathrm{O}(1)$-s (respectively $|\mathrm{O}(1)|$-s) in the same formula do not necessarily mean the same bounded sequence.

Proof of Proposition 2.1. First we prove (2.4) by induction. Note that by (2.3) the statement holds for $k = 0$. We suppose that it holds for $0, \ldots, k$, and show that it is also true for $k+1$. Using (2.1) it is easy to see that

$$
\begin{aligned}
&F_{0,\ldots,k,k+1}(z_0, \ldots, z_k, z_{k+1}) = \mathbb{E}\left(z_0^{X_0} \cdots z_k^{X_k} z_{k+1}^{X_{k+1}}\right) \\
&= \mathbb{E}\left(z_0^{X_0} \cdots z_k^{X_k} \, \mathbb{E}\left(z_{k+1}^{X_{k+1}} \mid X_0, \ldots, X_k\right)\right) = \mathbb{E}\left(z_0^{X_0} \cdots z_k^{X_k} \, \mathbb{E}\left(z_{k+1}^{X_{k+1}} \mid X_k\right)\right) \\
&= \mathbb{E}\left(z_0^{X_0} \cdots z_k^{X_k} \mathrm{e}^{\lambda(z_{k+1}-1)} (1 - \alpha + \alpha z_{k+1})^{X_k}\right).
\end{aligned}
$$

On the one hand, for any $z_0, \ldots, z_{k+1} \in \mathbb{C}$, by the assumption of the induction,

$$
\begin{aligned}
&F_{0,\ldots,k,k+1}(z_0, \ldots, z_k, z_{k+1}) = \mathrm{e}^{\lambda(z_{k+1}-1)} F_{0,\ldots,k}(z_0, \ldots, z_{k-1}, z_k(1 - \alpha + \alpha z_{k+1})) \\
&= \exp\left\{ \frac{\lambda}{1-\alpha} \left[(1-\alpha)(z_{k+1} - 1) + \sum_{0 \leqslant i \leqslant j \leqslant k-1} \alpha^{j-i} (z_i - 1) z_{i+1} \cdots z_{j-1} (z_j - 1) \right.\right. \\
&\qquad\qquad \left.\left. + \operatorname{Sum}_1 + z_k(1 - \alpha + \alpha z_{k+1}) - 1 \right] \right\},
\end{aligned}
$$

with

$$
\operatorname{Sum}_1 := \sum_{0 \leqslant i \leqslant k-1} \alpha^{k-i} (z_i - 1) z_{i+1} \cdots z_{k-1} [z_k(1 - \alpha + \alpha z_{k+1}) - 1].
$$

On the other hand, the right hand side of (2.4) for $k+1$ has the form

$$
\exp\left\{ \frac{\lambda}{1-\alpha} \left[\sum_{0 \leqslant i \leqslant j \leqslant k-1} \alpha^{j-i} (z_i - 1) z_{i+1} \cdots z_{j-1} (z_j - 1) + \operatorname{Sum}_2 + \operatorname{Sum}_3 \right] \right\},
$$

where

$$
\begin{aligned}
\operatorname{Sum}_2 &= \sum_{0 \leqslant i \leqslant k} \alpha^{k-i} (z_i - 1) z_{i+1} \cdots z_{k-1} (z_k - 1) \\
&= (z_k - 1) + \sum_{0 \leqslant i \leqslant k-1} \alpha^{k-i} (z_i - 1) z_{i+1} \cdots z_{k-1} (z_k - 1),
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Sum}_3 &= \sum_{0 \leqslant i \leqslant k+1} \alpha^{k+1-i} (z_i - 1) z_{i+1} \cdots z_k (z_{k+1} - 1) \\
&= (z_{k+1} - 1) + \alpha(z_k - 1)(z_{k+1} - 1) + \sum_{0 \leqslant i \leqslant k-1} \alpha^{k+1-i} (z_i - 1) z_{i+1} \cdots z_k (z_{k+1} - 1).
\end{aligned}
$$

Since

$$
\operatorname{Sum}_1 = \sum_{0 \leqslant i \leqslant k-1} \alpha^{k-i} (z_i - 1) z_{i+1} \cdots z_{k-1} (z_k - 1) + \sum_{0 \leqslant i \leqslant k-1} \alpha^{k+1-i} (z_i - 1) z_{i+1} \cdots z_k (z_{k+1} - 1),
$$

in order to show (2.4) for $k+1$, it is enough to check that

$$(1-\alpha)(z_{k+1}-1)+z_k(1-\alpha+\alpha z_{k+1})-1=(z_k-1)+(z_{k+1}-1)+\alpha(z_k-1)(z_{k+1}-1),$$

which holds trivially.

Now we prove (2.5). In formula (2.4), for fixed indices $0\leqslant i\leqslant j\leqslant k$ the term in the sum gives

$$\begin{aligned}&(z_i-1)z_{i+1}\cdots z_{j-1}(z_j-1)\\&\quad=(z_i\cdots z_j-1)-(z_i\cdots z_{j-1}-1)-(z_{i+1}\cdots z_j-1)+(z_{i+1}\cdots z_{j-1}-1),\end{aligned}$$

meaning that the sum consists of similar terms as in (2.5). We only have to show that the coefficients coincide in the formulas (2.5) and (2.4). In (2.5) the coefficient of $z_i\cdots z_j-1$ is $\lambda(1-\alpha)^{K_{i,j,k}}\alpha^{j-i}$. In (2.4) this term may appear multiple times, depending on the indices i and j. If $i=0$ and $j=k$, then it only appears once, with coefficient $\lambda\alpha^{j-i}/(1-\alpha)$, that is the same as in (2.5). However, if $i=0$ and $0\leqslant j\leqslant k-1$ in (2.5), then the term also appears when the indices are i and $j+1$ in (2.4), meaning that the coefficient is

$$\lambda\left(\frac{\alpha^{j-i}}{1-\alpha}-\frac{\alpha^{j+1-i}}{1-\alpha}\right)=\lambda\alpha^{j-i},$$

which is the same as in (2.5). Similarly, if $1\leqslant i\leqslant k$ and $j=k$ in (2.5), then the term also appears when the indices are $i-1$ and j in (2.4), meaning that the coefficient is

$$\lambda\left(\frac{\alpha^{j-i}}{1-\alpha}-\frac{\alpha^{j-(i-1)}}{1-\alpha}\right)=\lambda\alpha^{j-i},$$

which is the same as in (2.5). If $1\leqslant i\leqslant j\leqslant k-1$ in (2.5), then the term appears three more times, for the index pairs $(i-1,j)$, $(i,j+1)$, $(i-1,j+1)$ in (2.4), resulting the coefficient

$$\lambda\left(\frac{\alpha^{j-i}}{1-\alpha}-\frac{\alpha^{j-(i-1)}}{1-\alpha}-\frac{\alpha^{(j+1)-i}}{1-\alpha}+\frac{\alpha^{(j+1)-(i-1)}}{1-\alpha}\right)=\lambda\alpha^{j-i}\frac{1-2\alpha+\alpha^2}{1-\alpha}=\lambda\alpha^{j-i}(1-\alpha),$$

which is the same as in (2.5). This completes the proof. □

Proof of Proposition 3.1. The distribution of X_0 is a Poisson distribution with parameter $(1-\alpha)^{-1}\lambda$, thus $\operatorname{Cov}(X_0,X_0)=\operatorname{Var}(X_0)=(1-\alpha)^{-1}\lambda$. By (2.4), we have

$$F_{0,k}(x_0,x_k)=\mathbb{E}(x_0^{X_0}x_k^{X_k})=F_{0,\dots,k}(x_0,1,\dots,1,x_k)=\mathrm{e}^{(1-\alpha)^{-1}\lambda[\alpha^k(x_0-1)(x_k-1)+(x_0-1)+(x_k-1)]}$$

for all $x_0,x_k\in\mathbb{R}$ and $k\in\mathbb{N}$. Consequently,

$$\mathbb{E}(X_0X_k)=\frac{\partial^2F_{0,k}(x_0,x_k)}{\partial x_0\partial x_k}\bigg|_{(x_0,x_k)=(1,1)}=\frac{\lambda\alpha^k}{1-\alpha}+\frac{\lambda^2}{(1-\alpha)^2},\qquad k\in\mathbb{N},$$

since

$$\frac{\partial^2F_{0,k}(x_0,x_k)}{\partial x_0\partial x_k}=F_{0,k}(x_0,x_k)\frac{\lambda^2}{(1-\alpha)^2}(\alpha^k(x_0-1)+1)(\alpha^k(x_k-1)+1)+F_{0,k}(x_0,x_k)\frac{\lambda}{1-\alpha}\alpha^k.$$

Hence we obtain the formula for $\operatorname{Cov}(X_0, X_k)$. The statement follows from the multidimensional central limit theorem. Due to the continuous mapping theorem, it is sufficient to show the convergence $N^{-1/2}(S_0^{(N)}, S_1^{(N)}, \ldots, S_k^{(N)}) \xrightarrow{\mathcal{D}} (\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_k)$ as $N \to \infty$ for all $k \in \mathbb{Z}_+$. For all $k \in \mathbb{Z}_+$, the random vectors $\big(X_0^{(j)} - \frac{\lambda}{1-\alpha}, X_1^{(j)} - \frac{\lambda}{1-\alpha}, \ldots, X_k^{(j)} - \frac{\lambda}{1-\alpha}\big)$, $j \in \mathbb{N}$, are independent, identically distributed having zero expectation vector and covariances

$$\operatorname{Cov}(X_{\ell_1}^{(j)}, X_{\ell_2}^{(j)}) = \operatorname{Cov}(X_0^{(j)}, X_{|\ell_2-\ell_1|}^{(j)}) = \frac{\lambda\alpha^{|\ell_2-\ell_1|}}{1-\alpha}, \qquad j \in \mathbb{N}, \quad \ell_1, \ell_2 \in \{0, 1, \ldots, k\},$$

following from the strict stationarity of $X^{(j)}$ and from the form of $\operatorname{Cov}(X_0, X_k)$. □

Proof of Proposition 3.2. It is known that

$$M_k := X_k - \mathbb{E}(X_k \mid \mathcal{F}_{k-1}^X) = X_k - (X_{k-1}\,\mathbb{E}(\xi_{1,1}) + \mathbb{E}(\varepsilon_1)) = X_k - \alpha X_{k-1} - \lambda, \qquad k \in \mathbb{N},$$

are martingale differences with respect to the filtration $\mathcal{F}_k^X := \sigma(X_0, \ldots, X_k)$, $k \in \mathbb{Z}_+$ with

$$\mathbb{E}(M_k^2 \mid \mathcal{F}_{k-1}^X) = X_{k-1}\operatorname{Var}(\xi_{1,1}) + \operatorname{Var}(\varepsilon_1) = \alpha(1-\alpha)X_{k-1} + \lambda, \qquad k \in \mathbb{N}. \tag{5.1}$$

The functional martingale central limit theorem can be applied, see, e.g., Jacod and Shiryaev [12, Theorem VIII.3.33]. Indeed, by ergodicity, for each $t \in \mathbb{R}_+$, we have

$$\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} \mathbb{E}(M_k^2 \mid \mathcal{F}_{k-1}^X) \xrightarrow{\text{a.s.}} \Big(\alpha(1-\alpha)\frac{\lambda}{1-\alpha} + \lambda\Big)t = (1+\alpha)\lambda t \qquad \text{as } n \to \infty.$$

Moreover, the conditional Lyapunov condition holds, namely, again by ergodicity,

$$\frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor} \mathbb{E}(M_k^4 \mid \mathcal{F}_{k-1}^X) \xrightarrow{\text{a.s.}} 0 \qquad \text{as } n \to \infty,$$

since there exists a second order polynomial P such that $E(M_k^4 \mid \mathcal{F}_{k-1}) = P(X_{k-1})$, $k \in \mathbb{N}$, see Nedényi [20, Formula (8)], or Barczy et al. [2, Lemma A.2, part (ii)] together with the decomposition $M_k = \sum_{j=1}^{X_{k-1}}(\xi_{k,j} - \mathbb{E}(\xi_{k,j})) + (\varepsilon_k - \mathbb{E}(\varepsilon_k))$, $k \in \mathbb{N}$. Hence we obtain

$$\bigg(\frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor nt\rfloor} M_k\bigg)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} \sqrt{\lambda(1+\alpha)}\,B \qquad \text{as } n \to \infty.$$

We have $X_k = \alpha X_{k-1} + M_k + \lambda$, $k \in \mathbb{N}$, thus $\mathbb{E}(X_k) = \alpha\,\mathbb{E}(X_{k-1}) + \lambda$, $k \in \mathbb{N}$, and hence $X_k - \mathbb{E}(X_k) = \alpha(X_{k-1} - \mathbb{E}(X_{k-1})) + M_k$, $k \in \mathbb{N}$, yielding

$$X_k - \mathbb{E}(X_k) = \alpha^k(X_0 - \mathbb{E}(X_0)) + \sum_{j=1}^{k}\alpha^{k-j}M_j, \qquad k \in \mathbb{N}.$$

Consequently, for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (X_k - \mathbb{E}(X_k)) &= \frac{1}{\sqrt{n}} (X_0 - \mathbb{E}(X_0)) \sum_{k=1}^{\lfloor nt \rfloor} \alpha^k + \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{j=1}^{k} \alpha^{k-j} M_j \\ &= (X_0 - \mathbb{E}(X_0)) \frac{\alpha - \alpha^{\lfloor nt \rfloor + 1}}{(1-\alpha)\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} M_j \sum_{k=j}^{\lfloor nt \rfloor} \alpha^{k-j} \\ &= (X_0 - \mathbb{E}(X_0)) \frac{\alpha - \alpha^{\lfloor nt \rfloor + 1}}{(1-\alpha)\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} M_j \frac{1 - \alpha^{\lfloor nt \rfloor - j + 1}}{1-\alpha},\end{aligned}$$

implying the statement using Slutsky's lemma. Indeed, $n^{-1/2} \sum_{j=1}^{\lfloor nt \rfloor} \alpha^{\lfloor nt \rfloor - j + 1} M_j$ converges in L_1 and hence in probability to 0 as $n \to \infty$, since, by (5.1),

$$\mathbb{E}(|M_j|) \leqslant \sqrt{\mathbb{E}(M_j^2)} = \sqrt{\alpha(1-\alpha)\,\mathbb{E}(X_{j-1}) + \lambda} = \sqrt{\lambda(1+\alpha)},$$

and hence,

$$\mathbb{E}\left(\left| \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \alpha^{\lfloor nt \rfloor - j + 1} M_j \right| \right) \leqslant \frac{\sqrt{\lambda(1+\alpha)}}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \alpha^{\lfloor nt \rfloor - j + 1} = \frac{\sqrt{\lambda(1+\alpha)}}{\sqrt{n}} \frac{\alpha(1 - \alpha^{\lfloor nt \rfloor})}{1-\alpha} \to 0$$

as $n \to \infty$. □

Proof of Theorem 3.3. For all $N, m \in \mathbb{N}$ and all $t_1, \ldots, t_m \in \mathbb{R}_+$, by Proposition 3.2 and by the continuity theorem, we have

$$\frac{1}{\sqrt{n}} (S_{t_1}^{(N,n)}, \ldots, S_{t_m}^{(N,n)}) \stackrel{\mathcal{D}}{\longrightarrow} \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} \sum_{j=1}^{N} (B_{t_1}^{(j)}, \ldots, B_{t_m}^{(j)}) \qquad \text{as } n \to \infty,$$

where $B^{(j)} = (B_t^{(j)})_{t \in \mathbb{R}_+}$, $j \in \{1, \ldots, N\}$, are independent standard Wiener processes. Since

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} (B_{t_1}^{(j)}, \ldots, B_{t_m}^{(j)}) \stackrel{\mathcal{D}}{=} (B_{t_1}, \ldots, B_{t_m}), \qquad N \in \mathbb{N},$$

we obtain the first convergence.

For all $n \in \mathbb{N}$ and for all $t_1, \ldots, t_m \in \mathbb{R}_+$ with $t_1 < \ldots < t_m$, $m \in \mathbb{N}$, by Proposition 3.1 and by the continuous mapping theorem, we have

$$\begin{aligned}\frac{1}{N^{1/2}} \big(S_{t_1}^{(N,n)}, \ldots, S_{t_m}^{(N,n)}\big) &\stackrel{\mathcal{D}}{\longrightarrow} \left(\sum_{k=1}^{\lfloor nt_1 \rfloor} \mathcal{X}_k, \ldots, \sum_{k=1}^{\lfloor nt_m \rfloor} \mathcal{X}_k \right) \\ &\stackrel{\mathcal{D}}{=} \mathcal{N}_m\left(\mathbf{0}, \operatorname{Var}\left(\sum_{k=1}^{\lfloor nt_1 \rfloor} \mathcal{X}_k, \ldots, \sum_{k=1}^{\lfloor nt_m \rfloor} \mathcal{X}_k \right) \right), \qquad N \to \infty,\end{aligned}$$

where $(\mathcal{X}_k)_{k\in\mathbb{Z}_+}$ is the stationary Gaussian process given in Proposition 3.1 and

$$\operatorname{Var}\left(\sum_{k=1}^{\lfloor nt_1\rfloor}\mathcal{X}_k,\ldots,\sum_{k=1}^{\lfloor nt_m\rfloor}\mathcal{X}_k\right)=\left(\frac{\lambda}{1-\alpha}\sum_{k=1}^{\lfloor nt_i\rfloor}\sum_{\ell=1}^{\lfloor nt_j\rfloor}\alpha^{|\ell-k|}\right)_{i,j\in\{1,\ldots,m\}}.$$

By the continuity theorem, for all $\theta_1,\ldots,\theta_m\in\mathbb{R}$, $m\in\mathbb{N}$, we conclude

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{j=1}^{m}\theta_j n^{-1/2}N^{-1/2}S_{t_j}^{(N,n)}\right\}\right)\\
&=\exp\left\{-\frac{\lambda}{2n(1-\alpha)}\sum_{i=1}^{m}\sum_{j=1}^{m}\theta_i\theta_j\sum_{k=1}^{\lfloor nt_i\rfloor}\sum_{\ell=1}^{\lfloor nt_j\rfloor}\alpha^{|\ell-k|}\right\}\\
&\to\exp\left\{-\frac{(1+\alpha)\lambda}{2(1-\alpha)^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\theta_i\theta_j(t_i\wedge t_j)\right\}
\end{aligned}$$

as $n\to\infty$. Indeed, for all $s,t\in\mathbb{R}_+$ with $s\leqslant t$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}\sum_{\ell=1}^{\lfloor nt\rfloor}\alpha^{|\ell-k|}&=\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}\sum_{\ell=1}^{k-1}\alpha^{k-\ell}+\frac{\lfloor ns\rfloor}{n}+\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}\sum_{\ell=k+1}^{\lfloor nt\rfloor}\alpha^{\ell-k}\\
&=\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}\frac{\alpha-\alpha^k}{1-\alpha}+\frac{\lfloor ns\rfloor}{n}+\frac{1}{n}\sum_{k=1}^{\lfloor ns\rfloor}\frac{\alpha-\alpha^{\lfloor nt\rfloor-k+1}}{1-\alpha}\\
&=\frac{1}{n(1-\alpha)}\left(\lfloor ns\rfloor\alpha-\alpha\frac{1-\alpha^{\lfloor ns\rfloor}}{1-\alpha}\right)+\frac{\lfloor ns\rfloor}{n}\\
&\quad+\frac{1}{n(1-\alpha)}\left(\lfloor ns\rfloor\alpha-\alpha^{\lfloor nt\rfloor-\lfloor ns\rfloor+1}\frac{1-\alpha^{\lfloor ns\rfloor}}{1-\alpha}\right)\\
&=\frac{1+\alpha}{1-\alpha}\frac{\lfloor ns\rfloor}{n}-\frac{\alpha}{(1-\alpha)^2n}(1+\alpha^{\lfloor nt\rfloor-\lfloor ns\rfloor})(1-\alpha^{\lfloor ns\rfloor})\to\frac{1+\alpha}{1-\alpha}s
\end{aligned}\tag{5.2}$$

as $n\to\infty$. This implies the second convergence. □

Proof of Proposition 4.1. We have

$$\mathbb{E}\left(X_k-\frac{\lambda}{1-\alpha}\right)=\mathbb{E}\left[\mathbb{E}\left(X_k-\frac{\lambda}{1-\alpha}\,\Big|\,\alpha\right)\right]=0,\qquad k\in\mathbb{Z}_+,$$

and hence, for all $k\in\mathbb{Z}_+$,

$$\begin{aligned}
&\operatorname{Cov}\left(X_0-\frac{\lambda}{1-\alpha},X_k-\frac{\lambda}{1-\alpha}\right)=\mathbb{E}\left[\left(X_0-\frac{\lambda}{1-\alpha}\right)\left(X_k-\frac{\lambda}{1-\alpha}\right)\right]\\
&=\mathbb{E}\left\{\mathbb{E}\left[\left(X_0-\frac{\lambda}{1-\alpha}\right)\left(X_k-\frac{\lambda}{1-\alpha}\right)\,\Big|\,\alpha\right]\right\}=\mathbb{E}\left(\frac{\lambda\alpha^k}{1-\alpha}\right),
\end{aligned}\tag{5.3}$$

where we applied (3.2). Now the statement follows from the multidimensional central limit theorem in the same way as in the proof of Proposition 3.1. □

Proof of Proposition 4.2. For each $n \in \mathbb{N}$ and each $t \in \mathbb{R}_+$, put

$$\widetilde{T}_t^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{S}_k^{(1)}.$$

For each $m \in \mathbb{N}$, each $t_1, \ldots, t_m \in \mathbb{R}_+$, and each bounded continuous function $g : \mathbb{R}^m \to \mathbb{R}$, we have

$$\mathbb{E}(g(\widetilde{T}_{t_1}^{(n)}, \ldots, \widetilde{T}_{t_m}^{(n)})) = \int_0^1 \mathbb{E}(g(\widetilde{T}_{t_1}^{(n)}, \ldots, \widetilde{T}_{t_m}^{(n)}) \mid \alpha = a) \, \mathbb{P}_\alpha(\mathrm{d}a)$$

$$= \int_0^1 \mathbb{E}\biggl(g\biggl(\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt_1 \rfloor} \Bigl(X_k - \frac{\lambda}{1-a} \Bigr), \ldots, \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt_m \rfloor} \Bigl(X_k - \frac{\lambda}{1-a} \Bigr) \biggr) \biggm| \alpha = a \biggr) \, \mathbb{P}_\alpha(\mathrm{d}a).$$

Proposition 3.2, the portmanteau theorem and the boundedness of g justify the usage of the dominated convergence theorem, and we obtain

$$\lim_{n\to\infty} \mathbb{E}(g(\widetilde{T}_{t_1}^{(n)}, \ldots, \widetilde{T}_{t_m}^{(n)})) = \int_0^1 \mathbb{E}\biggl(g\biggl(\frac{\sqrt{\lambda(1+a)}}{1-a} B_{t_1}, \ldots, \frac{\sqrt{\lambda(1+a)}}{1-a} B_{t_m} \biggr) \biggr) \, \mathbb{P}_\alpha(\mathrm{d}a)$$

$$= \int_0^1 \mathbb{E}\biggl(g\biggl(\frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B_{t_1}, \ldots, \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B_{t_m} \biggr) \biggm| \alpha = a \biggr) \, \mathbb{P}_\alpha(\mathrm{d}a)$$

$$= \mathbb{E}\biggl(g\biggl(\frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B_{t_1}, \ldots, \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B_{t_m} \biggr) \biggr),$$

hence we obtain the statement by the portmanteau theorem. □

Proof of Proposition 4.3. For all $k \in \mathbb{Z}_+$, by the strict stationarity of $(X_k)_{k\in\mathbb{Z}_+}$ and (5.3), we have

$$\begin{aligned} \operatorname{Cov}(X_0, X_k) &= \mathbb{E}\left[\left(X_0 - \mathbb{E}\left(\frac{\lambda}{1-\alpha} \right) \right) \left(X_k - \mathbb{E}\left(\frac{\lambda}{1-\alpha} \right) \right) \right] \\ &= \mathbb{E}\left[\left(X_0 - \frac{\lambda}{1-\alpha} \right) \left(X_k - \frac{\lambda}{1-\alpha} \right) \right] + \mathbb{E}\left[\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha} \right) \right)^2 \right] \\ &= \lambda \, \mathbb{E}\left(\frac{\alpha^k}{1-\alpha} \right) + \lambda^2 \operatorname{Var}\left(\frac{1}{1-\alpha} \right), \end{aligned} \tag{5.4}$$

since

$$\begin{aligned} &\mathbb{E}\left[\left(X_k - \frac{\lambda}{1-\alpha} \right) \left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha} \right) \right) \right] \\ &= \mathbb{E}\left\{ \mathbb{E}\left[\left(X_k - \frac{\lambda}{1-\alpha} \right) \left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha} \right) \right) \biggm| \alpha \right] \right\} \\ &= \mathbb{E}\left\{ \left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha} \right) \right) \mathbb{E}\left(X_k - \frac{\lambda}{1-\alpha} \biggm| \alpha \right) \right\} = 0 \end{aligned}$$

for all $k \in \mathbb{Z}_+$.

The statement follows from the multidimensional central limit theorem as in the proof of Proposition 3.1. Indeed, for all $k \in \mathbb{Z}_+$, the random vectors

$$\left(X_0^{(j)} - \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right), X_1^{(j)} - \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right), \ldots, X_k^{(j)} - \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right)\right), \quad j \in \mathbb{N},$$

are independent, identically distributed having zero expectation vector and covariances

$$\operatorname{Cov}(X_{\ell_1}^{(j)}, X_{\ell_2}^{(j)}) = \operatorname{Cov}(X_0^{(j)}, X_{|\ell_2-\ell_1|}^{(j)}) = \lambda\,\mathbb{E}\left(\frac{\alpha^{|\ell_2-\ell_1|}}{1-\alpha}\right) + \lambda^2 \operatorname{Var}\left(\frac{1}{1-\alpha}\right)$$

for $j \in \mathbb{N}$ and $\ell_1, \ell_2 \in \{0, 1, \ldots, k\}$, following from the strict stationarity of $X^{(j)}$ and from the form of $\operatorname{Cov}(X_0, X_k)$ given in (5.4). □

Proof of Proposition 4.4. We have a decomposition $S_k^{(1)} = \widetilde{S}_k^{(1)} + R_k^{(1)}$, $k \in \mathbb{Z}_+$, with

$$R_k^{(1)} := \mathbb{E}(X_k^{(1)} \mid \alpha^{(1)}) - \mathbb{E}(X_k^{(1)}) = \frac{\lambda}{1-\alpha^{(1)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(1)}}\right), \qquad k \in \mathbb{Z}_+.$$

We have

$$\begin{aligned}\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} R_k^{(1)}\right)_{t\in\mathbb{R}_+} &= \left(\frac{\lfloor nt\rfloor}{n}\left(\frac{\lambda}{1-\alpha^{(1)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(1)}}\right)\right)\right)_{t\in\mathbb{R}_+}\\ &\stackrel{\mathcal{D}_{\mathrm{f}}}{\longrightarrow} \left(\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\right)t\right)_{t\in\mathbb{R}_+}\end{aligned}$$

as $n \to \infty$. Moreover, by Proposition 4.2, $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\big(n^{-1/2}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{S}_k^{(1)}\big)_{t\in\mathbb{R}_+}$ exists, hence

$$\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{S}_k^{(1)}\right)_{t\in\mathbb{R}_+} \stackrel{\mathcal{D}_{\mathrm{f}}}{\longrightarrow} 0 \qquad \text{as } n \to \infty,$$

implying that for all $m \in \mathbb{N}$ and all $t_1, \ldots, t_m \in \mathbb{R}_+$, we have

$$\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt_1\rfloor}\widetilde{S}_k^{(1)}, \ldots, \frac{1}{n}\sum_{k=1}^{\lfloor nt_m\rfloor}\widetilde{S}_k^{(1)}\right) \stackrel{\mathbb{P}}{\longrightarrow} 0 \qquad \text{as } n \to \infty.$$

By Slutsky's lemma we conclude the statement. □

First proof of Theorem 4.7. By Remark 4.5, condition $\beta \in (0,1)$ implies $\mathbb{E}\big(\frac{1}{1-\alpha}\big) < \infty$. Hence, by Proposition 4.1 and the continuous mapping theorem, it suffices to show that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\left(\frac{1}{n^{1-\frac{\beta}{2}}}\sum_{k=1}^{\lfloor nt\rfloor}\widetilde{\mathcal{Y}}_k\right)_{t\in\mathbb{R}_+} = \sqrt{\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)}}\mathcal{B}_{1-\frac{\beta}{2}}. \tag{5.5}$$

We are going to apply Theorem 4.3 in Beran et al. [3] with $m = 1$ for the strictly stationary Gaussian process $\left(\widetilde{\mathcal{Y}}_k/\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}\right)_{k\in\mathbb{Z}_+}$, where, by (4.4),

$$\operatorname{Var}(\widetilde{\mathcal{Y}}_0) = \lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right), \qquad \operatorname{Cov}(\widetilde{\mathcal{Y}}_0, \widetilde{\mathcal{Y}}_k) = \lambda\,\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right), \qquad k \in \mathbb{Z}_+,$$

hence

$$\operatorname{Cov}\left(\frac{\widetilde{\mathcal{Y}}_0}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}}, \frac{\widetilde{\mathcal{Y}}_k}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}}\right) = \frac{\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)}, \qquad k \in \mathbb{Z}_+.$$

In order to check the conditions of Theorem 4.3 in Beran et al. [3], first we show that

$$k^\beta\,\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right) = k^\beta \int_0^1 a^k (1-a)^{\beta-1}\psi(a)\,\mathrm{d}a \to \psi_1\Gamma(\beta) \qquad \text{as } k\to\infty, \tag{5.6}$$

meaning that the covariance function of the process $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ is regularly varying with index $-\beta$. First note that, by Stirling's formula,

$$\begin{aligned}\lim_{k\to\infty} k^\beta \int_0^1 a^k(1-a)^{\beta-1}\psi_1\,\mathrm{d}a &= \lim_{k\to\infty}\psi_1\frac{k^\beta\Gamma(k+1)}{\Gamma(k+\beta+1)}\Gamma(\beta)\\ &= \psi_1\Gamma(\beta)\lim_{k\to\infty}\sqrt{\frac{k}{k+\beta}}\left(\frac{k}{k+\beta}\right)^{k+\beta}\mathrm{e}^\beta = \psi_1\Gamma(\beta).\end{aligned}$$

Next, for arbitrary $\delta\in(0,\psi_1)$, there exists $\varepsilon\in(0,1)$ such that $|\psi(a)-\psi_1|\leqslant\delta$ for all $a\in[1-\varepsilon,1)$, and hence

$$k^\beta\int_{1-\varepsilon}^1 a^k(1-a)^{\beta-1}|\psi(a)-\psi_1|\,\mathrm{d}a \leqslant \delta\sup_{k\in\mathbb{N}} k^\beta\int_0^1 a^k(1-a)^{\beta-1}\,\mathrm{d}a$$

can be arbitrary small. Further, observe

$$\begin{aligned}k^\beta\int_0^{1-\varepsilon} a^k(1-a)^{\beta-1}\psi(a)\,\mathrm{d}a &\leqslant \frac{k^\beta(1-\varepsilon)^k}{\varepsilon}\int_0^{1-\varepsilon}(1-a)^\beta\psi(a)\,\mathrm{d}a\\ &\leqslant \frac{k^\beta(1-\varepsilon)^k}{\varepsilon}\int_0^1(1-a)^\beta\psi(a)\,\mathrm{d}a = \frac{k^\beta(1-\varepsilon)^k}{\varepsilon}\to 0 \qquad \text{as } k\to\infty.\end{aligned}$$

In a similar way, we have

$$\begin{aligned}k^\beta\int_0^{1-\varepsilon} a^k(1-a)^{\beta-1}\psi_1\,\mathrm{d}a &\leqslant \psi_1 k^\beta(1-\varepsilon)^k\int_0^{1-\varepsilon}(1-a)^{\beta-1}\,\mathrm{d}a\\ &\leqslant \psi_1 k^\beta(1-\varepsilon)^k\int_0^1(1-a)^{\beta-1}\,\mathrm{d}a = \psi_1\,\frac{k^\beta(1-\varepsilon)^k}{\beta}\to 0 \qquad \text{as } k\to\infty,\end{aligned}$$

hence

$$k^\beta\int_0^{1-\varepsilon} a^k(1-a)^{\beta-1}|\psi(a)-\psi_1|\,\mathrm{d}a \to 0 \qquad \text{as } k\to\infty,$$

implying (5.6). Applying (5.6), we conclude

$$k^\beta \operatorname{Cov}\left(\frac{\widetilde{\mathcal{Y}}_0}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}}, \frac{\widetilde{\mathcal{Y}}_k}{\sqrt{\operatorname{Var}(\widetilde{\mathcal{Y}}_0)}}\right) = k^\beta \frac{\mathbb{E}\left(\frac{\alpha^k}{1-\alpha}\right)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)} \to \frac{\psi_1\Gamma(\beta)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)}$$

as $k \to \infty$. Consequently, by Theorem 4.3 in Beran et al. [3],

$$\left(\frac{1}{n^{1-\frac{\beta}{2}} L_1(n)^{1/2}} \sum_{k=1}^{\lfloor nt \rfloor} \frac{\widetilde{\mathcal{Y}}_k}{\sqrt{\lambda\,\mathbb{E}\left(\frac{1}{1-\alpha}\right)}}\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} \mathcal{Z}_{1,1-\frac{\beta}{2}} \stackrel{\mathcal{D}}{=} \mathcal{B}_{1-\frac{\beta}{2}}, \qquad \text{as } n \to \infty,$$

where $\mathcal{Z}_{1,1-\frac{\beta}{2}}$ is the Hermite-Rosenblatt process defined in Definition 3.24 of Beran et al. [3], and

$$L_1(n) = \frac{\psi_1\Gamma(\beta)}{\mathbb{E}\left(\frac{1}{1-\alpha}\right)} C_1, \qquad n \in \mathbb{N}, \qquad \text{with} \qquad C_1 = \frac{2}{(1-\beta)(2-\beta)}.$$

The fact that the Hermite-Rosenblatt process $\mathcal{Z}_{1,1-\frac{\beta}{2}}$ coincides in law with $\mathcal{B}_{1-\frac{\beta}{2}}$ is shown in Beran et al. [3], see Definition 3.23, the representation in formula (3.111), and page 195 of [3] for details. Hence we obtain the statement. □

Second proof of Theorem 4.7. As in the first proof of Theorem 4.7, it suffices to show (5.5). As for every $n \in \mathbb{N}$ the process $n^{-1+\beta/2} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{\mathcal{Y}}_k$, $t \in \mathbb{R}_+$, is Gaussian, so is the limit process. Also, it is clear that both processes have zero mean. Therefore, it suffices to show that the covariance function of $n^{-1+\beta/2} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{\mathcal{Y}}_k$, $t \in \mathbb{R}_+$, converges to that of the limit process in (5.5). By (4.4), the covariance function of $n^{-1+\beta/2} \sum_{k=1}^{\lfloor nt \rfloor} \widetilde{\mathcal{Y}}_k$, $t \in \mathbb{R}_+$, for any $t_1, t_2 \in \mathbb{R}_+$ takes the form

$$\operatorname{Cov}\left(n^{-1+\beta/2} \sum_{k=1}^{\lfloor nt_1 \rfloor} \widetilde{\mathcal{Y}}_k, n^{-1+\beta/2} \sum_{\ell=1}^{\lfloor nt_2 \rfloor} \widetilde{\mathcal{Y}}_\ell\right) = n^{-2+\beta} \lambda\,\mathbb{E}\left(\sum_{k=1}^{\lfloor nt_1 \rfloor} \sum_{\ell=1}^{\lfloor nt_2 \rfloor} \frac{\alpha^{|k-\ell|}}{1-\alpha}\right).$$

By (5.2), for time points $0 \leqslant t_1 \leqslant t_2$, we get

$$\begin{aligned}
\frac{1}{1-\alpha} \sum_{k=1}^{\lfloor nt_1 \rfloor} \sum_{\ell=1}^{\lfloor nt_2 \rfloor} \alpha^{|k-\ell|} &= \frac{(1-\alpha^2)\lfloor nt_1 \rfloor - \alpha\left(1 - \alpha^{\lfloor nt_2 \rfloor} - \alpha^{\lfloor nt_1 \rfloor} + \alpha^{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}\right)}{(1-\alpha)^3} \\
&= \frac{\alpha(\alpha^{\lfloor nt_2 \rfloor} - 1) + \lfloor nt_2 \rfloor (1-\alpha^2)/2}{(1-\alpha)^3} + \frac{\alpha(\alpha^{\lfloor nt_1 \rfloor} - 1) + \lfloor nt_1 \rfloor (1-\alpha^2)/2}{(1-\alpha)^3} \\
&\quad - \frac{\alpha(\alpha^{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor} - 1) + (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor)(1-\alpha^2)/2}{(1-\alpha)^3}.
\end{aligned} \tag{5.7}$$

We are going to show that for any $0 \leqslant t_1 \leqslant t_2$ we get

$$\begin{aligned}
&n^{-2+\beta}\,\mathbb{E}\left(\frac{\alpha(\alpha^{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor} - 1) + (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor)(1-\alpha^2)/2}{(1-\alpha)^3}\right) \\
&\to \psi_1 \int_0^\infty \left(\mathrm{e}^{-y(t_2-t_1)} - 1 + y(t_2-t_1)\right) y^{\beta-3}\,\mathrm{d}y = \frac{\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)} (t_2-t_1)^{2-\beta},
\end{aligned}$$

as $n \to \infty$, where the equality follows with repeated partial integration. This will imply that the limit of the sequence of the covariance functions in question is

$$\frac{2\lambda\psi_1\Gamma(\beta)}{(2-\beta)(1-\beta)} \, \frac{t_1^{2-\beta} + t_2^{2-\beta} - |t_2 - t_1|^{2-\beta}}{2},$$

that is the covariance function of a fractional Brownian motion with parameter $1-\beta/2$ multiplied by $\sqrt{2\lambda\psi_1\Gamma(\beta)/((2-\beta)(1-\beta))}$, as desired.

By substituting $a = 1 - y/n$ we get that

$$\begin{aligned}
&n^{-2+\beta}\,\mathbb{E}\left(\frac{\alpha(\alpha^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1)+(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)(1-\alpha^2)/2}{(1-\alpha)^3}\right)\\
&= n^{-2+\beta}\int_0^1 \frac{a(a^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1)+(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)(1-a^2)/2}{(1-a)^3}(1-a)^\beta\psi(a)\,\mathrm{d}a\\
&= n^{-2+\beta}\int_0^n \Bigg[\left(1-\frac{y}{n}\right)\left(\left(1-\frac{y}{n}\right)^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1\right)\\
&\qquad\qquad + (\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)\left(1-\left(1-\frac{y}{n}\right)^2\right)\frac{1}{2}\Bigg]\left(\frac{y}{n}\right)^{\beta-3}\psi\left(1-\frac{y}{n}\right)\frac{\mathrm{d}y}{n}\\
&= \int_0^n D_n(y)\,\mathrm{d}y
\end{aligned}$$

with

$$D_n(y) := \left[\left(1-\frac{y}{n}\right)\left(\left(1-\frac{y}{n}\right)^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1\right)+\frac{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}{n}\,y\left(1-\frac{y}{2n}\right)\right]y^{\beta-3}\psi\left(1-\frac{y}{n}\right)$$

for $y \in [0,n]$. First note that, for any $\varepsilon \in (0,1)$ and $n > 1/\varepsilon$, we have

$$\begin{aligned}
\left|\int_{n\varepsilon}^n D_n(y)\,\mathrm{d}y\right| &\leqslant \int_{n\varepsilon}^n \left(1\cdot 2+\left(t_2-t_1+\frac{1}{n}\right)y\right)y^{\beta-3}\psi\left(1-\frac{y}{n}\right)\mathrm{d}y\\
&\leqslant \int_{n\varepsilon}^n (1\cdot 2y+(t_2-t_1+1)y)y^{\beta-3}\psi\left(1-\frac{y}{n}\right)\mathrm{d}y\\
&= (t_2-t_1+3)\int_0^{1-\varepsilon}\big(n(1-a)\big)^{\beta-2}\psi(a)n\,\mathrm{d}a \qquad (5.8)\\
&= (t_2-t_1+3)n^{\beta-1}\int_0^{1-\varepsilon}(1-a)^{\beta-2}\psi(a)\,\mathrm{d}a\\
&\leqslant (t_2-t_1+3)n^{\beta-1}\varepsilon^{-2}\int_0^1(1-a)^\beta\psi(a)\,\mathrm{d}a \to 0, \qquad n\to\infty.
\end{aligned}$$

We are going to show that

$$\int_0^{n\varepsilon} D_n(y)\,\mathrm{d}y \to \psi_1\int_0^\infty \left(\mathrm{e}^{-y(t_2-t_1)}-1+y(t_2-t_1)\right)y^{\beta-3}\,\mathrm{d}y, \qquad n\to\infty.$$

The pointwise convergence is evident, and we can give a dominating integrable function proving the above convergence.

Note that

$$\left(1-\frac{y}{n}\right)^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor} = 1-(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)\frac{y}{n}+\sum_{k=2}^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}\binom{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}{k}\left(-\frac{y}{n}\right)^k,$$

where for any $y\in[0,1]$ we have

$$\begin{aligned}
&\left|\sum_{k=2}^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}\binom{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}{k}\left(-\frac{y}{n}\right)^k\right| \\
&\leqslant y^2\sum_{k=2}^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}\frac{y^{k-2}}{k!}\frac{(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)\cdots(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor-k+1)}{n^k} \\
&\leqslant y^2\sum_{k=0}^{\infty}\frac{(t_2-t_1+1)^k}{k!}=y^2\mathrm{e}^{t_2-t_1+1}.
\end{aligned}$$

Choose $\varepsilon\in(0,1)$ such that for every $x\in(1-\varepsilon,1)$ we have $\psi(x)\leqslant 2\psi_1$. Then for any $n>1/\varepsilon$, applying Bernoulli's inequality, we obtain

$$\begin{aligned}
\int_0^1|D_n(y)|\,\mathrm{d}y &= \int_0^1\Bigg|\left[\left(1-\frac{y}{n}\right)^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1+(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)\frac{y}{n}\right] \\
&\qquad -\frac{y}{n}\left[\left(1-\frac{y}{n}\right)^{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}-1+\frac{y(\lfloor nt_2\rfloor-\lfloor nt_1\rfloor)}{2n}\right]\Bigg|y^{\beta-3}\psi\left(1-\frac{y}{n}\right)\mathrm{d}y \\
&\leqslant \int_0^1\left(y^2\mathrm{e}^{t_2-t_1+1}+\frac{y}{n}\left(y(t_2-t_1+1)+y\frac{t_2-t_1+1}{2}\right)\right)y^{\beta-3}\psi\left(1-\frac{y}{n}\right)\mathrm{d}y \\
&\leqslant 2\psi_1(\mathrm{e}^{t_2-t_1+1}+2(t_2-t_1+1))\int_0^1 y^{\beta-1}\,\mathrm{d}y<\infty.
\end{aligned}$$

Similarly to (5.8), for any $n>1/\varepsilon$, one gets

$$\begin{aligned}
&\int_1^{n\varepsilon}|D_n(y)|\,\mathrm{d}y\leqslant\int_1^{n\varepsilon}(2+t_2-t_1+1)y^{\beta-2}\psi\left(1-\frac{y}{n}\right)\mathrm{d}y \\
&\leqslant(t_2-t_1+3)2\psi_1\int_1^{n\varepsilon}y^{\beta-2}\,\mathrm{d}y\leqslant(t_2-t_1+3)2\psi_1\int_1^{\infty}y^{\beta-2}\,\mathrm{d}y<\infty.
\end{aligned}$$

So the function

$$2\psi_1(\mathrm{e}^{t_2-t_1+1}+2(t_2-t_1+1))y^{\beta-1}\mathbb{1}_{[0,1)}(y)+(t_2-t_1+3)2\psi_1y^{\beta-2}\mathbb{1}_{[1,\infty)}(y)$$

can be chosen as a dominating integrable function. □

Proof of Theorem 4.8. To prove this limit theorem it is enough to show that for any $n \in \mathbb{N}$,

$$\mathcal{D}_\mathrm{f}\text{-}\lim_{N\to\infty} N^{-\frac{1}{2(1+\beta)}} \widetilde{S}^{(N,n)} = (\lfloor nt \rfloor V_{2(1+\beta)})_{t\in\mathbb{R}_+}.$$

For this, by the continuous mapping theorem, it is enough to verify that for any $m \in \mathbb{N}$,

$$N^{-\frac{1}{2(1+\beta)}} \sum_{j=1}^{N} \left(X_1^{(j)} - \frac{\lambda}{1-\alpha^{(j)}}, \ldots, X_m^{(j)} - \frac{\lambda}{1-\alpha^{(j)}} \right) \stackrel{\mathcal{D}}{\longrightarrow} V_{2(1+\beta)}(1,\ldots,1)$$

as $N \to \infty$. So, by the continuity theorem, we have to check that for any $m \in \mathbb{N}$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$ the convergence

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{ \mathrm{i} \sum_{k=1}^{m} \theta_k \left(N^{-\frac{1}{2(1+\beta)}} \sum_{j=1}^{N} \left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}} \right) \right) \right\} \right) \\
&\qquad = \mathbb{E}\left(\exp\left\{ \mathrm{i} N^{-\frac{1}{2(1+\beta)}} \sum_{j=1}^{N} \sum_{k=1}^{m} \theta_k \left(X_k^{(j)} - \frac{\lambda}{1-\alpha^{(j)}} \right) \right\} \right) \\
&\qquad = \left[\mathbb{E}\left(\exp\left\{ \mathrm{i} N^{-\frac{1}{2(1+\beta)}} \sum_{k=1}^{m} \theta_k \left(X_k - \frac{\lambda}{1-\alpha} \right) \right\} \right) \right]^N \\
&\qquad \to \mathbb{E}\left(\mathrm{e}^{\mathrm{i} \sum_{k=1}^{m} \theta_k V_{2(1+\beta)}} \right) = \mathrm{e}^{-K_\beta |\sum_{k=1}^{m} \theta_k|^{2(1+\beta)}} \qquad \text{as } N \to \infty
\end{aligned}$$

holds. Note that it suffices to show

$$\Theta_N := N \left[1 - \mathbb{E}\left(\exp\left\{ \mathrm{i} N^{-\frac{1}{2(1+\beta)}} \sum_{k=1}^{m} \theta_k \left(X_k - \frac{\lambda}{1-\alpha} \right) \right\} \right) \right] \to K_\beta \left| \sum_{k=1}^{m} \theta_k \right|^{2(1+\beta)}$$

as $N \to \infty$, since it implies that $(1 - \Theta_N/N)^N \to \mathrm{e}^{-K_\beta |\sum_{k=1}^{m} \theta_k|^{2(1+\beta)}}$ as $N \to \infty$. By applying (2.4) to the left hand side, we get

$$\begin{aligned}
\Theta_N &= N\,\mathbb{E}\Big[1 - F_{0,\ldots,m-1}\Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_m} \,\Big|\, \alpha \Big) \mathrm{e}^{-\mathrm{i}N^{-\frac{1}{2(1+\beta)}} \frac{\lambda}{1-\alpha} \sum_{k=1}^{m} \theta_k} \Big] \\
&= N\,\mathbb{E}\left[1 - \mathrm{e}^{\frac{\lambda}{1-\alpha} A_N(\alpha)} \right] = N \int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} A_N(a)} \right) \psi(a)(1-a)^\beta \,\mathrm{d}a,
\end{aligned}$$

where $F_{0,\ldots,m-1}(z_0,\ldots,z_{m-1} \,|\, \alpha) := \mathbb{E}(z_0^{X_0} z_1^{X_1} \cdots z_{m-1}^{X_{m-1}} \,|\, \alpha)$, $z_0,\ldots,z_{m-1} \in \mathbb{C}$, and

$$\begin{aligned}
A_N(a) := &- \frac{\mathrm{i}(\theta_1 + \cdots + \theta_m)}{N^{\frac{1}{2(1+\beta)}}} \\
&+ \sum_{1 \leqslant \ell \leqslant j \leqslant m} a^{j-\ell} \big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1 \big) \mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}(\theta_{\ell+1}+\cdots+\theta_{j-1})} \big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_j} - 1 \big)
\end{aligned}$$

for $a \in [0,1]$. Let us show that for any $\varepsilon \in (0,1)$ we have $\sup_{a\in(0,1-\varepsilon)} |NA_N(a)| \to 0$ as $N \to \infty$. Using (B.2), for any $\varepsilon \in (0,1)$ we get

$$\begin{aligned}
\sup_{a\in(0,1-\varepsilon)} N|A_N(a)| &= \sup_{a\in(0,1-\varepsilon)} N\Bigg|\sum_{k=1}^m \Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k} - 1 - \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k\Big) \\
&\quad + \sum_{1\leqslant \ell<j\leqslant m} a^{j-\ell}\Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1\Big)\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}(\theta_{\ell+1}+\cdots+\theta_{j-1})}\Big(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_j} - 1\Big)\Bigg| \\
&\leqslant N\left(\sum_{k=1}^m N^{-\frac{1}{1+\beta}}\frac{\theta_k^2}{2} + \sum_{1\leqslant \ell<j\leqslant m} N^{-\frac{1}{1+\beta}}|\theta_\ell||\theta_j|\right) = N^{\frac{\beta}{1+\beta}}\frac{\left(\sum_{k=1}^m|\theta_k|\right)^2}{2} \to 0
\end{aligned}$$

as $N\to\infty$, since $\beta/(1+\beta) < 0$. Therefore, by Lemma B.2, substituting $a = 1 - z^{-1}N^{-\frac{1}{1+\beta}}$, the statement of the theorem will follow from

$$\begin{aligned}
&\limsup_{N\to\infty} N\int_{1-\varepsilon}^1 \Big|1 - \mathrm{e}^{\frac{\lambda}{1-a}A_N(a)}\Big|(1-a)^\beta\,\mathrm{d}a \\
&= \limsup_{N\to\infty}\int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^\infty \Big|1 - \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}}A_N\left(1-z^{-1}N^{-\frac{1}{1+\beta}}\right)}\Big| z^{-(2+\beta)}\,\mathrm{d}z < \infty
\end{aligned} \tag{5.9}$$

for all $\varepsilon\in(0,1)$ and

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\limsup_{N\to\infty}\left|N\int_{1-\varepsilon}^1\Big(1 - \mathrm{e}^{\frac{\lambda}{1-a}A_N(a)}\Big)(1-a)^\beta\,\mathrm{d}a - I\right| \\
&= \lim_{\varepsilon\downarrow 0}\limsup_{N\to\infty}\left|\int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^\infty\Big(1 - \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}}A_N\left(1-z^{-1}N^{-\frac{1}{1+\beta}}\right)}\Big)z^{-(2+\beta)}\,\mathrm{d}z - I\right| = 0
\end{aligned} \tag{5.10}$$

with

$$\begin{aligned}
I &:= \int_0^\infty\Big(1 - \mathrm{e}^{-\frac{\lambda z}{2}\left(\sum_{k=1}^m\theta_k\right)^2}\Big)z^{-(2+\beta)}\,\mathrm{d}z \\
&= \left(\frac{\lambda}{2}\left|\sum_{k=1}^m\theta_k\right|^2\right)^{1+\beta}\int_0^\infty(1-\mathrm{e}^{-z})z^{-(2+\beta)}\,\mathrm{d}z = \psi_1^{-1}K_\beta\left|\sum_{k=1}^m\theta_k\right|^{2(1+\beta)},
\end{aligned}$$

where the last equality is justified by Lemma 2.2.1 in Zolotarev [38] (be careful for the misprint in [38]: a negative sign is superfluous) or by Li [17, formula (1.28)]. Next we check (5.9) and (5.10).

By Taylor expansion,

$$\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1 = \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell + N^{-\frac{1}{1+\beta}}\,\mathrm{O}(1) = N^{-\frac{1}{2(1+\beta)}}\,\mathrm{O}(1),$$

$$\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1 - \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell = -N^{-\frac{1}{1+\beta}}\frac{\theta_\ell^2}{2} + N^{-\frac{3}{2(1+\beta)}}\,\mathrm{O}(1)$$

for all $\ell \in \{1, \dots, m\}$, resulting

$$\lambda z N^{\frac{1}{1+\beta}} A_N\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right) = -\frac{\lambda z\bigl(\sum_{k=1}^m \theta_k\bigr)^2}{2} + \frac{z\,\mathrm{O}(1)}{N^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{N^{\frac{1}{1+\beta}}} \tag{5.11}$$

for $z > N^{-\frac{1}{1+\beta}}$. Indeed, for $z > N^{-\frac{1}{1+\beta}}$, we have

$$\begin{aligned}
&A_N\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right)\\
&= \sum_{k=1}^m \bigl(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k} - 1 - \mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_k\bigr)\\
&\quad + \sum_{1\leqslant \ell < j \leqslant m} \left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right)^{j-\ell} \bigl(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_\ell} - 1\bigr)\,\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}(\theta_{\ell+1}+\cdots+\theta_{j-1})}\bigl(\mathrm{e}^{\mathrm{i}N^{-\frac{1}{2(1+\beta)}}\theta_j} - 1\bigr)\\
&= \sum_{k=1}^m \left(-\frac{\theta_k^2}{2N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}}\right)\\
&\quad + \sum_{1\leqslant \ell < j \leqslant m} \left(1 + \frac{\mathrm{O}(1)}{zN^{\frac{1}{1+\beta}}}\right)\left(\frac{\mathrm{i}\theta_\ell}{N^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{N^{\frac{1}{1+\beta}}}\right)\left(1 + \frac{\mathrm{O}(1)}{N^{\frac{1}{2(1+\beta)}}}\right)\left(\frac{\mathrm{i}\theta_j}{N^{\frac{1}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{N^{\frac{1}{1+\beta}}}\right)\\
&= -\frac{\sum_{k=1}^m \theta_k^2}{2N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}} - \frac{\sum_{1\leqslant \ell<j\leqslant m}\theta_\ell\theta_j}{N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{zN^{\frac{2}{1+\beta}}}\\
&= -\frac{\bigl(\sum_{k=1}^m \theta_k\bigr)^2}{2N^{\frac{1}{1+\beta}}} + \frac{\mathrm{O}(1)}{N^{\frac{3}{2(1+\beta)}}} + \frac{\mathrm{O}(1)}{zN^{\frac{2}{1+\beta}}},
\end{aligned}$$

since by Bernoulli's inequality

$$\left|\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right)^{j-\ell} - 1\right| \leqslant \frac{j-\ell}{zN^{\frac{1}{1+\beta}}} \leqslant \frac{m}{zN^{\frac{1}{1+\beta}}},$$

yielding that

$$\left(1 - \frac{1}{zN^{\frac{1}{1+\beta}}}\right)^{j-\ell} = 1 + \frac{\mathrm{O}(1)}{zN^{\frac{1}{1+\beta}}}.$$

By (5.11), for $z \in [1, \infty)$ and for large enough N we have

$$\begin{aligned}
\lambda z N^{\frac{1}{1+\beta}} \operatorname{Re} A_N\bigl(1 - z^{-1}N^{-\frac{1}{1+\beta}}\bigr) &= -\frac{\lambda z(\sum_{k=1}^m \theta_k)^2}{2}\left(1 - \frac{\operatorname{Re}\mathrm{O}(1)}{N^{\frac{1}{2(1+\beta)}}}\right) + \frac{\operatorname{Re}\mathrm{O}(1)}{N^{\frac{1}{1+\beta}}}\\
&\leqslant -\frac{\lambda z(\sum_{k=1}^m \theta_k)^2}{4} + \frac{|\,\mathrm{O}(1)|}{N^{\frac{1}{1+\beta}}} \leqslant 0,
\end{aligned}$$

hence we obtain

$$\begin{aligned}
&\int_1^\infty \left|1 - \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} A_N(1 - z^{-1}N^{-\frac{1}{1+\beta}})}\right| z^{-(\beta+2)}\,\mathrm{d}z\\
&\leqslant \int_1^\infty \left(1 + \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} \operatorname{Re} A_N(1 - z^{-1}N^{-\frac{1}{1+\beta}})}\right) z^{-(\beta+2)}\,\mathrm{d}z \leqslant 2\int_1^\infty z^{-(\beta+2)}\,\mathrm{d}z < \infty.
\end{aligned} \tag{5.12}$$

Again by (5.11), for $\varepsilon \in (0,1)$, $z \in \big(\varepsilon^{-1} N^{-\frac{1}{1+\beta}}, 1\big]$ and for large enough N, we have

$$\begin{aligned}\Big|\lambda z N^{\frac{1}{1+\beta}} A_N \Big(1 - z^{-1} N^{-\frac{1}{1+\beta}}\Big)\Big| &\leqslant \frac{\lambda z (\sum_{k=1}^m \theta_k)^2}{2} + \frac{z|\operatorname{O}(1)|}{N^{\frac{1}{2(1+\beta)}}} + \frac{|\operatorname{O}(1)|}{N^{\frac{1}{1+\beta}}} \\ &\leqslant z\bigg(\frac{\lambda(\sum_{k=1}^m \theta_k)^2}{2} + \frac{|\operatorname{O}(1)|}{N^{\frac{1}{2(1+\beta)}}} + \varepsilon|\operatorname{O}(1)|\bigg) \leqslant z|\operatorname{O}(1)| \leqslant |\operatorname{O}(1)|,\end{aligned}$$

since $N^{-\frac{1}{1+\beta}} < z\varepsilon$. Hence, using (B.3), we obtain

$$\begin{aligned}&\int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^1 \Big|1 - \mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} A_N \left(1 - z^{-1} N^{-\frac{1}{1+\beta}}\right)}\Big| z^{-(2+\beta)} \,\mathrm{d}z \\ &\leqslant \int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^1 \Big|\lambda z N^{\frac{1}{1+\beta}} A_N \big(1 - z^{-1} N^{-\frac{1}{1+\beta}}\big)\Big| \mathrm{e}^{\left|\lambda z N^{\frac{1}{1+\beta}} A_N \left(1 - z^{-1} N^{-\frac{1}{1+\beta}}\right)\right|} z^{-(2+\beta)} \,\mathrm{d}z \\ &\leqslant |\operatorname{O}(1)| \mathrm{e}^{|\operatorname{O}(1)|} \int_0^1 z^{-(1+\beta)} \,\mathrm{d}z < \infty,\end{aligned}$$

which, together with (5.12), imply (5.9).

Now we turn to prove (5.10). By (B.1), we have

$$\begin{aligned}&\bigg|\int_0^{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-\frac{\lambda z}{2}(\sum_{k=1}^m \theta_k)^2}\Big) z^{-(2+\beta)} \,\mathrm{d}z\bigg| \leqslant \int_0^{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}} \frac{\lambda z (\sum_{k=1}^m \theta_k)^2}{2} z^{-(2+\beta)} \,\mathrm{d}z \\ &= \frac{\lambda(\sum_{k=1}^m \theta_k)^2}{2} \int_0^{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}} z^{-(1+\beta)} \,\mathrm{d}z = \frac{\lambda(\sum_{k=1}^m \theta_k)^2}{2(-\beta)} \left(\frac{1}{\varepsilon N^{\frac{1}{1+\beta}}}\right)^{-\beta} \to 0\end{aligned}$$

as $N \to \infty$, hence (5.10) reduces to check that $\lim_{\varepsilon \downarrow 0} \limsup_{N\to\infty} I_{N,\varepsilon} = 0$, where

$$I_{N,\varepsilon} := \int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^\infty \Big[\mathrm{e}^{\lambda z N^{\frac{1}{1+\beta}} A_N (1 - z^{-1} N^{-\frac{1}{1+\beta}})} - \mathrm{e}^{-\frac{\lambda z}{2}(\sum_{k=1}^m \theta_k)^2}\Big] z^{-(2+\beta)} \,\mathrm{d}z.$$

Applying again (5.11), we obtain

$$|I_{N,\varepsilon}| \leqslant \int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^\infty \mathrm{e}^{-\frac{\lambda z}{2}(\sum_{k=1}^m \theta_k)^2} \Big|\mathrm{e}^{zN^{-\frac{1}{2(1+\beta)}} \operatorname{O}(1) + N^{-\frac{1}{1+\beta}} \operatorname{O}(1)} - 1\Big| z^{-(2+\beta)} \,\mathrm{d}z.$$

Here, for $\varepsilon \in (0,1)$ and $z \in (\varepsilon^{-1} N^{-\frac{1}{1+\beta}}, \infty)$, we have

$$\Big|zN^{-\frac{1}{2(1+\beta)}} \operatorname{O}(1) + N^{-\frac{1}{1+\beta}} \operatorname{O}(1)\Big| \leqslant z\Big(N^{-\frac{1}{2(1+\beta)}} + \varepsilon\Big)|\operatorname{O}(1)|,$$

and hence, by (B.3), we get

$$\begin{aligned}\Big|\mathrm{e}^{zN^{-\frac{1}{2(1+\beta)}} \operatorname{O}(1) + N^{-\frac{1}{1+\beta}} \operatorname{O}(1)} - 1\Big| &\leqslant \Big|zN^{-\frac{1}{2(1+\beta)}} \operatorname{O}(1) + N^{-\frac{1}{1+\beta}} \operatorname{O}(1)\Big| \mathrm{e}^{\left|zN^{-\frac{1}{2(1+\beta)}} \operatorname{O}(1) + N^{-\frac{1}{1+\beta}} \operatorname{O}(1)\right|} \\ &\leqslant z\Big(N^{-\frac{1}{2(1+\beta)}} + \varepsilon\Big)|\operatorname{O}(1)| \,\mathrm{e}^{z\left(N^{-\frac{1}{2(1+\beta)}} + \varepsilon\right)|\operatorname{O}(1)|}.\end{aligned}$$

Consequently, for large enough N and small enough $\varepsilon \in (0,1)$,

$$|I_{N,\varepsilon}| \leqslant \big(N^{-\frac{1}{2(1+\beta)}} + \varepsilon\big)|\operatorname{O}(1)| \int_{\varepsilon^{-1}N^{-\frac{1}{1+\beta}}}^{\infty} \mathrm{e}^{-\frac{\lambda z}{2}(\sum_{k=1}^m \theta_k)^2 + z\left(N^{-\frac{1}{2(1+\beta)}}+\varepsilon\right)|\operatorname{O}(1)|} z^{-(1+\beta)}\,\mathrm{d}z$$
$$\leqslant \big(N^{-\frac{1}{2(1+\beta)}} + \varepsilon\big)|\operatorname{O}(1)| \int_0^{\infty} \mathrm{e}^{-\frac{\lambda z}{4}(\sum_{k=1}^m \theta_k)^2} z^{-(1+\beta)}\,\mathrm{d}z,$$

that gets arbitrarily close to zero as N approaches infinity and ε tends to 0, since the integral is finite due to the fact that

$$\Gamma(-\beta)\left(\frac{\lambda}{4}\left(\sum_{k=1}^m \theta_k\right)^2\right)^{\beta} \mathrm{e}^{-\lambda z(\sum_{k=1}^m \theta_k)^2/4} z^{-(1+\beta)}, \qquad z > 0,$$

is the density function of a Gamma distributed random variable with parameters $-\beta$ and $\lambda(\sum_{k=1}^m \theta_k)^2/4$. This yields (5.10) completing the proof. □

Proof of Theorem 4.9. Similarly as in the proof of Theorem 4.8, it suffices to show that for any $m \in \mathbb{N}$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$ we have the convergence

$$N\left[1 - \mathbb{E}\left(\exp\left\{\frac{\mathrm{i}}{\sqrt{N\log N}}\sum_{k=1}^m \theta_k\left(X_k - \frac{\lambda}{1-\alpha}\right)\right\}\right)\right] \to \frac{\lambda\psi_1}{2}\left(\sum_{k=1}^m \theta_k\right)^2$$

as $N \to \infty$. By applying (2.4), the left hand side equals

$$N\,\mathbb{E}\left[1 - F_{0,\ldots,m-1}\left(\mathrm{e}^{\frac{\mathrm{i}\theta_1}{\sqrt{N\log N}}}, \ldots, \mathrm{e}^{\frac{\mathrm{i}\theta_m}{\sqrt{N\log N}}} \,\Big|\, \alpha\right) \mathrm{e}^{-\frac{\mathrm{i}\lambda(\theta_1+\cdots+\theta_m)}{(1-\alpha)\sqrt{N\log N}}}\right]$$
$$= N\,\mathbb{E}\left[1 - \mathrm{e}^{\frac{\lambda}{1-\alpha}B_N(\alpha)}\right] = N\int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a}B_N(a)}\right)\psi(a)\,\mathrm{d}a$$

with

$$B_N(a) := \sum_{k=1}^m \left(\mathrm{e}^{\frac{\mathrm{i}\theta_k}{\sqrt{N\log N}}} - 1 - \frac{\mathrm{i}\theta_k}{\sqrt{N\log N}}\right)$$
$$+ \sum_{1\leqslant \ell < j \leqslant m} a^{j-\ell}\big(\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{\sqrt{N\log N}}} - 1\big)\mathrm{e}^{\frac{\mathrm{i}(\theta_{\ell+1}+\cdots+\theta_{j-1})}{\sqrt{N\log N}}}\big(\mathrm{e}^{\frac{\mathrm{i}\theta_j}{\sqrt{N\log N}}} - 1\big), \qquad a \in [0,1].$$

Just like in the proof of Theorem 4.8 it is easy to see that for any $\varepsilon \in (0,1)$ we have

$$\sup_{a\in(0,1-\varepsilon)} |NB_N(a)| \leqslant \frac{(\sum_{k=1}^m \theta_k)^2}{2\log N} \to 0$$

as $N \to \infty$. Therefore, by Lemma B.2, substituting $a = 1 - z/N$, the statement of the theorem will follow from

$$\limsup_{N\to\infty} N\int_{1-\varepsilon}^1 \left|1 - \mathrm{e}^{\frac{\lambda}{1-a}B_N(a)}\right| \mathrm{d}a = \limsup_{N\to\infty} \int_0^{\varepsilon N} \left|1 - \mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\right| \mathrm{d}z < \infty, \tag{5.13}$$

and

$$\lim_{N\to\infty} N\int_{1-\varepsilon}^{1}\Bigl(1-\mathrm{e}^{\frac{\lambda}{1-a}B_N(a)}\Bigr)\,\mathrm{d}a = \lim_{N\to\infty}\int_0^{\varepsilon N}\Bigl(1-\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\Bigr)\,\mathrm{d}z = \frac{\lambda}{2}\left(\sum_{k=1}^m \theta_k\right)^2 \tag{5.14}$$

for all $\varepsilon\in(0,1)$. Next we check (5.13) and (5.14).

Using Taylor expansions, similarly as in the proof of Theorem 4.8, we get

$$\frac{\lambda N}{z}B_N\Bigl(1-\frac{z}{N}\Bigr) = -\frac{\lambda(\sum_{k=1}^m\theta_k)^2}{2z\log N} + \frac{\mathrm{O}(1)}{zN^{1/2}(\log N)^{3/2}} + \frac{\mathrm{O}(1)}{N\log N}. \tag{5.15}$$

Indeed, for $z\in[0,N]$ we have

$$\begin{aligned}
B_N\Bigl(1-\frac{z}{N}\Bigr) &= \sum_{k=1}^m\Bigl(\mathrm{e}^{\frac{\mathrm{i}\theta_k}{\sqrt{N\log N}}}-1-\frac{\mathrm{i}\theta_k}{\sqrt{N\log N}}\Bigr)\\
&\quad+\sum_{1\leqslant\ell<j\leqslant m}\Bigl(1-\frac{z}{N}\Bigr)^{j-\ell}\bigl(\mathrm{e}^{\frac{\mathrm{i}\theta_\ell}{\sqrt{N\log N}}}-1\bigr)\,\mathrm{e}^{\frac{\mathrm{i}(\theta_{\ell+1}+\cdots+\theta_{j-1})}{\sqrt{N\log N}}}\bigl(\mathrm{e}^{\frac{\mathrm{i}\theta_j}{\sqrt{N\log N}}}-1\bigr)\\
&=\sum_{k=1}^m\Bigl(-\frac{\theta_k^2}{2N\log N}+\frac{\mathrm{O}(1)}{(N\log N)^{3/2}}\Bigr)\\
&\quad+\sum_{1\leqslant\ell<j\leqslant m}\Bigl(1+\frac{z\,\mathrm{O}(1)}{N}\Bigr)\Bigl(\frac{\mathrm{i}\theta_\ell}{\sqrt{N\log N}}+\frac{\mathrm{O}(1)}{N\log N}\Bigr)\\
&\qquad\times\Bigl(1+\frac{\mathrm{O}(1)}{\sqrt{N\log N}}\Bigr)\Bigl(\frac{\mathrm{i}\theta_j}{\sqrt{N\log N}}+\frac{\mathrm{O}(1)}{N\log N}\Bigr)\\
&=-\frac{\sum_{k=1}^m\theta_k^2}{2N\log N}+\frac{\mathrm{O}(1)}{(N\log N)^{3/2}}-\frac{\sum_{1\leqslant\ell<j\leqslant m}\theta_\ell\theta_j}{N\log N}+\frac{\mathrm{O}(1)}{(N\log N)^{3/2}}+\frac{z\,\mathrm{O}(1)}{N^2\log N}\\
&=-\frac{\bigl(\sum_{k=1}^m\theta_k\bigr)^2}{2N\log N}+\frac{\mathrm{O}(1)}{(N\log N)^{3/2}}+\frac{z\,\mathrm{O}(1)}{N^2\log N},
\end{aligned}$$

since, by Bernoulli's inequality,

$$\left|\Bigl(1-\frac{z}{N}\Bigr)^{j-\ell}-1\right|\leqslant(j-\ell)\frac{z}{N}\leqslant m\frac{z}{N},$$

yielding that

$$\Bigl(1-\frac{z}{N}\Bigr)^{j-\ell}=1+\frac{z}{N}\,\mathrm{O}(1).$$

By (5.15), for $z\in\bigl(0,\frac{1}{\log N}\bigr)$ and for large enough N we have

$$\begin{aligned}
\frac{\lambda N}{z}\,\mathrm{Re}\,B_N\Bigl(1-\frac{z}{N}\Bigr) &= -\frac{\lambda(\sum_{k=1}^m\theta_k)^2}{2z\log N}\left(1-\frac{\mathrm{Re}\,\mathrm{O}(1)}{\sqrt{N\log N}}\right)+\frac{\mathrm{Re}\,\mathrm{O}(1)}{N\log N}\\
&\leqslant -\frac{\lambda(\sum_{k=1}^m\theta_k)^2}{4z\log N}+\frac{|\,\mathrm{O}(1)|}{N\log N}\leqslant-\frac{\lambda(\sum_{k=1}^m\theta_k)^2}{4}+\frac{|\,\mathrm{O}(1)|}{N\log N},
\end{aligned}$$

hence we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{\log N}} \left|1-\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\right| \mathrm{d}z \leqslant \int_0^{\frac{1}{\log N}} \left(1+\mathrm{e}^{\frac{\lambda N}{z}\operatorname{Re} B_N(1-\frac{z}{N})}\right) \mathrm{d}z \\
&\leqslant \frac{1}{\log N}\left(1+\exp\left\{-\frac{\lambda\left(\sum_{k=1}^m \theta_k\right)^2}{4}+\frac{|\,\mathrm{O}(1)|}{N\log N}\right\}\right) \to 0 \qquad \text{as} \quad N\to\infty.
\end{aligned} \tag{5.16}$$

Note that

$$\frac{1}{\log N}\int_{\frac{1}{\log N}}^{\varepsilon N} \frac{1}{z}\,\mathrm{d}z = \frac{\log\varepsilon+\log N+\log\log N}{\log N} \to 1 \qquad \text{as} \quad N\to\infty, \tag{5.17}$$

$$\frac{1}{\log N}\int_{\frac{1}{\log N}}^{\varepsilon N} \frac{1}{z^2}\,\mathrm{d}z = \frac{\varepsilon N\log N-1}{\varepsilon N\log N} \to 1 \qquad \text{as} \quad N\to\infty. \tag{5.18}$$

By (5.15), for all $z\in\left(\frac{1}{\log N},\varepsilon N\right)$, we have

$$\left|\frac{\lambda N}{z}B_N\left(1-\frac{z}{N}\right)\right| \leqslant \frac{\lambda(\sum_{k=1}^m\theta_k)^2}{2z\log N}+\frac{|\,\mathrm{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\,\mathrm{O}(1)|}{N\log N} = |\,\mathrm{O}(1)|. \tag{5.19}$$

Thus, by (B.3) and (5.17), we get

$$\begin{aligned}
&\limsup_{N\to\infty}\int_{\frac{1}{\log N}}^{\varepsilon N}\left|1-\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\right|\mathrm{d}z \\
&\leqslant \limsup_{N\to\infty}\int_{\frac{1}{\log N}}^{\varepsilon N}\left|\frac{\lambda N}{z}B_N(1-\frac{z}{N})\right|\mathrm{e}^{\left|\frac{\lambda N}{z}B_N(1-\frac{z}{N})\right|}\,\mathrm{d}z \\
&\leqslant \limsup_{N\to\infty}\mathrm{e}^{|\,\mathrm{O}(1)|}\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\frac{\lambda(\sum_{k=1}^m\theta_k)^2}{2z\log N}+\frac{|\,\mathrm{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\,\mathrm{O}(1)|}{N\log N}\right]\mathrm{d}z<\infty,
\end{aligned}$$

which, together with (5.16), imply (5.13).

Now we turn to prove (5.14). By (5.16), the convergence (5.14) reduces to prove that

$$\left|\int_{\frac{1}{\log N}}^{\varepsilon N}\left(1-\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}\right)\mathrm{d}z-\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2}\right| \to 0 \qquad \text{as} \quad N\to\infty.$$

Using (5.17), it is enough to check that

$$\left|\int_{\frac{1}{\log N}}^{\varepsilon N}\left(\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}-1+\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}\right)\mathrm{d}z\right| \to 0 \qquad \text{as} \quad N\to\infty.$$

By applying (B.4), (5.15) and (5.19), for large enough N we get

$$\begin{aligned}
&\left|\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\left(\mathrm{e}^{\frac{\lambda N}{z}B_N(1-\frac{z}{N})}-1\right)+\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}\right]\mathrm{d}z\right| \\
&\leqslant\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\frac{1}{2}\left|\frac{\lambda N}{z}B_N\Big(1-\frac{z}{N}\Big)\right|^2\mathrm{e}^{\left|\frac{\lambda N}{z}B_N(1-\frac{z}{N})\right|}+\left|\frac{\lambda N}{z}B_N\Big(1-\frac{z}{N}\Big)+\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}\right|\right]\mathrm{d}z \\
&\leqslant\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\frac{1}{2}\left(\frac{\lambda\left(\sum_{k=1}^m\theta_k\right)^2}{2z\log N}+\frac{|\,\mathrm{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\,\mathrm{O}(1)|}{N\log N}\right)^2\mathrm{e}^{|\,\mathrm{O}(1)|}\right. \\
&\qquad\qquad\left.+\frac{|\,\mathrm{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\,\mathrm{O}(1)|}{N\log N}\right]\mathrm{d}z \\
&\leqslant\int_{\frac{1}{\log N}}^{\varepsilon N}\left[\frac{3}{2}\left(\frac{|\,\mathrm{O}(1)|}{z^2(\log N)^2}+\frac{|\,\mathrm{O}(1)|}{z^2N(\log N)^3}+\frac{|\,\mathrm{O}(1)|}{N^2(\log N)^2}\right)+\frac{|\,\mathrm{O}(1)|}{zN^{1/2}(\log N)^{3/2}}+\frac{|\,\mathrm{O}(1)|}{N\log N}\right]\mathrm{d}z,
\end{aligned}$$

which converges to 0 as $N\to\infty$ using (5.17) and (5.18). This yields (5.14) completing the proof. □

First proof of Theorem 4.10. By Proposition 4.2, we have

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor}(X_k^{(1)}-\mathbb{E}(X_k^{(1)}\,|\,\alpha^{(1)}))\right)_{t\in\mathbb{R}_+}=\frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and α is a random variable having a density function of the form (4.5) with $\beta\in(-1,1)$ and $\psi_1\in(0,\infty)$, and being independent of B. Let $\mathcal{W}_t:=\frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha}B_t$, $t\in\mathbb{R}_+$, and $(\mathcal{W}_t^{(i)})_{t\in\mathbb{R}_+}$, $i\in\mathbb{N}$, be its independent copies. It remains to prove that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}\left(N^{-\frac{1}{1+\beta}}\sum_{i=1}^N\mathcal{W}_t^{(i)}\right)_{t\in\mathbb{R}_+}=\mathcal{Y}_{1+\beta}.$$

Using the continuity theorem and the continuous mapping theorem, it is enough to prove that for all $m\in\mathbb{N}$, $\theta_1,\ldots,\theta_m\in\mathbb{R}$ and $0=:t_0<t_1<t_2<\cdots<t_m$,

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{j=1}^m\theta_j\left(N^{-\frac{1}{1+\beta}}\sum_{i=1}^N(\mathcal{W}_{t_j}^{(i)}-\mathcal{W}_{t_{j-1}}^{(i)})\right)\right\}\right)=\left[\mathbb{E}\left(\exp\left\{\mathrm{i}N^{-\frac{1}{1+\beta}}\sum_{j=1}^m\theta_j(\mathcal{W}_{t_j}-\mathcal{W}_{t_{j-1}})\right\}\right)\right]^N \\
&\to\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{j=1}^m\theta_j(\mathcal{Y}_{1+\beta}(t_j)-\mathcal{Y}_{1+\beta}(t_{j-1}))\right\}\right)=\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{j=1}^m\theta_j\sqrt{Y_{(1+\beta)/2}}(B_{t_j}-B_{t_{j-1}})\right\}\right) \\
&=\mathbb{E}\left(\exp\left\{-\frac{1}{2}Y_{(1+\beta)/2}\sum_{j=1}^m\theta_j^2(t_j-t_{j-1})\right\}\right)=\exp\left\{-k_\beta\left(\frac{1}{2}\sum_{j=1}^m\theta_j^2(t_j-t_{j-1})\right)^{\frac{1+\beta}{2}}\right\}=\mathrm{e}^{-k_\beta\,\omega^{\frac{1+\beta}{2}}}
\end{aligned}$$

as $N \to \infty$, where $\omega := \frac{1}{2}\sum_{j=1}^m \theta_j^2(t_j - t_{j-1})$. Note that, using the independence of α and B, it suffices to show

$$\begin{aligned}
\Psi_N &:= N\left[1 - \mathbb{E}\left(\exp\left\{\mathrm{i}N^{-\frac{1}{1+\beta}}\sum_{j=1}^m \theta_j(\mathcal{W}_{t_j} - \mathcal{W}_{t_{j-1}})\right\}\right)\right] \\
&= N\left[1 - \mathbb{E}\left(\exp\left\{-\frac{1}{2}N^{-\frac{2}{1+\beta}}\lambda(1+\alpha)(1-\alpha)^{-2}\sum_{j=1}^m \theta_j^2(t_j - t_{j-1})\right\}\right)\right] \\
&= N\int_0^1 \left(1 - \mathrm{e}^{-\omega N^{-\frac{2}{1+\beta}}\lambda(1+a)(1-a)^{-2}}\right)\psi(a)(1-a)^\beta \,\mathrm{d}a \to k_\beta\,\omega^{\frac{1+\beta}{2}}
\end{aligned}$$

as $N \to \infty$, since it implies that $(1 - \Psi_N/N)^N \to \mathrm{e}^{-k_\beta\,\omega^{\frac{1+\beta}{2}}}$ as $N \to \infty$. For all $\varepsilon \in (0,1)$,

$$\sup_{a\in(0,1-\varepsilon)} \left|-N\omega N^{-\frac{2}{1+\beta}}(1+a)(1-a)^{-1}\right| = \omega N^{\frac{-1+\beta}{1+\beta}}(2-\varepsilon)\varepsilon^{-1} \to 0$$

as $N \to \infty$. Therefore, by Lemma B.2, substituting $a = 1 - N^{-\frac{1}{1+\beta}}y$, the statement of the theorem will follow from

$$\begin{aligned}
&\limsup_{N\to\infty} N\int_{1-\varepsilon}^1 \left|1 - \mathrm{e}^{-\omega N^{-\frac{2}{1+\beta}}\lambda(1+a)(1-a)^{-2}}\right|(1-a)^\beta \,\mathrm{d}a \\
&= \limsup_{N\to\infty} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \left|1 - \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\right| y^\beta \,\mathrm{d}y < \infty,
\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}
&\lim_{N\to\infty} N\int_{1-\varepsilon}^1 \left(1 - \mathrm{e}^{-\omega N^{-\frac{2}{1+\beta}}\lambda(1+a)(1-a)^{-2}}\right)(1-a)^\beta \,\mathrm{d}a \\
&= \lim_{N\to\infty} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \left(1 - \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\right) y^\beta \,\mathrm{d}y = \psi_1^{-1}k_\beta\,\omega^{\frac{1+\beta}{2}}
\end{aligned} \tag{5.21}$$

for all $\varepsilon \in (0,1)$. Next we prove (5.20) and (5.21).

For all $N \in \mathbb{N}$ and $\varepsilon \in (0,1)$, using (B.1), we have

$$\begin{aligned}
\int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \left|1 - \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}}y)y^{-2}}\right| y^\beta \,\mathrm{d}y &\leqslant \int_0^\infty \left|1 - \mathrm{e}^{-2\omega\lambda y^{-2}}\right| y^\beta \,\mathrm{d}y \\
&\leqslant \int_0^1 y^\beta \,\mathrm{d}y + 2\omega\lambda\int_1^\infty y^{\beta-2}\,\mathrm{d}y < \infty,
\end{aligned}$$

hence we obtain (5.20).

Now we turn to prove (5.21). For all $\varepsilon \in (0,1)$, we have

$$\left|\int_{\varepsilon N^{\frac{1}{1+\beta}}}^\infty \left(1 - \mathrm{e}^{-2\omega\lambda y^{-2}}\right) y^\beta \,\mathrm{d}y\right| \leqslant 2\omega\lambda\int_{\varepsilon N^{\frac{1}{1+\beta}}}^\infty y^{\beta-2}\,\mathrm{d}y = \frac{2\omega\lambda}{1-\beta}(\varepsilon N^{\frac{1}{1+\beta}})^{\beta-1} \to 0 \tag{5.22}$$

as $N \to \infty$. Further, using (B.3),

$$\left| \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}} y)y^{-2}}\Big) y^\beta \, \mathrm{d}y - \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-2\omega\lambda y^{-2}}\Big) y^\beta \, \mathrm{d}y \right|$$

$$\leqslant \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \left| \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}} y)y^{-2}} - \mathrm{e}^{-2\omega\lambda y^{-2}} \right| y^\beta \, \mathrm{d}y$$

$$= \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \mathrm{e}^{-2\omega\lambda y^{-2}} \left| \mathrm{e}^{\omega\lambda N^{-\frac{1}{1+\beta} a} y^{-1}} - 1 \right| y^\beta \, \mathrm{d}y$$

$$\leqslant \omega\lambda N^{-\frac{1}{1+\beta}} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \mathrm{e}^{-2\omega\lambda y^{-2}} \mathrm{e}^{\omega\lambda N^{-\frac{1}{1+\beta}} y^{-1}} y^{\beta-1} \, \mathrm{d}y$$

$$\leqslant \omega\lambda N^{-\frac{1}{1+\beta}} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \mathrm{e}^{-(2-\varepsilon)\omega\lambda y^{-2}} y^{\beta-1} \, \mathrm{d}y \leqslant \omega\lambda N^{-\frac{1}{1+\beta}} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} y^{\beta-1} \, \mathrm{d}y$$

$$= \omega\lambda N^{-\frac{1}{1+\beta}} \frac{(\varepsilon N^{\frac{1}{1+\beta}})^\beta}{\beta} = \omega\lambda \frac{\varepsilon^\beta N^{\frac{\beta-1}{1+\beta}}}{\beta} \to 0 \qquad \text{as} \quad N \to \infty,$$

hence, using (5.22), we conclude

$$\lim_{N\to\infty} \int_0^{\varepsilon N^{\frac{1}{1+\beta}}} \Big(1 - \mathrm{e}^{-\omega\lambda(2-N^{-\frac{1}{1+\beta}} y)y^{-2}}\Big) y^\beta \, \mathrm{d}y = \int_0^\infty \Big(1 - \mathrm{e}^{-2\omega\lambda y^{-2}}\Big) y^\beta \, \mathrm{d}y$$
$$= \frac{1}{2} (2\omega\lambda)^{\frac{1+\beta}{2}} \int_0^\infty (1 - \mathrm{e}^{-u}) u^{-\frac{3+\beta}{2}} \, \mathrm{d}u = \psi_1^{-1} k_\beta \, \omega^{\frac{1+\beta}{2}},$$

where the last equality follows by Lemma 2.2.1 in Zolotarev [38], thus we obtain (5.21). For the characteristic function of $Y_{(1+\beta)/2}$, see the second proof. □

Second proof of Theorem 4.10. By Proposition 4.2, we have

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \left(n^{-1/2} \sum_{k=1}^{\lfloor nt \rfloor} (X_k^{(1)} - \mathbb{E}(X_k^{(1)} \mid \alpha^{(1)})) \right)_{t\in\mathbb{R}_+} = \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and α is a random variable having a density function of the form (4.5) with $\beta \in (-1, 1)$ and $\psi_1 \in (0, \infty)$, and being independent of B. Hence it remains to prove that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (T_t^{(N)})_{t\in\mathbb{R}_+} = \big(\sqrt{Y_{(1+\beta)/2}} \, B_t\big)_{t\in\mathbb{R}_+},$$

where

$$T_t^{(N)} := \frac{1}{N^{\frac{1}{1+\beta}}} \sum_{j=1}^N \frac{\sqrt{\lambda(1+\alpha^{(j)})}}{1-\alpha^{(j)}} B_t^{(j)}, \qquad t \in \mathbb{R}_+, \qquad N \in \mathbb{N},$$

and $\alpha^{(j)}$, $j \in \mathbb{N}$, and $B^{(j)}$, $j \in \mathbb{N}$, are independent copies of α and B, respectively, being independent of each other as well. By the continuous mapping theorem, it is enough to show that for all $m \in \mathbb{N}$ and $0 =: t_0 \leqslant t_1 < t_2 < \cdots < t_m$,

$$\Big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\Big) \xrightarrow{\mathcal{D}} \big(\sqrt{Y_{(1+\beta)/2}}(B_{t_1} - B_{t_0}), \ldots, \sqrt{Y_{(1+\beta)/2}}(B_{t_m} - B_{t_{m-1}})\big)$$

as $N \to \infty$. By the portmanteau theorem, it is enough to check that for all $m \in \mathbb{N}$, $0 = t_0 \leqslant t_1 < t_2 < \cdots < t_m$, and for all bounded and continuous functions $g : \mathbb{R}^m \to \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\Big(g\big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\big)\Big) \\
&\qquad \to \mathbb{E}\left(g\left(\sqrt{Y_{(1+\beta)/2}}(B_{t_1} - B_{t_0}), \ldots, \sqrt{Y_{(1+\beta)/2}}(B_{t_m} - B_{t_{m-1}})\right)\right)
\end{aligned}$$

as $N \to \infty$. Since

$$\begin{aligned}
&\mathbb{E}\Big(g\big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\big)\Big) \\
&= \mathbb{E}\Big[\mathbb{E}\Big[g\big(T_{t_1}^{(N)} - T_{t_0}^{(N)}, \ldots, T_{t_m}^{(N)} - T_{t_{m-1}}^{(N)}\big) \,\Big|\, \alpha^{(j)},\, j \in \mathbb{N}\Big]\Big] \\
&= \mathbb{E}\left[g\left(\sqrt{N^{-\frac{2}{1+\beta}} \sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}}(\widetilde{B}_{t_1} - \widetilde{B}_{t_0}), \ldots, \sqrt{N^{-\frac{2}{1+\beta}} \sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}}(\widetilde{B}_{t_m} - \widetilde{B}_{t_{m-1}})\right)\right] \\
&= \mathbb{E}\left[h\left(N^{-\frac{2}{1+\beta}} \sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}, \widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_m}\right)\right],
\end{aligned}$$

where $(\widetilde{B}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process independent of $\alpha^{(j)}$, $j \in \mathbb{N}$, and $h : \mathbb{R}^{m+1} \to \mathbb{R}$ is an appropriate bounded and continuous function. Hence it is enough to prove that

$$\frac{1}{N^{\frac{2}{1+\beta}}} \sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2} \xrightarrow{\mathcal{D}} Y_{(1+\beta)/2} \qquad \text{as } N \to \infty, \tag{5.23}$$

i.e., it suffices to show that $\frac{\lambda(1+\alpha)}{(1-\alpha)^2}$ belongs to the domain of normal attraction of the $\frac{1+\beta}{2}$-stable law of $Y_{(1+\beta)/2}$. Indeed, then, by the continuity theorem,

$$\left(N^{-\frac{2}{1+\beta}} \sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}, \widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_m}\right) \xrightarrow{\mathcal{D}} (Y_{(1+\beta)/2}, \widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_m}) \qquad \text{as } N \to \infty,$$

where we additionally suppose that $(\widetilde{B}_t)_{t\in\mathbb{R}_+}$ is independent of $Y_{(1+\beta)/2}$ as well. Hence, using again the portmanteau theorem,

$$\begin{aligned}
&\mathbb{E}\left[h\left(N^{-\frac{2}{1+\beta}} \sum_{j=1}^N \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2}, \widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_m}\right)\right] \\
&\qquad \to \mathbb{E}\left[h\big(Y_{(1+\beta)/2}, \widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_m}\big)\right] \\
&\qquad\quad = \mathbb{E}\left[g\Big(\sqrt{Y_{(1+\beta)/2}}(\widetilde{B}_{t_1} - \widetilde{B}_{t_0}), \ldots, \sqrt{Y_{(1+\beta)/2}}(\widetilde{B}_{t_m} - \widetilde{B}_{t_{m-1}})\Big)\right]
\end{aligned}$$

as $N \to \infty$, as desired. Note that

$$\lim_{x\to-\infty} |x|^{\frac{1+\beta}{2}} \mathbb{P}\bigg(\frac{\lambda(1+\alpha)}{(1-\alpha)^2} < x\bigg) = 0,$$

and

$$\lim_{x\to\infty} x^{\frac{1+\beta}{2}} \mathbb{P}\bigg(\frac{\lambda(1+\alpha)}{(1-\alpha)^2} > x\bigg) = \frac{\psi_1(2\lambda)^{\frac{1+\beta}{2}}}{1+\beta}.$$

Indeed, the first convergence follows immediately due to the positivity of $\frac{\lambda(1+\alpha)}{(1-\alpha)^2}$, and using that

$$\frac{\lambda(1+\alpha)}{(1-\alpha)^2} = 2\lambda\left(\left(\frac{1}{1-\alpha}-\frac{1}{4}\right)^2 - \frac{1}{16}\right)$$

and $\frac{1}{1-\alpha} \geqslant 1$, we have for all $x > 0$,

$$\frac{\lambda(1+\alpha)}{(1-\alpha)^2} > x \qquad \Longleftrightarrow \qquad \alpha > 1 - \frac{1}{\frac{1}{4}+\sqrt{\frac{x}{2\lambda}+\frac{1}{16}}} =: 1-\widetilde{h}(\lambda, x),$$

and hence

$$\begin{aligned}
x^{\frac{1+\beta}{2}} \mathbb{P}\bigg(\frac{\lambda(1+\alpha)}{(1-\alpha)^2} > x\bigg) &= x^{\frac{1+\beta}{2}} \int_{1-\widetilde{h}(\lambda,x)}^1 (1-a)^\beta \psi(a)\,\mathrm{d}a \\
&= \int_0^{\sqrt{x}\widetilde{h}(\lambda,x)} y^\beta \psi\Big(1-\frac{y}{\sqrt{x}}\Big)\,\mathrm{d}y \to \psi_1 \int_0^{\sqrt{2\lambda}} y^\beta\,\mathrm{d}y = \frac{\psi_1(2\lambda)^{\frac{1+\beta}{2}}}{1+\beta} \qquad \text{as } x\to\infty,
\end{aligned}$$

as desired. Indeed, one can use the dominated convergence theorem, since there exists $\varepsilon \in (0,1)$ such that $|\psi(x)-\psi_1| < 2\psi_1$ for all $x \in (1-\varepsilon, 1)$, and if $y \in (0, \sqrt{x}\widetilde{h}(\lambda,x))$, then $1-\frac{y}{\sqrt{x}} \in (1-\widetilde{h}(\lambda,x), 1)$ and hence, for large enough x, we get

$$\psi\Big(1-\frac{y}{\sqrt{x}}\Big) \leqslant 3\psi_1, \qquad y \in (0, \sqrt{x}\widetilde{h}(\lambda,x)).$$

Since $\sqrt{x}\widetilde{h}(\lambda,x) \leqslant \sqrt{2\lambda}$, $x \in \mathbb{R}_{++}$, this yields that $3\psi_1 y^\beta \mathbb{1}_{[0,\sqrt{2\lambda}]}(y)$, $y \in \mathbb{R}_+$, serves as an integrable dominating function for large enough x. Consequently (5.23) holds, see, e.g., Puplinskaitė and Surgailis [27, Remark 2.1]. Indeed, the characteristic function of the random variable $Y_{(1+\beta)/2}$ takes the form

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Y_{(1+\beta)/2}}) \\
&= \exp\left\{-|\theta|^{\frac{1+\beta}{2}} \frac{\Gamma(2-\frac{1+\beta}{2})}{1-\frac{1+\beta}{2}} \frac{\psi_1(2\lambda)^{\frac{1+\beta}{2}}}{1+\beta}\left(\cos\left(\frac{\pi(1+\beta)}{4}\right) - \mathrm{i}\,\mathrm{sign}(\theta)\sin\left(\frac{\pi(1+\beta)}{4}\right)\right)\right\} \\
&= \exp\left\{-|\theta|^{\frac{1+\beta}{2}} \Gamma\Big(1-\frac{1+\beta}{2}\Big) \frac{\psi_1(2\lambda)^{\frac{1+\beta}{2}}}{1+\beta}\left(\cos\left(\mathrm{sign}(\theta)\frac{\pi(1+\beta)}{4}\right) - \mathrm{i}\sin\left(\mathrm{sign}(\theta)\frac{\pi(1+\beta)}{4}\right)\right)\right\} \\
&= \exp\left\{-k_\beta|\theta|^{\frac{1+\beta}{2}} \mathrm{e}^{-\mathrm{i}\,\mathrm{sign}(\theta)\frac{\pi(1+\beta)}{4}}\right\}, \qquad \theta \in \mathbb{R}.
\end{aligned}$$

This can be also seen using, for example, Theorem C.3 in Zolotarev [38] (with the choices $\alpha = \frac{1+\beta}{2}$, $\beta = 1$, $\gamma = 0$ and $\lambda = k_\beta$). $\Box$

Proof of Theorem 4.11. Since $\mathbb{E}\big(\frac{1}{1-\alpha}\big) < \infty$, by Proposition 4.1, we have

$$\frac{1}{\sqrt{N}} \widetilde{S}^{(N)} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \widetilde{\mathcal{Y}} \qquad \text{as} \ \ N \to \infty,$$

where the strictly stationary Gaussian process $(\widetilde{\mathcal{Y}}_k)_{k\in\mathbb{Z}_+}$ is given in Proposition 4.1. Consequently, by the continuous mapping theorem, for all $n \in \mathbb{N}$, we get

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} (nN)^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = \left(n^{-1/2} \sum_{k=1}^{\lfloor nt\rfloor} \widetilde{\mathcal{Y}}_k \right)_{t\in\mathbb{R}_+},$$

and hence it remains to prove that

$$\left(n^{-1/2} \sum_{k=1}^{\lfloor nt\rfloor} \widetilde{\mathcal{Y}}_k \right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}_{\mathrm{f}}} \sigma B \qquad \text{as} \ \ n \to \infty.$$

Since the processes $\big(n^{-1/2} \sum_{k=1}^{\lfloor nt\rfloor} \widetilde{\mathcal{Y}}_k\big)_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, and σB are zero mean Gaussian processes, it suffices to show that the covariance function of $\big(n^{-1/2} \sum_{k=1}^{\lfloor nt\rfloor} \widetilde{\mathcal{Y}}_k\big)_{t\in\mathbb{R}_+}$ converges pointwise to that of σB as $n \to \infty$. For all $0 \leqslant t_1 \leqslant t_2$,

$$\begin{aligned} \operatorname{Cov}\left(n^{-1/2} \sum_{k=1}^{\lfloor nt_1\rfloor} \widetilde{\mathcal{Y}}_k, n^{-1/2} \sum_{k=1}^{\lfloor nt_2\rfloor} \widetilde{\mathcal{Y}}_k \right) &= \frac{\lambda}{n} \mathbb{E}\left(\sum_{k=1}^{\lfloor nt_1\rfloor} \sum_{\ell=1}^{\lfloor nt_2\rfloor} \frac{\alpha^{|k-\ell|}}{1-\alpha} \right) \\ \to \lambda \mathbb{E}\left(\frac{1+\alpha}{(1-\alpha)^2} \right) \min(t_1, t_2) &= \operatorname{Cov}(\sigma B_{t_1}, \sigma B_{t_2}) \qquad \text{as} \ \ n \to \infty, \end{aligned}$$

since one can use the decomposition (5.7) together with

$$\frac{1}{n} \mathbb{E}\left(\frac{\alpha(\alpha^{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor} - 1) + (\lfloor nt_2\rfloor - \lfloor nt_1\rfloor)(1-\alpha^2)/2}{(1-\alpha)^3} \right) \to (t_2 - t_1)\, \mathbb{E}\left(\frac{1+\alpha}{2(1-\alpha)^2} \right)$$

as $n \to \infty$. Indeed, by the dominated convergence theorem,

$$\frac{1}{n} \mathbb{E}\left(\frac{\alpha(\alpha^{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor} - 1)}{(1-\alpha)^3} \right) \to 0 \qquad \text{as} \ \ n \to \infty,$$

where the pointwise convergence follows by

$$\left| \frac{\alpha(\alpha^{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor} - 1)}{(1-\alpha)^3} \right| \leqslant \frac{1}{(1-\alpha)^3},$$

and $(t_2 - t_1 + 1)\frac{\alpha}{(1-\alpha)^2}$ serves as an integrable dominating function, since, by Remark 4.5, $\mathbb{E}\left(\frac{\alpha}{(1-\alpha)^2}\right) < \infty$, and

$$\begin{aligned} \frac{1}{n} \left| \frac{\alpha(\alpha^{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor} - 1)}{(1-\alpha)^3} \right| &= \frac{\alpha(1 + \alpha + \alpha^2 + \cdots + \alpha^{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor - 1})}{n(1-\alpha)^2} \\ &\leqslant \frac{\alpha(\lfloor nt_2\rfloor - \lfloor nt_1\rfloor)}{n(1-\alpha)^2} \leqslant (t_2 - t_1 + 1)\frac{\alpha}{(1-\alpha)^2}. \end{aligned}$$

For the second convergence, first note that, by Proposition 4.2, we have

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt\rfloor} (X_k^{(1)} - \mathbb{E}(X_k^{(1)} \mid \alpha^{(1)})) \right)_{t\in\mathbb{R}_+} = \frac{\sqrt{\lambda(1+\alpha)}}{1-\alpha} B,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and α is a random variable having a density function of the form (4.5) with $\beta \in (1,\infty)$ and $\psi_1 \in (0,\infty)$, and being independent of B. Hence it remains to prove that

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\sqrt{\lambda(1+\alpha^{(j)})}}{1-\alpha^{(j)}} B^{(j)} = \sigma B,$$

where $\alpha^{(j)}$, $j \in \mathbb{N}$, and $B^{(j)}$, $j \in \mathbb{N}$, are independent copies of α and B, respectively, being independent of each other as well. Similarly to the second proof of Theorem 4.10, it is enough to show that

$$\frac{1}{N} \sum_{j=1}^{N} \frac{\lambda(1+\alpha^{(j)})}{(1-\alpha^{(j)})^2} \xrightarrow{\mathcal{D}} \sigma^2 \qquad \text{as } N \to \infty.$$

This readily follows by the strong law of large numbers, since $\mathbb{E}\big(\frac{\lambda(1+\alpha)}{(1-\alpha)^2}\big) < \infty$ due to Remark 4.5. $\square$

Proof of Theorem 4.12. We have a decomposition

$$S_t^{(N,n)} = R_t^{(N,n)} + \widetilde{S}_t^{(N,n)}, \qquad t \in \mathbb{R}_+, \tag{5.24}$$

with

$$R_t^{(N,n)} := \sum_{j=1}^{N} \sum_{k=1}^{\lfloor nt\rfloor} (\mathbb{E}(X_k^{(j)} \mid \alpha^{(j)}) - \mathbb{E}(X_k^{(j)})) = \lfloor nt\rfloor \sum_{j=1}^{N} \left(\frac{\lambda}{1-\alpha^{(j)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}} \right) \right)$$

for $t \in \mathbb{R}_+$. By Theorem 4.7, for each $n \in \mathbb{N}$, $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)}$ exists, hence

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} N^{-\frac{1}{1+\beta}} \widetilde{S}^{(N,n)} = \mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} N^{\frac{\beta-1}{2(1+\beta)}} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = 0. \tag{5.25}$$

The distribution of the random variable $\lambda(1-\alpha)^{-1} - \mathbb{E}(\lambda(1-\alpha)^{-1})$ belongs to the domain of attraction of an $(1+\beta)$-stable distribution. Indeed, we have

$$\begin{aligned}
&\lim_{x\to\infty} x^{1+\beta}\, \mathbb{P}\left(\frac{\lambda}{1-\alpha} - \mathbb{E}\left(\frac{\lambda}{1-\alpha} \right) > x \right) \\
&= \lim_{x\to\infty} x^{1+\beta}\, \mathbb{P}\left(\alpha > 1 - \frac{1}{\lambda^{-1}x + \mathbb{E}((1-\alpha)^{-1})} \right) \\
&= \lim_{x\to\infty} \frac{1}{x^{-(1+\beta)}} \int_{1-(\lambda^{-1}x+\mathbb{E}((1-\alpha)^{-1}))^{-1}}^{1} \psi(a)(1-a)^{\beta}\, \mathrm{d}a \\
&= \lim_{x\to\infty} \frac{-\psi(1-(\lambda^{-1}x+\mathbb{E}((1-\alpha)^{-1}))^{-1})(\lambda^{-1}x+\mathbb{E}((1-\alpha)^{-1}))^{-\beta-2}\lambda^{-1}}{-(1+\beta)x^{-(1+\beta)-1}} = \frac{\psi_1\lambda^{1+\beta}}{1+\beta}
\end{aligned}$$

by L'Hôpital's rule. Further, using that $\mathbb{P}(\lambda(1-\alpha)^{-1}>0)=1$,

$$\lim_{x\to-\infty}|x|^{1+\beta}\,\mathbb{P}\left(\frac{\lambda}{1-\alpha}-\mathbb{E}\left(\frac{\lambda}{1-\alpha}\right)\leqslant x\right)=\lim_{x\to-\infty}|x|^{1+\beta}\cdot 0=0.$$

Consequently, for each $n\in\mathbb{N}$,

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} N^{-\frac{1}{1+\beta}}R^{(N,n)}=\big(\lfloor nt\rfloor Z_{1+\beta}\big)_{t\in\mathbb{R}_+},$$

see, e.g., Puplinskaitė and Surgailis [27, Remark 2.1]. Indeed, the characteristic function of the random variable $Z_{1+\beta}$ takes the form

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Z_{1+\beta}})\\
&=\exp\left\{-|\theta|^{1+\beta}\frac{\Gamma(2-(1+\beta))}{1-(1+\beta)}\frac{\psi_1\lambda^{1+\beta}}{1+\beta}\left(\cos\left(\frac{\pi(1+\beta)}{2}\right)-\mathrm{i}\,\mathrm{sign}(\theta)\sin\left(\frac{\pi(1+\beta)}{2}\right)\right)\right\}\\
&=\exp\left\{-|\theta|^{1+\beta}\frac{\Gamma(1-\beta)}{-\beta}\frac{\psi_1\lambda^{1+\beta}}{1+\beta}\mathrm{e}^{-\mathrm{i}\,\mathrm{sign}(\theta)\frac{\pi(1+\beta)}{2}}\right\}\\
&=\exp\left\{-|\theta|^{1+\beta}\omega_\beta(\theta)\right\},\qquad \theta\in\mathbb{R}.
\end{aligned}$$

Together with (5.25), we obtain the first convergence.

By Theorem 4.10, for each $N\in\mathbb{N}$, $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}}\widetilde{S}^{(N,n)}$ exists and hence

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1}\widetilde{S}^{(N,n)}=\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-\frac{1}{2}}n^{-\frac{1}{2}}\widetilde{S}^{(N,n)}=0,$$

and

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1}R^{(N,n)}=\left(t\sum_{j=1}^N\left(\frac{\lambda}{1-\alpha^{(j)}}-\mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}}\right)\right)\right)_{t\in\mathbb{R}_+}.$$

Based on the above considerations, using the decomposition (5.24) as well, we obtain the second convergence. □

Proof of Theorem 4.13. First note that, since $\beta>1$, by Remark 4.5, $\operatorname{Var}((1-\alpha)^{-1})<\infty$. Hence, by the central limit theorem, for each $n\in\mathbb{N}$,

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} N^{-\frac{1}{2}}R^{(N,n)}=\big(\lfloor nt\rfloor W_{\lambda^2\operatorname{Var}((1-\alpha)^{-1})}\big)_{t\in\mathbb{R}_+}.$$

Consequently,

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1}N^{-\frac{1}{2}}R^{(N,n)}=(W_{\lambda^2\operatorname{Var}((1-\alpha)^{-1})}t)_{t\in\mathbb{R}_+}.$$

By Theorem 4.11, $\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty}(nN)^{-\frac{1}{2}}\widetilde{S}^{(N,n)}$ exists, hence

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-1}N^{-\frac{1}{2}}\widetilde{S}^{(N,n)}=\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty}\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} n^{-\frac{1}{2}}(nN)^{-\frac{1}{2}}\widetilde{S}^{(N,n)}=0.$$

Using the decomposition (5.24), we have the first convergence.

Similarly, for each $N \in \mathbb{N}$,

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1} R^{(N,n)} = \left(\sum_{j=1}^{N} \left(\frac{\lambda}{1-\alpha^{(j)}} - \mathbb{E}\left(\frac{\lambda}{1-\alpha^{(j)}} \right) \right) t \right)_{t\in\mathbb{R}_+},$$

and, by the central limit theorem,

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{2}} R^{(N,n)} = (W_{\lambda^2 \operatorname{Var}((1-\alpha)^{-1})} t)_{t\in\mathbb{R}_+}.$$

By Theorem 4.11, we also have

$$\mathcal{D}_{\mathrm{f}}\text{-}\lim_{N\to\infty} \mathcal{D}_{\mathrm{f}}\text{-}\lim_{n\to\infty} n^{-1} N^{-\frac{1}{2}} \widetilde{S}^{(N,n)} = 0,$$

which yields the second convergence using the decomposition (5.24) as well. □

Appendices

A Non-Markov property of the randomized INAR(1) model

The aim of this appendix is to show that the randomized INAR(1) process $(X_k)_{k\in\mathbb{Z}_+}$ defined in Section 4 does not have the Markov property provided that α is non-degenerate. We show that if α is non-degenerate, then

$$\mathbb{P}(X_2 = 0 \,|\, X_1 = 1, X_0 = 0) \neq \mathbb{P}(X_2 = 0 \,|\, X_1 = 1),$$

implying our statement. By the strict stationarity of $(X_k)_{k\in\mathbb{Z}_+}$, the conditional independence of $\xi_{1,1}$, ε_1 and X_0 given α, and (4.1)–(4.3), we have

$$\begin{aligned}
\mathbb{P}(X_2 = 0 \,|\, X_1 = 1) &= \mathbb{P}(X_1 = 0 \,|\, X_0 = 1) = \frac{\mathbb{P}(X_1 = 0, X_0 = 1)}{\mathbb{P}(X_0 = 1)} = \frac{\mathbb{P}(\xi_{1,1} = 0, \varepsilon_1 = 0, X_0 = 1)}{\mathbb{P}(X_0 = 1)} \\
&= \frac{\int_0^1 \mathbb{P}(\xi_{1,1} = 0, \varepsilon_1 = 0, X_0 = 1 \,|\, \alpha = a)\, \mathbb{P}_\alpha(\mathrm{d}a)}{\int_0^1 \mathbb{P}(X_0 = 1 \,|\, \alpha = a)\, \mathbb{P}_\alpha(\mathrm{d}a)} \\
&= \frac{\int_0^1 \mathbb{P}(\xi_{1,1} = 0 \,|\, \alpha = a)\, \mathbb{P}(\varepsilon_1 = 0 \,|\, \alpha = a)\, \mathbb{P}(X_0 = 1 \,|\, \alpha = a)\, \mathbb{P}_\alpha(\mathrm{d}a)}{\int_0^1 \mathbb{P}(X_0 = 1 \,|\, \alpha = a)\, \mathbb{P}_\alpha(\mathrm{d}a)} \\
&= \frac{\int_0^1 (1-a) \mathrm{e}^{-\lambda} \frac{\lambda}{1-a} \mathrm{e}^{-\frac{\lambda}{1-a}}\, \mathbb{P}_\alpha(\mathrm{d}a)}{\int_0^1 \frac{\lambda}{1-a} \mathrm{e}^{-\frac{\lambda}{1-a}}\, \mathbb{P}_\alpha(\mathrm{d}a)} = \frac{\int_0^1 \mathrm{e}^{-\lambda-\frac{\lambda}{1-a}}\, \mathbb{P}_\alpha(\mathrm{d}a)}{\int_0^1 \frac{1}{1-a} \mathrm{e}^{-\frac{\lambda}{1-a}}\, \mathbb{P}_\alpha(\mathrm{d}a)}.
\end{aligned}$$

Similarly, we have

$$\mathbb{P}(X_2 = 0 \,|\, X_1 = 1, X_0 = 0) = \frac{\mathbb{P}(X_2 = 0, X_1 = 1, X_0 = 0)}{\mathbb{P}(X_1 = 1, X_0 = 0)} = \frac{\mathbb{P}(\xi_{2,1} = 0, \varepsilon_2 = 0, \varepsilon_1 = 1, X_0 = 0)}{\mathbb{P}(\varepsilon_1 = 1, X_0 = 0)}$$

$$= \frac{\int_0^1 (1-a)\mathrm{e}^{-\lambda}\lambda \mathrm{e}^{-\lambda}\,\mathrm{e}^{-\frac{\lambda}{1-a}}\,\mathbb{P}_\alpha(\mathrm{d}a)}{\int_0^1 \lambda\mathrm{e}^{-\lambda}\,\mathrm{e}^{-\frac{\lambda}{1-a}}\,\mathbb{P}_\alpha(\mathrm{d}a)} = \frac{\int_0^1 (1-a)\mathrm{e}^{-\lambda-\frac{\lambda}{1-a}}\,\mathbb{P}_\alpha(\mathrm{d}a)}{\int_0^1 \mathrm{e}^{-\frac{\lambda}{1-a}}\,\mathbb{P}_\alpha(\mathrm{d}a)}.$$

By Cauchy–Schwarz's inequality, we have

$$\left(\int_0^1 \mathrm{e}^{-\frac{\lambda}{1-a}}\,\mathbb{P}_\alpha(\mathrm{d}a)\right)^2 \leqslant \int_0^1 (1-a)\mathrm{e}^{-\frac{\lambda}{1-a}}\,\mathbb{P}_\alpha(\mathrm{d}a)\int_0^1 \frac{1}{1-a}\mathrm{e}^{-\frac{\lambda}{1-a}}\,\mathbb{P}_\alpha(\mathrm{d}a),$$

and equality holds if and only if there exists some positive constant $C > 0$ such that $(1-a)\mathrm{e}^{-\frac{\lambda}{1-a}} = C\frac{1}{1-a}\mathrm{e}^{-\frac{\lambda}{1-a}}$ $\mathbb{P}_\alpha$-almost every $a \in (0,1)$, which is equivalent to the fact that there exists $C \in (0,1)$ such that $\mathbb{P}_\alpha$ is the Dirac measure concentrated on the point $1-\sqrt{C}$. Consequently, $\mathbb{P}(X_2 = 0 \,|\, X_1 = 1, X_0 = 0) \geqslant \mathbb{P}(X_2 = 0 \,|\, X_1 = 1)$ and equality holds if and only if $\mathbb{P}_\alpha$ is a Dirac measure concentrated on some point in $(0,1)$, i.e., α is degenerate. Hence if α is non-degenerate, then the randomized INAR(1) process $(X_k)_{k\in\mathbb{Z}_+}$ does not have the Markov property. If α is degenerate, then $(X_k)_{k\in\mathbb{Z}_+}$ is a usual INAR(1) model being a Markov chain.

B Approximations of the exponential function and some of its integrals

In this appendix we collect some useful approximations of the exponential function and some of its integrals.

We will frequently use the following the well-known inequalities:

(B.1) $$1 - \mathrm{e}^{-x} \leqslant x, \qquad x \in \mathbb{R},$$

(B.2) $$|\mathrm{e}^{\mathrm{i}u} - 1| \leqslant |u|, \qquad |\mathrm{e}^{\mathrm{i}u} - 1 - \mathrm{i}u| \leqslant u^2/2, \qquad u \in \mathbb{R}.$$

The next lemma is about how the inequalities in (B.2) change if we replace $u \in \mathbb{R}$ by an arbitrary complex number.

B.1 Lemma. *We have*

(B.3) $$|\mathrm{e}^z - 1| \leqslant |z|\mathrm{e}^{|z|}, \qquad z \in \mathbb{C},$$

(B.4) $$|\mathrm{e}^z - 1 - z| \leqslant \frac{|z|^2}{2}\mathrm{e}^{|z|}, \qquad z \in \mathbb{C}.$$

Proof. For any $z \in \mathbb{C}$ we have

$$|\mathrm{e}^z - 1| = \left| z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \right| \leqslant |z| \left(1 + \frac{|z|}{2!} + \frac{|z|^2}{3!} + \ldots \right)$$

$$\leqslant |z| \left(1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \ldots \right) = |z| \mathrm{e}^{|z|},$$

$$|\mathrm{e}^z - 1 - z| = \left| \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \right| \leqslant \frac{|z|^2}{2} \left(1 + \frac{|z|}{3} + \frac{|z|^2}{3 \cdot 4} + \ldots \right)$$

$$\leqslant \frac{|z|^2}{2} \left(1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \ldots \right) = \frac{|z|^2}{2} \mathrm{e}^{|z|},$$

since $3 \cdot 4 \cdots (n+2) \geqslant n!$ for any $n \in \mathbb{N}$. □

B.2 Lemma. *Suppose that* $(0,1) \ni x \mapsto \psi(x)(1-x)^\beta$ *is a probability density, where* ψ *is a function on* $(0,1)$ *having a limit* $\lim_{x \uparrow 1} \psi(x) = \psi_1 \in (0, \infty)$ *(and necessarily* $\beta \in (-1, \infty)$*). For all* $a \in (0,1)$*, let* $(z_N(a))_{N \in \mathbb{N}}$ *be a sequence of complex numbers such that*

$$\lim_{N \to \infty} \sup_{a \in (0, 1-\varepsilon)} |N z_N(a)| = 0 \qquad \textit{for all } \varepsilon \in (0,1), \tag{B.5}$$

$$\limsup_{N \to \infty} N \int_{1-\varepsilon_0}^1 \left| 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right| (1-a)^\beta \, \mathrm{d}a < \infty \qquad \textit{for some } \varepsilon_0 \in (0,1),$$

$$\lim_{\varepsilon \downarrow 0} \limsup_{N \to \infty} \left| N \int_{1-\varepsilon}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) (1-a)^\beta \, \mathrm{d}a - I \right| = 0$$

with some $I \in \mathbb{C}$*. Then*

$$\lim_{N \to \infty} N \int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \psi(a)(1-a)^\beta \, \mathrm{d}a = \psi_1 I.$$

Proof. Using dominated convergence theorem, first we check that

$$\lim_{N \to \infty} N \int_0^{1-\varepsilon} \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \psi(a)(1-a)^\beta \, \mathrm{d}a = 0 \qquad \text{for all } \varepsilon \in (0,1). \tag{B.6}$$

By applying (B.3) and using (B.5), for any $\varepsilon \in (0,1)$ and $a \in (0, 1-\varepsilon)$, we get

$$\left| N \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \right| \leqslant N \left| \frac{\lambda}{1-a} z_N(a) \right| \mathrm{e}^{\left| \frac{\lambda}{1-a} z_N(a) \right|} \to 0 \tag{B.7}$$

as $N \to \infty$. Further, if $\varepsilon \in (0,1)$ and $a \in (0, 1-\varepsilon)$, then

$$\left| N \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \right| \leqslant \frac{\lambda}{\varepsilon} \sup_{N \in \mathbb{N}} \sup_{a \in (0,1-\varepsilon)} |N z_N(a)| \, \mathrm{e}^{\frac{\lambda}{\varepsilon} \sup_{N \in \mathbb{N}} \sup_{a \in (0,1-\varepsilon)} |z_N(a)|} =: C_\varepsilon,$$

where $C_\varepsilon \in \mathbb{R}_+$. Since $\int_0^1 \psi(a)(1-a)^\beta \, \mathrm{d}a = 1$, we have

$$\left| N \int_0^{1-\varepsilon} \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right) \psi(a)(1-a)^\beta \, \mathrm{d}a \right| \leqslant \int_0^{1-\varepsilon} C_\varepsilon \psi(a)(1-a)^\beta \, \mathrm{d}a < \infty.$$

Therefore, $(0,1-\varepsilon) \ni a \mapsto C_\varepsilon \psi(a)(1-a)^\beta$ serves as a dominating integrable function. Thus the pointwise convergence in (B.7) results (B.6). Moreover, for all $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
&\left| N \int_0^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) \psi(a)(1-a)^\beta \, \mathrm{d}a - \psi_1 I \right| \\
&\qquad \leqslant \left| N \int_0^{1-\varepsilon} \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) \psi(a)(1-a)^\beta \, \mathrm{d}a \right| \\
&\qquad\quad + \left| N \int_{1-\varepsilon}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) (\psi(a) - \psi_1)(1-a)^\beta \, \mathrm{d}a \right| \\
&\qquad\quad + \psi_1 \left| N \int_{1-\varepsilon}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) (1-a)^\beta \, \mathrm{d}a - I \right|,
\end{aligned}$$

where

$$\begin{aligned}
&\left| N \int_{1-\varepsilon}^1 \left(1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)}\right) (\psi(a) - \psi_1)(1-a)^\beta \, \mathrm{d}a \right| \\
&\leqslant N \sup_{a \in [1-\varepsilon, 1)} |\psi(a) - \psi_1| \int_{1-\varepsilon}^1 \left| 1 - \mathrm{e}^{\frac{\lambda}{1-a} z_N(a)} \right| (1-a)^\beta \, \mathrm{d}a,
\end{aligned}$$

with $\sup_{a \in [1-\varepsilon,1)} |\psi(a) - \psi_1| \to 0$ as $\varepsilon \downarrow 0$, by the assumption. First taking $\limsup_{N\to\infty}$ and then $\varepsilon \downarrow 0$, using (B.6), we obtain the statement. □

C A representation of fractional Brownian motion due to Pilipauskaitė and Surgailis [23]

We recall an integral representation of the fractional Brownian motion with Hurst parameter in $\left(\frac{1}{2}, 1\right)$ due to Pilipauskaitė and Surgailis [23] in order to connect our results with the ones in Pilipauskaitė and Surgailis [23] and in Puplinskaitė and Surgailis [26], [27].

For all $\beta \in (0,1)$ let us consider the stochastic process given by

(C.1) $$\widetilde{\mathcal{B}}_{1-\frac{\beta}{2}}(t) := \int_{\mathbb{R}_+ \times \mathbb{R}} (f(x, t-s) - f(x, -s)) \, Z(\mathrm{d}x, \mathrm{d}s), \qquad t \in \mathbb{R}_+,$$

where

(C.2) $$f(x,t) := \begin{cases} (1 - \mathrm{e}^{-xt})/x & \text{if } x \in \mathbb{R}_{++} \text{ and } t \in \mathbb{R}_{++}, \\ 0 & \text{otherwise,} \end{cases}$$

with respect to a Gaussian random measure $Z(\mathrm{d}x, \mathrm{d}s)$ on $\mathbb{R}_+ \times \mathbb{R}$ with zero mean, variance $\nu(\mathrm{d}x, \mathrm{d}s) := (2-\beta)(1-\beta)/\Gamma(\beta) x^\beta \, \mathrm{d}x \, \mathrm{d}s$ and characteristic function $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta Z(A)}) = \mathrm{e}^{-\theta^2 \nu(A)/2}$ for each Borel set $A \subset \mathbb{R}_+ \times \mathbb{R}$ with $\nu(A) < \infty$ and $\theta \in \mathbb{R}$. Note that, by Pilipauskaitė and

Surgailis [23, page 1014], $(\widetilde{\mathcal{B}}_{1-\frac{\beta}{2}}(t))_{t\in\mathbb{R}_+}$ is a fractional Brownian motion multiplied by some constant. In what follows we check that this constant is in fact one. It suffices to show that the variance of the process defined in (C.1) at time 1 is 1. By (C.1) and (C.2) (see also formula (2.4) in Pilipauskaitė and Surgailis [23]) one can easily see that the variance of $\widetilde{\mathcal{B}}_{1-\frac{\beta}{2}}(1)$ takes the form

$$\mathbb{E}(\widetilde{\mathcal{B}}_{1-\frac{\beta}{2}}(1)^2) = \frac{(2-\beta)(1-\beta)}{\Gamma(\beta)} \int_0^\infty \int_{-\infty}^\infty (f(x,1-s) - f(x,-s))^2 \, x^\beta \, \mathrm{d}s \, \mathrm{d}x,$$

where, for $x \in \mathbb{R}_{++}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}
&\int_{-\infty}^\infty \big(f(x,t-s) - f(x,-s)\big)^2 x^\beta \, \mathrm{d}s \\
&\quad = \int_{-\infty}^0 \left(\frac{1-\mathrm{e}^{-x(t-s)}}{x} - \frac{1-\mathrm{e}^{-x(-s)}}{x}\right)^2 x^\beta \, \mathrm{d}s + \int_0^t \left(\frac{1-\mathrm{e}^{-x(t-s)}}{x}\right)^2 x^\beta \, \mathrm{d}s \\
&\quad = \int_{-\infty}^0 \mathrm{e}^{2xs}(1-\mathrm{e}^{-xt})^2 x^{\beta-2} \, \mathrm{d}s + \int_0^t (1-\mathrm{e}^{-x(t-s)})^2 x^{\beta-2} \, \mathrm{d}s \\
&\quad = \frac{1}{2}(1-\mathrm{e}^{-xt})^2 x^{\beta-3} + \int_0^t (1-\mathrm{e}^{-x(t-s)})^2 x^{\beta-2} \, \mathrm{d}s.
\end{aligned}$$

Hence, with repeated partial integration, we have

$$\begin{aligned}
\mathbb{E}(\widetilde{\mathcal{B}}_{1-\frac{\beta}{2}}(1)^2) &= \frac{(2-\beta)(1-\beta)}{\Gamma(\beta)} \int_0^\infty \left[\frac{1}{2}(1-\mathrm{e}^{-x})^2 x^{\beta-3} + \int_0^1 (1-\mathrm{e}^{-x(1-s)})^2 x^{\beta-2} \, \mathrm{d}s\right] \mathrm{d}x \\
&= \frac{(2-\beta)(1-\beta)}{\Gamma(\beta)} \int_0^\infty \big(\mathrm{e}^{-x} - 1 + x\big) \, x^{\beta-3} \, \mathrm{d}x \\
&= \frac{(2-\beta)(1-\beta)}{\Gamma(\beta)} \frac{1}{2-\beta} \int_0^\infty \big(-\mathrm{e}^{-x} + 1\big) \, x^{\beta-2} \, \mathrm{d}x \\
&= \frac{(2-\beta)(1-\beta)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta)}{(2-\beta)(1-\beta)} = 1,
\end{aligned}$$

as desired.

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References

[1] M.A. Al-Osh and A.A. Alzaid. First-order integer-valued autoregressive (INAR(1)) process. *J. Time Ser. Anal.*, 8(3):261–275, 1987.

[2] M. Barczy, M. Ispány, and G. Pap. Asymptotic behavior of conditional least squares estimators for unstable integer-valued autoregressive models of order 2. *Scand. J. Stat.*, 41(4):866–892, 2014.

[3] J. Beran, Y. Feng, S. Ghosh, and R. Kulik. *Long-Memory Processes. Probabilistic Properties and Statistical Methods*. Springer, Heidelberg, 2013.

[4] J. Beran, M. Schützner, and S. Ghosh. From short to long memory: aggregation and estimation. *Comput. Statist. Data Anal.*, 54(11):2432–2442, 2010.

[5] D. Celov, R. Leipus, and A. Philippe. Time series aggregation, disaggregation, and long memory. *Liet. Mat. Rink.*, 47(4):466–481, 2007.

[6] C. Dombry and I. Kaj. The on-off network traffic model under intermediate scaling. *Queueing Syst.*, 69(1):29–44, 2011.

[7] J.H. Foster and J.A. Williamson. Limit theorems for the Galton–Watson process with time-dependent immigration. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 20:227–235, 1971.

[8] R. Gaigalas and I. Kaj. Convergence of scaled renewal processes and a packet arrival model. *Bernoulli*, 9(4):671–703, 2003.

[9] E. Gonçalves and Ch. Gouriéroux. Agrégation de processus autorégressifs d'ordre 1. *Ann. Économ. Statist.*, (12):127–149, 1988.

[10] C.W.J. Granger. Long memory relationships and the aggregation of dynamic models. *J. Econometrics*, 14(2):227–238, 1980.

[11] E. Iglói and G. Terdik. Long-range dependence through gamma-mixed Ornstein–Uhlenbeck process. *Electron. J. Probab.*, 4:no. 16, 33 pp. (electronic), 1999.

[12] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[13] M. Jirak. Limit theorems for aggregated linear processes. *Adv. in Appl. Probab.*, 45(2):520–544, 2013.

[14] Norman L. Johnson, Samuel Kotz, and N. Balakrishnan. *Discrete multivariate distributions*. Wiley Series in Probability and Statistics: Applied Probability and Statistics. John Wiley & Sons, Inc., New York, 1997. A Wiley-Interscience Publication.

[15] E.S. Key. Limiting distributions and regeneration times for multitype branching processes with immigration in a random environment. *Ann. Probab.*, 15(1):344–353, 1987.

[16] N.N. Leonenko, V. Savani, and A.A. Zhigljavsky. Autoregressive negative binomial processes. *Ann. I.S.U.P.*, 51(1-2):25–47, 2007.

[17] Z. Li. *Measure-valued branching Markov processes*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[18] E. McKenzie. Some simple models for discrete variate time series. *JAWRA Journal of the American Water Resources Association*, 21(4):645–650, 1985.

[19] T. Mikosch, S. Resnick, H. Rootzén, and A. Stegeman. Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.*, 12(1):23–68, 2002.

[20] F. Nedényi. Conditional least squares estimators for multitype Galton–Watson processes. *Acta Sci. Math. (Szeged)*, 81(1-2):325–348, 2015.

[21] F. Nedényi and G. Pap. Iterated scaling limits for aggregation of random coefficient AR(1) and INAR(1) processes. *Statist. Probab. Lett.*, 118:16–23, 2016.

[22] G. Oppenheim and M.-C. Viano. Aggregation of random parameters Ornstein-Uhlenbeck or AR processes: some convergence results. *J. Time Ser. Anal.*, 25(3):335–350, 2004.

[23] V. Pilipauskaitė and D. Surgailis. Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. *Stochastic Process. Appl.*, 124(2):1011–1035, 2014.

[24] V. Pilipauskaitė and D. Surgailis. Joint aggregation of random-coefficient AR(1) processes with common innovations. *Statist. Probab. Lett.*, 101:73–82, 2015.

[25] V. Pipiras, M.S. Taqqu, and J.B. Levy. Slow, fast and arbitrary growth conditions for renewal-reward processes when both the renewals and the rewards are heavy-tailed. *Bernoulli*, 10(1):121–163, 2004.

[26] D. Puplinskaitė and D. Surgailis. Aggregation of random-coefficient AR(1) process with infinite variance and common innovations. *Lith. Math. J.*, 49(4):446–463, 2009.

[27] D. Puplinskaitė and D. Surgailis. Aggregation of a random-coefficient AR(1) process with infinite variance and idiosyncratic innovations. *Adv. in Appl. Probab.*, 42(2):509–527, 2010.

[28] P.M. Robinson. Statistical inference for a random coefficient autoregressive model. *Scand. J. Statist.*, 5(3):163–168, 1978.

[29] E. Seneta. Functional equations and the Galton-Watson process. *Adv. in Appl. Probab.*, 1:1–42, 1969.

[30] A.N. Shiryaev. *Probability*, volume 95 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996. Translated from the first (1980) Russian edition by R.P. Boas.

[31] F.W. Steutel and K. van Harn. Discrete analogues of self-decomposability and stability. *Ann. Probab.*, 7(5):893–899, 1979.

[32] D.W. Stroock. *Probability Theory. An Analytic View.* Cambridge University Press, Cambridge, second edition, 2011.

[33] M.S. Taqqu, W. Willinger, and R. Sherman. Proof of a fundamental result in self-similar traffic modeling. *ACM SIGCOMM Computer Communication Review*, 27(2):5–23, 1997.

[34] Ch.H. Weiß. Thinning operations for modeling time series of counts—a survey. *AStA Adv. Stat. Anal.*, 92(3):319–341, 2008.

[35] W. Willinger, M.S. Taqqu, R. Sherman, and D.V. Wilson. Self-similarity through high-variability: statistical analysis of ethernet lan traffic at the source level. *IEEE/ACM Transactions on Networking*, 5(1):71–86, 1997.

[36] P. Zaffaroni. Contemporaneous aggregation of linear dynamic models in large economies. *J. Econometrics*, 120(1):75–102, 2004.

[37] H. Zheng, I.V. Basawa, and S. Datta. First-order random coefficient integer-valued autoregressive processes. *J. Statist. Plann. Inference*, 137(1):212–229, 2007.

[38] V.M. Zolotarev. *One-Dimensional Stable Distributions*, volume 65 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1986. Translated from the Russian by H.H. McFaden, Translation edited by Ben Silver.