THE UNSTABLE SET OF A PERIODIC ORBIT FOR DELAYED POSITIVE FEEDBACK

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Abstract In the paper [Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback, JDDE 23 (2011), no. 4, 727–790], we have constructed large-amplitude periodic orbits for an equation with delayed monotone positive feedback. We have shown that the unstable sets of the large-amplitude periodic orbits constitute the global attractor besides spindle-like structures. In this paper we focus on a large-amplitude periodic orbit $\mathcal{O}_p$ with two Floquet multipliers outside the unit circle, and we intend to characterize the geometric structure of its unstable set $\mathcal{W}^u(\mathcal{O}_p)$. We prove that $\mathcal{W}^u(\mathcal{O}_p)$ is a three-dimensional C^1-submanifold of the phase space and admits a smooth global graph representation. Within $\mathcal{W}^u(\mathcal{O}_p)$, there exist heteroclinic connections from $\mathcal{O}_p$ to three different periodic orbits. These connecting sets are two-dimensional C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$ and homeomorphic to the two-dimensional open annulus. They form C^1-smooth separatrices in the sense that they divide the points of $\mathcal{W}^u(\mathcal{O}_p)$ into three subsets according to their ω-limit sets.

Key words Delay differential equation, Positive feedback, Periodic orbit, Unstable set, Floquet theory, Poincaré map, Invariant manifold, Lyapunov functional, Transversality

Suggested running head The unstable set of a periodic orbit

AMS Subject Classification 34K13, 34K19, 37C70, 37D05, 37L25, 37L45

1. Introduction

Consider the delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)), \tag{1.1}$$

where μ is a positive constant and $f : \mathbb{R} \to \mathbb{R}$ is a smooth monotone nonlinearity.

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The natural phase space for Eq. (1.1) is $C = C([-1,0],\mathbb{R})$ equipped with the supremum norm. For any $\varphi \in C$, there is a unique solution $x^{\varphi} : [-1,\infty) \to \mathbb{R}$ of (1.1). For each $t \geq 0$, $x_t^{\varphi} \in C$ is defined by $x_t^{\varphi}(s) = x^{\varphi}(t+s)$, $-1 \leq s \leq 0$. Then the map

$$\Phi : [-1,\infty) \times C \ni (t,\varphi) \mapsto x_t^{\varphi} \in C$$

is a continuous semiflow.

In [8], the authors of this paper have studied Eq. (1.1) under the subsequent hypothesis:

(H1) $\mu > 0$, $f \in C^1(\mathbb{R},\mathbb{R})$ with $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$, and

$$\xi_{-2} < \xi_{-1} < \xi_0 = 0 < \xi_1 < \xi_2$$

are five consecutive zeros of $\mathbb{R} \ni \xi \mapsto -\mu\xi + f(\xi) \in \mathbb{R}$ with $f'(\xi_j) < \mu < f'(\xi_k)$ for $j \in \{-2,0,2\}$ and $k \in \{-1,1\}$ (see Fig. 1).

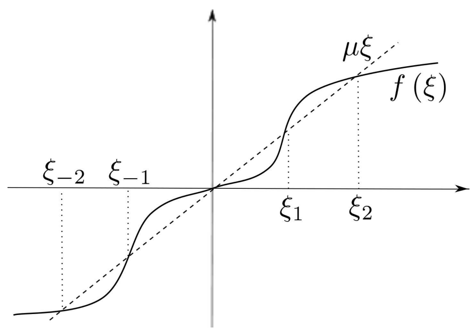

FIGURE 1. A feedback function satisfying condition (H1).

Under hypothesis (H1), $\hat{\xi}_j \in C$, defined by $\hat{\xi}_j(s) = \xi_j$, $-1 \leq s \leq 0$, is an equilibrium point of Φ for all $j \in \{-2,-1,0,1,2\}$, furthermore $\hat{\xi}_{-2}$, $\hat{\xi}_0$ and $\hat{\xi}_2$ are stable, and $\hat{\xi}_{-1}$ and $\hat{\xi}_1$ are unstable. By the monotonicity property of f, the subsets

$$C_{-2,2} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1,0]\},$$

$$C_{-2,0} = \{\varphi \in C : \xi_{-2} \leq \varphi(s) \leq 0 \text{ for all } s \in [-1,0]\},$$

$$C_{0,2} = \{\varphi \in C : 0 \leq \varphi(s) \leq \xi_2 \text{ for all } s \in [-1,0]\}$$

of the phase space C are positively invariant under the semiflow Φ (see Proposition 2.4 in Section 2).

Let $\mathcal{A}$, $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ denote the global attractors of the restrictions $\Phi|_{[0,\infty)\times C_{-2,2}}$, $\Phi|_{[0,\infty)\times C_{-2,0}}$ and $\Phi|_{[0,\infty)\times C_{0,2}}$, respectively. If (H1) holds and $\xi_{-2}, \xi_{-1}, 0, \xi_1, \xi_2$ are the only zeros of $-\mu\xi + f(\xi)$, then $\mathcal{A}$ is the global attractor of Φ. The structures of $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ are (at least partially) well understood, see e.g. [5, 6, 7, 9, 10, 11]. $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ admit Morse decompositions [18]. Further technical conditions regarding

f ensure that $\mathcal{A}_{-2,0}$ and $\mathcal{A}_{0,2}$ have spindle-like structures [5, 9, 10, 11]: $\mathcal{A}_{0,2}$ is the closure of the unstable set of $\hat{\xi}_1$ containing the equilibrium points $\hat{\xi}_0$, $\hat{\xi}_1$, $\hat{\xi}_2$, periodic orbits in $C_{0,2}$ and heteroclinic orbits among them. In other cases $\mathcal{A}_{0,2}$ is larger than the the closure of the unstable set of $\hat{\xi}_1$. The structure of $\mathcal{A}_{-2,0}$ is similar. See Fig. 2 for a simple situation.

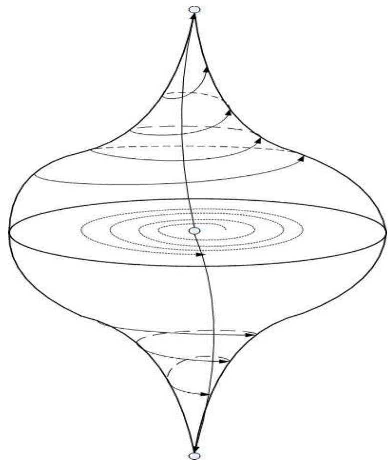

FIGURE 2. A spindle-like structure

The monograph [10] of Krisztin, Walther and Wu has addressed the question whether the equality $\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2}$ holds under hypothesis (H1). The authors of this paper have constructed an example in [8] so that (H1) holds, and Eq. (1.1) admits periodic orbits in $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$, that is, besides the spindle-like structures. The periodic solutions defining these periodic orbits oscillate slowly about 0 and have large amplitudes in the following sense.

A periodic solution $r : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) is called a large amplitude periodic solution if $r(\mathbb{R}) \supset (\xi_{-1}, \xi_1)$. A solution $r : \mathbb{R} \to \mathbb{R}$ is slowly oscillatory if for each t, the restriction $r|_{[t-1,t]}$ has one or two sign changes. Note that here slow oscillation is different from the usual one used for equations with negative feedback condition [2, 21]. A large-amplitude slowly oscillatory periodic solution $r : \mathbb{R} \to \mathbb{R}$ is abbreviated as an LSOP solution. We say that an LSOP solution $r : \mathbb{R} \to \mathbb{R}$ is normalized if $r(-1) = 0$, and for some $\eta > 0$, $r(s) > 0$ for all $s \in (-1, -1 + \eta)$.

The first main result of [8] is as follows.

Theorem A. *There exist μ and f satisfying (H1) such that Eq.* (1.1) *has exactly two normalized LSOP solutions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$. For the ranges of p and q, $(\xi_{-1}, \xi_1) \subset p(\mathbb{R}) \subset q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$ holds. The corresponding periodic orbits*

$$\mathcal{O}_p = \{p_t : t \in \mathbb{R}\} \text{ and } \mathcal{O}_q = \{q_t : t \in \mathbb{R}\}$$

are hyperbolic and unstable. $\mathcal{O}_p$ *admits two different Floquet multipliers outside the unit circle, which are real and simple.* $\mathcal{O}_q$ *has one real simple Floquet multiplier outside the unit circle.*

Note that although Theorem 1.1 in [8] does not mention that the Floquet multipliers found outside the unit circle are simple and real, these properties are verified in Section 4 of the same paper.

In the proof of the theorem, $\mu = 1$ and f is close to the step function

$$f^{K,0}(x) = \begin{cases} -K & \text{if } x < -1, \\ 0 & \text{if } |x| \leq 1, \\ K & \text{if } x > 1, \end{cases}$$

where $K > 0$ is chosen large enough.

In their paper [3], Fiedler, Rocha and Wolfrum considered a special class of one-dimensional parabolic partial differential equations and obtained a catalogue listing the possible structures of the global attractor. In particular, the result of Theorem A motivated Fiedler, Rocha and Wolfrum to find an analogous configuration for their equation. It is an interesting question whether all the structures found by them have counterparts in the theory of Eq. (1.1).

Let $\mathcal{W}^u(\mathcal{O}_p)$ and $\mathcal{W}^u(\mathcal{O}_q)$ denote the unstable sets of $\mathcal{O}_p$ and $\mathcal{O}_q$, respectively.

A solution $r : \mathbb{R} \to \mathbb{R}$ is called slowly oscillatory about ξ_k, $k \in \{-1, 1\}$, if $\mathbb{R} \ni t \mapsto r(t) - \xi_k \in \mathbb{R}$ admits one or two sign changes on each interval of length 1. As it is described by Proposition 2.7 in [8], f and μ in Theorem A are set so that there exist at least one periodic solution oscillating slowly about ξ_1 with range in $(0, \xi_2)$, furthermore there is a solution $x^1 : \mathbb{R} \to \mathbb{R}$ among such periodic solutions that has maximal range $x^1(\mathbb{R})$ in the sense that $x^1(\mathbb{R}) \supset x(\mathbb{R})$ for all periodic solutions x oscillating slowly about ξ_1 with range in $(0, \xi_2)$. Similarly, there exists a maximal periodic solution x^{-1} oscillating slowly about ξ_{-1} with range in $(\xi_{-2}, 0)$. Set

$$\mathcal{O}_1 = \left\{x_t^1 : t \in \mathbb{R}\right\} \text{ and } \mathcal{O}_{-1} = \left\{x_t^{-1} : t \in \mathbb{R}\right\}.$$

Let $\omega(\varphi)$ denote the ω-limit set of any $\varphi \in C$. Introduce the connecting sets

$$C_j^p = \left\{\varphi \in \mathcal{W}^u(\mathcal{O}_p) : \omega(\varphi) = \hat{\xi}_j\right\}, \qquad j \in \{-2, 0, 2\},$$

$$C_k^p = \{\varphi \in \mathcal{W}^u(\mathcal{O}_p) : \omega(\varphi) = \mathcal{O}_k\}, \qquad k \in \{-1, 1\},$$

and

$$C_q^p = \{\varphi \in \mathcal{W}^u(\mathcal{O}_p) : \omega(\varphi) = \mathcal{O}_q\}.$$

Sets C_j^q, $j \in \{-2, 2\}$, are defined analogously.

The next theorem has also been given in [8] and describes the dynamics in $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$.

Theorem B. *One may set μ and f satisfying (H1) such that the statement of Theorem A holds, and for the global attractor $\mathcal{A}$ we have the equality*

$$\mathcal{A} = \mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2} \cup \mathcal{W}^u\left(\mathcal{O}_p\right) \cup \mathcal{W}^u\left(\mathcal{O}_q\right).$$

Moreover, the dynamics on $\mathcal{W}^u\left(\mathcal{O}_p\right)$ and $\mathcal{W}^u\left(\mathcal{O}_q\right)$ is as follows. The connecting sets C_j^p, C_q^p, C_k^p, $j \in \{-2, 0, 2\}$, $k \in \{-1, 1\}$, are nonempty, and

$$\mathcal{W}^u\left(\mathcal{O}_p\right) = \mathcal{O}_p \cup C_{-2}^p \cup C_{-1}^p \cup C_0^p \cup C_1^p \cup C_2^p \cup C_q^p.$$

The connecting sets C_{-2}^q and C_2^q are nonempty, and

$$\mathcal{W}^u\left(\mathcal{O}_q\right) = \mathcal{O}_q \cup C_{-2}^q \cup C_2^q.$$

The system of heteroclinic connections is represented in Fig. 3.

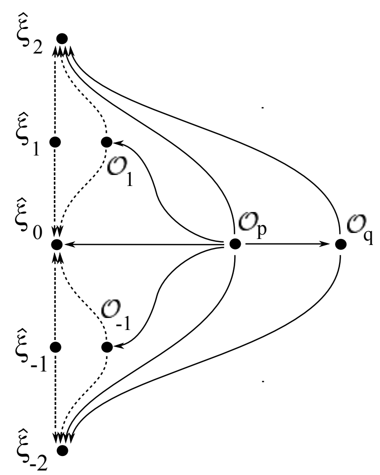

FIGURE 3. Connecting orbits: the dashed arrows represent heteroclinic connections in $\mathcal{A}_{-2,0}$ and in $\mathcal{A}_{0,2}$, while the solid ones represent connecting orbits given by Theorem B.

Hereinafter we fix $\mu = 1$ and set f in Eq. (1.1) so that Theorems A and B hold. The purpose of this paper is to characterize the geometrical properties of $\mathcal{W}^u\left(\mathcal{O}_p\right)$ and the connecting sets within $\mathcal{W}^u\left(\mathcal{O}_p\right)$.

We say that a subset W of C admits global graph representation, if there exists a splitting $C = G \oplus E$ with closed subspaces G and E of C, a subset U of G and a map $w : U \to E$ such that

$$W = \{\chi + w(\chi) : \chi \in U\}.$$

W is said to have a smooth global graph representation if in the above definition U is open in G and w is C^1-smooth on U. Note that in this case W is a C^1-submanifold of C in the usual sense with dimension $\dim G$, see e.g. the definition of Lang in [12]. W is said to admit a smooth global graph representation with boundary if G is n dimensional with some integer $n \geq 1$, U is the closure of an open set U^0, w is C^1-smooth on U^0, the boundary $\mathrm{bd}U$ of U in G is an $(n-1)$-dimensional C^1-submanifold of G, and all points of $\mathrm{bd}U$ have an open neighborhood in G on which w can be extended to a C^1-smooth function. In this case W is an n-dimensional C^1-submanifold of C with boundary in the usual sense [12].

The first result of this paper is the following.

Theorem 1.1. *$\mathcal{W}^u(\mathcal{O}_p)$, C^p_{-2}, C^p_0 and C^p_2 are three-dimensional C^1-submanifolds of C admitting smooth global graph representations.*

The next objects of our study are the connecting sets C^p_q, C^p_{-1}, C^p_1 containing the heteroclinic orbits from $\mathcal{O}_p$ to $\mathcal{O}_q$, $\mathcal{O}_{-1}$, $\mathcal{O}_1$, respectively. We actually get a detailed picture of the structure of $\mathcal{W}^u(\mathcal{O}_p)$ by characterizing the unions

$$S_{-1} = C^p_{-1} \cup \mathcal{O}_p \cup C^p_q \qquad \text{and} \qquad S_1 = C^p_1 \cup \mathcal{O}_p \cup C^p_q.$$

A solution $x : \mathbb{R} \to \mathbb{R}$ is said to oscillate about ξ_i, $i \in \{-2,-1,0,1,2\}$, if the set $x^{-1}(\xi_i) \subset \mathbb{R}$ is not bounded from above. It is a direct consequence of Theorem B that for $k \in \{-1,1\}$,

$$S_k = \{\varphi \in \mathcal{W}^u(\mathcal{O}_p) : x^\varphi \text{ oscillates about } \xi_k\}. \tag{1.2}$$

We say that a subset W of $\mathcal{W}^u(\mathcal{O}_p)$ is above S_k, $k \in \{-1,1\}$, if to each $\varphi \in W$ there corresponds an element ψ of S_k with $\psi \ll \varphi$ (that is, $\psi(s) < \varphi(s)$ for all $s \in [-1,0]$). Similarly, a subset W of $\mathcal{W}^u(\mathcal{O}_p)$ is below S_k, $k \in \{-1,1\}$, if for all $\varphi \in W$ there exists $\psi \in S_k$ with $\varphi \ll \psi$. W is between S_{-1} and S_1 if it is below S_1 and above S_{-1}.

Our main result offers geometrical and topological descriptions of C^p_q, C^p_{-1}, C^p_1, S_{-1} and S_1, and their closures in C. It shows that S_{-1} and S_1 separate the points of $\mathcal{W}^u(\mathcal{O}_p)$ into three groups according to their ω-limit sets. Thereby, S_{-1} and S_1 play a key role in the dynamics of the equation.

Theorem 1.2.
(i) The sets C^p_q, C^p_{-1}, C^p_1, S_{-1} and S_1 are two-dimensional C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$ with smooth global graph representations. They are homeomorphic to the open annulus

$$A^{(1,2)} = \left\{u \in \mathbb{R}^2 : 1 < |u| < 2\right\}.$$

(ii) The equalities

$$\overline{C^p_q} = \mathcal{O}_p \cup C^p_q \cup \mathcal{O}_q, \qquad \overline{C^p_k} = \mathcal{O}_p \cup C^p_k \cup \mathcal{O}_k$$

and

$$\overline{S_k} = \mathcal{O}_k \cup S_k \cup \mathcal{O}_q = \mathcal{O}_k \cup C_k^p \cup \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q$$

hold for both $k \in \{-1, 1\}$. *The sets* $\overline{C_q^p}$, $\overline{C_{-1}^p}$, $\overline{C_1^p}$, $\overline{S_{-1}}$ *and* $\overline{S_1}$ *admit smooth global graph representations with boundary, and thereby they are two-dimensional* C^1*-submanifolds of* C *with boundary. In addition, they are homeomorphic to the closed annulus*

$$A^{[1,2]} = \left\{ u \in \mathbb{R}^2 : 1 \leq |u| \leq 2 \right\}.$$

(iii) S_{-1} *and* S_1 *are separatrices in the sense that* C_2^p *is above* S_1, C_0^p *is between* S_{-1} *and* S_1, *furthermore* C_{-2}^p *is below* S_{-1}.

Fig. 4 visualizes the structure of the closure $\overline{\mathcal{W}^u(\mathcal{O}_p)}$ of $\mathcal{W}^u(\mathcal{O}_p)$ in C. To get an overview of the above results regarding $\mathcal{W}^u(\mathcal{O}_p)$, see the inner part of Fig. 4, drawn in black. We emphasize a particular consequence of Theorem 1.2: the tangent spaces of S_{-1} and S_1 coincide along $\mathcal{O}_p$, see Fig. 5.

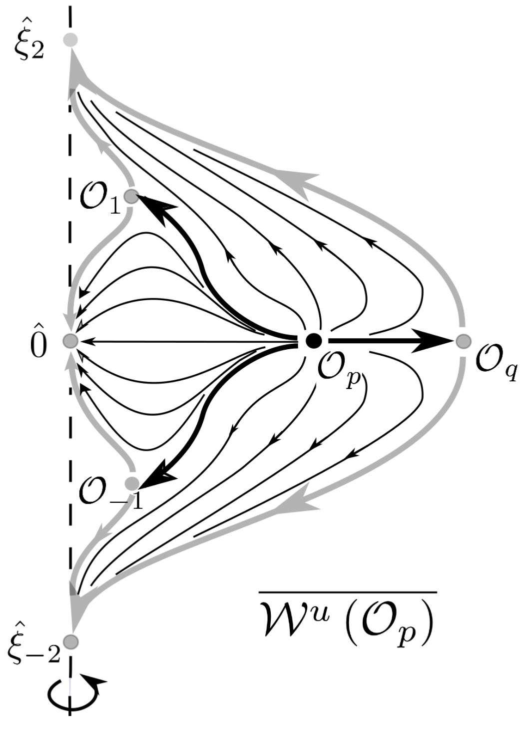

FIGURE 4. $\overline{\mathcal{W}^u(\mathcal{O}_p)}$ can be visualized as a "tulip" rotated around the vertical axis: the dots correspond to equilibria and periodic orbits, the thick arrows symbolize two-dimensional heteroclinic connecting sets, and the three groups of thin arrows represent three-dimensional connecting sets. The elements of $\mathcal{W}^u(\mathcal{O}_p)$ are drawn in black. Grey is used for the boundary of $\mathcal{W}^u(\mathcal{O}_p)$.

Let $\mathcal{W}^u(\mathcal{O}_1)$ and $\mathcal{W}^u(\mathcal{O}_{-1})$ denote the unstable sets of $\mathcal{O}_1$ and $\mathcal{O}_{-1}$, respectively, defined as the forward extension of a one-dimensional local unstable manifold of

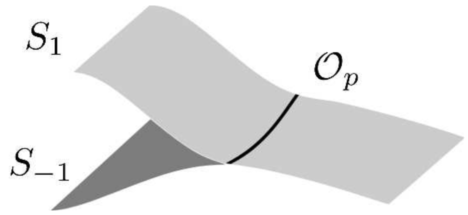

FIGURE 5. The tangent spaces of S_{-1} and S_1 coincide along $\mathcal{O}_p$.

a return map (corresponding to the only Floquet multiplier outside the unit circle which is real and simple), see (3.5). We expect $\mathcal{W}^u(\mathcal{O}_q)$, $\mathcal{W}^u(\mathcal{O}_{-1})$ and $\mathcal{W}^u(\mathcal{O}_1)$ to be two-dimensional C^1-submanifolds of C. We conjecture that for the closure $\overline{\mathcal{W}^u(\mathcal{O}_p)}$ of $\mathcal{W}^u(\mathcal{O}_p)$ in C, the equality

$$\overline{\mathcal{W}^u(\mathcal{O}_p)} = \mathcal{W}^u(\mathcal{O}_p) \cup \mathcal{W}^u(\mathcal{O}_q) \cup \mathcal{W}^u(\mathcal{O}_1) \cup \mathcal{W}^u(\mathcal{O}_{-1}) \cup \left\{\hat{\xi}_{-2}, \hat{0}, \hat{\xi}_2\right\}$$

holds, as it represented in Fig. 4. Moreover, all points of $\mathcal{W}^u(\mathcal{O}_q) \cup \mathcal{W}^u(\mathcal{O}_1) \cup \mathcal{W}^u(\mathcal{O}_{-1})$ have an open neighborhood on which the C^1-map in the graph representation of $\mathcal{W}^u(\mathcal{O}_p)$ can be smoothly extended.

It also remains an open question whether $\mathcal{A} \setminus (\mathcal{A}_{-2,0} \cup \mathcal{A}_{0,2})$ is homeomorphic to the three-dimensional body

$$\mathcal{B}_3((0,0,0),2) \setminus \{\mathcal{B}_3((0,0,1),1) \cup \mathcal{B}_3((0,0,-1),1)\} \subset \mathbb{R}^3,$$

where $\mathcal{B}_3((a_1,a_2,a_3),r)$ denotes the three-dimensional closed ball with center (a_1,a_2,a_3) and radius r.

The proofs of Theorems 1.1–1.2 apply general results on delay differential equations, the Floquet theory (Appendix VII of [10], [14]), results on local invariant manifolds for maps in Banach spaces (Appendices I-II of [10]), correspondences between different return maps (Appendices I and V of [10]), a result from transversality theory [1] and also a discrete Lyapunov functional of Mallet-Paret and Sell counting the sign changes of the elements of C (Appendix VI of [10], [16]).

This paper is organized as follows. Section 2 offers a general overview of the theoretical background and introduces the discrete Lyapunov functional. As the Floquet theory and certain results on local invariant manifolds of return maps play essential role in this work, Section 3 is devoted to the discussion of these concepts. Sections 4 and 5 contain the proofs of Theorems 1.1 and 1.2, respectively.

The proof of Theorem 1.1 in Section 4 takes advantage of the fact that the unstable set of a hyperbolic periodic orbit is the forward continuation of a local unstable manifold of a Poincaré map by the semiflow. In consequence, by using the smoothness of the local unstable manifold and the injectivity of the derivative of the solution operator, we prove that all points φ of $\mathcal{W}^u(\mathcal{O}_p)$ belong to a subset W_φ of $\mathcal{W}^u(\mathcal{O}_p)$ that is a three-dimensional C^1-submanifold of C. This means that $\mathcal{W}^u(\mathcal{O}_p)$ is an immersed submanifold of C. In general, an immersed submanifold is not necessarily an embedded submanifold of the phase space. In order to prove that $\mathcal{W}^u(\mathcal{O}_p)$ is embedded in C, we have to show that for any φ in $\mathcal{W}^u(\mathcal{O}_p)$, there is no sequence in $\mathcal{W}^u(\mathcal{O}_p)\backslash W_\varphi$ converging to φ. We define a projection π_3 from C into $\mathbb{R}^3$. Using well-known properties of the discrete Lyapunov functional, we show that π_3 is injective on $\mathcal{W}^u(\mathcal{O}_p)$ and on the tangent spaces of W_φ. This implies that $\pi_3 W_\varphi$ is open in $\mathbb{R}^3$. If a sequence $(\varphi^n)_{n=0}^\infty$ in $\mathcal{W}^u(\mathcal{O}_p)\backslash W_\varphi$ converges to φ as $n \to \infty$, then $\pi_3\varphi^n \to \pi_3\varphi$ as $n \to \infty$, and $\pi_3\varphi^n \in \pi_3 W_\varphi$ for all n large enough. The injectivity of π_3 on $\mathcal{W}^u(\mathcal{O}_p)$ then implies that $\varphi^n \in W_\varphi$, which is a contradiction. So $\mathcal{W}^u(\mathcal{O}_p)$ is a three-dimensional embedded C^1-submanifold of the phase space. The description of $\mathcal{W}^u(\mathcal{O}_p)$ is rounded up by giving a graph representation for $\mathcal{W}^u(\mathcal{O}_p)$ in order to present the simplicity of its structure. The smoothness of the sets C_{-2}^p, C_0^p and C_2^p then follows at once because they are open subsets of $\mathcal{W}^u(\mathcal{O}_p)$. We also obtain as an important consequence that the semiflow defined by the solution operator extends to a C^1-flow on $\mathcal{W}^u(\mathcal{O}_p)$ with injective derivatives.

The proof of Theorem 1.2 in Section 5 is built from several steps, and it is organized into five subsections.

In Subsection 5.1 we list preliminary results regarding the closure $\overline{S_k}$ of S_k in C, $k \in \{-1, 1\}$. We introduce in particular a projection π_2 from C into $\mathbb{R}^2$, and – using the special properties of the discrete Lyapunov functional – we show that π_2 is injective on $\overline{S_k}$. The injectivity of $\pi_2|_{\overline{S_k}}$ is already sufficient to give a two-dimensional graph representation for any subset W of $\overline{S_k}$ (without smoothness properties): there is an isomorphism $J_2 : \mathbb{R}^2 \to C$ such that $P_2 = J_2 \circ \pi_2 : C \to C$ is a projection onto a two-dimensional subspace G_2 of C, and there exists a map w_k defined on the image set $P_2\overline{S_k}$ with range in $P_2^{-1}(0)$ such that for any subset $W \subseteq S_k$,

$$W = \{\chi + w_k(\chi) : \chi \in P_2 W\}.$$

The smoothness of w_k and the properties of its domain $P_2\overline{S_k} \subset G_2$ are investigated later. Subsection 5.1 is closed with showing that $\pi_2|_{\overline{S_k}}$ is a homeomorphism onto its image, furthermore π_2 maps the nonzero tangent vectors of $\overline{S_k}$ to nonzero vectors in $\mathbb{R}^2$.

It is clear that $(\mathcal{O}_k \cup S_k \cup \mathcal{O}_q) \subseteq \overline{S_k}$ for both $k \in \{-1, 1\}$. The inclusion $\overline{S_k} \subseteq (\mathcal{O}_k \cup S_k \cup \mathcal{O}_q)$ is proved in Subsection 5.2 based on the previously obtained result that $\overline{S_k}$ is mapped injectively into $\mathbb{R}^2$. Then it follows easily that $\overline{C_k^p}$, $k \in \{-1, 1\}$, and $\overline{C_q^p}$ are not larger than the unions $\mathcal{O}_p \cup C_k^p \cup \mathcal{O}_k$ and $\mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q$, respectively.

It is a more challenging task to show that C_q^p and C_k^p, $k \in \{-1, 1\}$, are C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$ (as stated by Theorem 1.2.(i)). The proof of this assertion is contained in Subsection 5.3. It is partly based on transversality [1]; we verify that $\mathcal{W}^u(\mathcal{O}_p)$ intersects transversally a local center-stable manifold of a Poincaré return map at a point of $\mathcal{O}_k$ and a local stable manifold of a Poincaré return map at a point of $\mathcal{O}_q$, and thereby the intersections – subsets of C_q^p and C_k^p – are one-dimensional submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$. The main difficulty in this task is that the hyperbolicity of $\mathcal{O}_k$ is not known. Krisztin, Walther and Wu have proved transversality in a similar situation [10]. Then we apply techniques that already appeared in Section 4. The injectivity of the derivative of the flow induced by the solution operator on $\mathcal{W}^u(\mathcal{O}_p)$ guarantees that each point φ in C_q^p or C_k^p belongs to a "small" subset of C_q^p or C_k^p, respectively, that is a two-dimensional C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$. Therefore, C_q^p and C_k^p are immersed C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$. In order to prove that C_q^p and C_k^p are embedded in $\mathcal{W}^u(\mathcal{O}_p)$, we repeat an argument from the proof of Theorem 1.1 with π_2 in the role of π_3. Based on the property that C_q^p and C_k^p are C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$, we prove at the end of Subsection 5.3 that w_k is continuously differentiable on the open sets $P_2C_q^p$ and $P_2C_k^p$, i.e., the representations

$$C_q^p = \left\{\chi + w_k(\chi) : \chi \in P_2C_q^p\right\} \quad \text{and} \quad C_k^p = \left\{\chi + w_k(\chi) : \chi \in P_2C_k^p\right\}.$$

are smooth.

Next we verify in Subsection 5.4 that the images of C_q^p, C_k^p and S_k, $k \in \{-1, 1\}$, under π_2 are topologically equivalent to the open annulus, and the images of their closures are topologically equivalent to the closed annulus.

As

$$S_k = \{\chi + w_k(\chi) : \chi \in P_2S_k\} \quad \text{and} \quad P_2S_k = P_2C_k^p \cup P_2\mathcal{O}_p \cup P_2C_q^p,$$

we have a smooth representation for S_k if we show that P_2S_k is open in G_2 and w_k is smooth at the points of $P_2\mathcal{O}_p$. This is done in Subsection 5.5. It follows immediately that S_k is a C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$. Simultaneously, we verify that all points of $P_2\mathcal{O}_k \cup P_2\mathcal{O}_q$ have open neighborhoods on which w_k can be extended to C^1-functions. As $P_2\mathcal{O}_k \cup P_2\mathcal{O}_q$ is the boundary of $P_2\overline{S_k}$, this step guarantees that $\overline{S_k}$ has a smooth representation with boundary, and thereby $\overline{S_k}$ is a C^1-submanifold of C with boundary. The same reasonings yield the analogous results for $\overline{C_q^p}$ and $\overline{C_k^p}$. Summing up, the proofs of Theorem 1.2.(i) and (ii) are completed in Subsection 5.5.

It remains to show that S_{-1} and S_1 are indeed separatrices in the sense described by Theorem 1.2.(iii). It is easy to see that the assertion restricted to a local unstable manifold of $\mathcal{O}_p$ holds. Then we use the monotonicity of the semiflow to extend the statement for $\mathcal{W}^u(\mathcal{O}_p)$.

Several techniques applied here have already appeared in the monograph [10] of Krisztin, Walther and Wu. The novelty of this paper compared to [10] is that here we describe the unstable set of a periodic orbit, while [10] considers the unstable set of an equilibrium point.

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2. Preliminaries

We fix $\mu = 1$ and set f in Eq. (1.1) so that Theorems A and B hold. In this section we give a summary of the theoretical background. In particular, we discuss the differentiability of the semiflow, the basic properties of the global attractor, the discrete Lyapunov functional of Mallet-Paret and Sell, and we list some technical results. The discussion of the Floquet theory and the Poincaré return maps is left to the next section.

Phase space, solution, segment. The natural phase space for Eq. (1.1) is the Banach space $C = C([-1,0],\mathbb{R})$ of continuous real functions defined on $[-1,0]$ equipped with the supremum norm

$$\|\varphi\| = \sup_{-1\le s\le 0} |\varphi(s)|.$$

If J is an interval, $u : J \to \mathbb{R}$ is continuous and $[t-1,t] \subseteq J$, then the segment $u_t \in C$ is defined by $u_t(s) = u(t+s)$, $-1 \le s \le 0$.

Let C^1 denote the subspace of C containing the continuously differentiable functions. Then C^1 is also a Banach space with the norm $\|\varphi\|_{C^1} = \|\varphi\| + \|\varphi'\|$.

For all $\xi \in \mathbb{R}$, $\hat{\xi} \in C$ is defined by $\hat{\xi}(s) = \xi$ for all $s \in [-1,0]$.

A solution of Eq. (1.1) is either a continuous function on $[t_0-1,\infty)$, $t_0 \in \mathbb{R}$, which is differentiable for $t > t_0$ and satisfies equation Eq. (1.1) on (t_0,∞), or a continuously differentiable function on $\mathbb{R}$ satisfying the equation for all $t \in \mathbb{R}$. To all $\varphi \in C$, there corresponds a unique solution $x^\varphi : [-1,\infty) \to \mathbb{R}$ of Eq. (1.1) with $x_0^\varphi = \varphi$. On $(0,\infty)$, x^φ is given by the variation-of-constants formula for ordinary differential equations repeated on successive intervals of length 1:

$$x^\varphi(t) = e^{n-t}x^\varphi(n) + \int_n^t e^{s-t} f(x^\varphi(s-1))\,\mathrm{d}s \quad \text{for all } n \in \mathbb{N},\ n \le t \le n+1. \tag{2.1}$$

Semiflow. The solutions of Eq. (1.1) define the continuous semiflow

$$\Phi : \mathbb{R}^+ \times C \ni (t,\varphi) \mapsto x_t^\varphi \in C.$$

All maps $\Phi(t,\cdot): C \to C$, $t \geq 1$, are compact [4]. As $f' > 0$ on $\mathbb{R}$, all maps $\Phi(t,\cdot): C \to C$, $t \geq 0$, are injective [10]. It follows that for every $\varphi \in C$ there is at most one solution $x: \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $x_0 = \varphi$. Whenever such solution exists, we denote it also by x^{φ}.

For fixed $\varphi \in C$, the map $(1,\infty) \ni t \mapsto \Phi(t,\varphi) \in C$ is continuously differentiable with $D_1\Phi(t,\varphi)1 = \dot{x}_t^{\varphi}$ for all $t > 1$. For all $t \geq 0$ fixed, $C \ni \varphi \mapsto \Phi(t,\varphi) \in C$ is continuously differentiable, and $D_2\Phi(t,\varphi)\eta = v_t^{\eta}$, where $v^{\eta}: [-1,\infty) \to \mathbb{R}$ is the solution of the linear variational equation

$$\dot{v}(t) = -v(t) + f'(x^{\varphi}(t-1))\, v(t-1) \tag{2.2}$$

with $v_0^{\eta} = \eta$. So the restriction of Φ to the open set $(1,\infty) \times C$ is continuously differentiable.

Proposition 2.1. *Suppose that $\eta \in C$, $b: \mathbb{R} \to \mathbb{R}$ is positive, and the problem*

$$\begin{cases} \dot{v}(t) &= -v(t) + b(t)v(t-1) \\ v_0 &= \eta \end{cases}$$

has a solution v^{η} either on $[t_0 - 1, \infty)$ with $t_0 \leq 0$ or on $\mathbb{R}$ (i.e., there is a continuous function $v^{\eta}: [t_0-1,\infty) \to \mathbb{R}$ with $v_0^{\eta} = \eta$ that is differentiable and satisfies the equation for $t > t_0$, or there exists a differentiable function $v^{\eta}: \mathbb{R} \to \mathbb{R}$ with $v_0^{\eta} = \eta$ satisfying the equation for all real t, respectively). Then v^{η} is unique.

Proof. As the solution on $[0,\infty)$ is determined by a variation-of-constants formula analogous to (2.1), the uniqueness in forward time is clear. For $t < 0$, the uniqueness follows from $v(t-1) = (\dot{v}(t) + v(t))/b(t)$. □

In particular, the solution operator $D_2\Phi(t,\varphi)$ corresponding to the variational equation (2.2) is injective for all $\varphi \in C$ and $t \geq 0$.

A function $\hat{\xi} \in C$ is an equilibrium point (or stationary point) of Φ if and only if $\hat{\xi}(s) = \xi$ for all $-1 \leq s \leq 0$ with $\xi \in \mathbb{R}$ satisfying $-\xi + f(\xi) = 0$. Then $x^{\hat{\xi}}(t) = \xi$ for all $t \in \mathbb{R}$. As it is described in Chapter 2 of [10], condition $f'(\xi) < 1$ implies that $\hat{\xi}$ is stable and locally attractive. If $f'(\xi) > 1$, then $\hat{\xi}$ is unstable. So hypothesis (H1) with $\mu = 1$ implies that $\hat{\xi}_{-2}$, $\hat{\xi}_0$ and $\hat{\xi}_2$ are stable, and $\hat{\xi}_{-1}$ and $\hat{\xi}_1$ are unstable.

Limit sets. If $\varphi \in C$ and $x^{\varphi}: [-1,\infty) \to \mathbb{R}$ is a bounded solution of Eq. (1.1), then the ω-limit set

$$\omega(\varphi) = \{\psi \in C : \text{there exists a sequence } (t_n)_0^{\infty} \text{ in } [0,\infty) \text{ with } t_n \to \infty \text{ and } \Phi(t_n,\varphi) \to \psi \text{ as } n \to \infty\}$$

is nonempty, compact, connected and invariant. For a solution $x:\mathbb{R}\to\mathbb{R}$ such that $x|_{(-\infty,0]}$ is bounded, the α-limit set

$$\begin{aligned}\alpha\left(x\right)=\{&\psi\in C:\text{ there exists a sequence }\left(t_n\right)_0^\infty\text{ in }\mathbb{R}\\&\text{with }t_n\to-\infty\text{ and }x_{t_n}\to\psi\text{ as }n\to\infty\}\end{aligned}$$

is also nonempty, compact, connected and invariant.

According to the Poincaré–Bendixson theorem of Mallet-Paret and Sell [17], for all

$$\varphi\in C_{-2,2}=\{\varphi\in C:\ \xi_{-2}\leq\varphi\left(s\right)\leq\xi_2\text{ for all }s\in[-1,0]\},$$

the set $\omega\left(\varphi\right)$ is either a single nonconstant periodic orbit, or for each $\psi\in\omega\left(\varphi\right)$,

$$\alpha\left(x^\psi\right)\cup\omega\left(\psi\right)\subseteq\left\{\hat{\xi}_{-2},\hat{\xi}_{-1},\hat{\xi}_0,\hat{\xi}_1,\hat{\xi}_2\right\}.$$

An analogous result holds for $\alpha\left(x\right)$ in case x is defined on $\mathbb{R}$ and $\{x_t:\ t\leq0\}\subset C_{-2,2}$.

By Theorem 4.1 in Chapter 5 of [20], there is an open and dense set of initial functions in $C_{-2,2}$ so that the corresponding solutions converge to equilibria.

Note that there is no homoclinic orbit to $\hat{\xi}_j$, $j\in\{-2,0,2\}$, as these equilibria are stable. It follows from Proposition 3.1 in [7] that there exists no homoclinic orbits to the unstable equilibria $\hat{\xi}_{-1}$ and $\hat{\xi}_1$.

The global attractor. The global attractor $\mathcal{A}$ of the restriction $\Phi|_{[0,\infty)\times C_{-2,2}}$ is a nonempty, compact set in C, that is invariant in the sense that $\Phi\left(t,\mathcal{A}\right)=\mathcal{A}$ for all $t\geq0$, and that attracts bounded sets in the sense that for every bounded set $B\subset C_{-2,2}$ and for every open set $U\supset\mathcal{A}$, there exists $t\geq0$ with $\Phi\left([t,\infty)\times B\right)\subset U$. Global attractors are uniquely determined [4]. It can be shown that

$$\begin{aligned}\mathcal{A}=\{&\varphi\in C_{-2,2}:\text{ there is a bounded solution }x:\mathbb{R}\to\mathbb{R}\\&\text{of Eq. (1.1) so that }\varphi=x_0\},\end{aligned}$$

see [9, 14, 18].

The compactness of $\mathcal{A}$, its invariance property and the injectivity of the maps $\Phi\left(t,\cdot\right):C\to C$, $t\geq0$, combined permit to verify that the map

$$[0,\infty)\times\mathcal{A}\ni\left(t,\varphi\right)\mapsto\Phi\left(t,\varphi\right)\in\mathcal{A}$$

extends to a continuous flow $\Phi_{\mathcal{A}}:\mathbb{R}\times\mathcal{A}\to\mathcal{A}$; for every $\varphi\in\mathcal{A}$ and for all $t\in\mathbb{R}$ we have $\Phi_{\mathcal{A}}\left(t,\varphi\right)=x_t^\varphi$ with the uniquely determined solution $x^\varphi:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) satisfying $x_0^\varphi=\varphi$.

Note that we have $\mathcal{A}=\Phi\left(1,\mathcal{A}\right)\subset C^1$; $\mathcal{A}$ is a closed subset of C^1. Using the flow $\Phi_{\mathcal{A}}$ and the continuity of the map

$$C\ni\varphi\mapsto\Phi\left(1,\varphi\right)\in C^1,$$

one obtains that C and C^1 define the same topology on $\mathcal{A}$.

A discrete Lyapunov functional. Following Mallet-Paret and Sell in [16], we use a discrete Lyapunov functional $V : C \setminus \{\hat{0}\} \to 2\mathbb{N} \cup \{\infty\}$. For $\varphi \in C \setminus \{\hat{0}\}$, set $sc(\varphi) = 0$ if $\varphi \geq \hat{0}$ or $\varphi \leq \hat{0}$ (i.e., $\varphi(s) \geq 0$ for all $s \in [-1,0]$ or $\varphi(s) \leq 0$ for all $s \in [-1,0]$, respectively), otherwise define

$$sc(\varphi) = \sup\Big\{k \in \mathbb{N} \setminus \{0\} : \text{ there exist a strictly increasing sequence } (s_i)_0^k \subseteq [-1,0] \text{ with } \varphi(s_{i-1})\varphi(s_i) < 0 \text{ for } i \in \{1,2,..,k\}\Big\}.$$

Then set

$$V(\varphi) = \begin{cases} sc(\varphi), & \text{if } sc(\varphi) \text{ is even or } \infty, \\ sc(\varphi)+1, & \text{if } sc(\varphi) \text{ is odd.} \end{cases}$$

Also define

$$\begin{aligned} R \;=\; & \{\varphi \in C^1 : \varphi(0) \neq 0 \text{ or } \dot{\varphi}(0)\varphi(-1) > 0, \\ & \varphi(-1) \neq 0 \text{ or } \dot{\varphi}(-1)\varphi(0) < 0, \text{ all zeros of } \varphi \text{ are simple}\}. \end{aligned}$$

V has the following lower semi-continuity and continuity property (for a proof, see [10, 16]).

Lemma 2.2. *For each $\varphi \in C \setminus \{\hat{0}\}$ and $(\varphi_n)_0^\infty \subset C \setminus \{\hat{0}\}$ with $\varphi_n \to \varphi$ as $n \to \infty$, $V(\varphi) \leq \liminf_{n\to\infty} V(\varphi_n)$. For each $\varphi \in R$ and $(\varphi_n)_0^\infty \subset C^1 \setminus \{\hat{0}\}$ with $\|\varphi_n - \varphi\|_{C^1} \to 0$ as $n \to \infty$, $V(\varphi) = \lim_{n\to\infty} V(\varphi_n) < \infty$.*

The next result explains why V is called a Lyapunov functional (for a proof, see [10, 16] again). For an interval $J \subset \mathbb{R}$, we use the notation

$$J + [-1,0] = \{t \in \mathbb{R} : t = t_1 + t_2 \text{ with } t_1 \in J,\, t_2 \in [-1,0]\}.$$

Lemma 2.3. *Assume that $\mu \geq 0$, $J \subset \mathbb{R}$ is an interval, $a : J \to \mathbb{R}$ is positive and continuous, $z : J + [-1,0] \to \mathbb{R}$ is continuous, $z(t) \neq 0$ for some $t \in J + [-1,0]$, and z is differentiable on J. Suppose that*

$$\dot{z}(t) = -\mu z(t) + a(t) z(t-1) \tag{2.3}$$

holds for all $t > \inf J$ in J. Then the following statements hold.
(i) If $t_1, t_2 \in J$ with $t_1 < t_2$, then $V(z_{t_1}) \geq V(z_{t_2})$.
(ii) If $t, t-2 \in J$, $z(t-1) = z(t) = 0$, then either $V(z_t) = \infty$ or $V(z_{t-2}) > V(z_t)$.
(iii) If $t \in J$, $t-3 \in J$, and $V(z_{t-3}) = V(z_t) < \infty$, then $z_t \in R$.

If f is a C^1-smooth function with $f' > 0$ on $\mathbb{R}$, $x, \hat{x} : J + [-1,0] \to \mathbb{R}$ are solutions of Eq. (1.1) and $c \in \mathbb{R} \setminus \{0\}$, then Lemma 2.3 can be applied for $z = (x - \hat{x})/c$ with the positive continuous function

$$a : J \ni t \mapsto \int_0^1 f'(sx(t-1) + (1-s)\hat{x}(t-1))\,\mathrm{d}s \in [0,\infty).$$

Further notations and preliminary results. A solution x is oscillatory about an equilibrium $\hat{\xi}$ (or a constant ξ) if $x^{-1}(\xi)$ is not bounded from above. It is slowly oscillatory about $\hat{\xi}$ (or ξ) if $t \to x(t) - \xi$ has one or two sign changes on each interval of length 1.

$B(\varphi, r)$, $\varphi \in C$, $r > 0$, denotes the open ball in C with center φ and radius r.

We use the notation $S^1_{\mathbb{C}}$ for the set $\{z \in \mathbb{C} : |z| = 1\}$.

For a simple closed curve $c : [a,b] \to \mathbb{R}^2$, $\mathrm{int}(c[a,b])$ and $\mathrm{ext}(c[a,b])$ denote the interior and exterior, i.e., the bounded and unbounded components of $\mathbb{R}^2 \setminus c([a,b])$, respectively. We use the same notations for closed curves $c : [a,b] \to G_2$, where G_2 is any two-dimensional real Banach space.

We say $\varphi \leq \psi$ for $\varphi, \psi \in C$ if $\varphi(s) \leq \psi(s)$ for all $s \in [-1,0]$. Relation $\varphi < \psi$ holds if $\varphi \leq \psi$ and $\varphi \neq \psi$. In addition, $\varphi \ll \psi$ if $\varphi(s) < \psi(s)$ for all $s \in [-1,0]$. Relations "$\geq$", "$>$" and "$\gg$" are defined analogously.

The semiflow Φ is monotone in the following sense.

Proposition 2.4. *If $\varphi, \psi \in C$ with $\varphi \leq \psi$ ($\varphi \geq \psi$), then $x^\varphi_t \leq x^\psi_t$ $\left(x^\varphi_t \geq x^\psi_t\right)$ for all $t \geq 0$. If $\varphi < \psi$ ($\varphi > \psi$), then $x^\varphi_t \ll x^\psi_t$ $\left(x^\varphi_t \gg x^\psi_t\right)$ for all $t \geq 2$. If $\varphi \ll \psi$ ($\varphi \gg \psi$), then $x^\varphi_t \ll x^\psi_t$ $\left(x^\varphi_t \gg x^\psi_t\right)$ for all $t \geq 0$.*

The assertion follows easily from the variation-of-constant formula. For a proof we refer to [20]. Note that Proposition 2.4 guarantees the positive invariance of $C_{-2,0}$, $C_{0,2}$ and $C_{-2,2}$.

The periodic solutions have nice monotonicity properties (see Theorem 7.1 in [17]) as follows.

Proposition 2.5. *Suppose $r : \mathbb{R} \to \mathbb{R}$ is a periodic solution of Eq.* (1.1) *with minimal period $\omega > 0$. Then r is of monotone type in the following sense: if $t_0 < t_1 < t_0 + \omega$ are fixed so that $r(t_0) = \min_{t\in\mathbb{R}} r(t)$ and $r(t_1) = \max_{t\in\mathbb{R}} r(t)$, then $\dot{r}(t) > 0$ for $t \in (t_0, t_1)$ and $\dot{r}(t) < 0$ for $t \in (t_1, t_0 + \omega)$.*

We also need the next technical results. The first one is the direct consequence of Lemmas VI.4, VI.5 and VI.6 in [10].

Lemma 2.6. *Let $\mu \geq 0$, $\alpha_0 > 0$ and $\alpha_1 \geq \alpha_0$. Let sequences of continuous real functions a^n on $\mathbb{R}$ and continuously differentiable real functions z^n on $\mathbb{R}$, $n \geq 0$, be given such that for all $n \geq 0$, $\alpha_0 \leq a^n(t) \leq \alpha_1$ for all $t \in \mathbb{R}$, $z^n(t) \neq 0$ for some $t \in \mathbb{R}$, $V(z^n_t) \leq 2$ for all $t \in \mathbb{R}$, and z^n satisfies*

$$\dot{z}^n(t) = -\mu z^n(t) + a^n(t) z^n(t-1)$$

on $\mathbb{R}$. Let a further continuous real function a on $\mathbb{R}$ be given so that $a^n \to a$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$. Then a continuously differentiable function $z : \mathbb{R} \to \mathbb{R}$ and a subsequence $(z^{n_k})_{k=0}^{\infty}$ of $(z^n)_{n=0}^{\infty}$ can be given such that $z^{n_k} \to z$ and $\dot{z}^{n_k} \to \dot{z}$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$, moreover

$$\dot{z}(t) = -\mu z(t) + a(t) z(t-1)$$

for all $t \in \mathbb{R}$.

The subsequent result shows that Lyapunov functionals can be used effectively to show that solutions of linear equations cannot decay too fast at ∞. For a proof, see Lemma VI.3 in [10].

Lemma 2.7. *Let $\mu \geq 0$, $\alpha_0 > 0$ and $\alpha_1 \geq \alpha_0$. Assume that $t_0 \in \mathbb{R}$, $a : [t_0 - 5, t_0] \to \mathbb{R}$ is continuous with $\alpha_0 \leq a(t) \leq \alpha_1$ for all $t \in [t_0 - 5, t_0]$, $z : [t_0 - 6, t_0] \to \mathbb{R}$ is continuous, differentiable for $t_0 - 5 < t \leq t_0$ and satisfies (2.3) for $t_0 - 5 < t \leq t_0$. In addition, assume that $z_{t_0-5} \neq 0$ and $V(z_{t_0-5}) \leq 2$. Then there exists $K = K(\mu, \alpha_0, \alpha_1) > 0$ such that*

$$\|z_{t_0-1}\| \leq K \|z_{t_0}\|.$$

The last result of this section is Lemma I.8 in [10]. It will be used to abbreviate proofs of smoothness of submanifolds.

Proposition 2.8. *Let g be a C^1-map from an m-dimensional C^1-manifold M into a C^1-manifold N modeled over a Banach space. If for some $p \in M$, the derivative $Dg(p)$ of g at p is injective, then p has an open neighborhood U in M so that for all open sets V in U, $g(V)$ is an m-dimensional C^1-submanifold of N.*

3. Floquet multipliers and a Poincaré return map

In this section we give a brief introduction to the Floquet theory regarding periodic solutions which are slowly oscillatory about an equilibrium. Then we define a Poincaré map and collect the most important properties of its local invariant manifolds. At last we apply these results to p, q, x^1 and x^{-1}. The section is closed by showing that the unstable space of the monodromy operator corresponding to the periodic orbit $\mathcal{O}_k$ is one-dimensional for both $k \in \{-1, 1\}$.

3.1. **Floquet multipliers.** Suppose $r : \mathbb{R} \to \mathbb{R}$ is a periodic solution of Eq. (1.1) with minimal period $\omega > 0$. If r is slowly oscillatory about an equilibrium (as p, q, x^1 or x^{-1} are), then Proposition 2.5 implies that $\omega \in (1, 2)$. Assume that this is the case.

Consider the period map $Q = \Phi(\omega, \cdot)$ with fixed point r_0 and its derivative $M = D_2\Phi(\omega, r_0)$ at r_0. Then $M\varphi = u^{\varphi}_{\omega}$ for all $\varphi \in C$, where $u^{\varphi} : [-1, \infty) \to \mathbb{R}$ is the solution of the linear variational equation

$$\dot{u}(t) = -u(t) + f'\left(r\left(t-1\right)\right)u\left(t-1\right) \tag{3.1}$$

with $u_0^\varphi = \varphi$. M is called the monodromy operator.

M is a compact operator, 0 belongs to its spectrum $\sigma = \sigma\left(M\right)$, and its eigenvalues of finite multiplicity – the so called Floquet multipliers – form $\sigma\left(M\right) \setminus \{0\}$. The importance of M lies in the fact that we obtain information about the stability properties of the orbit $\mathcal{O}_r = \{r_t : t \in \mathbb{R}\}$ from $\sigma\left(M\right)$.

As $\dot{r}$ is a nonzero solution of the variational equation (3.1), 1 is a Floquet multiplier with eigenfunction $\dot{r}_0$. The periodic orbit $\mathcal{O}_r$ is said to be hyperbolic if the generalized eigenspace of M corresponding to the eigenvalue 1 is one-dimensional, furthermore there are no Floquet multipliers on the unit circle besides 1.

The paper [16] of Mallet-Paret and Sell and Appendix VII of the monograph [10] of Krisztin, Walther and Wu confirm the subsequent properties. $\mathcal{O}_r$ has a real Floquet multiplier $\lambda_1 > 1$ with a strictly positive eigenvector v_1. The realified generalized eigenspace $C_{<\lambda_1}$ associated with the spectral set $\{z \in \sigma : |z| < \lambda_1\}$ satisfies

$$C_{<\lambda_1} \cap V^{-1}\left(0\right) = \emptyset. \tag{3.2}$$

Let $C_{\leq\rho}$, $\rho > 0$, denote the realified generalized eigenspace of M associated with the spectral set $\{z \in \sigma : |z| \leq \rho\}$. The set

$$\left\{\rho \in (0,\infty) : \sigma\left(M\right) \cap \rho S^1_{\mathbb{C}} \neq \emptyset,\ C_{\leq\rho} \cap V^{-1}\left(\{0,2\}\right) = \emptyset\right\}$$

is nonempty and has a maximum r_M. Then

$$C_{\leq r_M} \cap V^{-1}\left(\{0,2\}\right) = \emptyset, \quad C_{r_M<} \setminus \left\{\hat{0}\right\} \subset V^{-1}\left(\{0,2\}\right) \text{ and } \dim C_{r_M<} \leq 3, \tag{3.3}$$

where $C_{r_M<}$ is the realified generalized eigenspace of M associated with the nonempty spectral set $\{z \in \sigma : |z| > r_M\}$. It will easily follow from the results of this paper that $\dim C_{r_M<} = 3$ for the periodic solutions p, q, x^{-1} and x^1, see Remark 3.7. Recently Mallet-Paret and Nussbaum have shown that the equality $\dim C_{r_M<} = 3$ holds in general [15].

Let C_s, C_c and C_u be the closed subspaces of C chosen so that $C = C_s \oplus C_c \oplus C_u$, C_s, C_c and C_u are invariant under M, and the spectra $\sigma_s\left(M\right)$, $\sigma_c\left(M\right)$ and $\sigma_u\left(M\right)$ of the induced maps $C_s \ni x \mapsto Mx \in C_s$, $C_c \ni x \mapsto Mx \in C_c$, and $C_u \ni x \mapsto Mx \in C_u$ are contained in $\{\mu \in \mathbb{C} : |\mu| < 1\}$, $\{\mu \in \mathbb{C} : |\mu| = 1\}$ and $\{\mu \in \mathbb{C} : |\mu| > 1\}$, respectively.

As $\mathcal{O}_r$ has a real Floquet multiplier $\lambda_1 > 1$, C_u is nontrivial.

C_c is also nontrivial because $\dot{r}_0 \in C_c$. It is easy to see that the monotonicity property of r described in Proposition 2.5 and $\omega \in (1,2)$ imply the existence of $t \in \mathbb{R}$ with $V\left(\dot{r}_t\right) = 2$. As $\mathbb{R} \ni t \to \dot{r}_t \in C$ is periodic, and $\mathbb{R} \ni t \to V\left(\dot{r}_t\right)$ is monotone decreasing by Lemma 2.3, it follows that $V(\dot{r}_t) = 2$ for all real t. In particular, $V(\dot{r}_0) = 2$. Hence (3.3) gives that $r_M < 1$, moreover (3.2) and (3.3) together give that $C_c \setminus \{\hat{0}\} \subset V^{-1}(2)$. The nontriviality of C_u and $\dim C_{r_M<} \leq 3$ in addition imply that C_c is at most two-dimensional in our case:

$$C_c = \begin{cases} \mathbb{R}\dot{r}_0, & \text{if } \mathcal{O}_r \text{ is hyperbolic,} \\ \mathbb{R}\dot{r}_0 \oplus \mathbb{R}\xi, & \text{otherwise,} \end{cases}$$

where $\xi \in C_c \setminus \mathbb{R}\dot{r}_0$ provided that $\mathcal{O}_r$ is nonhyperbolic.

3.2. **A Poincaré return map.** As above, let $r : \mathbb{R} \to \mathbb{R}$ be any periodic solution of Eq. (1.1) which oscillates slowly about an equilibrium, and let $\omega \in (1, 2)$ denote its minimal period.

Fix a $\xi \in C_c \setminus \mathbb{R}\dot{r}_0$ in case O_r is nonhyperbolic and define

$$Y = \begin{cases} C_s \oplus C_u, & \text{if } \mathcal{O}_r \text{ is hyperbolic,} \\ C_s \oplus \mathbb{R}\xi \oplus C_u, & \text{if } \mathcal{O}_r \text{ is nonhyperbolic.} \end{cases}$$

Then $Y \subset C$ is a hyperplane with codimension 1. Choose e^* to be a continuous linear functional with null space $(e^*)^{-1}(0) = Y$. The Hahn–Banach theorem guarantees the existence of e^*. As $D_1\Phi(\omega, r_0)\, 1 = \dot{r}_0 \notin Y$, and thus $e^*(D_1\Phi(\omega, r_0)\, 1) \neq 0$, the implicit function theorem can be applied to the map

$$(t, \varphi) \mapsto e^*(\Phi(t, \varphi) - r_0)$$

in a neighborhood of (ω, r_0). It yields a convex bounded open neighborhood N of r_0 in C, $\varepsilon \in (0, \omega)$ and a C^1-map $\gamma : N \to (\omega - \varepsilon, \omega + \varepsilon)$ with $\gamma(r_0) = \omega$ so that for each $(t, \varphi) \in (\omega - \varepsilon, \omega + \varepsilon) \times N$, the segment x_t^{φ} belongs to $r_0 + Y$ if and only if $t = \gamma(\varphi)$ (see [2], Appendix I in [10], [13]). In addition, by continuity we may assume that $D_1\Phi(\gamma(\varphi), \varphi)\, 1 \notin Y$ for all $\varphi \in N$. The Poincaré return map P_Y is defined by

$$P_Y : N \cap (r_0 + Y) \ni \varphi \mapsto \Phi(\gamma(\varphi), \varphi) \in r_0 + Y.$$

Then P_Y is continuously differentiable with fixed point r_0.

It is convenient to have a formula not only for the derivative $DP_Y(\varphi)$ of P_Y at $\varphi \in N \cap (r_0 + Y)$, but also for the derivatives of the iterates of P_Y. For all φ in the domain of P_Y^j, $j \geq 1$, set

$$\gamma_j(\varphi) = \Sigma_{k=0}^{j-1} \gamma\left(P_Y^k(\varphi)\right).$$

Then

$$DP_Y^j(\varphi)\eta = D_1\Phi(\gamma_j(\varphi), \varphi)\, \gamma_j'(\varphi)\eta + D_2\Phi(\gamma_j(\varphi), \varphi)\eta$$

for all $\eta \in Y$. Differentiation of the equation $e^*(\Phi(\gamma_j(\varphi), \varphi) - r_0) = 0$ yields that

$$\gamma_j'(\varphi)\eta = -\frac{e^*(D_2\Phi(\gamma_j(\varphi), \varphi)\eta)}{e^*(D_1\Phi(\gamma_j(\varphi), \varphi)\, 1)},$$

and therefore

$$(3.4) \qquad DP_Y^j(\varphi)\,\eta = D_2\Phi\left(\gamma_j(\varphi),\varphi\right)\eta - \frac{e^*\left(D_2\Phi\left(\gamma_j(\varphi),\varphi\right)\eta\right)}{e^*\left(D_1\Phi\left(\gamma_j(\varphi),\varphi\right)1\right)} D_1\Phi\left(\gamma_j(\varphi),\varphi\right)1$$

for all $\eta \in Y$.

Let $\sigma(P_Y)$ and $\sigma(M)$ denote the spectra of $DP_Y(r_0): Y \to Y$ and the monodromy operator, respectively. We obtain the following result from Theorem XIV.4.5 in [2].

Lemma 3.1.
(i) $\sigma(P_Y) \setminus \{0,1\} = \sigma(M) \setminus \{0,1\}$, *and for every* $\lambda \in \sigma(M) \setminus \{0,1\}$, *the projection along* $\mathbb{R}\dot{r}_0$ *onto* Y *defines an isomorphism from the realified generalized eigenspace of* λ *and* M *onto the realified generalized eigenspace of* λ *and* $DP_Y(r_0)$.
(ii) If the generalized eigenspace $G(1,M)$ *associated with* 1 *and* M *is one-dimensional, then* $1 \notin \sigma(P_Y)$.
(iii) If $\dim G(1,M) > 1$, *then* $1 \in \sigma(P_Y)$, *and the realified generalized eigenspaces* $G_{\mathbb{R}}(1,M)$ *and* $G_{\mathbb{R}}(1,P_Y)$ *associated with* 1 *and* M *and with* 1 *and* $DP_Y(r_0)$, *respectively, satisfy*

$$G_{\mathbb{R}}(1,P_Y) = Y \cap G_{\mathbb{R}}(1,M) \quad \text{and} \quad G_{\mathbb{R}}(1,M) = \mathbb{R}\dot{r}_0 \oplus G_{\mathbb{R}}(1,P_Y).$$

In our case, the special choice of Y implies the following corollary.

Corollary 3.2.
(i) C_s *and* C_u *are invariant under* $DP_Y(r_0)$, *and the spectra* $\sigma_s(P_Y)$ *and* $\sigma_u(P_Y)$ *of the induced maps* $C_s \ni x \mapsto DP_Y(r_0)x \in C_s$ *and* $C_u \ni x \mapsto DP_Y(r_0)x \in C_u$ *are contained in* $\{\mu \in \mathbb{C} : |\mu| < 1\}$ *and* $\{\mu \in \mathbb{C} : |\mu| > 1\}$, *respectively.*
(ii) If M *has an eigenfunction* v *corresponding to a simple eigenvalue* $\lambda \in \sigma(M) \setminus \{0,1\}$, *then* v *is an eigenfunction of* $DP_Y(r_0)$ *corresponding to the same eigenvalue.*
(iii) If $\mathcal{O}_r$ *is nonhyperbolic, then* ξ *is an eigenfunction of* $DP_Y(r_0)$, *and it corresponds to an eigenvalue with absolute value* 1.

In particular, if λ_1 is a simple Floquet multiplier, then the strictly positive eigenfunction v_1 of M corresponding to λ_1 is also an eigenfunction of $DP_Y(r_0)$ corresponding to λ_1.

In case $\mathcal{O}_r$ is hyperbolic, then according to Theorem I.3 in Appendix I of [10], there exist convex open neighborhoods N_s, N_u of $\hat{0}$ in C_s, C_u, respectively, and a C^1-map $w_u : N_u \to C_s$ with range in N_s so that $w_u(\hat{0}) = \hat{0}$, $Dw_u(\hat{0}) = 0$, and the submanifold

$$\mathcal{W}^u_{loc}(P_Y, r_0) = \{r_0 + \chi + w_u(\chi) : \chi \in N_u\}$$

of $r_0 + Y$ is equal to the set

$$\{\varphi \in r_0 + N_s + N_u : \text{ there is a trajectory } (\varphi_n)_{-\infty}^0 \text{ of } P_Y \text{ with } \varphi_0 = \varphi \text{ such that } \varphi_n \in r_0 + N_s + N_u \text{ for all } n \leq 0 \text{ and } \varphi_n \to r_0 \text{ as } n \to -\infty\}.$$

$\mathcal{W}^u_{loc}(P_Y, r_0)$ is called a local unstable manifold of P_Y at r_0.

The unstable set of the orbit $\mathcal{O}_r$ is defined as the forward extension of $\mathcal{W}^u_{loc}(P_Y, r_0)$ in time:

$$\mathcal{W}^u(\mathcal{O}_r) = \Phi([0,\infty) \times \mathcal{W}^u_{loc}(P_Y, r_0)). \tag{3.5}$$

If $\mathcal{O}_r$ is hyperbolic, then

$$\mathcal{W}^u(\mathcal{O}_r) = \{x_0 : \ x : \mathbb{R} \to \mathbb{R} \text{ is a solution of (1.1), } \alpha(x) \text{ exists and } \alpha(x) = \mathcal{O}_r\}.$$

If $\mathcal{O}_r$ is hyperbolic, then by Theorem I.2 in [10], there are convex open neighborhoods N_s, N_u of $\hat{0}$ in C_s, C_u, respectively, and a C^1-map $w_s : N_s \to C_u$ with range in N_u such that $w_s(\hat{0}) = \hat{0}$, $Dw_s(\hat{0}) = 0$, and

$$\mathcal{W}^s_{loc}(P_Y, r_0) = \{r_0 + \chi + w_s(\chi) : \ \chi \in N_s\}$$

is equal to

$$\{\varphi \in r_0 + N_s + N_u : \text{ there is a trajectory } (\varphi_n)_0^\infty \text{ of } P_Y \text{ in } r_0 + N_s + N_u \text{ with } \varphi_0 = \varphi \text{ and } \varphi_n \to r_0 \text{ as } n \to \infty\}.$$

$\mathcal{W}^s_{loc}(P_Y, r_0)$ is a local stable manifold of P_Y at r_0. It is a C^1-submanifold of $r_0 + Y$ with codimension $\dim C_u$, and it is a C^1-submanifold of C with codimension $\dim C_u + 1$.

In case $\mathcal{O}_r$ is nonhyperbolic, we need a local center-stable manifold $\mathcal{W}^{sc}_{loc}(P_Y, r_0)$ of P_Y at r_0. According to Theorem II.1 in [10], there exist convex open neighborhoods N_{sc} and N_u of $\hat{0}$ in $C_s \oplus \mathbb{R}\xi$ and C_u, respectively, and a C^1-map $w_{sc} : N_{sc} \to C_u$ such that $w_{sc}(\hat{0}) = \hat{0}$, $Dw_{sc}(\hat{0}) = 0$, $w_{sc}(N_{sc}) \subset N_u$ and the local center-stable manifold

$$\mathcal{W}^{sc}_{loc}(P_Y, r_0) = \{r_0 + \chi + w_{sc}(\chi) : \ \chi \in N_{sc}\}$$

satisfies

$$\bigcap_{n=0}^{\infty} P_Y^{-1}(r_0 + N_{sc} + N_u) \subset \mathcal{W}^{sc}_{loc}(P_Y, r_0).$$

Note that $\mathcal{W}^{sc}_{loc}(P_Y, r_0)$ is also a C^1-submanifold of $r_0 + Y$ with codimension $\dim C_u$, and it is a C^1-submanifold of C with codimension $\dim C_u + 1$.

Proposition 3.3. *One may choose the neighborhoods N_s and N_{sc} so small in the definitions of $\mathcal{W}^s_{loc}(P_Y, r_0)$, $\mathcal{W}^{sc}_{loc}(P_Y, r_0)$, respectively, such that for all φ in $\mathcal{W}^s_{loc}(P_Y, r_0) \cap \mathcal{A}$ and in $\mathcal{W}^{sc}_{loc}(P_Y, r_0) \cap \mathcal{A}$, $\dot{\varphi} \notin Y$ and $V(\dot{\varphi}) \geq 2$. Analogously, one may suppose that $\dot{\varphi} \notin Y$ for all $\varphi \in \mathcal{W}^u_{loc}(P_Y, r_0) \cap \mathcal{A}$.*

Proof. Recall that the C-norm and the C^1-norm are equivalent on the global attractor $\mathcal{A}$. Hence for all $\varphi \in \mathcal{A}$ with small $\|\varphi - r_0\|$, $\dot{\varphi} \notin Y$ follows from $\dot{r}_0 \notin Y$, furthermore $V(\dot{\varphi}) \geq 2$ follows from $V(\dot{r}_0) = 2$ and the lower semicontinuity of V. □

The next result is an immediate consequence of Proposition I.7 in [10] combined with characterizations of the local stable and center-stable manifolds given by Theorems I.2 and II.1 in [10].

Proposition 3.4. *Let $\mathcal{W}$ denote a local stable manifold $\mathcal{W}^s_{loc}(P_Y, r_0)$ if $\mathcal{O}_r$ is hyperbolic, and let $\mathcal{W}$ be a local center-stable manifold $\mathcal{W}^{sc}_{loc}(P_Y, r_0)$ otherwise. Let $\varphi \in C$ be given such that $\Phi(t, \varphi) \to \mathcal{O}_r$ as $t \to \infty$. Then there exist $T \geq 0$ and a trajectory $(\varphi^n)_{n=0}^{\infty}$ of P_Y in $\mathcal{W}$ such that $\varphi^0 = \Phi(T, \varphi)$ and $\varphi^n \to r_0$ as $n \to \infty$.*

3.3. **Examples.** Consider the case when r is the LSOP solution p given by Theorem A. Theorem A states that $\mathcal{O}_p$ is hyperbolic, and has two real and simple Floquet multipliers outside the unit circle. Hence $C_c = \mathbb{R}\dot{p}_0$ and

$$C_u = \{c_1 v_1 + c_2 v_2 : c_1, c_2 \in \mathbb{R}\},$$

where v_1 is a positive eigenfunction corresponding to M and the leading real eigenvalue $\lambda_1 > 1$, and v_2 is an eigenfunction corresponding to M and the eigenvalue λ_2 with $1 < \lambda_2 < \lambda_1$. For the solution $u^{v_2} : [-1, \infty) \to \mathbb{R}$ of the linear variational equation (3.1) with initial segment v_2, $V(u^{v_2}_t) = 2$ for all $t \geq 0$. For both $i \in \{1, 2\}$, λ_i is an eigenvalue of $DP_Y(p_0)$ with the eigenvector v_i.

The local unstable manifold $\mathcal{W}^u_{loc}(P_Y, p_0)$ of the Poincaré map P_Y at p_0 is a two-dimensional C^1-submanifold of $p_0 + Y$.

We will use the subsequent technical result.

Proposition 3.5. *One may choose N_u so small that the tangent space $T_\varphi \mathcal{W}^u_{loc}(P_Y, p_0)$ has a strictly positive element for all $\varphi \in \mathcal{W}^u_{loc}(P_Y, p_0)$.*

Proof. By decreasing N_u if necessary, we can achieve that $v_1 + Dw_u(\chi) v_1 \gg \hat{0}$ for all $\chi \in N_u$, where v_1 is a fixed positive eigenfunction corresponding to the leading eigenvalue λ_1 of $DP_Y(p_0)$. Let $\varphi \in \mathcal{W}^u_{loc}(P_Y, p_0)$ be arbitrary and choose $\chi^\varphi \in N_u$ with $\varphi = p_0 + \chi^\varphi + w_u(\chi^\varphi)$. Then for all t in an open interval $I \subset \mathbb{R}$ containing 0, $\gamma(t) = p_0 + \chi^\varphi + t v_1 + w_u(\chi^\varphi + t v_1)$ is defined. Moreover, $\gamma : I \to \mathcal{W}^u_{loc}(P_Y, p_0)$ is a C^1-curve with $\gamma(0) = \varphi$ and

$$T_\varphi \mathcal{W}^u_{loc}(P_Y, p_0) \ni \gamma'(0) = v_1 + Dw_u(\chi^\varphi) v_1 \gg \hat{0}.$$

□

We plan to consider other periodic orbits oscillating slowly about an equilibrium, but keep the same notations for simplicity (ω for the minimal period, P_Y for the

Poincaré map, λ_i, $i \geq 1$, for the Floquet multipliers, v_i, $i \geq 1$, for eigenvectors, and so on). It will be clear from the context which periodic orbit we refer to.

Theorem A gives a second LSOP solution $q : \mathbb{R} \to \mathbb{R}$. $\mathcal{O}_q$ is hyperbolic, and it has exactly one simple Floquet multiplier outside the unit circle, which is real and greater than 1. This leading eigenvalue will be also denoted by λ_1, but it differs from the leading Floquet multiplier of $\mathcal{O}_p$. To λ_1 there corresponds a positive eigenfunction v_1 (different from the previous v_1). Hence for $r = q$, $C_c = \mathbb{R}\dot{q}_0$ and $C_u = \mathbb{R}v_1$. The local stable manifold $\mathcal{W}^s_{loc}(P_Y, q_0)$ of P_Y at q_0 is a C^1-submanifold of $q_0 + Y$ with codimension 1, and a C^1-submanifold of C with codimension 2. We have the tangent space $T_{q_0}\mathcal{W}^s_{loc}(P_Y, q_0) = C_s$ at q_0 in $q_0 + Y$.

Recall that there exist periodic solutions $x^1 : \mathbb{R} \to \mathbb{R}$ and $x^{-1} : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) oscillating slowly about ξ_1 and ξ_{-1} with ranges in $(0, \xi_2)$ and $(\xi_{-2}, 0)$, respectively, so that the ranges $x^1(\mathbb{R})$ and $x^{-1}(\mathbb{R})$ are maximal in the sense that $x^1(\mathbb{R}) \supset x(\mathbb{R})$ for all periodic solutions x oscillating slowly about ξ_1 with ranges in $(0, \xi_2)$; and analogously for x^{-1}. We do not know whether the corresponding periodic orbits, $\mathcal{O}_1$ and $\mathcal{O}_{-1}$, are hyperbolic or not.

Proposition 3.6. *For both periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_{-1}$, $\dim C_u = 1$.*

Proof. We give a proof for $\mathcal{O}_1$. As $\mathcal{O}_1$ has a Floquet multiplier $\lambda_1 > 1$, it is clear that $\dim C_u \geq 1$.

Let $\mathcal{W}$ denote the local stable manifold $\mathcal{W}^s_{loc}(P_Y, x^1_0)$ if $\mathcal{O}_1$ is hyperbolic, and let $\mathcal{W}$ be the local center-stable manifold $\mathcal{W}^{sc}_{loc}(P_Y, x^1_0)$ otherwise. Then $\mathcal{W}$ is a C^1-submanifold of $x^1_0 + Y$ with $T_{x^1_0}\mathcal{W} = C_s$ if $\mathcal{O}_1$ is hyperbolic, and with $T_{x^1_0}\mathcal{W} = C_s \oplus \mathbb{R}\xi$ if $\mathcal{O}_1$ is nonhyperbolic.

By Theorem B, there exists $\eta \in \mathcal{W}^u(\mathcal{O}_p)$ so that $x^\eta_t \to \mathcal{O}_1$ as $t \to \infty$. Then Proposition 3.4 guarantees the existence of a sequence $(t_n)^\infty_{n=0}$ in $\mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$ such that $x^\eta_{t_n} \in \mathcal{W} \setminus \{x^1_0\}$ for all $n \geq 0$ and $x^\eta_{t_n} \to x^1_0$ as $n \to \infty$.

We introduce the notation $y^n : \mathbb{R} \to \mathbb{R}$, $n \geq 0$, for the function obtained from x^η by time shift so that $y^n_0 = x^\eta_{t_n}$. Then $y^n(t) \to x^1(t)$ as $n \to \infty$ for all $t \in \mathbb{R}$ by the continuity of the flow $\Phi_{\mathcal{A}}$. Since x^η is a bounded solution of Eq. (1.1), the solutions y^n are uniformly bounded on $\mathbb{R}$, and Eq. (1.1) gives a uniform bound for their derivatives. By applying the Arzelà–Ascoli theorem successively on the intervals $[-j, j]$, $j \geq 1$, we obtain strictly increasing maps $\chi_j : \mathbb{N} \to \mathbb{N}$, $1 \leq j \in \mathbb{N}$, so that for every integer $j \geq 1$, the subsequence $\left(y^{\chi_1 \circ \ldots \circ \chi_j(k)}\right)^\infty_{k=0}$ converges uniformly on $[-j, j]$. By diagonalization, set $\chi(k) = \chi_1 \circ \ldots \circ \chi_k(k)$ and consider the subsequence $(y^{n_k})^\infty_{k=0} = \left(y^{\chi(k)}\right)^\infty_{k=0}$. Then $y^{n_k} \to x^1$ as $k \to \infty$ uniformly on all compact subsets of $\mathbb{R}$.

Define

$$z^k(t) = \frac{y^{n_k}(t) - x^1(t)}{\left\| x^\eta_{t_{n_k}} - x^1_0 \right\|} \quad \text{for all } k \geq 0 \text{ and } t \in \mathbb{R}.$$

Then z^k, $k \geq 0$, satisfies the equation $\dot{z}^k(t) = -z^k(t) + a_k(t) z^k(t-1)$ on $\mathbb{R}$, where the coefficient function a_k is defined by

$$a_k : \mathbb{R} \ni t \mapsto \int_0^1 f'\left(s y^{n_k}(t-1) + (1-s) x^1(t-1)\right) \mathrm{d}s \in \mathbb{R}^+, \quad k \geq 0.$$

Note that there are constants $\alpha_1 \geq \alpha_0 > 0$ independent of k and t such that $\alpha_0 \leq a_k(t) \leq \alpha_1$ for all $k \geq 0$ and $t \in \mathbb{R}$, moreover, $a_k \to a$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$, where

$$a : \mathbb{R} \ni t \mapsto f'\left(x^1(t-1)\right) \in \mathbb{R}^+.$$

In addition, observe that for all $k \geq 0$ and $t \in \mathbb{R}$, $z^k_t \neq \hat{0}$ because $y^{n_k}_0 = x^\eta_{t_{n_k}} \neq x^1_0$ and the flow $\Phi_{\mathcal{A}}$ is injective. Hence $V\left(z^k_t\right)$ is defined and equals 2 for all $k \geq 0$ and $t \in \mathbb{R}$ by Proposition 8.3 in [8]. Lemma 2.6 then implies the existence of a continuously differentiable function $z : \mathbb{R} \to \mathbb{R}$ and a subsequence $\left(z^{k_l}\right)_{l=0}^\infty$ of $\left(z^k\right)_{k=0}^\infty$ such that $z^{k_l} \to z$ and $\dot{z}^{k_l} \to \dot{z}$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$, moreover

$$\dot{z}(t) = -z(t) + a(t) z(t-1) \tag{3.6}$$

for all real t.

We claim that $z_0 \neq \hat{0}$ and

$$z_0 \in T_{x^1_0}\mathcal{W} = \begin{cases} C_s, & \text{if } \mathcal{O}_r \text{ is hyperbolic,} \\ C_s \oplus \mathbb{R}\xi, & \text{otherwise.} \end{cases}$$

Consider the map $w = w_s$ if $\mathcal{O}_1$ is hyperbolic, and the map $w = w_{sc}$ otherwise. Choose $\chi^l \in T_{x^1_0}\mathcal{W}$, $l \geq 0$, with $\chi^l \to \hat{0}$ as $l \to \infty$ so that $x^\eta_{t_{n_{k_l}}} = x^1_0 + \chi^l + w\left(\chi^l\right)$ for all $l \geq 0$. Then

$$z_0 = \lim_{l \to \infty} z^{k_l}_0 = \lim_{l \to \infty} \frac{x^\eta_{t_{n_{k_l}}} - x^1_0}{\left\| x^\eta_{t_{n_{k_l}}} - x^1_0 \right\|} = \lim_{l \to \infty} \frac{\chi^l + w\left(\chi^l\right)}{\left\| \chi^l + w\left(\chi^l\right) \right\|}.$$

As z_0 is the limit of unit vectors, it is clearly nontrivial. $Dw\left(\hat{0}\right) = 0$ implies that $\lim_{l \to \infty} w\left(\chi^l\right) / \left\|\chi^l\right\| = \hat{0}$ and thus

$$\lim_{l \to \infty} \frac{w\left(\chi^l\right)}{\left\| \chi^l + w\left(\chi^l\right) \right\|} = \lim_{l \to \infty} \frac{\frac{w\left(\chi^l\right)}{\left\|\chi^l\right\|}}{\left\| \frac{\chi^l}{\left\|\chi^l\right\|} + \frac{w\left(\chi^l\right)}{\left\|\chi^l\right\|} \right\|} = \hat{0}$$

and

$$\lim_{l\to\infty}\frac{\left\|\chi^l\right\|}{\left\|\chi^l+w\left(\chi^l\right)\right\|}=\lim_{l\to\infty}\frac{1}{\left\|\frac{\chi^l}{\left\|\chi^l\right\|}+\frac{w\left(\chi^l\right)}{\left\|\chi^l\right\|}\right\|}=1.$$

We obtain that

$$\underbrace{\frac{\chi^l+w\left(\chi^l\right)}{\left\|\chi^l+w\left(\chi^l\right)\right\|}}_{\substack{\downarrow\\ z_0}}=\frac{\chi^l}{\left\|\chi^l\right\|}\underbrace{\frac{\left\|\chi^l\right\|}{\left\|\chi^l+w\left(\chi^l\right)\right\|}}_{\substack{\downarrow\\ 1}}+\underbrace{\frac{w\left(\chi^l\right)}{\left\|\chi^l+w\left(\chi^l\right)\right\|}}_{\substack{\downarrow\\ 0}}$$

as $l\to\infty$. Then the limit $\lim_{l\to\infty}\chi^l/\left\|\chi^l\right\|$ necessarily exists too, and

$$z_0=\lim_{l\to\infty}\frac{\chi^l+w\left(\chi^l\right)}{\left\|\chi^l+w\left(\chi^l\right)\right\|}=\lim_{l\to\infty}\frac{\chi^l}{\left\|\chi^l\right\|}\in T_{x_0^1}\mathcal{W}\subset Y.$$

Since $V\left(z_0^{k_l}\right)=2$ for all $l\geq 0$, the lower-semicontinuity of V proved in Lemma 2.2 implies that $V\left(z_0\right)\leq\liminf_{l\to\infty}V\left(z_0^{k_l}\right)=2$. Recall that $\dot{x}_0^1\in C_c$ also belongs to $V^{-1}\left(\{0,2\}\right)$, moreover, $\dot{x}_0^1\notin Y$. Thus $\dot{x}_0^1$ and z_0 are linearly independent elements of $\left(C_s\oplus C_c\right)\cap V^{-1}\left(\{0,2\}\right)$. In consequence, result (3.3) gives that C_u is at most one-dimensional.

The proof is analogous for $\mathcal{O}_{-1}$. □

The previous result implies that if $\mathcal{O}_k$, $k\in\{-1,1\}$, is hyperbolic, then the local stable manifold $\mathcal{W}^s_{loc}\left(P_Y,x_0^k\right)$ of P_Y at x_0^k is a C^1-submanifold of x_0^k+Y with codimension 1 and with tangent space $T_{x_0^k}\mathcal{W}^s_{loc}\left(P_Y,x_0^k\right)=C_s$ at x_0^k. It is a C^1-submanifold of C with codimension 2.

Similarly, if $\mathcal{O}_k$, $k\in\{-1,1\}$, is nonhyperbolic, then the local center-stable manifold $\mathcal{W}^{sc}_{loc}\left(P_Y,x_0^k\right)$ of P_Y at x_0^k is a C^1-submanifold of x_0^k+Y with codimension 1 and with tangent space $T_{x_0^k}\mathcal{W}^{sc}_{loc}\left(P_Y,x_0^k\right)=C_s\oplus\mathbb{R}\xi$ at x_0^k. It is also a C^1-submanifold of C with codimension 2.

Remark 3.7. We see from the proof of Proposition 3.6 that for $r=x^k$, $k\in\{-1,1\}$, $C_{r_M<}$ admits at least three linearly independent elements: $v_1\in C_u$, $\dot{x}_0^k\in C_c$ and $z_0\in C_s\oplus C_c$. As C_{r_M} is at most three-dimensional by (3.3), we conclude that $\dim C_{r_M<}=3$. A similar reasoning confirms the same equality for $r=q$. It is obvious that the dimension of $C_{r_M<}$ is maximal also in the case $r=p$, as $\mathcal{O}_p$ has two Floquet-multipliers outside the unit circle. These observations are in accordance with the recent result [15] of Mallet-Paret and Nussbaum stating that $\dim C_{r_M<}=3$ in more general situations.

4. The Proof of Theorem 1.1

Note that each φ in the unstable set $\mathcal{W}^u(\mathcal{O}_p)$ arises in the form $\varphi = \Phi(t,\psi)$, where $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$ and $t > 1$. Indeed,

$$\mathcal{W}^u(\mathcal{O}_r) = \Phi([0,\infty) \times \mathcal{W}^u_{loc}(P_Y, r_0)), \tag{3.5}$$

and from each $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$ we can start a backward trajectory $(\psi^n)^0_{-\infty}$ of P_Y in $\mathcal{W}^u_{loc}(P_Y, p_0)$ converging to p_0 as $n \to -\infty$. As the first part of the proof of Theorem 1.1, we are going to show in Proposition 4.1 that for all $t > 1$ and $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$, $\varphi = \Phi(t,\psi)$ belongs to a subset $W_{t,\psi,\varepsilon}$ of $\mathcal{W}^u(\mathcal{O}_p)$ that is a three-dimensional submanifold of C. This implies that $\mathcal{W}^u(\mathcal{O}_p)$ is an immersed submanifold of C. The proof of Proposition 4.1 is based on (3.5), the differentiability of $\Phi|_{(1,\infty)\times C}$ and the injectivity of $D_2\Phi(t,\varphi)$ for $t \geq 0$.

However, it does not follow immediately that $\mathcal{W}^u(\mathcal{O}_p)$ is an embedded C^1-submanifold of C. We also need to show for any $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ the existence of a ball B in C centered at φ such that

$$\mathcal{W}^u(\mathcal{O}_p) \cap B = W_{t,\psi,\varepsilon} \cap B. \tag{4.1}$$

To do this, we will give a sequence of further auxiliary results right after Proposition 4.1. We will introduce a projection π_3 from C into $\mathbb{R}^3$, and use the special properties of the Lyapunov functional V to show that π_3 is injective on $\mathcal{W}^u(\mathcal{O}_p)$ and on the tangent spaces of $W_{t,\psi,\varepsilon}$. These results will easily imply (4.1).

Afterwards we offer a smooth global graph representation for $\mathcal{W}^u(\mathcal{O}_p)$ in order to indicate the simplicity of its structure. The smoothness of the sets C^p_{-2}, C^p_0 and C^p_2 then follows at once because they are open subsets of $\mathcal{W}^u(\mathcal{O}_p)$. At last we show that the semiflow induced by the solution operator Φ extends to a C^1-flow on $\mathcal{W}^u(\mathcal{O}_p)$. This property will be applied later in the proof of Theorem 1.2.

Proposition 4.1. *To each $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$ and $t > 1$, there corresponds an $\varepsilon = \varepsilon(\psi, t) \in (0, t-1)$ so that the subset*

$$W_{t,\psi,\varepsilon} = \Phi((t-\varepsilon, t+\varepsilon) \times (\mathcal{W}^u_{loc}(P_Y, p_0) \cap B(\psi,\varepsilon)))$$

of $\mathcal{W}^u(\mathcal{O}_p)$ is a three-dimensional C^1-submanifold of C.

Proof. It is clear from (3.5) that $W_{t,\psi,\varepsilon}$ defined as above is a subset of $\mathcal{W}^u(\mathcal{O}_p)$ for all $\varepsilon \in (0, t-1)$.

Consider the three-dimensional C^1-submanifold $(1,\infty) \times \mathcal{W}^u_{loc}(P_Y, p_0)$ of $\mathbb{R} \times C$ and the continuously differentiable map

$$\Sigma : (1,\infty) \times \mathcal{W}^u_{loc}(P_Y, p_0) \ni (s,\varphi) \mapsto \Phi(s,\varphi) \in C.$$

It suffices to show by Proposition 2.8 that for all $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$ and $t > 1$, the derivative $D\Sigma(t,\psi)$ is injective on the tangent space $T_{(t,\psi)}((1,\infty) \times \mathcal{W}^u_{loc}(P_Y, p_0)) = \mathbb{R} \times T_\psi \mathcal{W}^u_{loc}(P_Y, p_0)$. This space is spanned by the tangent vectors of the following curves at 0:

$$(-1,1) \ni s \mapsto (t+s,\psi) \quad \text{and} \quad (-1,1) \ni s \mapsto (t,\gamma_i(s)), \, i \in \{1,2\},$$

where

$$\gamma_i : (-1,1) \to \mathcal{W}^u_{loc}(P_Y, p_0) \text{ is a } C^1\text{-curve,}$$
$$\gamma_i(0) = \psi \text{ and } D\gamma_i(0) = \eta_i \text{ for both } i \in \{1,2\},$$

with η_1 and η_2 forming a basis of the two-dimensional tangent space $T_\psi \mathcal{W}^u_{loc}(P_Y, p_0)$. As $\eta_1 \in Y$, $\eta_2 \in Y$ and $\dot{\psi} \notin Y$ by Proposition 3.3, the vectors η_1, η_2 and $\dot{\psi}$ are linearly independent. Clearly,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Sigma(t+s,\psi)|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi(t+s,\psi)|_{s=0} = D_1\Phi(t,\psi)1 = \dot{x}^\psi_t = D_2\Phi(t,\psi)\dot{\psi}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\Sigma(t,\gamma_i(s))|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi(t,\gamma_i(s))|_{s=0} = D_2\Phi(t,\psi)\eta_i, \qquad i \in \{1,2\}.$$

As $D_2\Phi(t,\psi) : C \to C$ is injective (see Section 2) and η_1, η_2 and $\dot{\psi}$ are linearly independent, we deduce that the range $D\Sigma(t,\psi)(\mathbb{R} \times T_\psi \mathcal{W}^u_{loc}(P_Y, p_0))$ is three-dimensional, and thus $D\Sigma(t,\psi)$ is injective. □

Next we characterize $\mathcal{W}^u(\mathcal{O}_p)$ and its tangent vectors in terms of oscillation frequencies.

Proposition 4.2. *For all $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ and $\psi \in \mathcal{W}^u(\mathcal{O}_p)$ with $\varphi \neq \psi$, $V(\psi - \varphi) \leq 2$.*

Proof. We distinguish three cases:
(i) both $\varphi \in \mathcal{O}_p$ and $\psi \in \mathcal{O}_p$;
(ii) $\varphi \in \mathcal{O}_p$ and $\psi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ (or vice verse);
(iii) both $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ and $\psi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$.

Let $\omega > 1$ denote the minimal period of p. It is easy to deduce from Proposition 2.5 that

$$V(p_\tau - p_\sigma) = 2 \text{ for all } \tau \in [0,\omega) \text{ and } \sigma \in [0,\omega) \text{ with } \tau \neq \sigma. \tag{4.2}$$

Hence the statement holds in case (i).

Case (ii). By definition, there exist $\sigma \in [0,\omega)$ and $(t_n)_0^\infty \subset \mathbb{R}$ so that $t_n \to -\infty$ and $x^\psi_{t_n} \to p_\sigma$ as $n \to \infty$. As $x^\varphi_{t_n} \in \mathcal{O}_p$ for all $n \geq 0$, we may also assume by compactness that $x^\varphi_{t_n} \to p_\tau$ as $n \to \infty$ for some $\tau \in [0,\omega)$. As the C-norm and C^1-norm are equivalent on the global attractor, $x^\psi_{t_n} \to p_\sigma$ and $x^\varphi_{t_n} \to p_\tau$ as $n \to \infty$ also in C^1-norm.

By Lemma 2.3 (iii) and property (4.2), $p_\sigma - p_\tau \in R$ for all $\tau \in [0,\omega)$ and $\sigma \in [0,\omega)$ with $\tau \neq \sigma$. Hence if $\sigma \neq \tau$, then Lemma 2.2 implies that

$$2 = V\left(p_\sigma - p_\tau\right) = \lim_{n\to\infty} V\left(x^{\psi}_{t_n} - x^{\varphi}_{t_n}\right).$$

By the monotonicity of V we conclude that $V\left(x^{\psi}_{t} - x^{\varphi}_{t}\right) \leq 2$ for all real t. If $\sigma = \tau$, then for all $\varepsilon > 0$ small, $\sigma + \varepsilon \neq \tau$ and $x^{\psi}_{t_n+\varepsilon} \to p_{\sigma+\varepsilon}$ as $n \to \infty$ both in C-norm and C^1-norm. Therefore by Lemma 2.2 and by our previous reasoning,

$$V\left(x^{\psi}_{t} - x^{\varphi}_{t}\right) \leq \liminf_{\varepsilon\to 0+} V\left(x^{\psi}_{t+\varepsilon} - x^{\varphi}_{t}\right) \leq 2$$

for all $t \in \mathbb{R}$. In particular, $V\left(\psi - \varphi\right) \leq 2$.

We omit the proof of case (iii), as it is analogous to the one given for (ii). □

As it is stated in the next proposition, the tangent vectors of $\mathcal{W}^u\left(\mathcal{O}_p\right)$ have at most two sign changes. This result is a direct consequence of Proposition 4.2.

Proposition 4.3. *Assume $\varphi \in \mathcal{W}^u\left(\mathcal{O}_p\right)$, $\gamma : (-1,1) \to C$ is a C^1-curve with $\gamma(0) = \varphi$, and $(s_n)_0^\infty$ is a sequence in $(-1,1)\setminus\{0\}$ so that $s_n \to 0$ as $n\to\infty$ and $\gamma(s_n) \in \mathcal{W}^u\left(\mathcal{O}_p\right)$ for all $n \geq 0$. Also assume that $\gamma'(0) \neq \hat{0}$. Then $V\left(\gamma'(0)\right) \leq 2$.*

Proof. By Proposition 4.2,

$$V\left(\frac{\gamma(s_n) - \gamma(0)}{s_n}\right) \leq 2$$

for all sufficiently large $n \geq 0$ (for all n with $\gamma(s_n) \neq \gamma(0)$). Since $\left(\gamma(s_n) - \gamma(0)\right)/s_n \to \gamma'(0)$ in C as $n \to \infty$, the statement follows from the lower semi-continuity property of V presented by Lemma 2.2. □

In order to get more information on the unstable set $\mathcal{W}^u\left(\mathcal{O}_p\right)$, we project it into the three-dimensional Euclidean space. Introduce the linear map

$$\pi_3 : C \ni \varphi \mapsto \left(\varphi(0), \varphi(-1), \mathcal{I}(\varphi)\right) \in \mathbb{R}^3,$$

where $\mathcal{I}(\varphi) = \int_{-1}^{0} \varphi(s)\,\mathrm{d}s$. The next statement can be obtained also from Proposition 4.2.

Proposition 4.4. *π_3 is injective on $\mathcal{W}^u\left(\mathcal{O}_p\right)$.*

Proof. Suppose that there exist $\varphi \in \mathcal{W}^u\left(\mathcal{O}_p\right)$ and $\psi \in \mathcal{W}^u\left(\mathcal{O}_p\right)$ so that $\varphi \neq \psi$ and $\pi_3\varphi = \pi_3\psi$. Consider the solutions $x^\varphi : \mathbb{R} \to \mathbb{R}$ and $x^\psi : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1). The segments x^{φ}_t and x^{ψ}_t belong to $\mathcal{W}^u\left(\mathcal{O}_p\right)$, and the injectivity of the semiflow Φ implies that $x^{\varphi}_t \neq x^{\psi}_t$ for all $t \in \mathbb{R}$. Hence $V\left(x^{\varphi}_t - x^{\psi}_t\right) \leq 2$ for all $t \in \mathbb{R}$ by Proposition 4.2. Since $\varphi(0) - \psi(0) = \varphi(-1) - \psi(-1) = 0$, Lemma 2.3 (ii) gives that

$$V\left(\varphi - \psi\right) < V\left(x^{\varphi}_{-2} - x^{\psi}_{-2}\right) \leq 2,$$

that is $V(\varphi-\psi)=0$ and $\varphi\leq\psi$ or $\psi\leq\varphi$. Using $\mathcal{I}(\varphi)=\mathcal{I}(\psi)$ we conclude that $\varphi=\psi$, which contradicts our initial assumption. □

We also need to know how π_3 acts on the tangent vectors of $\mathcal{W}^u(\mathcal{O}_p)$.

Proposition 4.5. *If $\gamma:(-1,1)\to C$ is a C^1-curve with range in $\mathcal{W}^u(\mathcal{O}_p)$ and $\gamma'(0)\neq\hat{0}$, then $\pi_3\gamma'(0)\neq(0,0,0)$.*

Proof. Let $\gamma:(-1,1)\to C$ be a C^1-curve with range in $\mathcal{W}^u(\mathcal{O}_p)$ and with $\gamma'(0)\neq\hat{0}$. Let $x:\mathbb{R}\to\mathbb{R}$ be the unique solution of Eq. (1.1) with $x_0=\gamma(0)\in\mathcal{W}^u(\mathcal{O}_p)$, and set $a:\mathbb{R}\ni t\mapsto f'(x(t-1))\in\mathbb{R}^+$.

1. We claim that the problem

$$\begin{cases}\dot{y}(t)=-y(t)+a(t)y(t-1), & t\in\mathbb{R},\\ y_0=\gamma'(0)\end{cases}$$

has a unique solution $y:\mathbb{R}\to\mathbb{R}$.

Fix a sequence $(s_n)_{n=0}^{\infty}$ in $(-1,1)\setminus\{0\}$ with $s_n\to0$ as $n\to\infty$. As $\gamma'(0)\neq\hat{0}$, we may assume that $\gamma(s_n)\neq\gamma(0)$ for all $n\geq0$. Consider the solutions $x^n=x^{\gamma(s_n)}:\mathbb{R}\to\mathbb{R}$. Then $x^n_t\in\mathcal{W}^u(\mathcal{O}_p)$ for all $n\geq0$ and $t\in\mathbb{R}$, furthermore $x^n(t)\to x(t)$ as $n\to\infty$ for all $t\in\mathbb{R}$ by the continuity of the flow $\Phi_{\mathcal{A}}$. Since all their segments belong to the bounded global attractor, the solutions x^n are uniformly bounded on $\mathbb{R}$, and Eq. (1.1) gives a uniform bound for their derivatives. Therefore by applying the Arzelà–Ascoli theorem successively on the intervals $[-j,j]$, $j\geq1$, and by using a diagonalization process, we obtain that $(x^n)_{n=0}^{\infty}$ has a subsequence $(x^{n_k})_{k=0}^{\infty}$ such that the convergence $x^{n_k}\to x$ is uniform on all compact subsets of $\mathbb{R}$. Set

$$y^k(t)=\frac{x^{n_k}(t)-x(t)}{s_{n_k}}\quad\text{for all } k\geq0 \text{ and } t\in\mathbb{R}.$$

Then for all $k\geq0$ and $t\in\mathbb{R}$, $y^k_t\neq\hat{0}$ by the injectivity of the flow $\Phi_{\mathcal{A}}$, and $V\left(y^k_t\right)\leq2$ by Proposition 4.2. In addition, y^k, $k\geq0$, satisfies the equation $\dot{y}^k(t)=-y^k(t)+a_k(t)y^k(t-1)$ on $\mathbb{R}$, where

$$a_k:\mathbb{R}\ni t\mapsto\int_0^1 f'(sx^{n_k}(t-1)+(1-s)x(t-1))\,\mathrm{d}s\in\mathbb{R}^+,\quad k\geq0.$$

It is clear that there are constants $\alpha_1\geq\alpha_0>0$ independent of k and t such that $\alpha_0\leq a_k(t)\leq\alpha_1$ for all $k\geq0$ and $t\in\mathbb{R}$. Also note that $a_k\to a$ as $k\to\infty$ uniformly on compact subsets of $\mathbb{R}$. Therefore by Lemma 2.6, there exist a continuously differentiable function $y:\mathbb{R}\to\mathbb{R}$ and a subsequence $\left(y^{k_l}\right)_{l=0}^{\infty}$ of $\left(y^k\right)_{k=0}^{\infty}$ such that $y^{k_l}\to y$ and $\dot{y}^{k_l}\to\dot{y}$ as $k\to\infty$ uniformly on compact subsets of $\mathbb{R}$, moreover

$$\dot{y}(t)=-y(t)+a(t)y(t-1)\tag{4.3}$$

for all real t. It is clear from the construction that

$$y_0 = \lim_{l\to\infty} \frac{x_0^{n_{k_l}} - x_0}{s_{n_{k_l}}} = \lim_{l\to\infty} \frac{\gamma\left(s_{n_{k_l}}\right) - \gamma(0)}{s_{n_{k_l}}} = \gamma'(0).$$

The uniqueness of y is guaranteed by Proposition 2.1.

2. Next we claim that $(-1,1) \ni s \mapsto \Phi_{\mathcal{A}}(-2, \gamma(s))$ is differentiable at $s=0$, and

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi_{\mathcal{A}}(-2, \gamma(s))\,|_{s=0} = y_{-2}.$$

If this is not true, then there exists a sequence $(s_n)_{n=0}^{\infty}$ in $(-1,1) \setminus \{0\}$ with $s_n \to 0$ as $n \to \infty$ such that for all $n \geq 0$,

$$\frac{\Phi_{\mathcal{A}}(-2, \gamma(s_n)) - \Phi_{\mathcal{A}}(-2, \gamma(0))}{s_n}$$

remains outside a fixed neighborhood of y_{-2} in C. So to verify the claim, it suffices to show that any sequence $(s_n)_{n=0}^{\infty}$ in $(-1,1) \setminus \{0\}$ with $s_n \to 0$ as $n \to \infty$ admits a subsequence $(s_{n_l})_{l=0}^{\infty}$ for which

$$\frac{\Phi_{\mathcal{A}}(-2, \gamma(s_{n_l})) - \Phi_{\mathcal{A}}(-2, \gamma(0))}{s_{n_l}} \to y_{-2} \quad \text{as } l \to \infty.$$

Indeed, by repeating the reasoning in the first part of the proof word by word, one can show that the sequence $(x^n)_{n=0}^{\infty}$ formed by the solutions $x^n = x^{\gamma(s_n)} : \mathbb{R} \to \mathbb{R}$, $n \geq 0$, has a subsequence $(x^{n_l})_{l=0}^{\infty}$ such that $(x^{n_l} - x)/s_{n_l} \to y$ as $l \to \infty$ uniformly on compact subsets of $\mathbb{R}$. In particular,

$$y_{-2} = \lim_{l\to\infty} \frac{x_{-2}^{n_l} - x_{-2}}{s_{n_l}} = \lim_{l\to\infty} \frac{\Phi_{\mathcal{A}}(-2, \gamma(s_{n_l})) - \Phi_{\mathcal{A}}(-2, \gamma(0))}{s_{n_l}}.$$

3. So y_{-2} is a tangent vector of $\mathcal{W}^u(\mathcal{O}_p)$ at x_{-2}, and thus $V(y_{-2}) \leq 2$ by Proposition 4.3.

4. To prove the assertion indirectly, suppose that

$$\gamma'(0)(0) = \gamma'(0)(-1) = \mathcal{I}(\gamma'(0)) = 0.$$

Then as $y(0) = \gamma'(0)(0) = 0$ and $y(-1) = \gamma'(0)(-1) = 0$, $V(\gamma'(0)) < V(y_{-2}) \leq 2$ by Lemma 2.3 (ii). So $V(\gamma'(0)) = 0$, that is $\gamma'(0) \geq \hat{0}$ or $\gamma'(0) \leq \hat{0}$. As we have also assumed that $\mathcal{I}(\gamma'(0)) = 0$, necessarily $\gamma'(0) = \hat{0}$ follows, a contradiction. The proof is complete. □

Now we can verify Theorem 1.1.

Proof of Theorem 1.1.
1. The proof of the assertion that $\mathcal{W}^u(\mathcal{O}_p)$ is a three-dimensional C^1-submanifold of C. All $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ can be written in form $\varphi = \Phi(t, \psi)$, where $t > 1$ and $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$. This property follows from relation (3.5) and the fact that to

each $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$, there corresponds a trajectory $(\psi^n)^0_{-\infty}$ of P_Y in $\mathcal{W}^u_{loc}(P_Y, p_0)$ with $\psi^0 = \psi$ and $\psi^n \to p_0$ as $n \to -\infty$. Hence Proposition 4.1 guarantees the existence of $\varepsilon > 0$ so that the subset

$$W_{t,\psi,\varepsilon} = \Phi\left((t-\varepsilon, t+\varepsilon) \times \left(\mathcal{W}^u_{loc}(P_Y, p_0) \cap B(\psi, \varepsilon)\right)\right)$$

of $\mathcal{W}^u(\mathcal{O}_p)$ containing φ is a three-dimensional C^1-submanifold of C.

To show that $\mathcal{W}^u(\mathcal{O}_p)$ is a three-dimensional C^1-submanifold of C, it suffices to exclude for all $t > 1$ and $\psi \in \mathcal{W}^u_{loc}(P_Y, p_0)$ the existence of a sequence $(\varphi^n)^\infty_{n=0}$ in $\mathcal{W}^u(\mathcal{O}_p)$ so that $\varphi^n \notin W_{t,\psi,\varepsilon}$ for $n \geq 0$ and $\varphi^n \to \varphi = \Phi(t,\psi)$ as $n \to \infty$. According to Proposition 4.5, $D\pi_3(\varphi) = \pi_3$ is injective on the three-dimensional tangent space $T_\varphi W_{t,\psi,\varepsilon}$, i.e. it defines an isomorphism from $T_\varphi W_{t,\psi,\varepsilon}$ onto $\mathbb{R}^3$. Thus the inverse mapping theorem yields a constant $\delta > 0$ such that the restriction of π_3 to $W_{t,\psi,\varepsilon} \cap B(\varphi, \delta)$ is a diffeomorphism from $W_{t,\psi,\varepsilon} \cap B(\varphi, \delta)$ onto an open set U in $\mathbb{R}^3$. If a sequence $(\varphi^n)^\infty_{n=0}$ in $\mathcal{W}^u(\mathcal{O}_p)$ converges to φ as $n \to \infty$, then $\pi_3\varphi^n \to \pi_3\varphi$ as $n \to \infty$, and $\pi_3\varphi^n \in U$ for all sufficiently large n. The injectivity of π_3 on $\mathcal{W}^u(\mathcal{O}_p)$ verified in Proposition 4.4 then implies that $\varphi^n \in W_{t,\psi,\varepsilon}$.

2. Graph representation for $\mathcal{W}^u(\mathcal{O}_p)$. Choose $\varphi_j \in C$ such that $\pi_3\varphi_j = e_j$, $j \in \{1,2,3\}$, where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. This is possible as $\pi_3 : C \ni \varphi \mapsto (\varphi(0), \varphi(-1), \mathcal{I}(\varphi)) \in \mathbb{R}^3$ is injective on the 3-dimensional tangent spaces of $\mathcal{W}^u(\mathcal{O}_p)$, and hence it is surjective. Clearly φ_1, φ_2 and φ_3 are linearly independent.

Let $J_3 : \mathbb{R}^3 \to C$ be the injective linear map for which $J_3 e_j = \varphi_j$, $j \in \{1,2,3\}$, and let $P_3 = J_3 \circ \pi_3$. Then $P_3 : C \to C$ is continuous, linear and $P_3\varphi_j = \varphi_j$ for all $j \in \{1,2,3\}$. In consequence, $P_3 \circ P_3 = P_3$, which means that P_3 is a projection. The space

$$G_3 = P_3 C = \{c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 : \ c_1, c_2, c_3 \in \mathbb{R}\}$$

is 3-dimensional, and with $E = P_3^{-1}(0)$, we have $C = G_3 \oplus E$. As the restriction of P_3 to $\mathcal{W}^u(\mathcal{O}_p)$ is injective, the inverse P_3^{-1} of the map $\mathcal{W}^u(\mathcal{O}_p) \ni \varphi \mapsto P_3\varphi \in G_3$ exists. At last, introduce the map

$$w : P_3\mathcal{W}^u(\mathcal{O}_p) \ni \chi \mapsto (\mathrm{id} - P_3) \circ P_3^{-1}(\chi) \in E.$$

Then

$$\mathcal{W}^u(\mathcal{O}_p) = \{\chi + w(\chi) : \ \chi \in P_3\mathcal{W}^u(\mathcal{O}_p)\}.$$

It remains to show that $U_3 = P_3\mathcal{W}^u(\mathcal{O}_p)$ is open in G_3 and w is C^1-smooth. Let $\chi \in P_3\mathcal{W}^u(\mathcal{O}_p)$ be arbitrary. Then $\chi = P_3\varphi$ with some $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$. As the restriction of π_3 to $T_\varphi\mathcal{W}(\mathcal{O}_p)$ is injective, $DP_3(\varphi) = P_3$ defines an isomorphism from $T_\varphi\mathcal{W}(\mathcal{O}_p)$ to G_3. Consequently the inverse mapping theorem implies that an $\varepsilon > 0$ can be given such that P_3 maps $\mathcal{W}(\mathcal{O}_p) \cap B(\varphi, \varepsilon)$ one-to-one onto an open neighborhood $U \subset U_3$ of χ in G_3, P_3 is invertible on $\mathcal{W}(\mathcal{O}_p) \cap B(\varphi, \varepsilon)$, and the inverse $\tilde{P}_3^{-1}$ of the map

$$\mathcal{W}(\mathcal{O}_p) \cap B(\varphi, \varepsilon) \ni \varphi \mapsto P_3\varphi \in U$$

is C^1-smooth. As

$$w(\chi) = (\mathrm{id} - P_3) \circ P_3^{-1}(\chi) = (\mathrm{id} - P_3) \circ \tilde{P}_3^{-1}(\chi)$$

for all $\chi \in U$, the restriction of w to U is C^1-smooth.

3. The characterization of C_j^p, $j \in \{-2, 0, 2\}$. Since the basin of attraction of a stable equilibrium is open in C, the connecting set C_j^p, $j \in \{-2, 0, 2\}$, is an open subset of $\mathcal{W}^u(\mathcal{O}_p)$. It follows immediately that C_j^p, $j \in \{-2, 0, 2\}$, is a three-dimensional C^1-submanifold of C and

$$C_j^p = \left\{\chi + w(\chi) : \chi \in P_3 C_j^p\right\}$$

for all $j \in \{-2, 0, 2\}$. □

As $\mathcal{W}^u(\mathcal{O}_p)$ is a C^1-submanifold of C, it makes sense to investigate the differentiability of the map

$$\Phi_{\mathcal{W}^u(\mathcal{O}_p)} : \mathbb{R} \times \mathcal{W}^u(\mathcal{O}_p) \ni (t, \varphi) \mapsto \Phi_{\mathcal{A}}(t, \varphi) \in \mathcal{W}^u(\mathcal{O}_p).$$

Suppose that η_1 η_2 and η_3 form a basis of the three-dimensional tangent space $T_\varphi \mathcal{W}^u(\mathcal{O}_p)$ of $\mathcal{W}^u(\mathcal{O}_p)$ at some $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$. Then for all $t \in \mathbb{R}$, the tangent space $T_{(t,\varphi)}(\mathbb{R} \times \mathcal{W}^u(\mathcal{O}_p))$ of $\mathbb{R} \times \mathcal{W}^u(\mathcal{O}_p)$ at (t, φ) is spanned by the tangent vectors of the following curves at 0:

$$(-1, 1) \ni s \mapsto (t + s, \varphi) \quad \text{and} \quad (-1, 1) \ni s \mapsto (t, \gamma_i(s)), \ i \in \{1, 2, 3\},$$

where $\gamma_i : (-1, 1) \to \mathcal{W}^u(\mathcal{O}_p)$ is a C^1-curve with $\gamma_i(0) = \varphi$ and $D\gamma_i(0) = \eta_i$ for all $i \in \{1, 2, 3\}$.

We are going to apply the following assertion in the proof of Theorem 1.2.(i).

Proposition 4.6. *The flow $\Phi_{\mathcal{W}^u(\mathcal{O}_p)}$ is C^1-smooth. For all $t \in \mathbb{R}$ and $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$,*

$$\frac{d}{ds}\Phi_{\mathcal{W}^u(\mathcal{O}_p)}(t + s, \varphi)\,|_{s=0} = \dot{x}_t^\varphi. \tag{4.4}$$

For all $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ and $\eta \in T_\varphi \mathcal{W}^u(\mathcal{O}_p)$, the variational equation

$$\dot{v}(t) = -v(t) + f'(x^\varphi(t - 1))\, v(t - 1) \tag{2.2}$$

has a unique solution $v^\eta : \mathbb{R} \to \mathbb{R}$ with $v_0^\eta = \eta$. If $t \in \mathbb{R}$ and $\gamma : (-1, 1) \to \mathcal{W}^u(\mathcal{O}_p)$ is a C^1-curve with $\gamma(0) = \varphi$ and $\gamma'(0) = \eta$, then

$$\frac{d}{ds}\Phi_{\mathcal{W}^u(\mathcal{O}_p)}(t, \gamma(s))\,|_{s=0} = v_t^\eta. \tag{4.5}$$

Proof. 1. To prove the smoothness of $\Phi_{\mathcal{W}^u(\mathcal{O}_p)}$, it is sufficient to show that for all $t \in \mathbb{R}$, the map

$$(t,\infty) \times \mathcal{W}^u(\mathcal{O}_p) \ni (s,\varphi) \mapsto \Phi_{\mathcal{A}}(s,\varphi) \in \mathcal{W}^u(\mathcal{O}_p) \tag{4.6}$$

is continuously differentiable.

Let $t \in \mathbb{R}$ be given, and introduce the map

$$A_t : \mathcal{W}^u(\mathcal{O}_p) \ni \varphi \mapsto \Phi_{\mathcal{A}}(t,\varphi) \in \mathcal{W}^u(\mathcal{O}_p).$$

For $t \geq 0$, A_t is clearly C^1-smooth as $\Phi(t,\cdot)$ is C^1-smooth and maps $\mathcal{W}^u(\mathcal{O}_p)$ into $\mathcal{W}^u(\mathcal{O}_p)$. For $t < 0$, the smoothness of A_t follows from the smoothness of the map $\Phi(-t,\cdot)$, the injectivity of its derivative, the inclusion $\Phi(-t,\mathcal{W}^u(\mathcal{O}_p)) \subset \mathcal{W}^u(\mathcal{O}_p)$ and the inverse mapping theorem.

For all $(s,\varphi) \in (t,\infty) \times \mathcal{W}^u(\mathcal{O}_p)$,

$$\Phi_{\mathcal{A}}(s,\varphi) = \Phi(s+1-t, \Phi_{\mathcal{A}}(t-1,\varphi)) = \Phi(s+1-t, A_{t-1}(\varphi)).$$

So the C^1-smoothness of the maps $\Phi|_{(1,\infty)\times C}$ and

$$(t,\infty) \times \mathcal{W}^u(\mathcal{O}_p) \ni (s,\varphi) \mapsto (s+1-t, A_{t-1}(\varphi)) \in (1,\infty) \times C$$

guarantee that (4.6) is also continuously differentiable.

2. Relation (4.4) is already known for $t > 1$. It can be easily obtained for $t \leq 1$ from the definition of the Fréchet derivative.

3. We already now that initial value problems corresponding to the variational equation (2.2) have unique solutions in forward time, moreover relation (4.5) holds for $t \geq 0$.

Fix $t < 0$. Note that if $\gamma : (-1,1) \to \mathcal{W}^u(\mathcal{O}_p)$ is a C^1-curve with $\gamma(0) = \varphi$ and $\gamma'(0) = \eta$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\mathcal{W}^u(\mathcal{O}_p)}(t,\gamma(s))|_{s=0} = DA_t(\varphi)\eta.$$

By part 1, the map A_t is a C^1-diffeomorphism with the inverse $A_t^{-1} = A_{-t}$. Hence for all $\eta \in T_\varphi\mathcal{W}^u(\mathcal{O}_p)$, $\chi = DA_t(\varphi)\eta$ exists and belongs to $T_{\Phi_{\mathcal{A}}(t,\varphi)}\mathcal{W}^u(\mathcal{O}_p)$. Then

$$\begin{aligned}\eta &= [DA_t(\varphi)]^{-1}\chi = DA_t^{-1}(\Phi_{\mathcal{A}}(t,\varphi))\chi = DA_{-t}(\Phi_{\mathcal{A}}(t,\varphi))\chi \\ &= D_2\Phi(-t,\Phi_{\mathcal{A}}(t,\varphi))\chi = u^\chi_{-t},\end{aligned}$$

where $u^\chi : [-1,\infty) \to \mathbb{R}$ is the solution of

$$\begin{aligned}\dot{u}(s) &= -u(s) + f'\left(x^{\Phi_{\mathcal{A}}(t,\varphi)}(s-1)\right)u(s-1) \\ &= -u(s) + f'(x^\varphi(t+s-1))u(s-1)\end{aligned}$$

with $u_0^{\chi} = \chi$. With transformation $v(s) = u(s-t)$ we obtain that the problem

$$\begin{cases} \dot{v}(s) = -v(s) + f'(x^{\varphi}(s-1))\, v(s-1) \\ v_0 = \eta \end{cases} \tag{4.7}$$

has a solution v^{η} on $[t-1, \infty)$ satisfying $v_t^{\eta} = \chi = DA_t(\varphi)\,\eta$. As this reasoning holds for any $t < 0$, we deduce – using Proposition 2.1 – that (4.7) admits a unique solution $v^{\eta} : \mathbb{R} \to \mathbb{R}$ with $v_t^{\eta} = DA_t(\varphi)\,\eta$ for any $t < 0$. This completes the proof of (4.5) for all $t \in \mathbb{R}$. □

The formula (4.5) plays a key role in the proof of the subsequent corollary.

Corollary 4.7. *For each fixed $t \in \mathbb{R}$, the derivative of the map*

$$A_t : \mathcal{W}^u(\mathcal{O}_p) \ni \varphi \mapsto \Phi_{\mathcal{W}^u(\mathcal{O}_p)}(t, \varphi) \in \mathcal{W}^u(\mathcal{O}_p)$$

at any $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ is injective on $T_{\varphi}\mathcal{W}^u(\mathcal{O}_p)$.

Proof. Suppose there exist $t \in \mathbb{R}$, $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ and $\eta \in T_{\varphi}\mathcal{W}^u(\mathcal{O}_p)$ with $\eta \neq \hat{0}$ such that $DA_t(\varphi)\,\eta = \hat{0}$. By the previous proposition, $DA_t(\varphi)\,\eta = v_t^{\eta}$, where $v^{\eta} : \mathbb{R} \to \mathbb{R}$ is the solution of (2.2) with $v_0^{\eta} = \eta$. So we assume that $v_t^{\eta} = \hat{0}$. Then the function $u : \mathbb{R} \to \mathbb{R}$ defined by $u(s) = v^{\eta}(t+s)$, $s \in \mathbb{R}$, is a nontrivial solution of the equation

$$\dot{u}(s) = -u(s) + f'\left(x^{\Phi_{\mathcal{W}^u(\mathcal{O}_p)}(t,\varphi)}(s-1)\right) u(s-1)$$

with $u_0 = \hat{0}$. This implies a contradiction to Proposition 2.1. □

5. The Proof of Theorem 1.2

Fix index $k \in \{-1, 1\}$ in the rest of the paper and consider the sets C_q^p, C_k^p and $S_k = C_k^p \cup \mathcal{O}_p \cup C_q^p$.

5.1 Preliminary results on $\overline{S_k}$

In this subsection we define a projection π_2 from C into $\mathbb{R}^2$ and show that π_2 is injective on the closure $\overline{S_k}$ of S_k in C, see Proposition 5.4. The proof of this assertion is based on the special properties of the discrete Lyapunov functional V. The injectivity of $\pi_2|_{\overline{S_k}}$ enables us to give a graph representation for $\overline{S_k}$ (without smoothness properties): there is an isomorphism $J_2 : \mathbb{R}^2 \to C$ such that $P_2 = J_2 \circ \pi_2 : C \to C$ is a projection onto a two-dimensional subspace G_2 of C, and a map $w_k : P_2\overline{S_k} \to P_2^{-1}(0)$ can be defined such that

$$\overline{S_k} = \left\{\chi + w_k(\chi) : \chi \in P_2\overline{S_k}\right\},$$

see Proposition 5.5. The differentiability of w_k and the properties of its domain $P_2\overline{S_k} \subset G_2$ are studied only in Subsections 5.3 and 5.5. We also show at the end of this subsection that $\pi_2|_{\overline{S_k}}$ is a homeomorphism onto its image (see Proposition 5.6), moreover π_2 maps the nonzero tangent vectors of $\overline{S_k}$ to nonzero vectors in $\mathbb{R}^2$ (see Proposition 5.7).

Clearly, S_k is invariant under $\Phi_{\mathcal{A}}$. Then it easily follows that $\overline{S_k}$ is invariant too. Indeed, let $\varphi \in \overline{S_k} \setminus S_k$ be arbitrary and choose a sequence $(\varphi_n)_{n=0}^{\infty}$ in S_k converging to φ as $n \to \infty$. As the global attractor $\mathcal{A}$ is closed, $\varphi \in \mathcal{A}$. By the continuity of the flow $\Phi_{\mathcal{A}}$ on $\mathbb{R} \times \mathcal{A}$, $S_k \ni x_t^{\varphi_n} \to x_t^{\varphi}$ as $n \to \infty$ for all $t \in \mathbb{R}$, which means that $\overline{S_k}$ is invariant under $\Phi_{\mathcal{A}}$.

By Theorem B,

$$S_k = \{\varphi \in \mathcal{W}^u(\mathcal{O}_p) : x^{\varphi} \text{ oscillates about } \xi_k\}. \tag{1.2}$$

Note that if x^{φ} is nonoscillatory about ξ_k for some $\varphi \in C$ (i.e. there exists $T \geq 0$ so that $x_T^{\varphi} \gg \hat{\xi}_k$ or $x_T^{\varphi} \ll \hat{\xi}_k$), then φ has an open neighborhood U_{φ} in C such that for all $\psi \in U_{\varphi}$, x^{ψ} is nonoscillatory about ξ_k. Hence it comes immediately from (1.2) that for all $\varphi \in \overline{S_k}$, x^{φ} oscillates about ξ_k.

The next result states that the stable set of the unstable equilibrium $\hat{\xi}_k$ contains only nonordered elements with respect to the pointwise ordering. The proof follows the first part of the proof of Proposition 3.1 in [10].

Proposition 5.1. *There exist no $\varphi \in C$ and $\psi \in C$ with $\varphi \ll \psi$ such that x_t^{φ} and x_t^{ψ} both converge to $\hat{\xi}_k$ as $t \to \infty$.*

Proof. Suppose that $\varphi \in C$, $\psi \in C$, $\varphi \ll \psi$ and both x_t^{φ}, x_t^{ψ} converge to $\hat{\xi}_k$ as $t \to \infty$. Then $y := x^{\psi} - x^{\varphi}$ is positive on $[-1, \infty)$ by Proposition 2.4, it satisfies

$$\dot{y}(t) = -y(t) + b(t)\, y(t-1)$$

for all $t > 0$, where

$$b : [0, \infty) \ni t \mapsto \int_0^1 f'\left(s x^{\psi}(t-1) + (1-s)\, x^{\varphi}(t-1)\right) \mathrm{d}s \in (0, \infty),$$

furthermore $b(t) \to f'(\xi_k)$ as $t \to \infty$. Since $f'(\xi_k) > 1$ by hypothesis (H1), the number $\varepsilon = (f'(\xi_k) - 1)\, e^{-1}/2$ is positive. So there exists $T \geq 0$ such that $b(t) \geq f'(\xi_k) - \varepsilon$ for all $t \geq T$. Observe that the positivity of y and b implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^t y(t)\right) = e^t b(t)\, y(t-1) > 0 \quad \text{for all } t > 0.$$

For this reason, $e^{t-1} y(t-1) < e^t y(t)$ for $t \geq 1$, and

$$\begin{aligned} \dot{y}(t) &\geq -y(t) + (f'(\xi_k) - \varepsilon)\, y(t-1) \\ &\geq -(1 + \varepsilon e)\, y(t) + f'(\xi_k)\, y(t-1) \end{aligned}$$

for all $t \geq T+1$. The choice of ε ensures that

$$1+\varepsilon e=\frac{1}{2}+\frac{1}{2}f'(\xi_k)<f'(\xi_k).$$

Hence the equation

$$\lambda+(1+\varepsilon e)=f'(\xi_k)e^{-\lambda}$$

has a positive real solution λ. Choose $\delta>0$ so that $y(t)>\delta e^{\lambda t}$ on $[T,T+1]$. The function $z(t)=\delta e^{\lambda t}$ is a solution of the equation

$$\dot{z}(t)=-(1+\varepsilon e)z(t)+f'(\xi_k)z(t-1)$$

on $\mathbb{R}$. Set $u=y-z$. Then $u_{T+1}\gg\hat{0}$ and

$$\dot{u}(t)\geq-(1+\varepsilon e)u(t)+f'(\xi_k)u(t-1) \text{ for all } t\geq T+1.$$

If there existed $t^*>T+1$ so that $u(t^*)=0$ and u is positive on $[T,t^*)$, then $\dot{u}(t^*)$ would be nonpositive. On the other hand, the inequality for u combined with $u(t^*)=0$ and $u(t^*-1)>0$ would yield that $\dot{u}(t^*)>0$. So $u(t)=y(t)-z(t)=y(t)-\delta e^{\lambda t}>0$ for all $t\geq T$, which contradicts the boundedness of y. □

The next proposition is the analogue of Proposition 3.1 in [10].

Proposition 5.2. *(Nonordering of $\overline{S_k}$) For all $\varphi,\psi\in C$ with $\varphi<\psi$, either $\varphi\in C\backslash\overline{S_k}$ or $\psi\in C\backslash\overline{S_k}$.*

Proof. If there are $\tilde{\varphi}\in\overline{S_k}$ and $\tilde{\psi}\in\overline{S_k}$ satisfying $\tilde{\varphi}<\tilde{\psi}$, then by Proposition 2.4 and the invariance of $\overline{S_k}$, $\varphi=x_2^{\tilde{\varphi}}\in\overline{S_k}$, $\psi=x_2^{\tilde{\psi}}\in\overline{S_k}$ and $\varphi\ll\psi$. Theorem 4.1 in Chapter 5 of [20] proves that there is an open and dense set of initial functions in $C_{-2,2}$ so that the corresponding solutions converge to equilibria. Hence there exist $\varphi^*\in C$ and $\psi^*\in C$ with $\varphi\ll\varphi^*\ll\psi^*\ll\psi$ such that both $x_t^{\varphi^*}$ and $x_t^{\psi^*}$ tend to equilibria as $t\to\infty$.

If $x_t^{\psi^*}\to\hat{\xi}$ as $t\to\infty$, where $\hat{\xi}$ is any equilibrium with $\xi>\xi_k$, then there exists $T>0$ such that $\hat{\xi}_k\ll x_T^{\psi^*}$. Then $\hat{\xi}_k\ll x_T^{\psi^*}\ll x_T^{\psi}$ by Proposition 2.4, which implies a contradiction to the fact that the elements of $\overline{S_k}$ oscillate about ξ_k. If $x_t^{\psi^*}\to\hat{\xi}\ll\hat{\xi}_k$ as $t\to\infty$, and there exists $T>0$ with $x_T^{\psi^*}\ll\hat{\xi}_k$, then $x_T^{\varphi}\ll x_T^{\psi^*}\ll\hat{\xi}_k$, which contradicts $\varphi\in\overline{S}_k$. Therefore, $\omega(\psi^*)=\left\{\hat{\xi}_k\right\}$. Similarly, $\omega(\varphi^*)=\left\{\hat{\xi}_k\right\}$. This is a contradiction to Proposition 5.1. □

Proposition 5.3. *If $\varphi\in\overline{S_k}$, $\psi\in\overline{S_k}$ and $\varphi\neq\psi$, then $V(\psi-\varphi)=2$.*

Proof. If $\varphi,\psi\in S_k$ and $\varphi\neq\psi$, then $V(\psi-\varphi)\leq 2$ by Proposition 4.2. The lower-semicontinuity of V (see Lemma 2.2) hence implies that $V(\psi-\varphi)\leq 2$ for all $\varphi,\psi\in\overline{S_k}$ satisfying $\varphi\neq\psi$. If $V(\psi-\varphi)=0$, then $\varphi<\psi$ or $\psi<\varphi$, which contradicts Proposition 5.2. □

The role of π_3 in the proof of Theorem 1.1 is now taken over by the linear map

$$\pi_2 : C \ni \varphi \mapsto (\varphi(0), \varphi(-1)) \in \mathbb{R}^2.$$

The next assertion is analogous to Proposition 4.4, and it will be used several times in the subsequent proofs.

Proposition 5.4. *π_2 is injective on $\overline{S_k}$.*

Proof. Suppose that there exist $\varphi \in \overline{S_k}$ and $\psi \in \overline{S_k}$ so that $\varphi \neq \psi$ and $\pi_2\varphi = \pi_2\psi$. Consider the solutions $x^\varphi : \mathbb{R} \to \mathbb{R}$ and $x^\psi : \mathbb{R} \to \mathbb{R}$. The invariance of $\overline{S_k}$ implies that $x_t^\varphi \in \overline{S_k}$ and $x_t^\psi \in \overline{S_k}$ for all $t \in \mathbb{R}$, and the the injectivity of the semiflow guarantees that $x_t^\varphi \neq x_t^\psi$ for all $t \in \mathbb{R}$. Hence $V\left(x_t^\varphi - x_t^\psi\right) = 2$ for all real t by Proposition 5.3. The initial assumption $\varphi(0) - \psi(0) = \varphi(-1) - \psi(-1) = 0$ and Lemma 2.3 (ii) however yield that

$$V(\varphi - \psi) < V\left(x_{-2}^\varphi - x_{-2}^\psi\right),$$

which is a contradiction. □

The injectivity of $\pi_2|_{\overline{S_k}}$ is sufficient to give a graph representation for $\overline{S_k}$.

Proposition 5.5. *$\overline{S_k}$ has a global graph representation: there exist a projection P_2 from C onto a two-dimensional subspace G_2 of C and a map $w_k : P_2\overline{S_k} \to P_2^{-1}(0)$ so that*

$$\overline{S_k} = \left\{\chi + w_k(\chi) : \chi \in P_2\overline{S_k}\right\}. \tag{5.1}$$

Proof. Let $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Let φ_1 and φ_2 be the linearly independent elements of C fixed in the proof of Theorem 1.1 with the property that $\pi_3\varphi_j = e_j$ for $j \in \{1, 2\}$. Define $J_2 : \mathbb{R}^2 \to C$ to be the injective linear map for which $J_2(1, 0) = \varphi_1$ and $J_2(0, 1) = \varphi_2$, and set $P_2 = J_2 \circ \pi_2 : C \to C$. Then P_2 is continuous, linear and $P_2\varphi_j = \varphi_j$ for both $j \in \{1, 2\}$. Hence $P_2 \circ P_2 = P_2$, and P_2 is a projection. The 2-dimensional image space

$$G_2 = P_2C = \{c_1\varphi_1 + c_2\varphi_2 : c_1, c_2 \in \mathbb{R}\}$$

is a subspace of G_3 and $C = G_2 \oplus P_2^{-1}(0)$. (Note that P_2 and G_2 are both independent of k.) As the restriction of P_2 to $\overline{S_k}$ is injective by Proposition 5.4, the inverse $\left(P_2|_{\overline{S_k}}\right)^{-1}$ of the map $\overline{S_k} \ni \varphi \mapsto P_2\varphi \in G_2$ exists. With the map

$$w_k : P_2\overline{S_k} \ni \chi \mapsto (\mathrm{id} - P_2) \circ \left(P_2|_{\overline{S_k}}\right)^{-1}(\chi) \in P_2^{-1}(0)$$

we have (5.1). □

The smoothness of this representation will be verified later. Observe that

$$w_{-1}|_{P_2\left(\overline{S_{-1}} \cap \overline{S_1}\right)} = w_1|_{P_2\left(\overline{S_{-1}} \cap \overline{S_1}\right)}.$$

Also note that now we have a global graph representation for any subset W of $\overline{S_k}$:

$$W = \{\chi + w_k(\chi) : \chi \in P_2 W\}.$$

Let $\pi_2^{-1} : \pi_2\left(\overline{S_k}\right) \to C$ be the inverse of the injective map $\overline{S_k} \ni \varphi \mapsto \pi_2\varphi \in \mathbb{R}^2$.

Proposition 5.6. *π_2^{-1} is Lipschitz-continuous.*

Proof. Suppose that π_2^{-1} is not Lipschitz-continuous, i.e., there are sequences of solutions $x^n : \mathbb{R} \to \mathbb{R}$ and $y^n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, so that $x_0^n \neq y_0^n$ for all $n \geq 0$, $x_0^n, y_0^n \in \overline{S_k}$ for all $n \geq 0$, and

$$\frac{|\pi_2(x_0^n - y_0^n)|_{\mathbb{R}^2}}{\|x_0^n - y_0^n\|} \to 0 \quad \text{as } n \to \infty.$$

By the compactness of $\overline{S_k}$, the solutions x^n and y^n are uniformly bounded, and Eq. (1.1) gives a uniform bound for their derivatives. Therefore we can use the Arzelà–Ascoli theorem successively on the intervals $[-j, j]$, $j \geq 1$, and apply a diagonalization process to get subsequences $(x^{n_m})_{m=0}^\infty$, $(y^{n_m})_{m=0}^\infty$ and continuous functions $x : \mathbb{R} \to \mathbb{R}$, $y : \mathbb{R} \to \mathbb{R}$ so that $x^{n_m} \to x$ and $y^{n_m} \to y$ as $m \to \infty$ uniformly on compact subsets of $\mathbb{R}$.

Set functions

$$z^m : \mathbb{R} \ni t \mapsto \frac{x^{n_m}(t) - y^{n_m}(t)}{\|x_0^{n_m} - y_0^{n_m}\|} \in \mathbb{R}, \quad m \in \mathbb{N}.$$

Then $V(z_t^m) = 2$ for all $m \geq 0$ and $t \in \mathbb{R}$ by Proposition 5.3, $\|z_0^m\| = 1$ for all $m \geq 0$, and

$$|\pi_2 z_0^m|_{\mathbb{R}^2} = \frac{|\pi_2(x_0^{n_m} - y_0^{n_m})|_{\mathbb{R}^2}}{\|x_0^{n_m} - y_0^{n_m}\|} \to 0 \quad \text{as } m \to \infty.$$

In addition, $\dot{z}^m(t) = -z^m(t) + a_m(t) z^m(t-1)$ for all $m \geq 0$ and $t \in \mathbb{R}$, where the coefficient functions

$$a_m : \mathbb{R} \ni t \mapsto \int_0^1 f'(s x^{n_m}(t-1) + (1-s) y^{n_m}(t-1))\, \mathrm{d}s \in \mathbb{R}^+, \quad m \geq 0,$$

converge to

$$a : \mathbb{R} \ni t \mapsto \int_0^1 f'(s x(t-1) + (1-s) y(t-1))\, \mathrm{d}s \in \mathbb{R}^+$$

uniformly on compact subsets of $\mathbb{R}$. It is also obvious that there are constants $\alpha_1 \geq \alpha_0 > 0$ such that $\alpha_0 \leq a_m(t) \leq \alpha_1$ for all $m \geq 0$ and $t \in \mathbb{R}$.

Therefore Lemma 2.6 guarantees the existence of a subsequence $(z^{m_l})_{l=0}^\infty$ of $(z^m)_{m=0}^\infty$ and a continuously differentiable function $z : \mathbb{R} \to \mathbb{R}$ such that $z^{m_l} \to z$ and $\dot{z}^{m_l} \to \dot{z}$ as $l \to \infty$ uniformly on compact subsets of $\mathbb{R}$, and z satisfies

$$\dot{z}(t) = -z(t) + a(t) z(t-1) \quad \text{for all } t \in \mathbb{R}.$$

It is clear that $\|z_0\| = 1$, and thus $z_0 \neq \hat{0}$. In addition, $\pi_2 z_0 = (0, 0)$.

By Lemma 2.2,

$$V\left(z_t\right) \leq \liminf_{l\to\infty} V\left(z_t^{m_l}\right) = 2 \quad \text{for all real } t.$$

Hence Lemma 2.3 (ii) and property $\pi_2 z_0 = (0,0)$ together give that $V\left(z_0\right) = 0$. As $t \mapsto V\left(z_t\right)$ is monotone nonincreasing, $V\left(z_3\right) = 0$. Lemma 2.3 (iii) then implies that z_3 belongs to the function class R, and the second statement of Lemma 2.2 gives that

$$0 = V\left(z_3\right) = \lim_{l\to\infty} V\left(z_3^{m_l}\right),$$

which contradicts $V\left(z_3^{m_l}\right) = 2$. □

We get the next result as a consequence, it is analogous to Proposition 4.5.

Proposition 5.7. *Suppose that* $\varphi \in \overline{S_k}$, $\gamma : (-1,1) \to C$ *is a* C^1*-curve with* $\gamma(0) = \varphi$, *and* $(s_n)_0^\infty$ *is a sequence in* $(-1,1)\setminus\{0\}$ *so that* $s_n \to 0$ *as* $n \to \infty$ *and* $\gamma(s_n) \in \overline{S_k}$ *for all* $n \geq 0$. *If* $\gamma'(0) \neq \hat{0}$, *then* $\pi_2\gamma'(0) \neq (0,0)$.

Proof. Let $K > 0$ be a Lipschitz-constant for π_2^{-1}. Proposition 5.6 guarantees that such K exists. Then

$$\left\| \frac{\gamma(s_n) - \gamma(0)}{s_n} \right\| \leq K \left| \frac{\pi_2\gamma(s_n) - \pi_2\gamma(0)}{s_n} \right|_{\mathbb{R}^2}$$

for all $n \geq 0$. Letting $n \to \infty$ we obtain that $\|\gamma'(0)\| \leq K\left|\pi_2\gamma'(0)\right|_{\mathbb{R}^2}$. Therefore if $\gamma'(0) \neq \hat{0}$, then $\pi_2\gamma'(0) \neq (0,0)$. □

5.2 The structure of $\overline{S_k}$

It is obvious from the definition of S_k that $(\mathcal{O}_k \cup S_k \cup \mathcal{O}_q) \subseteq \overline{S_k}$. The equality $\overline{S_k} = \mathcal{O}_k \cup S_k \cup \mathcal{O}_q$ is proved in this subsection based on the property that π_2 maps $\overline{S_k}$ injectively into $\mathbb{R}^2$. Then it will follow easily that $\overline{C_q^p} = \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q$ and $\overline{C_k^p} = \mathcal{O}_p \cup C_k^p \cup \mathcal{O}_k$.

Proposition 5.4 implies that π_2 maps periodic orbits with segments in $\overline{S_k}$ into simple closed curves in $\mathbb{R}^2$, and the images of different periodic orbits are disjoint curves in $\mathbb{R}^2$. Lemma 5.7 of [17] guarantees the same properties for all periodic orbits. So

$$\mathbb{R} \ni t \mapsto \pi_2 p_t \in \mathbb{R}^2,\ \mathbb{R} \ni t \mapsto \pi_2 q_t \in \mathbb{R}^2$$

and

$$\mathbb{R} \ni t \mapsto \pi_2 x_t^k \in \mathbb{R}^2$$

are pairwise disjoint simple closed curves.

It comes from Proposition 7.3 of [17] that if a periodic solution $r : \mathbb{R} \to \mathbb{R}$ oscillates about an equilibrium $\hat{\xi}$, then $\pi_2\hat{\xi} \in \operatorname{int}(\pi_2\mathcal{O}_r)$, where $\mathcal{O}_r = \{r_t :\ t \in \mathbb{R}\}$. If two periodic solutions $r_1 : \mathbb{R} \to \mathbb{R}$ and $r_2 : \mathbb{R} \to \mathbb{R}$ oscillate about an equilibrium

$\hat{\xi}$, then necessarily $\pi_2\mathcal{O}_{r_1} \subset \operatorname{int}(\pi_2\mathcal{O}_{r_2})$ or $\pi_2\mathcal{O}_{r_2} \subset \operatorname{int}(\pi_2\mathcal{O}_{r_1})$. These observations imply the subsequent results. Since p oscillates about $\hat{0}$, and x^k oscillates about $\hat{\xi}_k$, we have $\pi_2\hat{0} \in \operatorname{int}(\pi_2\mathcal{O}_p)$ and $\pi_2\hat{\xi}_k \in \operatorname{int}(\pi_2\mathcal{O}_k)$. Note that as either the minimum of p is smaller than the minimum of x^k or the maximum of p is greater than the maximum of x^k, it is impossible that $\pi_2\mathcal{O}_p \subset \operatorname{int}(\pi_2\mathcal{O}_k)$. As both p and x^k oscillate about $\hat{\xi}_k$, we obtain that $\pi_2\mathcal{O}_k \subset \operatorname{int}(\pi_2\mathcal{O}_p)$. From $p(\mathbb{R}) \subsetneq q(\mathbb{R})$ it follows that $\pi_2\mathcal{O}_p \subset \operatorname{int}(\pi_2\mathcal{O}_q)$. As $q(\mathbb{R}) \subset (\xi_{-2}, \xi_2)$, it is clear that $\pi_2\hat{\xi}_{-2}$ and $\pi_2\hat{\xi}_2$ belong to $\operatorname{ext}(\pi_2\mathcal{O}_q)$. See Fig. 6.

Let

$$A_k^p = \operatorname{ext}(\pi_2\mathcal{O}_k) \cap \operatorname{int}(\pi_2\mathcal{O}_p), \quad A_q^p = \operatorname{ext}(\pi_2\mathcal{O}_p) \cap \operatorname{int}(\pi_2\mathcal{O}_q)$$

and

$$A_{k,q} = \operatorname{ext}(\pi_2\mathcal{O}_k) \cap \operatorname{int}(\pi_2\mathcal{O}_q),$$

see Fig. 6. Then by the Schönflies theorem [19], A_k^p, A_q^p and $A_{k,q}$ are homeomorphic to the open annulus $A^{(1,2)} = \{u \in \mathbb{R}^2 : 1 < |u| < 2\}$. For the closures $\overline{A_k^p}$, $\overline{A_q^p}$ and $\overline{A_{k,q}}$ of A_k^p, A_q^p and $A_{k,q}$ in $\mathbb{R}^2$, respectively, we have

$$\overline{A_k^p} = A_k^p \cup \pi_2\mathcal{O}_k \cup \pi_2\mathcal{O}_p, \quad \overline{A_q^p} = A_q^p \cup \pi_2\mathcal{O}_p \cup \pi_2\mathcal{O}_q$$

and

$$\overline{A_{k,q}} = A_{k,q} \cup \pi_2\mathcal{O}_k \cup \pi_2\mathcal{O}_q.$$

Observe that for all $\varphi \in C_q^p$, $\pi_2\varphi \in A_q^p$ because $t \mapsto \pi_2 x_t^\varphi$ is continuous, $\pi_2 x_t^\varphi \to \pi_2\mathcal{O}_p$ as $t \to -\infty$, $\pi_2 x_t^\varphi \to \pi_2\mathcal{O}_q$ as $t \to \infty$, $\mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q \subset \overline{S_k}$, and π_2 is injective on $\overline{S_k}$. For the same reason, $\pi_2 C_k^p \subseteq A_k^p$. Then it is clear that $\pi_2\overline{C_q^p} = \overline{\pi_2 C_q^p} \subseteq \overline{A_q^p}$ and $\pi_2\overline{C_k^p} = \overline{\pi_2 C_k^p} \subseteq \overline{A_k^p}$. As $\mathcal{O}_p \subseteq \overline{C_q^p} \cap \overline{C_k^p}$, we conclude that

$$\pi_2\mathcal{O}_p \subseteq \pi_2\left(\overline{C_q^p} \cap \overline{C_k^p}\right) \subseteq \pi_2\overline{C_q^p} \cap \pi_2\overline{C_k^p} \subseteq \overline{A_q^p} \cap \overline{A_k^p} = \pi_2\mathcal{O}_p,$$

that is, $\pi_2\mathcal{O}_p = \pi_2\left(\overline{C_q^p} \cap \overline{C_k^p}\right)$. The injectivity of π_2 on $\overline{S_k}$ then implies that

$$\mathcal{O}_p = \overline{C_q^p} \cap \overline{C_k^p}. \tag{5.2}$$

We also obtain from $\pi_2 C_q^p \subseteq A_q^p$ and $\pi_2 C_k^p \subseteq A_k^p$ that

$$\pi_2 S_k = \pi_2 C_k^p \cup \pi_2\mathcal{O}_p \cup \pi_2 C_q^p \subseteq A_k^p \cup \pi_2\mathcal{O}_p \cup A_q^p = A_{k,q},$$

and hence $\pi_2\overline{S_k} = \overline{\pi_2 S_k} \subseteq \overline{A_{k,q}}$. Note that this means that $\hat{\xi}_k \notin \overline{S_k}$.

It has been already verified that for all $\varphi \in \overline{S_k}$, x^φ oscillates about ξ_k. We claim that this oscillation is slow.

Proposition 5.8. *$V\left(\varphi - \hat{\xi}_k\right) = 2$ for all $\varphi \in \overline{S_k}$.*

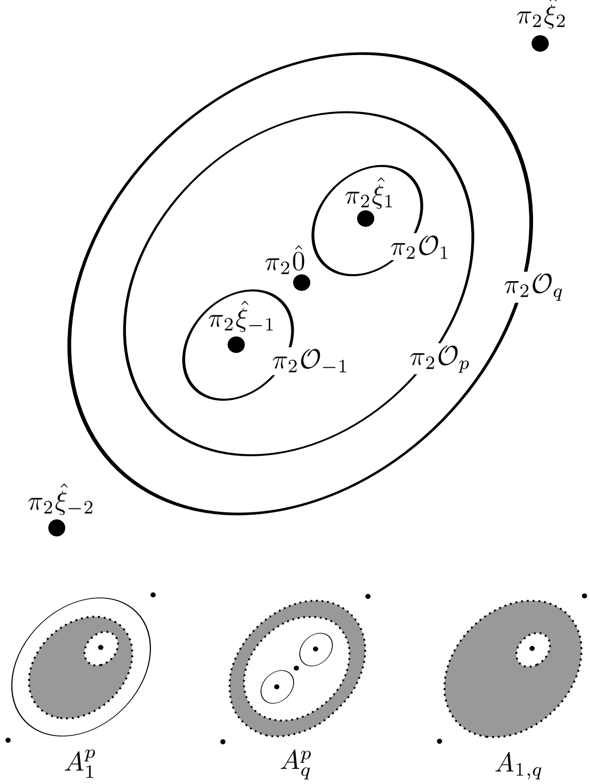

FIGURE 6. The images of the equilibria and the periodic orbits under π_2, and the definitions of the open sets A_1^p, A_q^p and $A_{1,q}$.

Proof. 1. First we prove the assertion for the elements of S_k. Choose an arbitrary element $\varphi \in S_k$ and a sequence $(t_n)_{n=0}^{\infty}$ with $t_n \to -\infty$ as $n \to \infty$ such that $x_{t_n}^{\varphi} \to p_0$ as $n \to \infty$. As the C-norm and C^1-norm are equivalent on the global attractor, $x_{t_n}^{\varphi} \to p_0$ as $n \to \infty$ also in C^1-norm. Note that p is slowly oscillatory about ξ_k (see Proposition 8.2 in [8]), i.e., $V\left(p_t - \hat{\xi}_k\right) = 2$ for all real t. Hence Lemma 2.3.(iii) gives that $p_0 - \hat{\xi}_k \in R$, and Lemma 2.2 implies that

$$2 = V\left(p_0 - \hat{\xi}_k\right) = \lim_{n\to\infty} V\left(x_{t_n}^{\varphi} - \hat{\xi}_k\right).$$

Then by the monotonicity of V (see Lemma 2.3.(i)), $V\left(x_t^{\varphi} - \hat{\xi}_k\right) \leq 2$ for all $t \in \mathbb{R}$. If $V\left(\varphi - \hat{\xi}_k\right) = 0$ and $\varphi < \hat{\xi}_k$ or $\varphi > \hat{\xi}_k$, then $x_2^{\varphi} \ll \hat{\xi}_k$ or $x_2^{\varphi} \gg \hat{\xi}_k$ by Proposition 2.4, which contradicts the fact that x^{φ} oscillates about ξ_k.

2. Now choose any $\varphi \in \overline{S_k}$ and fix a sequence $(\varphi_n)_{n=0}^{\infty}$ in S_k with $\varphi_n \to \varphi$ as $n \to \infty$. Since $\hat{\xi}_k \notin \overline{S_k}$, $V\left(\varphi - \hat{\xi}_k\right)$ is defined. The lower semi-continuity of V (see Lemma 2.2) and part 1 yield that $V\left(\varphi - \hat{\xi}_k\right) \leq \liminf_{n\to\infty} V\left(\varphi_n - \hat{\xi}_k\right) = 2$. Observe that assumption $V\left(\varphi - \hat{\xi}_k\right) = 0$ would lead to a contradiction just as in the previous step. So $V\left(\varphi - \hat{\xi}_k\right) = 2$ for all $\varphi \in \overline{S_k}$. □

Now we are ready to confirm the equalities regarding $\overline{C_k^p}$, $\overline{C_q^p}$ and $\overline{S_k}$ in Theorem 1.2.(ii).

Proposition 5.9. $\overline{S_k} = \mathcal{O}_k \cup S_k \cup \mathcal{O}_q = \mathcal{O}_k \cup C_k^p \cup \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q$.

Proof. Let us fix $k = 1$. It is clear from the definition of S_1 that $(\mathcal{O}_1 \cup \mathcal{O}_q) \subset \overline{S_1}$, and thus we only need to verify the inclusion $\overline{S_1} \setminus S_1 \subseteq (\mathcal{O}_1 \cup \mathcal{O}_q)$. Let $\varphi \in \overline{S_1} \setminus S_1$ be arbitrary.

It is an immediate consequence of the oscillation of x^φ about ξ_1 that $\varphi \notin \mathcal{W}^u(\mathcal{O}_p)$, otherwise φ would also belong to S_1 by (1.2). It is also obvious that $\varphi \notin \mathcal{A}_{-2,0}$. There are two possibilities by Theorem B: either $\varphi \in \mathcal{W}^u(\mathcal{O}_q) = \mathcal{O}_q \cup C_{-2}^q \cup C_2^q$ or $\varphi \in \mathcal{A}_{0,2}$. The solution x^φ cannot converge to any of the equilibria $\hat{\xi}_{-2}$, $\hat{\xi}_2$ because it oscillates about ξ_1. So if $\varphi \in \left(\overline{S_1} \setminus S_1\right) \cap \mathcal{W}^u(\mathcal{O}_q)$, then necessarily $\varphi \in \mathcal{O}_q$. It remains to show that the relation $\varphi \in \left(\overline{S_1} \setminus S_1\right) \cap \mathcal{A}_{0,2}$ implies that $\varphi \in \mathcal{O}_1$.

$\mathcal{A}_{0,2}$ is a compact and invariant subset of C, hence $\varphi \in \mathcal{A}_{0,2}$ implies that $x_t^\varphi \in \mathcal{A}_{0,2}$ for all real t, moreover $\alpha(x^\varphi)$ and $\omega(\varphi)$ are also subsets of $\mathcal{A}_{0,2}$. On the other hand, $\overline{S_1}$ is also compact and invariant, so $\alpha(x^\varphi) \cup \omega(\varphi) \subset \overline{S_1}$, and $V\left(\psi - \hat{\xi}_1\right) = 2$ for all $\psi \in \alpha(x^\varphi) \cup \omega(\varphi)$ by the previous proposition. The Poincaré–Bendixson Theorem (see Section 2) then implies that $\omega(\varphi)$ is either a periodic orbit in $\mathcal{A}_{0,2}$ oscillating slowly about ξ_1, or for each $\psi \in \omega(\varphi)$, $\alpha\left(x^\psi\right) = \omega(\psi) = \left\{\hat{\xi}_1\right\}$. As there are no homoclinic orbits to $\hat{\xi}_1$ (see Proposition 3.1 in [7]), $\omega(\varphi) = \left\{\hat{\xi}_1\right\}$ in the latter case. Similarly, $\alpha(x^\varphi)$ is either $\left\{\hat{\xi}_1\right\}$ or a periodic orbit in $\mathcal{A}_{0,2}$ oscillating slowly about ξ_1.

Recall that x^1 is defined so that the range $x^1(\mathbb{R})$ is maximal in the sense that $x^1(\mathbb{R}) \supset r(\mathbb{R})$ for all periodic solutions r oscillating slowly about ξ_1 with range in $(0, \xi_2)$. So if $r : \mathbb{R} \to \mathbb{R}$ is a periodic solution with segments in $\alpha(x^\varphi) \cup \omega(\varphi)$, then $\pi_2 r_t \in \pi_2\mathcal{O}_1 \cup \operatorname{int}(\pi_2\mathcal{O}_1)$ for all $t \in \mathbb{R}$. Recall that $\pi_2\hat{\xi}_1$ also belongs to $\operatorname{int}(\pi_2\mathcal{O}_1)$. Hence

$$\pi_2(\alpha(x^\varphi) \cup \omega(\varphi)) \subset \pi_2\mathcal{O}_1 \cup \operatorname{int}(\pi_2\mathcal{O}_1).$$

On the other hand,

$$\pi_2(\alpha(x^\varphi) \cup \omega(\varphi)) \subset \pi_2\overline{S_1} \subseteq \overline{A_{1,q}} \subset \mathbb{R}^2 \setminus \operatorname{int}(\pi_2\mathcal{O}_1).$$

It follows that $\pi_2(\alpha(x^\varphi) \cup \omega(\varphi)) \subseteq \pi_2\mathcal{O}_1$ and thus $\alpha(x^\varphi) = \omega(\varphi) = \mathcal{O}_1$. If x^φ is not the time translation of x^1, then this is only possible if the curve $t \to \pi_2 x_t^\varphi$ is self-intersecting, which contradicts the injectivity of π_2 on $\overline{S_1}$. Hence relation $\varphi \in \left(\overline{S_1} \setminus S_1\right) \cap \mathcal{A}_{0,2}$ implies that $\varphi \in \mathcal{O}_1$.

We have verified that each $\varphi \in \overline{S_1} \setminus S_1$ belongs to $\mathcal{O}_1 \cup \mathcal{O}_q$, that is

$$\overline{S_1} = \mathcal{O}_1 \cup C_1^p \cup \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q.$$

Handling the case $k = -1$ is completely analogous. $\square$

Corollary 5.10. *$\overline{S_{-1}} \cap \overline{S_1} = \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q$, $\overline{C_q^p} = \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q$ and $\overline{C_k^p} = \mathcal{O}_p \cup C_k^p \cup \mathcal{O}_k$.*

Proof. The first equality follows immediately from Proposition 5.9. The second and third equalities come from

$$\mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q \subseteq \overline{C_q^p} \subseteq \overline{S_k} = \mathcal{O}_k \cup C_k^p \cup \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q,$$

$$\mathcal{O}_k \cup C_k^p \cup \mathcal{O}_p \subseteq \overline{C_k^p} \subseteq \overline{S_k} = \mathcal{O}_k \cup C_k^p \cup \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q$$

and (5.2). $\square$

5.3 The smoothness of C_q^p and C_k^p

Suppose r is one of the periodic solutions q or x^k with minimal period $\omega > 1$, and let C_r^p be the heteroclinic connection from $\mathcal{O}_p$ to $O_r = \{r_t : t \in \mathbb{R}\}$.

Next we confirm that C_r^p is a C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$. First we verify that $\mathcal{W}^u(\mathcal{O}_p)$ intersects transversally a local stable or a local center-stable manifold of a Poincaré map at a point of $\mathcal{O}_r$. It follows that the intersection is a one-dimensional C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$. Then we apply the injectivity of the derivative of the flow induced by the solution operator on $\mathcal{W}^u(\mathcal{O}_p)$ (see Proposition 4.6 and Corollary 4.7) to confirm that each point φ in C_r^p belongs to a "small" subset W_φ of C_r^p that is a two-dimensional C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$. This means that C_r^p is an immersed C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$. In order to prove that C_r^p is embedded in $\mathcal{W}^u(\mathcal{O}_p)$, we have to show that for any φ in C_r^p, there is no sequence in $C_r^p \backslash W_\varphi$ converging to φ. According to results of Subsection 5.1, π_2 is injective on C_r^p and on the tangent spaces of C_r^p, which implies that $\pi_2 W_\varphi$ is open in $\mathbb{R}^2$. If a sequence $(\varphi^n)_{n=0}^\infty$ from the rest of the connecting set converges to φ as $n \to \infty$, then $\pi_2\varphi^n \to \pi_2\varphi$ as $n \to \infty$, and $\pi_2\varphi^n \in \pi_2 W_\varphi$ for all n large enough. The injectivity of π_2 on $\overline{S_k}$ then implies that $\varphi^n \in W_\varphi$, which is a contradiction. So C_r^p is embedded in $\mathcal{W}^u(\mathcal{O}_p)$. With the projection P_2 and the map w_k from Proposition 5.5,

$$C_r^p = \{\chi + w_k(\chi) : \chi \in P_2 C_r^p\}.$$

Using the previously obtained result that C_r^p is a C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$, we prove at the end of this subsection that w_k is continuously differentiable on the open set $P_2 C_r^p$, i.e., this representation for C_r^p is smooth.

Section 3 has introduced a hyperplane Y, a convex bounded open neighborhood N of r_0 in C, $\varepsilon \in (0, \omega)$ and a C^1-map $\gamma : N \to (\omega - \varepsilon, \omega + \varepsilon)$ with $\gamma(r_0) = \omega$ so that for each $(t, \varphi) \in (\omega - \varepsilon, \omega + \varepsilon) \times N$, the segment x_t^φ belongs to $r_0 + Y$ if and only if $t = \gamma(\varphi)$. A Poincaré return map P_Y has been defined as

$$P_Y : N \cap (r_0 + Y) \ni \varphi \mapsto \Phi(\gamma(\varphi), \varphi) \in r_0 + Y.$$

Let $\mathcal{W}$ denote a local stable manifold $\mathcal{W}^s_{loc}(P_Y, r_0)$ of P_Y at r_0 if $\mathcal{O}_r$ is hyperbolic, and let $\mathcal{W}$ be a local center-stable manifold $\mathcal{W}^{sc}_{loc}(P_Y, r_0)$ of P_Y at r_0 otherwise. By Section 3, $\mathcal{W}$ is a C^1-submanifold of $r_0 + Y$ with codimension 1, and it is a C^1-submanifold of C with codimension 2.

The subsequent proposition is an important step toward the proof of the assertion that C^p_q and C^p_k are two-dimensional C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$.

Proposition 5.11. *$\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W}$ is a one-dimensional C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$.*

Proof. 1. Theorem B and Proposition 3.4 imply that $\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W}$ is nonempty. It suffices to verify that the inclusion map $i : \mathcal{W}^u(\mathcal{O}_p) \ni \varphi \mapsto \varphi \in C$ and $\mathcal{W}$ are transversal. Then it follows that $i^{-1}(\mathcal{W}) = \mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W}$ is a C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$, furthermore it has the same codimension in $\mathcal{W}^u(\mathcal{O}_p)$ as $\mathcal{W}$ in C (see e.g. Corollary 17.2 in [1]). Accordingly we show that the inclusion map $i : \mathcal{W}^u(\mathcal{O}_p) \ni \varphi \mapsto \varphi \in C$ and $\mathcal{W}$ are transversal. This means that for all $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ with $\varphi = i(\varphi) \in \mathcal{W}$,

(i) the inverse image $(Di(\varphi))^{-1} T_{i(\varphi)}\mathcal{W} = T_\varphi \mathcal{W}^u(\mathcal{O}_p) \cap T_\varphi \mathcal{W}$ splits in $T_\varphi \mathcal{W}^u(\mathcal{O}_p)$ (it has a closed complementary subspace in $T_\varphi \mathcal{W}^u(\mathcal{O}_p)$), and

(ii) the space $Di(\varphi) T_\varphi \mathcal{W}^u(\mathcal{O}_p) = T_\varphi \mathcal{W}^u(\mathcal{O}_p)$ contains a closed complement to $T_{i(\varphi)}\mathcal{W} = T_\varphi \mathcal{W}$ in C.

Property (i) holds because $\dim T_\varphi \mathcal{W}^u(\mathcal{O}_p) = 3 < \infty$. In the following we confirm (ii).

2. Let $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W}$. First note that the invariance of $\mathcal{W}^u(\mathcal{O}_p)$ ensures that $\dot\varphi \in T_\varphi \mathcal{W}^u(\mathcal{O}_p)$. On the other hand, Proposition 3.3 gives that $\dot\varphi \notin Y$ can be assumed. Therefore $\dot\varphi \in T_\varphi \mathcal{W}^u(\mathcal{O}_p) \setminus T_\varphi \mathcal{W}$.

We claim that $T_\varphi \mathcal{W}^u(\mathcal{O}_p)$ contains a sign-preserving element χ. Let Z be the hyperplane in C with $C = \mathbb{R}\dot p_0 \oplus Z$ and define a Poincaré map P_Z on a neighborhood of p_0 in $p_0 + Z$ as in Section 3. (Here we use exceptionally the notation Z and P_Z to emphasize the difference from the above mentioned Y and P_Y.) Choose ψ from a local unstable manifold $\mathcal{W}^u_{loc}(P_Z, p_0)$ of P_Z such that $\varphi = \Phi(T, \psi)$ for some $T \geq 0$. This is possible by (3.5). Choose η to be a strictly positive vector in $T_\psi \mathcal{W}^u_{loc}(P_Z, p_0)$. Proposition 3.5 yields that the existence of such η may be supposed without loss of generality. Then $D_2\Phi(T, \psi)\eta \in T_\varphi \mathcal{W}^u(\mathcal{O}_p)$, and $D_2\Phi(T, \psi)\eta = u^\eta_T$, where $u^\eta : [-1, \infty) \to \mathbb{R}$ is the solution of the linear variational equation

$$\dot u(t) = -u(t) + f'\left(x^\psi(t-1)\right) u(t-1) \tag{5.3}$$

with $u^\eta_0 = \eta$. We claim that u is positive on $[-1, \infty)$. If this is not true, choose $t_0 > 0$ to be minimal with $u(t_0) = 0$. Then $\dot u(t_0) \leq 0$. On the other hand, the equation (5.3) and $u(t_0 - 1) > 0$ together yield that $\dot u(t_0) > 0$, which is a contradiction. Let χ be the positive vector $u^\eta_T \in T_\varphi \mathcal{W}^u(\mathcal{O}_p)$.

The vectors $\dot{\varphi}$ and χ are linearly independent because $V(\chi) = 0$ and we may assume by Proposition 3.3 that $V(\dot{\varphi}) \geq 2$.

3. As $T_\varphi \mathcal{W}$ is a subspace of C with codimension 2, it suffices to confirm that

$$T_\varphi \mathcal{W} \cap (\mathbb{R}\dot{\varphi} \oplus \mathbb{R}\chi) = \left\{\hat{0}\right\}.$$

Suppose that $a\dot{\varphi} + b\chi \in T_\varphi \mathcal{W} \backslash \left\{\hat{0}\right\}$ for some $a, b \in \mathbb{R}$. Then $b \neq 0$ as $\dot{\varphi} \notin T_\varphi \mathcal{W}$. Set $c = a/b$ and consider the vector $c\dot{\varphi} + \chi \in T_\varphi \mathcal{W} \backslash \left\{\hat{0}\right\}$. Let $v : [-1, \infty) \to \mathbb{R}$ be the solution of the linear variational equation

$$\dot{v}(t) = -v(t) + f'\left(x^\varphi (t-1)\right) v(t-1) \tag{2.2}$$

with $v_0 = \chi$, and let $x = x^\varphi$. As $\varphi \in \mathcal{W}$, $\gamma_j = \Sigma_{i=0}^{j-1} \gamma\left(P_Y^i(\varphi)\right)$ is defined for all $j \geq 1$, and $\gamma_j \to \infty$ as $j \to \infty$. Then by formula (3.4),

$$\begin{aligned} T_{P_Y^j(\varphi)} \mathcal{W} \ni DP_Y^j(\varphi)(c\dot{\varphi} + \chi) &= c\dot{x}_{\gamma_j} + v_{\gamma_j} - \frac{e^*\left(c\dot{x}_{\gamma_j} + v_{\gamma_j}\right)}{e^*\left(\dot{x}_{\gamma_j}\right)} \dot{x}_{\gamma_j} \\ &= v_{\gamma_j} - \frac{e^*\left(v_{\gamma_j}\right)}{e^*\left(\dot{x}_{\gamma_j}\right)} \dot{x}_{\gamma_j}. \end{aligned}$$

An application of Lemma 2.7 to the equation (2.2) and its strictly positive solution $v : [-1, \infty) \to \mathbb{R}$ gives constants $K > 0$ and $t \geq 1$ such that

$$\|v_{s-1}\| \leq K \|v_s\| \qquad \text{for all } s \geq t.$$

Equation (2.2) with this estimate then gives a uniform bound for the derivatives $\dot{v}_{\gamma_j} / \left\|v_{\gamma_j}\right\|$, $j \geq 1$. So by the Arzelà–Ascoli theorem, there exists a subsequence

$$\left(\frac{v_{\gamma_{j_n}}}{\left\|v_{\gamma_{j_n}}\right\|}\right)_{n=0}^{\infty}$$

converging to a nonnegative unit vector ρ as $n \to \infty$. As the C-norm and the C^1-norm are equivalent on $\mathcal{A}$, the convergence $x_{\gamma_j} = P_Y^j(\varphi) \to r_0$ implies that $\dot{x}_{\gamma_j} \to \dot{r}_0$ as $j \to \infty$. It follows that

$$\frac{1}{\left\|v_{\gamma_{j_n}}\right\|} DP_Y^{j_n}(\varphi)(c\dot{\varphi} + \chi) \in T_{P_Y^{j_n}(\varphi)} \mathcal{W}$$

converges to the vector

$$\rho - \frac{e^*(\rho)}{e^*(\dot{r}_0)} \dot{r}_0 \in T_{r_0} \mathcal{W} = \begin{cases} C_s, & \text{if } \mathcal{O}_r \text{ is hyperbolic,} \\ C_s \oplus \mathbb{R}\xi, & \text{if } \mathcal{O}_r \text{ is nonhyperbolic.} \end{cases}$$

As $T_{r_0} \mathcal{W} \subseteq C_{\leq 1}$ and $\dot{r}_0 \in C_{\leq 1}$, this means that $C_{\leq 1}$ has a nontrivial nonnegative element ρ. This is a contradiction since $\mathcal{O}_r$ has a Floquet multiplier $\lambda_1 > 1$ and $C_{<\lambda_1} \cap V^{-1}(0) = \emptyset$ by (3.2). □

Now we can verify a part of Theorem 1.2.(i).

Proposition 5.12. *C_q^p and C_k^p are both two-dimensional C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$.*

Proof. Define r, $\mathcal{W}$ and C_r^p as at the beginning of this subsection.

1. As a first step we confirm that to all $\varphi \in C_r^p$, one can give a subset W_φ of C_r^p so that W_φ is a two-dimensional C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$ and contains φ. Let $\varphi \in C_r^p$. Choose $T \geq 0$ such that $\psi = \Phi(T, \varphi) \in \mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W}$ and $\dot{\psi} \notin Y$. Propositions 3.3 and 3.4 guarantee that this is possible. Consider the two-dimensional C^1-submanifold $\mathbb{R} \times (\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W})$ of $\mathbb{R} \times \mathcal{W}^u(\mathcal{O}_p)$ and the map

$$\Sigma : \mathbb{R} \times (\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W}) \ni (t, \eta) \mapsto \Phi_{\mathcal{W}^u(\mathcal{O}_p)}(t, \eta) \in \mathcal{W}^u(\mathcal{O}_p).$$

Proposition 4.6 proves that Σ is C^1-smooth and gives formulas for its derivatives. Note that the derivative of the map

$$\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W} \ni \eta \mapsto \Phi_{\mathcal{W}^u(\mathcal{O}_p)}(-T, \eta) \in \mathcal{W}^u(\mathcal{O}_p)$$

at ψ is injective on $T_\psi(\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W})$ by Corollary 4.7. Also observe that $\dot{\psi} \notin Y$ implies that $\dot{\psi} \notin T_\psi(\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W})$. Using these two properties and a reasoning analogous to the one applied in Proposition 4.1, it is straightforward to show that $D\Sigma(-T, \psi)$ is injective on $\mathbb{R} \times T_\psi(\mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W})$. Thus there exists an $\varepsilon > 0$ by Proposition 2.8 such that the set

$$W_\varphi = \left\{ \Phi_{\mathcal{W}^u(\mathcal{O}_p)}(t, \eta) : t \in (-T - \varepsilon, -T + \varepsilon), \eta \in \mathcal{W}^u(\mathcal{O}_p) \cap \mathcal{W} \cap B(\psi, \varepsilon) \right\}$$

is a two-dimensional C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$. It is clear that $\varphi \in W_\varphi$. The invariance of C_r^p implies that $W_\varphi \subseteq C_r^p$.

2. To complete the proof, it suffices to exclude for all $\varphi \in C_r^p$ the existence of a sequence $(\varphi^n)_{n=0}^\infty$ in C_r^p so that $\varphi^n \notin W_\varphi$ for $n \geq 0$ and $\varphi^n \to \varphi$ as $n \to \infty$. By Proposition 5.7, $D\pi_2(\varphi) = \pi_2$ is injective on the two-dimensional tangent space $T_\varphi W_\varphi$, hence it defines an isomorphism from $T_\varphi W_\varphi$ onto $\mathbb{R}^2$. Therefore there exists $\tilde{\varepsilon} > 0$ such that the restriction of π_2 to $W_\varphi \cap B(\varphi, \tilde{\varepsilon})$ is a diffeomorphism from $W_\varphi \cap B(\varphi, \tilde{\varepsilon})$ onto an open set U in $\mathbb{R}^2$. If a sequence $(\varphi^n)_{n=0}^\infty$ in C_r^p converges to φ as $n \to \infty$, then $\pi_2 \varphi^n \to \pi_2 \varphi$ as $n \to \infty$, and $\pi_2 \varphi^n \in U$ for all n large enough. The injectivity of π_2 on $\overline{S_k}$ verified in Proposition 5.4 then implies that $\varphi^n \in W_\varphi$. $\square$

It is worth noting that the second part of the above proof confirms the following assertion.

Proposition 5.13. *$\pi_2 C_q^p$ and $\pi_2 C_k^p$ are open subsets of $\mathbb{R}^2$.*

We know from Proposition 5.5 that there exist a projection P_2 from C onto a two-dimensional subspace G_2 of C and a map $w_k : P_2 \overline{S_k} \to P_2^{-1}(0)$ so that

$$\overline{S_k} = \left\{ \chi + w_k(\chi) : \chi \in P_2 \overline{S_k} \right\}.$$

Then

$$C_q^p = \left\{\chi + w_k(\chi) : \chi \in P_2 C_q^p\right\} \quad \text{and} \quad C_k^p = \left\{\chi + w_k(\chi) : \chi \in P_2 C_k^p\right\}.$$

The next result implies that these representations of C_q^p and C_k^p are smooth.

Proposition 5.14. *$P_2 C_q^p$ and $P_2 C_k^p$ are open subsets of G_2, and w_k is continuously differentiable on $P_2 C_q^p \cup P_2 C_k^p$.*

Proof. The proof is based on the smoothness of C_q^p and C_k^p and applies an argument which is analogous to the one in the proof of Theorem 1.1.

Let C_r^p be any of the sets C_q^p and C_k^p. Let $\chi \in P_2 C_r^p$ be arbitrary, and choose $\varphi \in C_r^p$ so that $\chi = P_2\varphi$. As the restriction of π_2 to $T_\varphi C_r^p$ is injective, J_2 is an isomorphism and $P_2 = J_2 \circ \pi_2$, $DP_2(\varphi) = P_2$ defines an isomorphism from $T_\varphi C_r^p$ to G_2. The inverse mapping theorem implies that an $\varepsilon > 0$ can be given such that P_2 maps $C_r^p \cap B(\varphi, \varepsilon)$ one-to-one onto an open neighborhood $U \subset P_2 C_r^p$ of χ in G_2, P_2 is invertible on $C_r^p \cap B(\varphi, \varepsilon)$, and the inverse $\tilde{P}_2^{-1}$ of the map

$$C_r^p \cap B(\varphi, \varepsilon) \ni \varphi \mapsto P_2\varphi \in U$$

is C^1-smooth. As

$$w_k(\chi) = (\mathrm{id} - P_2) \circ \left(P_2|_{\overline{S_k}}\right)^{-1}(\chi) = (\mathrm{id} - P_2) \circ \tilde{P}_2^{-1}(\chi) \in P_2^{-1}(0)$$

for all $\chi \in U$, the restriction of w_k to U is C^1-smooth. □

5.4 C_q^p, C_k^p and S_k are homeomorphic to $A^{(1,2)}$, and their closures are homeomorphic to $A^{[1,2]}$

Recall that

$$A_q^p = \mathrm{ext}(\pi_2\mathcal{O}_p) \cap \mathrm{int}(\pi_2\mathcal{O}_q), \qquad A_k^p = \mathrm{ext}(\pi_2\mathcal{O}_k) \cap \mathrm{int}(\pi_2\mathcal{O}_p)$$

and

$$A_{k,q} = \mathrm{ext}(\pi_2\mathcal{O}_k) \cap \mathrm{int}(\pi_2\mathcal{O}_q).$$

We have already deduced that $\pi_2 C_q^p \subseteq A_q^p$ and $\pi_2 C_k^p \subseteq A_k^p$. As a result, $\pi_2 S_k \subseteq A_{k,q}$.

Proposition 5.15. *The map $\pi_2|_{\overline{S_k}}$ is a homeomorphism onto $\overline{A_{k,q}}$, furthermore $\pi_2 C_k^p = A_k^p$, $\pi_2 C_q^p = A_q^p$ and $\pi_2 S_k = A_{k,q}$.*

Proof. First we show that $\pi_2 C_q^p = A_q^p$. By Proposition 5.13, $\pi_2 C_q^p$ is open in A_q^p. We claim that $\pi_2 C_q^p$ is also closed in A_q^p. So assume that $(z_n)_{n=0}^\infty$ is a sequence in $\pi_2 C_q^p$ and $z_n \to z \in A_q^p$ as $n \to \infty$. Let $\varphi_n = \pi_2^{-1}(z_n) \in C_q^p$, $n \geq 0$. By Proposition 5.6, π_2^{-1} is Lipschitz-continuous. Thus $\{\varphi_n\}_{n=0}^\infty$ is a Cauchy-sequence in C_q^p and a $\varphi \in \overline{C_q^p}$ can be given such that $\varphi_n \to \varphi$ as $n \to \infty$, moreover, $\varphi = \pi_2^{-1}(z)$. It is clear that $\varphi \notin \mathcal{O}_p$ and $\varphi \notin \mathcal{O}_q$ because then $z = \pi_2\varphi \notin A_q^p$. Thus $\varphi \in \overline{C_q^p} \setminus (\mathcal{O}_p \cup \mathcal{O}_p) = C_q^p$

(here we use Corollary 5.10) and necessarily $z = \pi_2\varphi \in \pi_2 C_q^p$. In consequence, $\pi_2 C_q^p = A_q^p$.

It is analogous to verify that $\pi_2 C_k^p = A_k^p$. It follows immediately that

$$\pi_2 S_k = \pi_2 \left(C_k^p \cup \mathcal{O}_p \cup C_q^p \right) = A_k^p \cup \pi_2 \mathcal{O}_p \cup A_q^p = A_{k,q}$$

and

$$\pi_2 \overline{S_k} = \pi_2 \left(\mathcal{O}_k \cup S_k \cup \mathcal{O}_q \right) = \pi_2 \mathcal{O}_k \cup A_{k,q} \cup \pi_2 \mathcal{O}_q = \overline{A_{k,q}}.$$

As both $\pi_2|_{\overline{S_k}} : \overline{S_k} \to \mathbb{R}^2$ and $\pi_2^{-1} : \pi_2 \overline{S_k} \to C$ are continuous, we obtain that $\pi_2|_{\overline{S_k}}$ defines a homeomorphism from $\overline{S_k}$ onto $\overline{A_{k,q}}$. □

As a consequence we obtain that C_q^p, C_k^p, and S_k are homeomorphic to the open annulus

$$A^{(1,2)} = \left\{ u \in \mathbb{R}^2 : 1 < |u| < 2 \right\}.$$

Since the above proposition implies that $\pi_2 \overline{C_k^p} = \overline{A_k^p}$ and $\pi_2 \overline{C_q^p} = \overline{A_q^p}$, we also deduce that the closures $\overline{C_q^p}$, $\overline{C_k^p}$, and $\overline{S_k}$ are homeomorphic to the closed annulus

$$A^{[1,2]} = \left\{ u \in \mathbb{R}^2 : 1 \le |u| \le 2 \right\}.$$

Note that we have proven all the statements of Theorem 1.1.(i) regarding C_q^p and C_k^p (see propositions 5.12, 5.14 and 5.15). The smoothness of S_k is considered in the next subsection.

5.5 The smoothness of S_k, $\overline{C_q^p}$, $\overline{C_k^p}$ and $\overline{S_k}$

Now we can round up the proofs of Theorem 1.2.(i) and (ii).

Recall that

$$S_k = \{ \chi + w_k(\chi) : \chi \in P_2 S_k \}, \qquad P_2 S_k = P_2 C_k^p \cup P_2 \mathcal{O}_p \cup P_2 C_q^p$$

and w_k is continuously differentiable on the set $P_2 C_k^p \cup P_2 C_q^p$. Hence the smoothness of this representation for S_k is proved by showing that $P_2 S_k$ is open in G_2 and w_k is smooth at the points of $P_2 \mathcal{O}_p$. It follows at once that S_k is a two-dimensional C^1-submanifold of C. Since S_k is a subset of the three-dimensional C^1-submanifold $\mathcal{W}^u(\mathcal{O}_p)$, it is obvious that S_k is also a C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$.

We in addition show that $P_2 \mathcal{O}_k \cup P_2 \mathcal{O}_q$ is the boundary of $P_2 \overline{S_k}$, and all points of $P_2 \mathcal{O}_k \cup P_2 \mathcal{O}_q$ have open neighborhoods on which w_k can be extended to C^1-functions. This means that $\overline{S_k}$ has a smooth representation with boundary, and thus $\overline{S_k}$ is a two-dimensional C^1-submanifold of the phase space C with boundary. Similar reasonings yield the analogous results for $\overline{C_q^p}$ and $\overline{C_k^p}$.

Let $r : \mathbb{R} \to \mathbb{R}$ be any of the periodic solutions x^k, p or q shifted in time so that $r(0) = \xi_k$ and $\dot{r}(0) > 0$. As ξ_k belongs to the ranges of x^k, p or q, and ξ_k is not an extremum of them, the monotonicity property of periodic solutions in Proposition

2.5 implies that this choice of r is possible. Let $\omega > 0$ denote the minimal period of r. By Eq. (1.1),

$$f(r(-1)) = \dot{r}(0) + r(0) > \xi_k = f(\xi_k).$$

As f strictly increases, this means that $r(-1) > \xi_k$. Conversely, if there was $t_* \in (0, \omega)$ such that $r(t_*) = \xi_k$ and $r(t_* - 1) > \xi_k$, then

$$\dot{r}(t_*) = -r(t_*) + f(r(t_* - 1)) > -\xi_k + f(\xi_k) = 0$$

would follow, which would contradict Proposition 2.5. Therefore the half line $L_k = \{(\xi_k, x_2) \in \mathbb{R}^2 : x_2 > \xi_k\}$ and $\pi_2\mathcal{O}_r = \{\pi_2 r_t : t \in [0, \omega)\}$ have exactly one point in common: $(r(0), r(-1)) = (\xi_k, r(-1))$. See Fig. 7.

Choose $s_k, s_p, s_q > \xi_k$ so that

$$\{(\xi_k, s_k)\} = L_k \cap \pi_2\mathcal{O}_k, \quad \{(\xi_k, s_p)\} = L_k \cap \pi_2\mathcal{O}_p$$

and

$$\{(\xi_k, s_q)\} = L_k \cap \pi_2\mathcal{O}_q.$$

As s increases, $(\xi_k, \infty) \ni s \mapsto (\xi_k, s) \in \mathbb{R}^2$ first intersects $\pi_2\mathcal{O}_k$, then $\pi_2\mathcal{O}_p$ and finally $\pi_2\mathcal{O}_q$ because

$$(\xi_k, s) \to \pi_2\hat{\xi}_k = (\xi_k, \xi_k) \in \operatorname{int}(\pi_2\mathcal{O}_k) \text{ whenever } s \to \xi_k+,$$

$\pi_2\mathcal{O}_k \subset \operatorname{int}(\pi_2\mathcal{O}_p)$ and $\pi_2\mathcal{O}_p \subset \operatorname{int}(\pi_2\mathcal{O}_q)$. So $\xi_k < s_k < s_p < s_q$, as it is shown by Fig. 7.

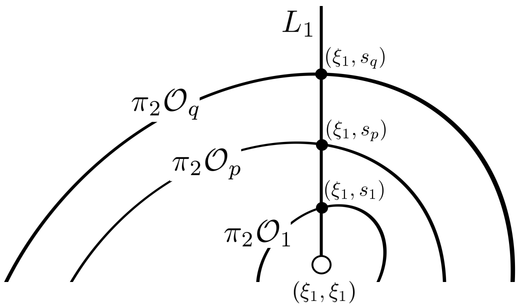

FIGURE 7. The definition of L_1, s_1, s_p and s_q in the case $k = 1$.

Consider the curve

$$h : [s_k, s_q] \ni s \mapsto \pi_2^{-1}(\xi_k, s) \in C.$$

Then h is Lipschitz-continuous and injective. By Proposition 5.15, $h([s_k, s_q]) \subset \overline{S_k}$. In detail,

$$h(s_k) \in \mathcal{O}_k, \quad h((s_k, s_p)) \subset C_k^p, \quad h(s_p) \in \mathcal{O}_p, \quad h((s_p, s_q)) \subset C_q^p, \quad \text{and} \quad h(s_q) \in \mathcal{O}_q.$$

According to the next result, h is C^1-smooth on $(s_1, s_q) \setminus \{s_p\}$.

Proposition 5.16. *$\pi_2^{-1}|_{\pi_2\left(C_q^p \cup C_k^p\right)}$ is C^1-smooth.*

Proof. We know from Proposition 5.13 that $\pi_2\left(C_q^p \cup C_k^p\right)$ is open in $\mathbb{R}^2$.

For all $x \in \pi_2\left(C_q^p \cup C_k^p\right)$, the graph representation of $C_q^p \cup C_k^p$ and the definition of P_2 together give that

$$\begin{aligned} C_q^p \cup C_k^p \ni \pi_2^{-1}(x) &= P_2\left(\pi_2^{-1}(x)\right) + w_k\left(P_2\left(\pi_2^{-1}(x)\right)\right) \\ &= J_2\left(\pi_2\left(\pi_2^{-1}(x)\right)\right) + w_k\left(J_2\left(\pi_2\left(\pi_2^{-1}(x)\right)\right)\right) \\ &= J_2(x) + w_k\left(J_2(x)\right). \end{aligned}$$

As J_2 defines an isomorphism from $\mathbb{R}^2$ to G_2, it is continuously differentiable. In addition, $J_2\left(\pi_2\left(C_q^p \cup C_k^p\right)\right) = P_2\left(C_q^p \cup C_k^p\right)$, and w_k is continuously differentiable on the open subset $P_2\left(C_q^p \cup C_k^p\right)$ of G_2 by Proposition 5.14. Hence the statement follows. ☐

As a next step, we show the smoothness of h at points s_k, s_p and s_q. We will need the following technical result, which is part of Proposition 8.5 in [10].

Proposition 5.17.

(i) Let $v : \mathbb{R} \to \mathbb{R}$ be a solution of Eq. (3.1) with $v_0 \neq \hat{0}$. If $V(v_t) = 2$ for all $t \in \mathbb{R}$, then $v_0 \in C_{r_M<} \cap C_{\leq 1}$.

(ii) For every $\varphi \in C_{r_M<} \cap C_{\leq 1} \setminus \{\hat{0}\}$, there is a solution $v : \mathbb{R} \to \mathbb{R}$ of Eq. (3.1) so that $v_0 = \varphi$ and $V(v_t) = 2$ for all $t \in \mathbb{R}$.

Proposition 5.18. *Let $* \in \{k, p, q\}$ and set $r : \mathbb{R} \to \mathbb{R}$ to be the periodic solution of Eq. (1.1) with $\pi_2 r_0 = (\xi_k, s_*)$.*

(i) There exists a unique continuously differentiable function $z = z^* : \mathbb{R} \to \mathbb{R}$ satisfying

$$(5.4) \qquad \begin{cases} \dot{z}(t) = -z(t) + f'(r(t-1))\, z(t-1), & t \in \mathbb{R}, \\ z(-1) = 1,\ z(0) = 0, & \\ V(z_t) = 2, & t \in \mathbb{R}. \end{cases}$$

(ii) For every $\varepsilon > 0$, there exists $\delta > 0$ so that for all $\chi \in [s_k, s_q]$, $\nu \in [s_k, s_q]$ with $|\chi - s_*| < \delta$, $|\nu - s_*| < \delta$ and $\chi \neq \nu$,

$$\left\| \frac{h(\chi) - h(\nu)}{\chi - \nu} - z_0 \right\| < \varepsilon.$$

(iii) z_0 and $\dot{r}_0$ are linearly independent.

Proof. 1. We prove that for all sequences $(\chi^n)_{n=0}^\infty$, $(\nu^n)_{n=0}^\infty$ in $[s_k, s_q]$ with $\chi^n \neq \nu^n$ for all $n \geq 0$ and $\chi^n \to s_*$, $\nu^n \to s_*$ as $n \to \infty$, there exist a strictly increasing sequence $(n_l)_{l=0}^\infty$ and a continuously differentiable function $z = z^* : \mathbb{R} \to \mathbb{R}$ so that

z is a solution of the equation in (5.4), and

$$\lim_{l\to\infty}\frac{h\left(\chi^{n_l}\right)-h\left(\nu^{n_l}\right)}{\chi^{n_l}-\nu^{n_l}}=z_0.$$

Consider the solutions $x^n:\mathbb{R}\to\mathbb{R}$ and $y^n:\mathbb{R}\to\mathbb{R}$ of Eq. (1.1) with $x_0^n=h\left(\chi^n\right)$ and $y_0^n=h\left(\nu^n\right)$ for all indices $n\geq 0$. Then $x^n\left(-1\right)=\chi^n$, $y^n\left(-1\right)=\nu^n$ and $x^n\left(0\right)=y^n\left(0\right)=\xi_k$ for all $n\geq 0$, moreover $x_t^n\in\overline{S_k}$ and $y_t^n\in\overline{S_k}$ for all $n\geq 0$ and $t\in\mathbb{R}$.

Introduce the functions

$$z^n=\frac{x^n-y^n}{\chi^n-\nu^n},\qquad n\geq 0.$$

It is clear that $z^n\left(0\right)=0$ and $z^n\left(-1\right)=1$ for all $n\geq 0$. By Proposition 5.3, $V\left(z_t^n\right)=2$ for all $n\geq 0$ and $t\in\mathbb{R}$. In addition, z^n, $n\geq 0$, satisfies the equation

$$\dot{z}^n\left(t\right)=-z^n\left(t\right)+b^n\left(t\right)z^n\left(t-1\right)$$

on $\mathbb{R}$, where the coefficient functions b^n are defined as

$$b^n:\mathbb{R}\ni t\mapsto\int_0^1 f'\left(sx^n\left(t-1\right)+\left(1-s\right)y^n\left(t-1\right)\right)\mathrm{d}s\in\left(0,\infty\right),\quad n\geq 0.$$

Since $\chi^n\to s_*$ and $\nu^n\to s_*$ as $n\to\infty$, $x_0^n\to r_0$ and $y_0^n\to r_0$ as $n\to\infty$. It follows that $b^n\to b$ as $n\to\infty$ uniformly on compact subsets of $\mathbb{R}$, where

$$b:\mathbb{R}\ni t\mapsto f'\left(r\left(t-1\right)\right)\in\left(0,\infty\right).$$

As the global attractor is bounded, there are constants $b_1>b_0>0$ so that $b_0<b^n\left(t\right)<b_1$ for all $n\geq 0$ and $t\in\mathbb{R}$. Thus Lemma 2.6 ensures the existence of a continuously differentiable function $z:\mathbb{R}\to\mathbb{R}$ and a subsequence $\left(z^{n_l}\right)_{l=0}^{\infty}$ of $\left(z^n\right)_{n=0}^{\infty}$ such that $z^{n_l}\to z$ and $\dot{z}^{n_l}\to\dot{z}$ as $l\to\infty$ uniformly on compact subsets of $\mathbb{R}$, and z is a solution of the equation in (5.4).

It is obvious that $z\left(0\right)=0$ and $z\left(-1\right)=1$.

By the first part of Lemma 2.2, $V\left(z_t\right)\leq 2$ for all real t. Suppose $V\left(z_{t^*}\right)=0$ for some $t^*\in\mathbb{R}$. Then $V\left(z_t\right)=0$ for all $t>t^*$ and $V\left(z_{t^*+3}\right)\in R$ by Lemma 2.3. The C^1-convergence of z^{n_l} to z and the second part of Lemma 2.2 then imply that $V\left(z_{t^*+3}^{n_l}\right)=0$ for all sufficiently large index l, which is contradiction. So $V\left(z_t\right)=2$ for all real t.

2. Suppose that $\hat{z}:\mathbb{R}\to\mathbb{R}$ is also a continuously differentiable function satisfying (5.4), and $z\neq\hat{z}$. Then Proposition 2.1 yields that $z_0\neq\hat{z}_0$. The function $d=z-\hat{z}$ is a solution of

$$\begin{cases}\dot{d}\left(t\right)=-d\left(t\right)+f'\left(r\left(t-1\right)\right)d\left(t-1\right), & t\in\mathbb{R},\\ d\left(-1\right)=d\left(0\right)=0.\end{cases}$$

Since $z_0, \hat{z}_0 \in C_{r_M<} \cap C_{\leq 1}$ by Proposition 5.17 (i), $d_0 \in C_{r_M<} \cap C_{\leq 1} \setminus \{\hat{0}\}$. Then it follows from Proposition 5.17 (ii) that $V(d_t) = 2$ for all $t \in \mathbb{R}$. So $d_0 \in R$ by Lemma 2.3.(iii), which is impossible as $d(-1) = d(0) = 0$.

These results imply both (i) and (ii).

3. Solution r has been defined to be a time translate of x^k, p or q with $r(0) = \xi_k$. Hence ξ_k is not an extremum of r, and thus $\dot{r}(0) \neq 0$ by Proposition 2.5. Consequently, $z_0 \notin \mathbb{R}\dot{r}_0 \setminus \{\hat{0}\}$, and z_0 and $\dot{r}_0$ are linearly independent. □

Corollary 5.19. *The function h is C^1-smooth on $[s_k, s_q]$.*

We extend the definition of h to the half line (ξ_k, ∞): we define $\hat{h} : (\xi_k, \infty) \to C$ as $\hat{h}(s) = h(s)$ for $s \in [s_k, s_q]$,

$$\hat{h}(s) = h(s_k) + (s - s_k) z_0^k \quad \text{for } s \in (\xi_k, s_k)$$

and

$$\hat{h}(s) = h(s_q) + (s - s_q) z_0^q \quad \text{for } s > s_q,$$

where z_0^k and z_0^q are given by Proposition 5.18. Then $\hat{h}$ is C^1-smooth with $\hat{h}'(s_k) = z_0^k$, $\hat{h}'(s_p) = z_0^p$, and $\hat{h}'(s_q) = z_0^q$. According to the choice of $s_k < s_p < s_q$ and Proposition 5.15,

(5.5)

$$\hat{h}(s_k) \in \mathcal{O}_k, \quad \hat{h}((s_k, s_p)) \subset C_k^p, \quad \hat{h}(s_p) \in \mathcal{O}_p, \quad \hat{h}((s_p, s_q)) \subset C_q^p \text{ and } \hat{h}(s_q) \in \mathcal{O}_q.$$

Observe that $\pi_2 \hat{h}(s) = (\xi_k, s)$ for all $s > \xi_k$, hence the map $(\xi_k, \infty) \ni s \mapsto \pi_2 \hat{h}(s) \in \mathbb{R}^2$ is injective on (ξ_k, ∞) and has range in $L_k = \{(\xi_k, x_2) \in \mathbb{R}^2 : x_2 > \xi_k\}$. So it follows from $\pi_2 \overline{S_k} = \overline{A_{k,q}}$ that

$$\hat{h}((\xi_k, s_k) \cup (s_q, \infty)) \cap \overline{S_k} = \emptyset. \tag{5.6}$$

Recall from Proposition 5.5 that there exist a projection P_2 from C onto a two-dimensional subspace G_2 of C and a map $w_k : P_2 \overline{S_k} \to P_2^{-1}(0)$ so that

$$\overline{S_k} = \left\{\chi + w_k(\chi) : \chi \in P_2 \overline{S_k}\right\}.$$

This induces a global graph representation for any subset W of $\overline{S_k}$:

$$W = \{\chi + w_k(\chi) : \chi \in P_2 W\}.$$

Since $J_2 : \mathbb{R}^2 \to G_2$ is an isomorphism and $P_2 = J_2 \circ \pi_2$, Proposition 5.15 shows that

$$P_2 C_k^p = \operatorname{ext}(P_2 \mathcal{O}_k) \cap \operatorname{int}(P_2 \mathcal{O}_p), \quad P_2 C_q^p = \operatorname{ext}(P_2 \mathcal{O}_p) \cap \operatorname{int}(P_2 \mathcal{O}_q),$$

$$P_2 S_k = \operatorname{ext}(P_2 \mathcal{O}_k) \cap \operatorname{int}(P_2 \mathcal{O}_q),$$

$P_2\mathcal{O}_k \cup P_2\mathcal{O}_q$ is the boundary of $P_2 S_k$, and hence

$$P_2\overline{S_k} = P_2\mathcal{O}_k \cup (\mathrm{ext}\,(P_2\mathcal{O}_k) \cap \mathrm{int}\,(P_2\mathcal{O}_q)) \cup P_2\mathcal{O}_q. \tag{5.7}$$

As $P_2\mathcal{O}_k$ and $P_2\mathcal{O}_q$ are the images of simple closed C^1-curves, the boundary $P_2\mathcal{O}_k \cup P_2\mathcal{O}_q$ of the domain $P_2\overline{S_k}$ of w_k is a one-dimensional C^1-submanifold of G_2. The next result shows that w_k is continuously differentiable at the points of $P_2\mathcal{O}_p$, and it is smooth at the points of $P_2\mathcal{O}_k \cup P_2\mathcal{O}_q$ in the sense that w_k can be extended to continuously differentiable functions on open neighborhoods of the boundary points.

Proposition 5.20.
(i) To each $\varphi \in \mathcal{O}_k \cup \mathcal{O}_q$ there corresponds an open neighborhood U of $P_2\varphi$ in G_2 and a continuously differentiable map $w_k^e : U \to P_2^{-1}(0)$ such that

$$w_k^e|_{U\cap P_2\overline{S_k}} = w_k|_{U\cap P_2\overline{S_k}}, \tag{5.8}$$

and $U \setminus \{P_2 x_t^{\varphi} : t \in \mathbb{R}\}$ is the union of open connected disjoint subsets U^+ and U^- of U with the following property:
$U^- \cap P_2\overline{S_k} = \emptyset$ and $U^+ \subset P_2 C_k^p$ if $\varphi \in \mathcal{O}_k$,
$U^- \subset P_2 C_q^p$ and $U^+ \cap P_2\overline{S_k} = \emptyset$ if $\varphi \in \mathcal{O}_q$.
(ii) The map w_k is continuously differentiable at the points of $P_2\mathcal{O}_p$. Each $\varphi \in \mathcal{O}_p$ has an open neighborhood U of $P_2\varphi$ in G_2 such that $U \setminus P_2\mathcal{O}_p$ is the union of open connected disjoint subsets U^+ and U^- of U with $U^- \subset P_2 C_k^p$ and $U^+ \subset P_2 C_q^p$.

Proof. The proof below verifies assertions (i) and (ii) simultaneously.

1. Let $r : \mathbb{R} \to \mathbb{R}$ be one of the periodic solutions x^k, p or q shifted in time so that $r(0) = \xi_k$ and $\dot{r}(0) > 0$ (that is $\pi_2 r_0 \in L_k$), and fix $* \in \{k, p, q\}$ accordingly. Set $s_* = r(-1)$. Let $\varphi \in \mathcal{O}_r = \{r_t : t \in \mathbb{R}\}$ and choose $T > 1$ so that $\varphi = \Phi(T, r_0)$. For all $0 < \varepsilon < \min\{T-1, s_k - \xi_k\}$, the map

$$a : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \ni (t, s) \mapsto \Phi\left(T + t, \hat{h}(s_* + s)\right) \in C$$

is C^1-smooth with

$$Da(0,0)\,\mathbb{R}^2 = \mathbb{R}\dot{\varphi} \oplus \mathbb{R} D_2\Phi(T, r_0)\, z_0^*,$$

where $z^* : \mathbb{R} \to \mathbb{R}$ is the solution of (5.4) given by Proposition 5.18. The vectors $\dot{\varphi} = D_2\Phi(T, r_0)\,\dot{r}_0$ and $D_2\Phi(T, r_0)\, z_0^*$ are linearly independent because $D_2\Phi(T, r_0)$ is injective, and $\dot{r}_0$ and z_0^* are linearly independent by Proposition 5.18 (iii).

Therefore Proposition 2.8 implies that for all small $\varepsilon > 0$, the sets

$$a((-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)), \quad a((-\varepsilon, \varepsilon) \times (-\varepsilon, 0)) \quad \text{and} \quad a((-\varepsilon, \varepsilon) \times (0, \varepsilon))$$

are two-dimensional C^1-submanifolds of C with

$$T_\varphi a((-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)) = Da(0,0)\,\mathbb{R}^2.$$

2. Set $E_1 = Da\,(0,0)\,\mathbb{R}^2$ and let E_2 be a closed complement of E_1 in C. We claim that for small $\varepsilon > 0$, there exist an open neighborhood N_ε of $\hat{0}$ in E_1 and a continuously differentiable function $b : N_\varepsilon \to E_2$ so that $b\left(\hat{0}\right) = 0$, $Db\left(\hat{0}\right) = 0$ and $a\left((-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\right)$ is the shifted graph of b:

$$a\left((-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\right) = \varphi + \{\chi + b\,(\chi) :\ \chi \in N_\varepsilon\}\,.$$

Let Pr_{E_1} denote the projection of C onto E_1 along E_2, and define $j : C \to C$ by $j\,(\chi) = \chi - \varphi$ for all $\chi \in C$. Then

$$D\left(\mathrm{Pr}_{E_1} \circ j \circ a\right)(0,0)\,\mathbb{R}^2 = \mathrm{Pr}_{E_1} \circ Da\,(0,0)\,\mathbb{R}^2 = E_1.$$

Hence the inverse function theorem guarantees that $\mathrm{Pr}_{E_1} \circ j \circ a$ is a local C^1-diffeomorphism, i.e., for small $\varepsilon > 0$, $\mathrm{Pr}_{E_1} \circ j \circ a$ maps $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)$ injectively onto an open neighborhood N_ε of $\hat{0}$ in E_1, and the inverse $\left(\mathrm{Pr}_{E_1} \circ j \circ a\right)^{-1}$ of $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon) \ni (t,s) \mapsto \mathrm{Pr}_{E_1} \circ j \circ a\,(t,s) \in N_\varepsilon$ is C^1-smooth. In consequence, Pr_{E_1} maps $j \circ a\left((-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\right)$ onto N_ε injectively, and there exists a map $b : N_\varepsilon \to E_2$ so that $b\left(\hat{0}\right) = 0$ and

$$j \circ a\left((-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\right) = \{\chi + b\,(\chi) :\ \chi \in N_\varepsilon\}\,.$$

The smoothness of b follows from

$$b = \left(\mathrm{id} - \mathrm{Pr}_{E_1}\right) \circ j \circ a \circ \left(\mathrm{Pr}_{E_1} \circ j \circ a\right)^{-1}.$$

$Db\left(\hat{0}\right) = 0$ because $Da\,(0,0)\,\mathbb{R}^2 = E_1$.

3. Next we show that the continuously differentiable map

$$c : E_1 \supset N_\varepsilon \ni \chi \mapsto P_2\left(\varphi + \chi + b\,(\chi)\right) \in G_2$$

is a local C^1-diffeomorphism.

Note that $Dc\left(\hat{0}\right)\chi = P_2\chi$ for all $\chi \in E_1$. So it suffices to confirm that $P_2|_{E_1}$ is injective. E_1 is spanned by the derivatives $D\gamma\,(0)\,1$ of the curves

$$\gamma : (-1,1) \ni s \mapsto a\,(c_1 s, c_2 s) \in C,$$

where $(c_1, c_2) \in \mathbb{R}^2$. From (5.5) and the invariance of $\overline{S_k}$ it follows that if $s_* + c_2 s \in [s_k, s_q]$, then $\gamma\,(s) \in \overline{S_k}$. Proposition 5.7 gives that $\pi_2\gamma'(0) \neq (0,0)$ if $\gamma'(0) \neq \hat{0}$. Thus $\pi_2|_{E_1}$ is injective. As J_2 is an isomorphism, $P_2 = J_2 \circ \pi_2$ is also injective on E_1.

In consequence, a positive constant ε_0 can be given such that c is a C^1-diffeomorphism from N_{ε_0} onto an open neighborhood U of $P_2\varphi$ in G_2. Define c^{-1} to be the inverse of $N_{\varepsilon_0} \ni \chi \mapsto c\,(\chi) \in U$.

The constant ε_0 can be chosen so that $\varepsilon_0 < \min\{T - 1, s_k - \xi_k, s_p - s_k, s_q - s_p\}$ also holds.

4. Notice that

$$U = P_2 a\left((-\varepsilon_0,\varepsilon_0)\times(-\varepsilon_0,\varepsilon_0)\right),$$

and set

$$\begin{aligned} U^- =& P_2 a\left((-\varepsilon_0,\varepsilon_0)\times(-\varepsilon_0,0)\right),\\ U^0 =& P_2 a\left((-\varepsilon_0,\varepsilon_0)\times\{0\}\right),\\ U^+ =& P_2 a\left((-\varepsilon_0,\varepsilon_0)\times(0,\varepsilon_0)\right). \end{aligned}$$

By steps 2 and 3 it is clear that P_2 restricted to $a\left((-\varepsilon_0,\varepsilon_0)\times(-\varepsilon_0,\varepsilon_0)\right)$ defines a C^1-diffeomorphism from $a\left((-\varepsilon_0,\varepsilon_0)\times(-\varepsilon_0,\varepsilon_0)\right)$ onto U. As $a\left((-\varepsilon_0,\varepsilon_0)\times(-\varepsilon_0,0)\right)$ and $a\left((-\varepsilon_0,\varepsilon_0)\times(0,\varepsilon_0)\right)$ are two-dimensional C^1-submanifolds of C, the arcwise connected sets U^- and U^+ are open in G_2.

As $\hat{h}\left(s_*\right)=r_0\in\mathcal{O}_r$, we have $a\left((-\varepsilon_0,\varepsilon_0)\times\{0\}\right)\subset\mathcal{O}_r$ and $U^0\subset P_2\mathcal{O}_r$. As $P_2\mathcal{O}_r$ is a one-dimensional C^1-submanifold of G_2, we may assume (by decreasing $\varepsilon_0>0$ if necessary) that

$$U^-\cap P_2\mathcal{O}_r=\emptyset \quad\text{and}\quad U^+\cap P_2\mathcal{O}_r=\emptyset. \tag{5.9}$$

5. Introduce the C^1-map

$$w_k^e : U\ni\eta\mapsto\varphi+c^{-1}(\eta)+b\left(c^{-1}(\eta)\right)-\eta\in C.$$

For all $\eta\in U$, $c^{-1}(\eta)\in N_{\varepsilon_0}$, and thus

$$\begin{aligned} P_2\left(\varphi+c^{-1}(\eta)+b\left(c^{-1}(\eta)\right)-\eta\right) &= P_2\left(\varphi+c^{-1}(\eta)+b\left(c^{-1}(\eta)\right)\right)-P_2\eta\\ &= c\left(c^{-1}(\eta)\right)-\eta=\hat{0}. \end{aligned}$$

So w_k^e maps U into $P_2^{-1}(0)$.

6. Assume that $\varphi\in\mathcal{O}_k$, that is r is the time translate of x^k, and $s_*=s_k$. Then the relations $\varepsilon_0<s_p-s_k$, $\hat{h}\left((s_k,s_p)\right)\subset C_k^p$ and the invariance of C_k^p guarantee that $U^+\subset P_2C_k^p\subset \operatorname{ext}\left(P_2\mathcal{O}_k\right)$. As $D_1P_2a(0,0)1$ and $D_2P_2a(0,0)1$ are linearly independent in G_2, the curve

$$(-\varepsilon_0,\varepsilon_0)\ni s\mapsto P_2a(0,s)$$

intersects transversally $P_2\mathcal{O}_k$ at $P_2\varphi=P_2a(0,0)$. Using this, (5.9) and that $P_2\mathcal{O}_k$ is a simple closed curve in G_2, it follows that the connected open sets U^- and U^+ belong to different connected components of $G_2\setminus P_2\mathcal{O}_k$. Then $U^+\subset\operatorname{ext}\left(P_2\mathcal{O}_k\right)$ implies that $U^-\subset\operatorname{int}\left(P_2\mathcal{O}_k\right)$. Now (5.7) can be applied to conclude that $U^-\cap P_2\overline{S_k}=\emptyset$.

In cases $\varphi\in\mathcal{O}_p$ and $\varphi\in\mathcal{O}_q$, it is similar to show that $U^-\subset P_2C_k^p$, $U^+\subset P_2C_q^p$ and $U^-\subset P_2C_q^p$, $U^+\cap P_2\overline{S_k}=\emptyset$, respectively. We omit the details.

7. It remains to confirm (5.8). Assume again that $\varphi\in\mathcal{O}_k$, that is r is the time translate of x^k, and $s_*=s_k$. Let $\eta\in U\cap P_2\overline{S_k}$ be arbitrary. As $U^-\cap P_2\overline{S_k}=\emptyset$ by

part 6, necessarily $\eta \in U^+ \cap U^0$. Then $\eta = P_2 a(t,s) = P_2 \Phi\left(T+t, \hat{h}(s_k+s)\right)$ for some $t \in (-\varepsilon_0, \varepsilon_0)$ and $s \in [0, \varepsilon_0)$. As $\hat{h}([s_k, s_k+\varepsilon_0)) \subset \hat{h}([s_k, s_q]) \subset \overline{S_k}$ and $\overline{S_k}$ is invariant, $a(t,s) \in \overline{S_k}$. Then due to the injectivity of P_2 on $\overline{S_k}$,

$$\eta + w_k(\eta) = a(t,s)$$

follows. On the other hand, we have

$$\eta + w_k^e(\eta) = \varphi + c^{-1}(\eta) + b\left(c^{-1}(\eta)\right) \in a\left((-\varepsilon_0, \varepsilon_0) \times (-\varepsilon_0, \varepsilon_0)\right),$$

and $P_2 w_k^e(\eta) = \hat{0}$. By the injectivity of P_2 on $a\left((-\varepsilon_0, \varepsilon_0) \times (-\varepsilon_0, \varepsilon_0)\right)$ and

$$P_2(\eta + w_k(\eta)) = P_2(\eta + w_k^e(\eta)) = \eta,$$

it follows that $w_k(\eta) = w_k^e(\eta)$.

Showing (5.8) in the cases $\varphi \in \mathcal{O}_p$ or $\varphi \in \mathcal{O}_q$ is analogous. □

Proof of Theorem 1.2.(i). We already know from Propositions 5.12, 5.14 and 5.15 that the connecting sets C_q^p and C_k^p are two-dimensional C^1-submanifolds of $\mathcal{W}^u(\mathcal{O}_p)$ with smooth global graph representations, furthermore C_q^p, C_k^p and S_k are homeomorphic to the open annulus $A^{(1,2)}$.

As $J_2 : \mathbb{R}^2 \to G_2$ is an isomorphism and $P_2 = J_2 \circ \pi_2$, Proposition 5.15 shows that $P_2 S_k$ is open in G_2. In addition, Propositions 5.14 and 5.20.(ii) together give that w_k is C^1-smooth on $P_2 S_k = P_2\left(C_k^p \cup \mathcal{O}_p \cup C_q^p\right)$. So the global graph representation

$$S_k = \{\chi + w_k(\chi) : \chi \in P_2 S_k\}$$

given for S_k is smooth. This property with $S_k \subset \mathcal{W}^u(\mathcal{O}_p)$ guarantees that S_k is a two-dimensional C^1-submanifold of $\mathcal{W}^u(\mathcal{O}_p)$ [12]. □

Proof of Theorem 1.2.(ii). Recall that Propositions 5.9 and 5.10 have confirmed the equalities

$$\overline{C_q^p} = \mathcal{O}_p \cup C_q^p \cup \mathcal{O}_q, \qquad \overline{C_k^p} = \mathcal{O}_p \cup C_k^p \cup \mathcal{O}_k$$

and

$$\overline{S_k} = \mathcal{O}_k \cup S_k \cup \mathcal{O}_q.$$

As $J_2 : \mathbb{R}^2 \to G_2$ is an isomorphism and $P_2 = J_2 \circ \pi_2$, Proposition 5.15 yields that $P_2 \overline{S_k}$ is the closure of the open set $P_2 S_k$, and its boundary is $P_2(\mathcal{O}_k \cup \mathcal{O}_q)$. The sets $P_2 \mathcal{O}_k$ and $P_2 \mathcal{O}_q$ are the images of simple closed C^1-curves, hence the boundary is a one-dimensional C^1-submanifold of G_2. By the proof of Theorem 1.2.(i), w_k is continuously differentiable on $P_2 S_k$. Proposition 5.20.(i) in addition verifies that all points of $P_2(\mathcal{O}_k \cup \mathcal{O}_q)$ have open neighborhoods in G_2 on which w_k can be extended to C^1-smooth functions. Summing up, the representation given for $\overline{S_k}$ is a two-dimensional smooth global graph representation with boundary. It is analogous to show that the induced representations of $\overline{C_q^p}$ and $\overline{C_k^p}$ are two-dimensional global graph representations with boundary, therefore we omit the details. It follows immediately that $\overline{C_q^p}$, $\overline{C_k^p}$ and $\overline{S_k}$ are two-dimensional C^1-submanifolds of C with boundary [12].

The assertion that $\overline{C_q^p}$, $\overline{C_k^p}$ and $\overline{S_k}$ are homeomorphic to the closed annulus $A^{[1,2]}$ follows from Proposition 5.15. □

5.6 S_1 *and* S_{-1} *are indeed separatrices*

To complete the proof of Theorem 1.2, it remains to show that S_{-1} and S_1 are separatrices in the sense that C_2^p is above S_1, C_0^p is between S_{-1} and S_1, furthermore C_{-2}^p is below S_{-1}. The underlying idea of the following proof is that the assertion restricted to a local unstable manifold $\mathcal{W}^u_{loc}(P_Y, p_0)$ is true, and the monotonicity of the semiflow can be used to extend the statement for $\mathcal{W}^u(\mathcal{O}_p)$.

Recall that for the periodic orbit $\mathcal{O}_p$, the unstable space C_u is two-dimensional:

$$C_u = \{c_1 v_1 + c_2 v_2 : \, c_1, c_2 \in \mathbb{R}\},$$

where v_1 is a positive eigenfunction corresponding to the leading real Floquet multiplier $\lambda_1 > 1$, and v_2 is an eigenfunction corresponding to the Floquet multiplier λ_2 with $1 < \lambda_2 < \lambda_1$. Also recall that a local unstable manifold $\mathcal{W}^u_{loc}(P_Y, p_0)$ of P_Y at p_0 is a graph of a C^1-map: there exist convex open neighborhoods N_s, N_u of $\hat{0}$ in C_s, C_u, respectively, and a C^1-map $w_u : N_u \to C_s$ with range in N_s so that $w_u(\hat{0}) = \hat{0}$, $Dw_u(\hat{0}) = 0$ and

$$\mathcal{W}^u_{loc}(P_Y, p_0) = \{p_0 + \chi + w_u(\chi) : \, \chi \in N_u\}.$$

Choose $\alpha \in (0,1)$ so small that $(-\alpha,\alpha) v_1 + (-\alpha,\alpha) v_2 \subset N_u$ and

$$\sup_{\chi \in (-\alpha,\alpha)v_1 + (-\alpha,\alpha)v_2} \|Dw_u(\chi)\| < \frac{1}{2}. \tag{5.10}$$

Introduce the sets

$$A_s = \{p_0 + \chi + w_u(\chi) : \, \chi \in (-\alpha,\alpha) v_1 + s v_2\} \subset \mathcal{W}^u_{loc}(P_Y, p_0), \quad s \in (-\alpha,\alpha).$$

The elements of A_s, $s \in (-\alpha,\alpha)$, are ordered pointwise. Indeed, if $s \in (-\alpha,\alpha)$ is fixed and $a, b \in (-\alpha,\alpha)$ are arbitrary with $a < b$, then (5.10) implies that

$$\left[b - a + \int_a^b Dw_u(uv_1 + sv_2)\,\mathrm{d}u \right] v_1 \gg \hat{0},$$

and thus

$$p_0 + (av_1 + sv_2) + w_u(av_1 + sv_2) \ll p_0 + (bv_1 + sv_2) + w_u(bv_1 + sv_2).$$

Introduce the subsets

$$A_s^{k,+} = \left\{ \varphi \in A_s : \, x_t^\varphi \gg \hat{\xi}_k \text{ for some } t \geq 0 \right\}$$

and

$$A_s^{k,-} = \left\{\varphi \in A_s : x_t^{\varphi} \ll \hat{\xi}_k \text{ for some } t \geq 0\right\}$$

of A_s for all $s \in (-\alpha, \alpha)$. Then $A_s^{k,+}$ and $A_s^{k,-}$ are open and disjoint in A_s. It is also clear from the monotonicity of the semiflow combined with the ordering of A_s that for any $\varphi^- \in A_s^{k,-}$ and $\varphi^+ \in A_s^{k,+}$, $\varphi^- \ll \varphi^+$.

We claim that there exists $\beta \in (0, \alpha]$ so that $A_s^{k,+}$ and $A_s^{k,-}$ are nonempty for all $s \in (-\beta, \beta)$. Choose

$$\eta_1 = p_0 - \frac{\alpha}{2}v_1 + w_u\left(-\frac{\alpha}{2}v_1\right) \in A_0 \text{ and } \eta_2 = p_0 + \frac{\alpha}{2}v_1 + w_u\left(\frac{\alpha}{2}v_1\right) \in A_0.$$

Then $\eta_1 \ll p_0 \ll \eta_2$. By Theorem 4.1 in Chapter 5 of [20], there is an open and dense set of initial functions in C so that the corresponding solutions converge to equilibria. In consequence, there exist $\eta_1^+, \eta_1^-, \eta_2^+, \eta_2^- \in C$ such that

$$\eta_1^- \ll \eta_1 \ll \eta_1^+ \ll p_0 \ll \eta_2^- \ll \eta_2 \ll \eta_2^+,$$

and for both $i = 1$ and $i = 2$, $x_t^{\eta_i^-}$ and $x_t^{\eta_i^+}$ converge to one of the equilibrium points as $t \to \infty$. Since $\max_{t\in\mathbb{R}} p(t) > \xi_1$, $\min_{t\in\mathbb{R}} p(t) < \xi_{-1}$ and

$$x_t^{\eta_1^-} \ll x_t^{\eta_1^+} \ll p_t \ll x_t^{\eta_2^-} \ll x_t^{\eta_2^+} \quad \text{for all } t \geq 0$$

by Proposition 2.4, we obtain that

$$x_t^{\eta_1^-} \to \hat{\xi}_{-2},\ x_t^{\eta_1^+} \to \hat{\xi}_{-2},\ x_t^{\eta_2^-} \to \hat{\xi}_2 \text{ and } x_t^{\eta_2^+} \to \hat{\xi}_2 \text{ as } t \to \infty.$$

Using again Proposition 2.4, we get that $x_t^{\eta_1} \to \hat{\xi}_{-2}$ and $x_t^{\eta_2} \to \hat{\xi}_2$ as $t \to \infty$, therefore $x_{t_1}^{\eta_1} \ll \hat{\xi}_k$ and $x_{t_2}^{\eta_2} \gg \hat{\xi}_k$ for some $t_1, t_2 \geq 0$. The continuity of the semiflow Φ implies that there exist open balls B_1, B_2 centered at η_1, η_2, respectively, such that $x_{t_1}^{\varphi} \ll \hat{\xi}_k$ for all $\varphi \in B_1$ and $x_{t_2}^{\varphi} \gg \hat{\xi}_k$ for all $\varphi \in B_2$. It follows that there exists $\beta \in (0, \alpha]$ so that $A_s^{k,+}$ and $A_s^{k,-}$ are nonempty for all $s \in (-\beta, \beta)$.

Summing up, $A_s^{k,+}$ and $A_s^{k,-}$ are open, disjoint and nonempty subsets of the connected A_s for all $s \in (-\beta, \beta)$. Consequently, the set $A_s \setminus (A_s^+ \cup A_s^-)$ is nonempty for all $s \in (-\beta, \beta)$, i.e., A_s has at least one element in S_k. On the other hand, the nonordering property of S_k stated in Proposition 5.2 implies that $A_s \cap S_k$ contains at most one element, i.e., $A_s \cap S_k$ is a singleton for all $s \in (-\beta, \beta)$.

Note that for any $s \in (-\beta, \beta)$, $\varphi^- \in A_s^{k,-}$, $\varphi^+ \in A_s^{k,+}$ and $\psi \in A_s \cap S_k$, $\varphi^- \ll \psi \ll \varphi^+$.

Also observe that if $(\varphi_n)_{-\infty}^0$ is a trajectory of P_Y in $\mathcal{W}_{loc}^u(P_Y, p_0)$ with $\varphi_n \to p_0$ as $n \to -\infty$, then for all indices with sufficiently large absolute value, $\varphi_n \in A_s$ for some $s \in (-\beta, \beta)$.

An element φ of $\mathcal{W}^u(\mathcal{O}_p)$ is said to be above S_k if $\psi \in S_k$ can be given with $\psi \ll \varphi$, and $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ is said to be below S_k if there exists $\psi \in S_k$ with $\varphi \ll \psi$. An element of $\mathcal{W}^u(\mathcal{O}_p)$ is between S_{-1} and S_1 if it is below S_1 and above S_{-1}.

A subset W of $\mathcal{W}^u(\mathcal{O}_p)$ is above (below) S_k, if all elements of W are above (below) S_k. A subset W of $\mathcal{W}^u(\mathcal{O}_p)$ is between S_{-1} and S_1 if it is below S_1 and above S_{-1}.

Proposition 5.21. *For each $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$, exactly one of the following cases holds:*
(i) $\varphi \in S_k$,
(ii) φ is above S_k,
(ii) φ is below S_k.

Proof. It is clear that $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$ cannot be below and above S_k at the same time because then there would exist $\psi_1, \psi_2 \in S_k$ with $\psi_1 \ll \varphi \ll \psi_2$, which would contradict Proposition 5.2. For the same reason, $\varphi \in S_k$ cannot be above (or below) S_k. So at most one of the above cases holds for all $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$.

Let $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus S_k$ be arbitrary. By (3.5) and the characterization of $\mathcal{W}^u_{loc}(P_Y, p_0)$, there exists a sequence $(t_n)_{-\infty}^0$ with $t_n \to -\infty$ as $n \to -\infty$ so that $\{x^\varphi_{t_n}\}_{-\infty}^0$ is a trajectory of P_Y in $\mathcal{W}^u_{loc}(P_Y, p_0)$ and $x^\varphi_{t_n} \to p_0$ as $n \to -\infty$. So an index $n^* \leq 0$ can be given with $t_{n^*} < 0$ such that $x^\varphi_{t_{n^*}} \in A_s$ for some $s \in (-\beta, \beta)$. Let ψ denote the single element of $A_s \cap S_k$. As the elements of A_s are ordered pointwise, we obtain that $x^\varphi_{t_{n^*}} \ll \psi$ or $x^\varphi_{t_{n^*}} \gg \psi$ or $x^\varphi_{t_{n^*}} = \psi$. Observe that $x^\varphi_{t_{n^*}} = \psi$ is impossible: as $\psi \in S_k$ and S_k is invariant, $x^\varphi_{t_{n^*}} = \psi$ would imply that $\varphi = x^\psi_{-t_{n^*}} \in S_k$, which contradicts the choice of φ. If $x^\varphi_{t_{n^*}} \ll \psi$, then the invariance of S_k and the monotonicity of the semiflow imply that $x^\psi_{-t_{n^*}} \in S_k$ and $\varphi \ll x^\psi_{-t_{n^*}}$, that is, φ is below S_k. If $x^\varphi_{t_{n^*}} \gg \psi$, then $\varphi \gg x^\psi_{-t_{n^*}}$ and φ is above S_k. □

Now we are able to complete the proof of Theorem 1.2.

Proof of Theorem 1.2.(iii). 1. First we show that for any $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, $\varphi \in C^p_2$ if and only if φ is above S_1.

Suppose that $\varphi \in C^p_2$. Then $x^\varphi_{t_1} \gg \hat{\xi}_1$ for some $t_1 > 0$. Choose $t_2 > 0$ in addition so that $x^\varphi_{-t_2} \in A_s$ for some $s \in (-\beta, \beta)$. Necessarily $x^\varphi_{-t_2} \in A^{1,+}_s$, and thereby $x^\varphi_{-t_2} \gg \psi$, where ψ is the single element of $A_s \cap S_1$. Then $x^\psi_{t_2} \in S_1$ and $\varphi \gg x^\psi_{t_2}$, that is, φ is above S_1.

Conversely, suppose that $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ is above S_1, and choose $\psi \in S_1$ with $\varphi \gg \psi$. Recall that there is an open and dense set of initial functions in C so that the corresponding solutions are convergent (Theorem 4.1 in Chapter 5 of [20]). Hence $\eta_1 \in C$, $\eta_2 \in C$ and $\eta_3 \in C$ can be given such that

$$\psi \ll \eta_1 \ll \eta_2 \ll \varphi \ll \eta_3,$$

furthermore $x^{\eta_1}_t$, $x^{\eta_2}_t$ and $x^{\eta_3}_t$ converge to equilibria as $t \to \infty$. By the monotonicity of the semiflow,

$$x^\psi_t \ll x^{\eta_1}_t \ll x^{\eta_2}_t \ll x^\varphi_t \ll x^{\eta_3}_t \quad \text{for all } t \geq 0, \tag{5.11}$$

hence the oscillation of x^{ψ} about $\hat{\xi}_1$ implies that $\omega(\eta_i)$ is either $\left\{\hat{\xi}_1\right\}$ or $\left\{\hat{\xi}_2\right\}$ for all $i \in \{1,2,3\}$. If $\omega(\eta_2) = \left\{\hat{\xi}_1\right\}$, then necessarily $\omega(\eta_1) = \omega(\eta_2) = \left\{\hat{\xi}_1\right\}$, which contradicts Proposition 5.1. So $\omega(\eta_2) = \left\{\hat{\xi}_2\right\}$. Then (5.11) guarantees that $x_t^{\eta_3} \to \hat{\xi}_2$ and thus $x_t^{\varphi} \to \hat{\xi}_2$ as $t \to \infty$.

2. It is similar to show that for any $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, $\varphi \in C_{-2}^p$ if and only if φ is below S_{-1}.

3. Relations $S_k = C_k^p \cup \mathcal{O}_p \cup C_q^p$, $k \in \{-1,1\}$, imply the equalities $C_q^p \cup \mathcal{O}_p = S_{-1} \cap S_1$ and $C_k^p = S_k \setminus S_{-k}$ for both $k \in \{-1,1\}$.

4. It remains to verify that for $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$, $\omega(\varphi) = \{\hat{0}\}$ if and only if φ is between S_{-1} and S_1. Recall that for both $k \in \{-1,1\}$ and each $\varphi \in \mathcal{W}^u(\mathcal{O}_p)$, φ is either below S_k, or it is above S_k, or it is an element of S_k. For this reason, $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ is between S_{-1} and S_1 if and only if all the following three properties hold: $\varphi \notin S_{-1} \cup S_1$, φ is not above S_1 and φ is not below S_{-1}. So by the above results, $\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \mathcal{O}_p$ is between S_{-1} and S_1 if and only if

$$\varphi \in \mathcal{W}^u(\mathcal{O}_p) \setminus \left\{\mathcal{O}_p \cup C_{-2}^p \cup C_{-1}^p \cup C_q^p \cup C_1^p \cup C_2^p\right\} = C_0^p.$$

$\square$

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