

Statistical inference for critical continuous state and continuous time branching processes with immigration

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Abstract We study asymptotic behavior of conditional least squares estimators for critical continuous state and continuous time branching processes with immigration based on discrete time (low frequency) observations.

Keywords Branching processes with immigration · Conditional least squares estimator

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1 Introduction

Under some mild moment condition (see (2.3)), a continuous state and continuous time branching process with immigration (CBI process) can be represented as a pathwise unique strong solution of the stochastic differential equation (SDE)

$$\begin{aligned} X_t &= X_0 + \int_0^t (a + BX_s) \, ds + \int_0^t \sqrt{2c \max\{0, X_s\}} \, dW_s \\ &+ \int_0^t \int_0^\infty \int_0^\infty z \mathbb{1}_{\{u \leq X_{s-}\}} \tilde{N}(ds, dz, du) + \int_0^t \int_0^\infty z M(ds, dz) \end{aligned} \tag{1.1}$$

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for $t \in [0, \infty)$, where $a, c \in [0, \infty)$, $B \in \mathbb{R}$, and $(W_t)_{t \geq 0}$ is a standard Wiener process, N and M are Poisson random measures on $(0, \infty)^3$ and on $(0, \infty)^2$ with intensity measures $ds\mu(dz)du$ and $ds\nu(dz)$, respectively, $\tilde{N}(ds, dz, du) := N(ds, dz, du) - ds\mu(dz)du$ is the compensated Poisson random measure corresponding to N , the branching jump measure μ and the immigration jump measure ν satisfy some moment conditions, and $(W_t)_{t \geq 0}$, N and M are independent, see Dawson and Li (Dawson and Li 2006, Theorems 5.1 and 5.2). The model is called subcritical, critical or supercritical if $B < 0$, $B = 0$ or $B > 0$, see Huang et al. (Huang et al 2011, page 1105). Based on discrete time (low frequency) observations $(X_k)_{k \in \{0, 1, \dots, n\}}$, $n \in \{1, 2, \dots\}$, Huang et al. (Huang et al 2011) derived weighted conditional least squares (CLS) estimator of (B, a) . Under some additional moment conditions, they showed the following results: in the subcritical case the estimator of (B, a) is asymptotically normal; in the critical case the estimator of B has a non-normal limit, but the asymptotic behavior of the estimator of a remained open; in the supercritical case the estimator of B is asymptotically normal with a random scaling, but the estimator of a is not weakly consistent.

Overbeck and Rydén (Overbeck and Rydén 1997) considered CLS and weighted CLS estimators for the well-known Cox–Ingersoll–Ross model, which is, in fact, a diffusion CBI process (without jump part), i.e., when $\mu = 0$ and $\nu = 0$ in (1.1). Based on discrete time observations $(X_k)_{k \in \{0, 1, \dots, n\}}$, $n \in \{1, 2, \dots\}$, they derived CLS estimator of (B, a, c) and proved its asymptotic normality in the subcritical case. Note that Li and Ma (Li and Ma 2015) started to investigate the asymptotic behaviour of the CLS and weighted CLS estimators of the parameters (B, a) in the subcritical case for a Cox–Ingersoll–Ross model driven by a stable noise, which is again a special CBI process (with jump part).

For simplicity, we suppose $X_0 = 0$. We suppose that c , μ and ν are known, and we derive the CLS estimator of (B, A) based on discrete time (low frequency) observations $(X_k)_{k \in \{1, \dots, n\}}$, $n \in \{1, 2, \dots\}$, where $A := a + \int_0^\infty z \nu(dz)$. In the critical case, i.e., when $B = 0$, under some moment conditions, we describe the asymptotic behavior of these CLS estimators as $n \rightarrow \infty$, provided that $a \neq 0$ or $\nu \neq 0$, see Theorem 3.1. We point out that the limit distributions are non-normal in general. In the present paper we do not investigate the asymptotic behavior of CLS estimators of (B, A) in the subcritical and supercritical cases, it could be the topic of separate papers.

2 CBI processes

Let \mathbb{Z}_+ , \mathbb{N} , \mathbb{R} , \mathbb{R}_+ and \mathbb{R}_{++} denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min\{x, y\}$ and $x^+ := \max\{0, x\}$. By $\|\mathbf{x}\|$ and $\|\mathbf{A}\|$, we denote the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^d$ and the induced matrix norm of a matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$, respectively. The null vector and the null matrix will be denoted by $\mathbf{0}$. By

$C_c^2(\mathbb{R}_+, \mathbb{R})$ we denote the set of twice continuously differentiable real-valued functions on \mathbb{R}_+ with compact support. Convergence in distribution and in probability will be denoted by $\xrightarrow{\mathcal{D}}$ and $\xrightarrow{\mathbb{P}}$, respectively.

Definition 2.1 A tuple (c, a, b, ν, μ) is called a set of admissible parameters if $c, a \in \mathbb{R}_+$, $b \in \mathbb{R}$, and ν and μ are Borel measures on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge z) \nu(dz) < \infty$ and $\int_0^\infty (z \wedge z^2) \mu(dz) < \infty$. \square

Theorem 2.2 Let (c, a, b, ν, μ) be a set of admissible parameters. Then there exists a unique conservative transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ acting on the Banach space (endowed with the supremum norm) of real-valued bounded Borel-measurable functions on the state space \mathbb{R}_+ such that its infinitesimal generator is

$$\begin{aligned} (\mathcal{G}f)(x) = & cx f''(x) + (a + bx) f'(x) + \int_0^\infty (f(x+z) - f(x)) \nu(dz) \\ & + x \int_0^\infty (f(x+z) - f(x) - f'(x)(1 \wedge z)) \mu(dz) \end{aligned} \quad (2.1)$$

for $f \in C_c^2(\mathbb{R}_+, \mathbb{R})$ and $x \in \mathbb{R}_+$. Moreover, the Laplace transform of the transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ has a representation

$$\int_0^\infty e^{-\lambda y} P_t(x, dy) = e^{-xv(t, \lambda) - \int_0^t \psi(v(s, \lambda)) ds}, \quad x \in \mathbb{R}_+, \quad \lambda \in \mathbb{R}_+, \quad t \in \mathbb{R}_+,$$

where, for any $\lambda \in \mathbb{R}_+$, the continuously differentiable function $\mathbb{R}_+ \ni t \mapsto v(t, \lambda) \in \mathbb{R}_+$ is the unique locally bounded solution to the differential equation

$$\partial_t v(t, \lambda) = -\varphi(v(t, \lambda)), \quad v(0, \lambda) = \lambda, \quad (2.2)$$

with

$$\varphi(\lambda) := c\lambda^2 - b\lambda + \int_0^\infty (e^{-\lambda z} - 1 + \lambda(1 \wedge z)) \mu(dz), \quad \lambda \in \mathbb{R}_+,$$

and

$$\psi(\lambda) := a\lambda + \int_0^\infty (1 - e^{-\lambda z}) \nu(dz), \quad \lambda \in \mathbb{R}_+.$$

Remark 2.3 This theorem is a special case of Theorem 2.7 of Duffie et al. (2003) with $m = 1$, $n = 0$ and zero killing rate. The unique existence of a locally bounded solution to the differential equation (2.2) is proved by Li (Li 2011, page 45). Here, we point out that the moment condition on μ given in Definition 2.1 (which is stronger than the one (2.11) in Definition 2.6 in Duffie et al. (2003)) ensures that the semigroup $(P_t)_{t \in \mathbb{R}_+}$ is conservative (we do not need the one-point compactification of \mathbb{R}_+^d), see Duffie et al. (Duffie et al 2003, Lemma 9.2) and Li (Li 2011, page 45). For the continuity of the function $\mathbb{R}_+ \times \mathbb{R}_+ \ni (t, \lambda) \mapsto v(t, \lambda)$, see Duffie et al. (Duffie

et al 2003, Proposition 6.4). Finally, we note that the infinitesimal generator (2.1) can be rewritten in another equivalent form

$$\begin{aligned} (\mathcal{G}f)(x) &= cx f''(x) + \left(a + \left(b + \int_1^\infty (z-1) \mu(dz) \right) x \right) f'(x) \\ &+ \int_0^\infty (f(x+z) - f(x)) \nu(dz) + x \int_0^\infty (f(x+z) - f(x) - zf'(x)) \mu(dz), \end{aligned}$$

where $b + \int_1^\infty (z-1) \mu(dz)$ is nothing else but B given in (2.5). \square

Definition 2.4 A conservative Markov process with state space \mathbb{R}_+ and with transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ given in Theorem 2.2 is called a CBI process with parameters (c, a, b, ν, μ) . The function $\mathbb{R}_+ \ni \lambda \mapsto \varphi(\lambda) \in \mathbb{R}$ is called its branching mechanism, and the function $\mathbb{R}_+ \ni \lambda \mapsto \psi(\lambda) \in \mathbb{R}_+$ is called its immigration mechanism. \square

Note that the branching mechanism depends only on the parameters c, b and μ , while the immigration mechanism depends only on the parameters a and ν .

Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $\mathbb{E}(X_0) < \infty$ and the moment condition

$$\int_1^\infty z \nu(dz) < \infty \quad (2.3)$$

holds. Then, by formula (3.4) in Barczy et al. Barczy et al (2015),

$$\mathbb{E}(X_t | X_0 = x) = e^{Bt} x + A \int_0^t e^{Bu} du, \quad x \in \mathbb{R}_+, \quad t \in \mathbb{R}_+, \quad (2.4)$$

where

$$B := b + \int_1^\infty (z-1) \mu(dz), \quad A := a + \int_0^\infty z \nu(dz). \quad (2.5)$$

Note that $B \in \mathbb{R}$ and $A \in \mathbb{R}_+$ due to (2.3). One can give probabilistic interpretations of the modified parameters B and A , namely, $e^B = \mathbb{E}(Y_1 | Y_0 = 1)$ and $A = \mathbb{E}(Z_1 | Z_0 = 0)$, where $(Y_t)_{t \in \mathbb{R}_+}$ and $(Z_t)_{t \in \mathbb{R}_+}$ are CBI processes with parameters $(c, 0, b, 0, \mu)$ and $(0, a, 0, \nu, 0)$, respectively, see formula (2.4). The processes $(Y_t)_{t \in \mathbb{R}_+}$ and $(Z_t)_{t \in \mathbb{R}_+}$ can be considered as *pure branching* (without immigration) and *pure immigration* (without branching) processes, respectively. Consequently, e^B and A may be called the branching and immigration mean, respectively. Moreover, by the help of the modified parameters B and A , the SDE (1.1) can be rewritten as

$$\begin{aligned} X_t &= X_0 + \int_0^t (A + BX_s) ds + \int_0^t \sqrt{2cX_s^+} dW_s \\ &+ \int_0^t \int_0^\infty \int_0^\infty z \mathbb{1}_{\{u \leq X_{s-}\}} \tilde{N}(ds, dz, du) + \int_0^t \int_0^\infty z \tilde{M}(ds, dz) \end{aligned} \quad (2.6)$$

for $t \in [0, \infty)$, where $\widetilde{M}(ds, dz) := M(ds, dz) - ds \mu(dz)$.

Next we will recall a convergence result for critical CBI processes.

A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is called *càdlàg* if it is right continuous with left limits. Let $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$ denote the space of all \mathbb{R} -valued càdlàg and continuous functions on \mathbb{R}_+ , respectively. Let $\mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R})$ denote the Borel σ -field in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ for the metric characterized by Jacod and Shiryaev (Jacod and Shiryaev 2003, VI.1.15) (with this metric $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is a complete and separable metric space). For \mathbb{R} -valued stochastic processes $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ and $(\mathcal{Y}_t^{(n)})_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, with càdlàg paths we write $\mathcal{Y}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{Y}$ as $n \rightarrow \infty$ if the distribution of $\mathcal{Y}^{(n)}$ on the space $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}))$ converges weakly to the distribution of \mathcal{Y} on the space $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}))$ as $n \rightarrow \infty$. Concerning the notation $\xrightarrow{\mathcal{D}}$ we note that if ξ and ξ_n , $n \in \mathbb{N}$, are random elements with values in a metric space (E, ρ) , then we also denote by $\xi_n \xrightarrow{\mathcal{D}} \xi$ the weak convergence of the distributions of ξ_n on the space $(E, \mathcal{B}(E))$ towards the distribution of ξ on the space $(E, \mathcal{B}(E))$ as $n \rightarrow \infty$, where $\mathcal{B}(E)$ denotes the Borel σ -algebra on E induced by the given metric ρ .

The following convergence theorem can be found in Huang et al. (Huang et al 2011, Theorem 2.3).

Theorem 2.5 *Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $X_0 = 0$, the moment conditions*

$$\int_1^\infty z^q \nu(dz) < \infty, \quad \int_1^\infty z^q \mu(dz) < \infty \quad (2.7)$$

hold with $q = 2$, and $B = 0$ (hence the process is critical). Then

$$(\mathcal{X}_t^{(n)})_{t \in \mathbb{R}_+} := (n^{-1} X_{\lfloor nt \rfloor})_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}} (\mathcal{Y}_t)_{t \in \mathbb{R}_+} \quad \text{as } n \rightarrow \infty \quad (2.8)$$

in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, where $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$d\mathcal{Y}_t = A dt + \sqrt{C\mathcal{Y}_t^+} d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad \mathcal{Y}_0 = 0, \quad (2.9)$$

where $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and

$$C := 2c + \int_0^\infty z^2 \mu(dz) \in \mathbb{R}_+. \quad (2.10)$$

Remark 2.6 The SDE (2.9) has a pathwise unique strong solution $(\mathcal{Y}_t^{(y)})_{t \in \mathbb{R}_+}$ for all initial values $\mathcal{Y}_0^{(y)} = y \in \mathbb{R}$, and if the initial value y is nonnegative, then $\mathcal{Y}_t^{(y)}$ is nonnegative for all $t \in \mathbb{R}_+$ with probability one, since $A \in \mathbb{R}_+$, see, e.g., Ikeda and Watanabe (Ikeda and Watanabe 1989, Chapter IV, Example 8.2). \square

Remark 2.7 Note that $C = 0$ if and only if $c = 0$ and $\mu = 0$, when the pathwise unique strong solution of (2.9) is the deterministic function $\mathcal{Y}_t = At$, $t \in \mathbb{R}_+$. Further, $C = \text{Var}(Y_1 | Y_0 = 1)$, see Proposition B.3, where $(Y_t)_{t \in \mathbb{R}_+}$ is a pure branching CBI process with parameters $(c, 0, b, 0, \mu)$. Clearly, C depends only on the branching mechanism. \square

3 Main results

Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that the moment condition (2.3) holds. For the sake of simplicity, we suppose $X_0 = 0$. In the sequel we also assume that $a \neq 0$ or $\nu \neq 0$ (i.e., the immigration mechanism is non-zero), equivalently, $A \neq 0$ (where A is defined in (2.5)), otherwise $X_t = 0$ for all $t \in \mathbb{R}_+$, following from (2.4). The parameter B can also be called the *criticality parameter*, since $(X_t)_{t \in \mathbb{R}_+}$ is critical if and only if $B = 0$.

For $k \in \mathbb{Z}_+$, let $\mathcal{F}_k := \sigma(X_0, X_1, \dots, X_k)$. Since $(X_k)_{k \in \mathbb{Z}_+}$ is a time-homogeneous Markov process, by (2.4),

$$\mathbb{E}(X_k | \mathcal{F}_{k-1}) = \mathbb{E}(X_k | X_{k-1}) = \varrho X_{k-1} + \mathcal{A}, \quad k \in \mathbb{N}, \quad (3.1)$$

where

$$\varrho := e^B \in \mathbb{R}_{++}, \quad \mathcal{A} := A \int_0^1 e^{Bs} ds \in \mathbb{R}_+. \quad (3.2)$$

Note that $\mathcal{A} = \mathbb{E}(X_1 | X_0 = 0)$, see (2.4). Note also that \mathcal{A} depends both on the branching and immigration mechanisms, although A depends only on the immigration mechanism. Let us introduce the sequence

$$M_k := X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1}) = X_k - \varrho X_{k-1} - \mathcal{A}, \quad k \in \mathbb{N}, \quad (3.3)$$

of martingale differences with respect to the filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}_+}$. By (3.3), the process $(X_k)_{k \in \mathbb{Z}_+}$ satisfies the recursion

$$X_k = \varrho X_{k-1} + \mathcal{A} + M_k, \quad k \in \mathbb{N}. \quad (3.4)$$

For each $n \in \mathbb{N}$, a CLS estimator $(\widehat{\varrho}_n, \widehat{\mathcal{A}}_n)$ of (ϱ, \mathcal{A}) based on a sample X_1, \dots, X_n can be obtained by minimizing the sum of squares

$$\sum_{k=1}^n (X_k - \varrho X_{k-1} - \mathcal{A})^2$$

with respect to (ϱ, \mathcal{A}) over \mathbb{R}^2 , and it has the form

$$\begin{aligned}\widehat{\varrho}_n &:= \frac{n \sum_{k=1}^n X_k X_{k-1} - \sum_{k=1}^n X_k \sum_{k=1}^n X_{k-1}}{n \sum_{k=1}^n X_{k-1}^2 - \left(\sum_{k=1}^n X_{k-1} \right)^2} \\ \widehat{\mathcal{A}}_n &:= \frac{\sum_{k=1}^n X_k \sum_{k=1}^n X_{k-1}^2 - \sum_{k=1}^n X_k X_{k-1} \sum_{k=1}^n X_{k-1}}{n \sum_{k=1}^n X_{k-1}^2 - \left(\sum_{k=1}^n X_{k-1} \right)^2}\end{aligned}\tag{3.5}$$

on the set

$$H_n := \left\{ \omega \in \Omega : n \sum_{k=1}^n X_{k-1}^2(\omega) - \left(\sum_{k=1}^n X_{k-1}(\omega) \right)^2 > 0 \right\},$$

see, e.g., Wei and Winnicki (Wei and Winnicki 1989, formulas (1.4), (1.5)). In the sequel we investigate the critical case. By Lemma C.1, $\mathbb{P}(H_n) \rightarrow 1$ as $n \rightarrow \infty$. Let us introduce the function $h : \mathbb{R}^2 \rightarrow \mathbb{R}_{++} \times \mathbb{R}$ by

$$h(B, A) := \left(e^B, A \int_0^1 e^{Bs} ds \right) = (\varrho, \mathcal{A}), \quad (B, A) \in \mathbb{R}^2.$$

Note that h is bijective having inverse

$$h^{-1}(\varrho, \mathcal{A}) = \left(\log(\varrho), \frac{\mathcal{A}}{\int_0^1 \varrho^s ds} \right) = (B, A), \quad (\varrho, \mathcal{A}) \in \mathbb{R}_{++} \times \mathbb{R}.$$

Theorem 3.4 will imply that the CLS estimator $\widehat{\varrho}_n$ of ϱ is weakly consistent, hence, for sufficiently large $n \in \mathbb{N}$ with probability converging to 1, $(\widehat{\varrho}_n, \widehat{\mathcal{A}}_n)$ falls into the set $\mathbb{R}_{++} \times \mathbb{R}$, and hence

$$(\widehat{\varrho}_n, \widehat{\mathcal{A}}_n) = \arg \min_{(\varrho, \mathcal{A}) \in \mathbb{R}_{++} \times \mathbb{R}} \sum_{k=1}^n (X_k - \varrho X_{k-1} - \mathcal{A})^2.$$

Thus one can introduce a natural estimator of (B, A) by applying the inverse of h to the CLS estimator of (ϱ, \mathcal{A}) , that is,

$$(\widehat{B}_n, \widehat{A}_n) := h^{-1}(\widehat{\varrho}_n, \widehat{\mathcal{A}}_n) = \left(\log(\widehat{\varrho}_n), \frac{\widehat{\mathcal{A}}_n}{\int_0^1 (\widehat{\varrho}_n)^s ds} \right), \quad n \in \mathbb{N},$$

on the set $\{\omega \in \Omega : (\widehat{\varrho}_n(\omega), \widehat{\mathcal{A}}_n(\omega)) \in \mathbb{R}_{++} \times \mathbb{R}\}$. We also obtain

$$(\widehat{B}_n, \widehat{A}_n) = \arg \min_{(B, A) \in \mathbb{R}^2} \sum_{k=1}^n \left(X_k - e^B X_{k-1} - A \int_0^1 e^{Bs} ds \right)^2 \tag{3.6}$$

for sufficiently large $n \in \mathbb{N}$ with probability converging to 1, hence (\hat{B}_n, \hat{A}_n) is the CLS estimator of (B, A) for sufficiently large $n \in \mathbb{N}$ with probability converging to 1. We would like to stress the point that the estimator (\hat{B}_n, \hat{A}_n) exists only for sufficiently large $n \in \mathbb{N}$ with probability converging to 1. However, as all our results are asymptotic, this will not cause a problem.

Theorem 3.1 *Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $X_0 = 0$, the moment conditions (2.7) hold with $q = 8$, $a \neq 0$ or $\nu \neq 0$, and $B = 0$ (hence the process is critical). Then the probability of the existence of the estimator (\hat{B}_n, \hat{A}_n) converges to 1 as $n \rightarrow \infty$ and*

$$\begin{bmatrix} n(\hat{B}_n - B) \\ \hat{A}_n - A \end{bmatrix} \xrightarrow{\mathcal{D}} \frac{1}{\int_0^1 \mathcal{Y}_t^2 dt - (\int_0^1 \mathcal{Y}_t dt)^2} \begin{bmatrix} \int_0^1 \mathcal{Y}_t d\mathcal{M}_t - \mathcal{M}_1 \int_0^1 \mathcal{Y}_t dt \\ \mathcal{M}_1 \int_0^1 \mathcal{Y}_t^2 dt - \int_0^1 \mathcal{Y}_t dt \int_0^1 \mathcal{Y}_t d\mathcal{M}_t \end{bmatrix} \quad (3.7)$$

as $n \rightarrow \infty$, where $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE (2.9), and $\mathcal{M}_t := \mathcal{Y}_t - At$, $t \in \mathbb{R}_+$.

If, in addition, $c = 0$ and $\mu = 0$ (hence the process is a pure immigration process), then

$$\begin{bmatrix} n^{3/2}(\hat{B}_n - B) \\ n^{1/2}(\hat{A}_n - A) \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2 \left(\mathbf{0}, \int_0^\infty z^2 \nu(dz) \begin{bmatrix} \frac{A^2}{3} & \frac{A}{2} \\ \frac{A}{2} & 1 \end{bmatrix}^{-1} \right) \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

Remark 3.2 By Remark 2.7, if $C = 0$, then $\mathcal{M}_t = 0$, $t \in \mathbb{R}_+$, further, by (3.7), $n(\hat{B}_n - B) \xrightarrow{\mathcal{D}} 0$ and $\hat{A}_n - A \xrightarrow{\mathcal{D}} 0$ as $n \rightarrow \infty$. \square

Remark 3.3 If $C \neq 0$ then the estimator \hat{A}_n is not consistent. The same holds for the discrete time analogues of A , for instance, the immigration mean of a critical Galton–Watson branching process with immigration, see Wei and Winnicki Wei and Winnicki (1990), or the innovation mean of a positive regular unstable INAR(2) process, see Barczy et al. Barczy et al (2014). \square

Theorem 3.1 will follow from the following statement.

Theorem 3.4 *Under the assumptions of Theorem 3.1, the probability of the existence of unique CLS estimator $(\hat{\varrho}_n, \hat{\mathcal{A}}_n)$ converges to 1 as $n \rightarrow \infty$ and*

$$\begin{bmatrix} n(\hat{\varrho}_n - \varrho) \\ \hat{\mathcal{A}}_n - \mathcal{A} \end{bmatrix} \xrightarrow{\mathcal{D}} \frac{1}{\int_0^1 \mathcal{Y}_t^2 dt - (\int_0^1 \mathcal{Y}_t dt)^2} \begin{bmatrix} \int_0^1 \mathcal{Y}_t d\mathcal{M}_t - \mathcal{M}_1 \int_0^1 \mathcal{Y}_t dt \\ \mathcal{M}_1 \int_0^1 \mathcal{Y}_t^2 dt - \int_0^1 \mathcal{Y}_t dt \int_0^1 \mathcal{Y}_t d\mathcal{M}_t \end{bmatrix} \quad (3.9)$$

as $n \rightarrow \infty$.

If, in addition, $c = 0$ and $\mu = 0$ (hence the process is a pure immigration process), then

$$\begin{bmatrix} n^{3/2}(\hat{\varrho}_n - \varrho) \\ n^{1/2}(\hat{\mathcal{A}}_n - \mathcal{A}) \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2 \left(\mathbf{0}, \int_0^\infty z^2 \nu(dz) \begin{bmatrix} \frac{A^2}{3} & \frac{A}{2} \\ \frac{A}{2} & 1 \end{bmatrix}^{-1} \right) \quad \text{as } n \rightarrow \infty. \quad (3.10)$$

Proof of Theorem 3.1 Before Theorem 3.1 we have already investigated the existence of $(\widehat{B}_n, \widehat{A}_n)$. Now we apply Lemma D.1 with $S = T = \mathbb{R}^2$, $C = \mathbb{R}^2$,

$$\xi_n = \begin{bmatrix} n(\widehat{\varrho}_n - \varrho) \\ \widehat{\mathcal{A}}_n - \mathcal{A} \end{bmatrix} = \begin{bmatrix} n(\widehat{\varrho}_n - 1) \\ \widehat{\mathcal{A}}_n - A \end{bmatrix},$$

$$\xi = \frac{1}{\int_0^1 \mathcal{Y}_t^2 dt - (\int_0^1 \mathcal{Y}_t dt)^2} \begin{bmatrix} \int_0^1 \mathcal{Y}_t d\mathcal{M}_t - \mathcal{M}_1 \int_0^1 \mathcal{Y}_t dt \\ \mathcal{M}_1 \int_0^1 \mathcal{Y}_t^2 dt - \int_0^1 \mathcal{Y}_t dt \int_0^1 \mathcal{Y}_t d\mathcal{M}_t \end{bmatrix},$$

with functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $f_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $n \in \mathbb{N}$, given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} x \\ y \end{bmatrix}, \quad (x, y) \in \mathbb{R}^2, \quad f_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} n \log(1 + \frac{x}{n}) \\ \frac{y + A}{\int_0^1 (1 + \frac{x}{n})^s ds} - A \end{bmatrix}$$

for $(x, y) \in \mathbb{R}^2$ with $x > -n$, and $f_n(x, y) := 0$ otherwise. We have $f_n(n(\widehat{\varrho}_n - 1), \widehat{\mathcal{A}}_n - A) = (n(\widehat{B}_n - B), \widehat{\mathcal{A}}_n - A)$ on the set $\{\omega \in \Omega : \widehat{\varrho}_n(\omega) \in \mathbb{R}_{++}\}$, and $f_n(x_n, y_n) \rightarrow f(x, y)$ as $n \rightarrow \infty$ if $(x_n, y_n) \rightarrow (x, y)$ as $n \rightarrow \infty$, since

$$\lim_{n \rightarrow \infty} \log\left(1 + \frac{x_n}{n}\right)^n = \log(e^x) = x,$$

and $\lim_{n \rightarrow \infty} \int_0^1 (1 + \frac{x_n}{n})^s ds = 1$, if $x_n \rightarrow x$ as $n \rightarrow \infty$, since the function $\mathbb{R}_{++} \ni u \mapsto \int_0^1 u^s ds \in \mathbb{R}$ is continuous. Consequently, (3.9) implies (3.7).

Next we apply Lemma D.1 with $S = T = \mathbb{R}^2$, $C = \mathbb{R}^2$,

$$\xi_n = \begin{bmatrix} n^{3/2}(\widehat{\varrho}_n - \varrho) \\ n^{1/2}(\widehat{\mathcal{A}}_n - \mathcal{A}) \end{bmatrix}, \quad \xi \stackrel{\mathcal{D}}{=} \mathcal{N}_2 \left(\mathbf{0}, \int_0^\infty z^2 \nu(dz) \begin{bmatrix} \frac{A^2}{3} & \frac{A}{2} \\ \frac{A}{2} & 1 \end{bmatrix}^{-1} \right),$$

with functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $f_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $n \in \mathbb{N}$, given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} x \\ y \end{bmatrix}, \quad (x, y) \in \mathbb{R}^2,$$

$$f_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{cases} \begin{bmatrix} n^{3/2} \log(1 + \frac{x}{n^{3/2}}) \\ n^{1/2} \left(\frac{n^{-1/2}y + A}{\int_0^1 (1 + \frac{x}{n^{3/2}})^s ds} - A \right) \end{bmatrix}, & (x, y) \in \mathbb{R}^2, \quad x > -n^{3/2}, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

We have again $f_n(x_n, y_n) \rightarrow f(x, y)$ as $n \rightarrow \infty$ if $(x_n, y_n) \rightarrow (x, y)$ as $n \rightarrow \infty$. Indeed,

$$n^{1/2} \left(\frac{n^{-1/2}y_n + A}{\int_0^1 (1 + \frac{x_n}{n^{3/2}})^s ds} - A \right) = \frac{y_n}{\int_0^1 (1 + \frac{x_n}{n^{3/2}})^s ds} + \frac{An^{1/2} \left(1 - \int_0^1 (1 + \frac{x_n}{n^{3/2}})^s ds \right)}{\int_0^1 (1 + \frac{x_n}{n^{3/2}})^s ds}$$

if $x_n > -n^{3/2}$. Moreover,

$$\begin{aligned} & \left| n^{1/2} \left(1 - \int_0^1 \left(1 + \frac{x_n}{n^{3/2}} \right)^s ds \right) - n^{1/2} \left(1 - \int_0^1 \left(1 + \frac{x}{n^{3/2}} \right)^s ds \right) \right| \\ &= n^{1/2} \left| \frac{x_n - x}{n^{3/2}} \int_0^1 s \left(1 + \frac{\theta_n}{n^{3/2}} \right)^{s-1} ds \right| \leq K \frac{|x_n - x|}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

with θ_n (depending on x_n and x) lying between x_n and x , and with some appropriate $K > 0$. Further, by L'Hospital's rule,

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{1/2} \left(1 - \int_0^1 \left(1 + \frac{x}{n^{3/2}} \right)^s ds \right) &= \lim_{h \rightarrow 0} \frac{1 - \int_0^1 (1 + h^3 x)^s ds}{h} \\ &= - \lim_{h \rightarrow 0} 3h^2 x \int_0^1 s (1 + h^3 x)^{s-1} ds = 0. \end{aligned}$$

Consequently, (3.10) implies (3.8). \square

Theorem 3.4 will follow from the following statements by the continuous mapping theorem and by Slutsky's lemma, see below.

Theorem 3.5 *Under the assumptions of Theorem 3.1, we have*

$$\sum_{k=1}^n \begin{bmatrix} n^{-2} X_{k-1} \\ n^{-3} X_{k-1}^2 \\ n^{-1} M_k \\ n^{-2} M_k X_{k-1} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \int_0^1 \mathcal{Y}_t dt \\ \int_0^1 \mathcal{Y}_t^2 dt \\ \mathcal{M}_1 \\ \int_0^1 \mathcal{Y}_t d\mathcal{M}_t \end{bmatrix} \quad \text{as } n \rightarrow \infty. \quad (3.11)$$

In case of $C = 0$ the third and fourth coordinates of the limit vector is 0 in Theorem 3.5, since $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is the deterministic function $\mathcal{Y}_t = At$, $t \in \mathbb{R}_+$ (see Remark 2.7), hence other scaling factors should be chosen for these coordinates, as given in the following theorem.

Theorem 3.6 *Suppose that the assumptions of Theorem 3.1 hold. If $C = 0$, then*

$$\begin{aligned} n^{-2} \sum_{k=1}^n X_{k-1} &\xrightarrow{\mathbb{P}} \frac{A}{2} \quad \text{as } n \rightarrow \infty, \\ n^{-3} \sum_{k=1}^n X_{k-1}^2 &\xrightarrow{\mathbb{P}} \frac{A^2}{3} \quad \text{as } n \rightarrow \infty, \\ \sum_{k=1}^n \begin{bmatrix} n^{-1/2} M_k \\ n^{-3/2} M_k X_{k-1} \end{bmatrix} &\xrightarrow{\mathcal{D}} \mathcal{N}_2 \left(\mathbf{0}, \int_0^\infty z^2 \nu(dz) \begin{bmatrix} 1 & \frac{A}{2} \\ \frac{A}{2} & \frac{A^2}{3} \end{bmatrix} \right) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Proof of Theorem 3.4 The statements about the existence of unique CLS estimators $(\hat{\varrho}_n, \hat{\mathcal{A}}_n)$ under the given conditions follow from Lemma C.1.

In order to derive (3.9) from Theorem 3.5, we can use the continuous mapping theorem. Indeed,

$$\begin{bmatrix} \hat{\varrho}_n - \varrho \\ \hat{\mathcal{A}}_n - \mathcal{A} \end{bmatrix} = \frac{1}{n \sum_{k=1}^n X_{k-1}^2 - \left(\sum_{k=1}^n X_{k-1} \right)^2} \begin{bmatrix} n \sum_{k=1}^n M_k X_{k-1} - \sum_{k=1}^n M_k \sum_{k=1}^n X_{k-1} \\ \sum_{k=1}^n M_k \sum_{k=1}^n X_{k-1}^2 - \sum_{k=1}^n M_k X_{k-1} \sum_{k=1}^n X_{k-1} \end{bmatrix}$$

on the set H_n . Moreover, since $A \neq 0$, by the SDE (2.9), we have $\mathbb{P}(\mathcal{Y}_t = 0, t \in [0, 1]) = 0$, which implies $\mathbb{P}(\int_0^1 \mathcal{Y}_t^2 dt > 0) = 1$. By Remark 2.6, $\mathbb{P}(\mathcal{Y}_t \geq 0, t \in \mathbb{R}_+) = 1$, and hence $\mathbb{P}(\int_0^1 \mathcal{Y}_t dt > 0) = 1$. Next we show $\mathbb{P}(\int_0^1 \mathcal{Y}_t^2 dt - (\int_0^1 \mathcal{Y}_t dt)^2 > 0) = 1$. We have $\int_0^1 \mathcal{Y}_t^2 dt - (\int_0^1 \mathcal{Y}_t dt)^2 = \int_0^1 (\mathcal{Y}_t - \int_0^1 \mathcal{Y}_s ds)^2 dt \geq 0$, and equality holds if and only if $\mathcal{Y}_t = \int_0^1 \mathcal{Y}_s ds$ for almost every $t \in [0, 1]$. Since \mathcal{Y} has continuous sample paths almost surely, $\mathbb{P}(\int_0^1 \mathcal{Y}_t^2 dt - (\int_0^1 \mathcal{Y}_t dt)^2 = 0) > 0$ holds if and only if $\mathbb{P}(\mathcal{Y}_t = \int_0^1 \mathcal{Y}_s ds, \forall t \in [0, 1]) > 0$. Hence, since $\mathcal{Y}_0 = 0$, this holds if and only if $P(\mathcal{Y}_t = 0, \forall t \in [0, 1]) > 0$, which is a contradiction due to our assumption $A \in \mathbb{R}_{++}$. Indeed, with the notations of the proof of Theorem 3.1 in Barczy et al. Barczy et al (2013), $\{\omega \in \Omega : Y_t(\omega) = 0, \forall t \in [0, 1]\} = \tilde{A}_1 \cap A_1 = \emptyset$. Consequently,

$$\begin{bmatrix} n(\hat{\varrho}_n - \varrho) \\ \hat{\mathcal{A}}_n - \mathcal{A} \end{bmatrix} \xrightarrow{\mathcal{D}} \frac{1}{\int_0^1 \mathcal{Y}_t^2 dt - (\int_0^1 \mathcal{Y}_t dt)^2} \begin{bmatrix} \int_0^1 \mathcal{Y}_t d\mathcal{M}_t - \mathcal{M}_1 \int_0^1 \mathcal{Y}_t dt \\ \mathcal{M}_1 \int_0^1 \mathcal{Y}_t^2 dt - \int_0^1 \mathcal{Y}_t dt \int_0^1 \mathcal{Y}_t d\mathcal{M}_t \end{bmatrix}$$

as $n \rightarrow \infty$, and we obtain (3.9).

If, in addition, $c = 0$ and $\mu = 0$, then we derive (3.10) from Theorem 3.6 applying the continuous mapping theorem and Slutsky's lemma. We have

$$\frac{1}{n^3} \sum_{k=1}^n X_{k-1}^2 - \left(\frac{1}{n^2} \sum_{k=1}^n X_{k-1} \right)^2 \xrightarrow{\mathbb{P}} \frac{A^2}{3} - \left(\frac{A}{2} \right)^2 = \frac{A^2}{12} \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$\begin{aligned} & n^{-4} \begin{bmatrix} n \sum_{k=1}^n M_k X_{k-1} - \sum_{k=1}^n M_k \sum_{k=1}^n X_{k-1} \\ \sum_{k=1}^n M_k \sum_{k=1}^n X_{k-1}^2 - \sum_{k=1}^n M_k X_{k-1} \sum_{k=1}^n X_{k-1} \end{bmatrix} \\ &= n^{-4} \begin{bmatrix} -n^{1/2} \sum_{k=1}^n X_{k-1} & n^{5/2} \\ n^{1/2} \sum_{k=1}^n X_{k-1}^2 & -n^{3/2} \sum_{k=1}^n X_{k-1} \end{bmatrix} \begin{bmatrix} n^{-1/2} \sum_{k=1}^n M_k \\ n^{-3/2} \sum_{k=1}^n M_k X_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} n^{-3/2} & 0 \\ 0 & n^{-1/2} \end{bmatrix} \begin{bmatrix} -\frac{\sum_{k=1}^n X_{k-1}}{n^2} & 1 \\ \frac{\sum_{k=1}^n X_{k-1}^2}{n^3} & -\frac{\sum_{k=1}^n X_{k-1}}{n^2} \end{bmatrix} \begin{bmatrix} n^{-1/2} \sum_{k=1}^n M_k \\ n^{-3/2} \sum_{k=1}^n M_k X_{k-1} \end{bmatrix}, \end{aligned}$$

hence, by Theorem 3.6 and Slutsky's lemma,

$$\begin{bmatrix} n^{3/2}(\hat{\varrho}_n - \varrho) \\ n^{1/2}(\hat{\mathcal{A}}_n - \mathcal{A}) \end{bmatrix} = \begin{bmatrix} n^{3/2} & 0 \\ 0 & n^{1/2} \end{bmatrix} \begin{bmatrix} \hat{\varrho}_n - \varrho \\ \hat{\mathcal{A}}_n - \mathcal{A} \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2(\mathbf{0}, \boldsymbol{\Sigma}),$$

as $n \rightarrow \infty$, where

$$\begin{aligned} \boldsymbol{\Sigma} &:= \left(\frac{12}{A^2} \right)^2 \int_0^\infty z^2 \nu(dz) \begin{bmatrix} -\frac{A}{2} & 1 \\ \frac{A^2}{3} & -\frac{A}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{A}{2} \\ \frac{A}{2} & \frac{A^2}{3} \end{bmatrix} \begin{bmatrix} -\frac{A}{2} & \frac{A^2}{3} \\ 1 & -\frac{A}{2} \end{bmatrix} \\ &= \left(\frac{12}{A^2} \right)^2 \int_0^\infty z^2 \nu(dz) \begin{bmatrix} \frac{A^2}{12} & -\frac{A^3}{24} \\ -\frac{A^3}{24} & \frac{A^4}{36} \end{bmatrix} = \frac{12}{A^2} \int_0^\infty z^2 \nu(dz) \begin{bmatrix} 1 & -\frac{A}{2} \\ -\frac{A}{2} & \frac{A^2}{3} \end{bmatrix}, \end{aligned}$$

and we obtain (3.10). \square

4 Proof of Theorem 3.5

Consider the sequence of stochastic processes

$$\mathcal{Z}_t^{(n)} := \begin{bmatrix} \mathcal{M}_t^{(n)} \\ \mathcal{N}_t^{(n)} \end{bmatrix} := \sum_{k=1}^{\lfloor nt \rfloor} \mathcal{Z}_k^{(n)} \quad \text{with} \quad \mathcal{Z}_k^{(n)} := \begin{bmatrix} n^{-1} M_k \\ n^{-2} M_k X_{k-1} \end{bmatrix}$$

for $t \in \mathbb{R}_+$ and $k, n \in \mathbb{N}$. Theorem 3.5 follows from the following theorem (this will be explained after Theorem 4.1).

Theorem 4.1 *Under the assumptions of Theorem 3.1, we have*

$$\mathcal{Z}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{Z}, \quad \text{as } n \rightarrow \infty, \quad (4.1)$$

where the process $(\mathcal{Z}_t)_{t \in \mathbb{R}_+}$ with values in \mathbb{R}^2 is the pathwise unique strong solution of the SDE

$$d\mathcal{Z}_t = \gamma(t, \mathcal{Z}_t) d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad (4.2)$$

with initial value $\mathcal{Z}_0 = \mathbf{0}$, where $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, and $\gamma : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$\gamma(t, \mathbf{x}) := \begin{bmatrix} C^{1/2} ((x_1 + At)^+)^{1/2} \\ C^{1/2} ((x_1 + At)^+)^{3/2} \end{bmatrix}, \quad t \in \mathbb{R}_+, \quad \mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2.$$

(Note that the statement of Theorem 4.1 holds even if $C = 0$.)

The SDE (4.2) has the form

$$d\mathcal{Z}_t = \begin{bmatrix} d\mathcal{M}_t \\ d\mathcal{N}_t \end{bmatrix} = \begin{bmatrix} C^{1/2} ((\mathcal{M}_t + At)^+)^{1/2} d\mathcal{W}_t \\ C^{1/2} ((\mathcal{M}_t + At)^+)^{3/2} d\mathcal{W}_t \end{bmatrix}, \quad t \in \mathbb{R}_+. \quad (4.3)$$

One can prove that the first equation of the SDE (4.3) has a pathwise unique strong solution $(\mathcal{M}_t^{(y_0)})_{t \in \mathbb{R}_+}$ with arbitrary initial value $\mathcal{M}_0^{(y_0)} = y_0 \in \mathbb{R}$. Indeed, it is equivalent to the existence of a pathwise unique strong solution of the SDE

$$d\mathcal{S}_t^{(y_0)} = A dt + C^{1/2} ((\mathcal{S}_t^{(y_0)})^+)^{1/2} d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad (4.4)$$

with initial value $\mathcal{S}_0^{(y_0)} = y_0$, since we have the correspondences

$$\mathcal{S}_t^{(y_0)} = \mathcal{M}_t^{(y_0)} + At, \quad \mathcal{M}_t^{(y_0)} = \mathcal{S}_t^{(y_0)} - At,$$

by Itô's formula. By Remark 2.6, the SDE (4.4) has a pathwise unique strong solution $(\mathcal{S}_t^{(y_0)})_{t \in \mathbb{R}_+}$ for all initial values $\mathcal{S}_0^{(y_0)} = y_0 \in \mathbb{R}$, and $(\mathcal{S}_t^{(y_0)})^+$ may be replaced by $\mathcal{S}_t^{(y_0)}$ for all $t \in \mathbb{R}_+$ in (4.4) provided that $y_0 \in \mathbb{R}_+$, hence $(\mathcal{M}_t + At)^+$ may be replaced by $\mathcal{M}_t + At$ for all $t \in \mathbb{R}_+$ in (4.3). Thus the SDE (4.2) has a pathwise unique strong solution with initial value $\mathcal{Z}_0 = \mathbf{0}$, and we have

$$\mathcal{Z}_t = \begin{bmatrix} \mathcal{M}_t \\ \mathcal{N}_t \end{bmatrix} = \begin{bmatrix} \int_0^t C^{1/2} (\mathcal{M}_s + As)^{1/2} d\mathcal{W}_s \\ \int_0^t (\mathcal{M}_s + As) d\mathcal{M}_s \end{bmatrix}, \quad t \in \mathbb{R}_+.$$

By continuous mapping theorem (see, e.g., the method of the proof of $\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{X}$ in Theorem 3.1 in Barczy et al. Barczy et al (2011)), one can easily derive

$$\begin{bmatrix} \mathcal{X}^{(n)} \\ \mathcal{Z}^{(n)} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \tilde{\mathcal{X}} \\ \mathcal{Z} \end{bmatrix}, \quad \text{as } n \rightarrow \infty, \quad (4.5)$$

where

$$\mathcal{X}_t^{(n)} = n^{-1} X_{\lfloor nt \rfloor}, \quad \tilde{\mathcal{X}}_t := \mathcal{M}_t + At, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

By Itô's formula and the first equation of the SDE (4.3) we obtain

$$d\tilde{\mathcal{X}}_t = A dt + C^{1/2} (\tilde{\mathcal{X}}_t^+)^{1/2} d\mathcal{W}_t, \quad t \in \mathbb{R}_+,$$

hence the process $(\tilde{\mathcal{X}}_t)_{t \in \mathbb{R}_+}$ satisfies the SDE (2.9). Consequently, $\tilde{\mathcal{X}} = \mathcal{Y}$. Next, by continuous mapping theorem, convergence (4.5) implies (3.11), see, e.g., the method of the proof of Proposition 3.1 in Barczy et al. Barczy et al (2010).

Proof of Theorem 4.1 In order to show convergence $\mathcal{Z}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{Z}$, we apply Theorem E.1 with the special choices $\mathcal{U} := \mathcal{Z}$, $\mathbf{U}_k^{(n)} := \mathbf{Z}_k^{(n)}$, $n, k \in \mathbb{N}$, $(\mathcal{F}_k^{(n)})_{k \in \mathbb{Z}_+} := (\mathcal{F}_k)_{k \in \mathbb{Z}_+}$ and the function γ which is defined in Theorem 4.1. Note that the discussion after Theorem 4.1 shows that the SDE (4.2) admits a pathwise unique strong solution $(\mathcal{Z}_t^z)_{t \in \mathbb{R}_+}$ for all initial values $\mathcal{Z}_0^z = \mathbf{z} \in \mathbb{R}^2$. Applying Cauchy–Schwarz inequality and Corollary B.5, one can check that $\mathbb{E}(\|\mathbf{U}_k^{(n)}\|^2) < \infty$ for all $n, k \in \mathbb{N}$.

Now we show that conditions (i) and (ii) of Theorem E.1 hold. The conditional variance has the form

$$\text{Var}(\mathbf{Z}_k^{(n)} | \mathcal{F}_{k-1}) = \text{Var}(M_k | \mathcal{F}_{k-1}) \begin{bmatrix} n^{-2} & n^{-3}X_{k-1} \\ n^{-3}X_{k-1} & n^{-4}X_{k-1}^2 \end{bmatrix}$$

for $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$, and

$$\gamma(s, \mathbf{Z}_s^{(n)})\gamma(s, \mathbf{Z}_s^{(n)})^\top = C \begin{bmatrix} \mathcal{M}_s^{(n)} + As & (\mathcal{M}_s^{(n)} + As)^2 \\ (\mathcal{M}_s^{(n)} + As)^2 & (\mathcal{M}_s^{(n)} + As)^3 \end{bmatrix}$$

for $s \in \mathbb{R}_+$, where we used that $(\mathcal{M}_s^{(n)} + As)^+ = \mathcal{M}_s^{(n)} + As$, $s \in \mathbb{R}_+$, $n \in \mathbb{N}$. Indeed, by (3.3), we get

$$\mathcal{M}_s^{(n)} + As = \frac{1}{n} \sum_{k=1}^{\lfloor ns \rfloor} (X_k - e^B X_{k-1} - \mathcal{A}) + As = \frac{1}{n} X_{\lfloor ns \rfloor} + \frac{ns - \lfloor ns \rfloor}{n} A \in \mathbb{R}_+ \quad (4.6)$$

for $s \in \mathbb{R}_+$, $n \in \mathbb{N}$, since $e^B = 1$ and $\mathcal{A} = A$.

In order to check condition (i) of Theorem E.1, we need to prove that for each $T > 0$, as $n \rightarrow \infty$,

$$\sup_{t \in [0, T]} \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \text{Var}(M_k | \mathcal{F}_{k-1}) - C \int_0^t (\mathcal{M}_s^{(n)} + As) ds \right| \xrightarrow{\mathbb{P}} 0, \quad (4.7)$$

$$\sup_{t \in [0, T]} \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1} \text{Var}(M_k | \mathcal{F}_{k-1}) - C \int_0^t (\mathcal{M}_s^{(n)} + As)^2 ds \right| \xrightarrow{\mathbb{P}} 0, \quad (4.8)$$

$$\sup_{t \in [0, T]} \left| \frac{1}{n^4} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^2 \text{Var}(M_k | \mathcal{F}_{k-1}) - C \int_0^t (\mathcal{M}_s^{(n)} + As)^3 ds \right| \xrightarrow{\mathbb{P}} 0. \quad (4.9)$$

First we show (4.7). By (4.6), $\int_0^t (\mathcal{M}_s^{(n)} + As) ds$ has the form

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor - 1} X_k + \frac{nt - \lfloor nt \rfloor}{n^2} X_{\lfloor nt \rfloor} + \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2}{2n^2} A.$$

By Proposition B.3 and $B = 0$,

$$\text{Var}(M_k | \mathcal{F}_{k-1}) = V X_{k-1} + V_0 = C X_{k-1} + V_0. \quad (4.10)$$

Thus, in order to show (4.7), it suffices to prove

$$n^{-2} \sup_{t \in [0, T]} X_{\lfloor nt \rfloor} \xrightarrow{\mathbb{P}} 0, \quad (4.11)$$

$$n^{-2} \sup_{t \in [0, T]} [\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^2] \rightarrow 0, \quad (4.12)$$

as $n \rightarrow \infty$. Using (B.5) with $(\ell, i) = (2, 1)$, we have (4.11). Clearly, (4.12) follows from $|nt - \lfloor nt \rfloor| \leq 1$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, thus we conclude (4.7).

Next we turn to prove (4.8). By (4.6),

$$\begin{aligned} \int_0^t (\mathcal{M}_s^{(n)} + sA)^2 ds &= \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor - 1} X_k^2 + \frac{1}{n^3} A \sum_{k=1}^{\lfloor nt \rfloor - 1} X_k + \frac{nt - \lfloor nt \rfloor}{n^3} X_{\lfloor nt \rfloor}^2 \\ &\quad + \frac{(nt - \lfloor nt \rfloor)^2}{n^3} A X_{\lfloor nt \rfloor} + \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^3}{3n^3} A^2. \end{aligned}$$

Recalling formula (4.10), we obtain

$$\sum_{k=1}^{\lfloor nt \rfloor} X_{k-1} \operatorname{Var}(M_k | \mathcal{F}_{k-1}) = C \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^2 + V_0 \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}. \quad (4.13)$$

Thus, in order to show (4.8), it suffices to prove

$$n^{-3} \sum_{k=1}^{\lfloor nt \rfloor} X_k \xrightarrow{\mathbb{P}} 0, \quad (4.14)$$

$$n^{-3/2} \sup_{t \in [0, T]} X_{\lfloor nt \rfloor} \xrightarrow{\mathbb{P}} 0, \quad (4.15)$$

$$n^{-3} \sup_{t \in [0, T]} [\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^3] \rightarrow 0 \quad (4.16)$$

as $n \rightarrow \infty$. Using (B.4) with $(\ell, i) = (2, 1)$, we have (4.14). By (B.5) with $(\ell, i) = (3, 1)$, we have (4.15). Clearly, (4.16) follows from $|nt - \lfloor nt \rfloor| \leq 1$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, thus we conclude (4.8).

Now we turn to check (4.9). Again by (4.6), we have

$$\begin{aligned} \int_0^t (\mathcal{M}_s^{(n)} + sA)^3 ds &= \frac{1}{n^4} \sum_{k=1}^{\lfloor nt \rfloor - 1} X_k^3 + \frac{3}{2n^4} A \sum_{k=1}^{\lfloor nt \rfloor - 1} X_k^2 + \frac{1}{n^4} A^2 \sum_{k=1}^{\lfloor nt \rfloor - 1} X_k \\ &\quad + \frac{nt - \lfloor nt \rfloor}{n^4} X_{\lfloor nt \rfloor}^3 + \frac{3(nt - \lfloor nt \rfloor)^2}{2n^4} A X_{\lfloor nt \rfloor}^2 \\ &\quad + \frac{(nt - \lfloor nt \rfloor)^3}{n^4} A^2 X_{\lfloor nt \rfloor} + \frac{\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^4}{4n^4} A^3. \end{aligned}$$

Recalling formula (4.10), we obtain

$$\sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^2 \operatorname{Var}(M_k | \mathcal{F}_{k-1}) = C \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^3 + V_0 \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^2. \quad (4.17)$$

Thus, in order to show (4.9), it suffices to prove

$$n^{-4} \sum_{k=1}^{\lfloor nT \rfloor} X_k^2 \xrightarrow{\mathbb{P}} 0, \quad (4.18)$$

$$n^{-4} \sum_{k=1}^{\lfloor nT \rfloor} X_k \xrightarrow{\mathbb{P}} 0, \quad (4.19)$$

$$n^{-4/3} \sup_{t \in [0, T]} X_{\lfloor nt \rfloor} \xrightarrow{\mathbb{P}} 0, \quad (4.20)$$

$$n^{-4} \sup_{t \in [0, T]} [\lfloor nt \rfloor + (nt - \lfloor nt \rfloor)^4] \rightarrow 0 \quad (4.21)$$

as $n \rightarrow \infty$. Using (B.4) with $(\ell, i) = (4, 2)$ and $(\ell, i) = (2, 1)$, we have (4.18) and (4.19), respectively. By (B.5) with $(\ell, i) = (4, 1)$, we have (4.20). Clearly, (4.21) follows again from $|nt - \lfloor nt \rfloor| \leq 1$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, thus we conclude (4.9). Note that the proof of (4.7)–(4.9) is essentially the same as the proof of (5.5)–(5.7) in Ispány et al. (2014).

Finally, we check condition (ii) of Theorem E.1, that is, the conditional Lindeberg condition

$$\sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\mathbf{Z}_k^{(n)}\| > \theta\}} \mid \mathcal{F}_{k-1}) \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty \quad (4.22)$$

for all $\theta > 0$ and $T > 0$. We have $\mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\mathbf{Z}_k^{(n)}\| > \theta\}} \mid \mathcal{F}_{k-1}) \leq \theta^{-2} \mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^4 \mid \mathcal{F}_{k-1})$ and

$$\|\mathbf{Z}_k^{(n)}\|^4 \leq 2 (n^{-4} + n^{-8} X_{k-1}^4) M_k^4.$$

Hence, for all $\theta > 0$ and $T > 0$, we have

$$\sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\mathbf{Z}_k^{(n)}\| > \theta\}}) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

since $\mathbb{E}(M_k^4) = O(k^2)$ and $\mathbb{E}(M_k^4 X_{k-1}^4) \leq \sqrt{\mathbb{E}(M_k^8) \mathbb{E}(X_{k-1}^8)} = O(k^6)$ by Corollary B.5. This yields (4.22). \square

We call the attention that our moment conditions (2.7) with $q = 8$ are used for applying Corollaries B.5 and B.6.

5 Proof of Theorem 3.6

The first two convergences in Theorem 3.6 follows from the following approximations.

Lemma 5.1 Suppose that the assumptions of Theorem 3.1 hold. If $C = 0$, then for each $T > 0$,

$$\sup_{t \in [0, T]} \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1} - A \frac{t^2}{2} \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty. \quad (5.1)$$

Proof We have

$$\left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1} - A \frac{t^2}{2} \right| \leq \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} |X_{k-1} - A(k-1)| + A \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (k-1) - \frac{t^2}{2} \right|,$$

where

$$\sup_{t \in [0, T]} \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (k-1) - \frac{t^2}{2} \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

hence, in order to show (5.1), it suffices to prove

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} |X_k - Ak| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty. \quad (5.2)$$

Recursion (3.4) yields $\mathbb{E}(X_k) = \mathbb{E}(X_{k-1}) + A$, $k \in \mathbb{N}$, with initial value $\mathbb{E}(X_0) = 0$, hence $\mathbb{E}(X_k) = Ak$, $k \in \mathbb{N}$. For the sequence

$$\tilde{X}_k := X_k - \mathbb{E}(X_k) = X_k - Ak, \quad k \in \mathbb{N}, \quad (5.3)$$

by (3.4), we get a recursion $\tilde{X}_k = \tilde{X}_{k-1} + M_k$, $k \in \mathbb{N}$, with initial value $\tilde{X}_0 = 0$. Applying Doob's maximal inequality (see, e.g., Revuz and Yor (Revuz and Yor 1999, Chapter II, Theorem 1.7)) for the martingale $\tilde{X}_n = \sum_{k=1}^n M_k$, $n \in \mathbb{N}$,

$$\mathbb{E} \left(\sup_{t \in [0, T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} M_k \right|^2 \right) \leq 4 \mathbb{E} \left(\left| \sum_{k=1}^{\lfloor nt \rfloor} M_k \right|^2 \right) = 4 \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(M_k^2) = O(n),$$

where we applied Corollary B.5. Consequently,

$$n^{-1} \max_{k \in \{1, \dots, \lfloor nt \rfloor\}} |X_k - Ak| = n^{-1} \max_{k \in \{1, \dots, \lfloor nt \rfloor\}} |\tilde{X}_k| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty. \quad (5.4)$$

Thus,

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} |X_k - Ak| \leq \frac{\lfloor nt \rfloor}{n^2} \max_{k \in \{1, \dots, \lfloor nt \rfloor\}} |X_k - Ak| \xrightarrow{\mathbb{P}} 0,$$

as $n \rightarrow \infty$, thus we conclude (5.2), and hence (5.1). \square

Lemma 5.2 Suppose that the assumptions of Theorem 3.1 hold. If $C = 0$, then for each $T > 0$,

$$\sup_{t \in [0, T]} \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^2 - A^2 \frac{t^3}{3} \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty. \quad (5.5)$$

Proof We have

$$\begin{aligned} & \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^2 - A^2 \frac{t^3}{3} \right| \\ & \leq \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} |X_{k-1}^2 - A^2(k-1)^2| + A^2 \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} (k-1)^2 - \frac{t^3}{3} \right|, \end{aligned}$$

where

$$\sup_{t \in [0, T]} \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} (k-1)^2 - \frac{t^3}{3} \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

hence, in order to show (5.5), it suffices to prove

$$\frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} |X_k^2 - A^2 k^2| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty. \quad (5.6)$$

We have

$$|X_k^2 - k^2 A^2| \leq |X_k - kA|^2 + 2kA|X_k - kA|,$$

hence, by (5.4),

$$\begin{aligned} & n^{-2} \max_{k \in \{1, \dots, \lfloor nt \rfloor\}} |X_k^2 - k^2 A^2| \\ & \leq \left(n^{-1} \max_{k \in \{1, \dots, \lfloor nt \rfloor\}} |X_k - kA| \right)^2 + \frac{2 \lfloor nt \rfloor}{n^2} A \max_{k \in \{1, \dots, \lfloor nt \rfloor\}} |X_k - kA| \xrightarrow{\mathbb{P}} 0, \end{aligned}$$

as $n \rightarrow \infty$. Thus,

$$\frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} |X_k^2 - k^2 A^2| \leq \frac{\lfloor nt \rfloor}{n^3} \max_{k \in \{1, \dots, \lfloor nt \rfloor\}} |X_k^2 - k^2 A^2| \xrightarrow{\mathbb{P}} 0,$$

as $n \rightarrow \infty$, and we conclude (5.6), and hence (5.5). \square

The proof of the third convergence in Theorem 3.6 is similar to the proof of Theorem 3.5. Consider the sequence of stochastic processes

$$\mathbf{Z}_t^{(n)} := \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{Z}_k^{(n)} \quad \text{with} \quad \mathbf{Z}_k^{(n)} := \begin{bmatrix} n^{-1/2} M_k \\ n^{-3/2} M_k X_{k-1} \end{bmatrix}$$

for $t \in \mathbb{R}_+$ and $k, n \in \mathbb{N}$. The proof of the third convergence in Theorem 3.6 follows from Lemmas 5.1 and 5.2, and the following theorem.

Theorem 5.3 *If $C = 0$ then*

$$\mathcal{Z}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{Z}, \quad \text{as } n \rightarrow \infty, \quad (5.7)$$

where the process $(\mathcal{Z}_t)_{t \in \mathbb{R}_+}$ with values in \mathbb{R}^2 is the pathwise unique strong solution of the SDE

$$d\mathcal{Z}_t = \gamma(t) \widetilde{\mathcal{W}}_t, \quad t \in \mathbb{R}_+, \quad (5.8)$$

with initial value $\mathcal{Z}_0 = \mathbf{0}$, where $(\widetilde{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$ is a 2-dimensional standard Wiener process, and $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}^{2 \times 2}$ is defined by

$$\gamma(t) := V_0 \begin{bmatrix} 1 & At \\ At & A^2 t^2 \end{bmatrix}^{1/2}, \quad t \in \mathbb{R}_+,$$

where $V_0 = \int_0^\infty z^2 \nu(dz)$.

The SDE (5.8) has a pathwise unique strong solution with initial value $\mathcal{Z}_0 = \mathbf{0}$, for which we have

$$\mathcal{Z}_t = V_0^{1/2} \int_0^t \begin{bmatrix} 1 & As \\ As & A^2 s^2 \end{bmatrix}^{1/2} d\widetilde{\mathcal{W}}_s, \quad t \in \mathbb{R}_+.$$

Proof of Theorem 5.3 We follow again the method of the proof of Theorem 4.1. The conditional variance has the form

$$\text{Var}(\mathcal{Z}_k^{(n)} | \mathcal{F}_{k-1}) = \text{Var}(M_k | \mathcal{F}_{k-1}) \begin{bmatrix} n^{-1} & n^{-2} X_{k-1} \\ n^{-2} X_{k-1} & n^{-3} X_{k-1}^2 \end{bmatrix}$$

for $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$. Moreover, $\gamma(s)\gamma(s)^\top$ takes the form

$$\gamma(s)\gamma(s)^\top = V_0 \begin{bmatrix} 1 & As \\ As & A^2 s^2 \end{bmatrix}, \quad s \in \mathbb{R}_+.$$

In order to check condition (i) of Theorem E.1, we need to prove only that for each $T > 0$,

$$\sup_{t \in [0, T]} \left| \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \text{Var}(M_k | \mathcal{F}_{k-1}) - V_0 \int_0^t s ds \right| \xrightarrow{\mathbb{P}} 0, \quad (5.9)$$

$$\sup_{t \in [0, T]} \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1} \text{Var}(M_k | \mathcal{F}_{k-1}) - V_0 A \int_0^t s ds \right| \xrightarrow{\mathbb{P}} 0, \quad (5.10)$$

$$\sup_{t \in [0, T]} \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} X_{k-1}^2 \text{Var}(M_k | \mathcal{F}_{k-1}) - V_0 A^2 \int_0^t s^2 ds \right| \xrightarrow{\mathbb{P}} 0, \quad (5.11)$$

as $n \rightarrow \infty$.

By Proposition B.3, the assumption $C = 0$ yields $\text{Var}(M_k | \mathcal{F}_{k-1}) = V_0 = \int_0^\infty z^2 \nu(dz)$, hence (5.9), (5.10) and (5.11) follow from Lemmas 5.1 and 5.2, respectively.

Finally, we check condition (ii) of Theorem E.1, that is, the conditional Lindeberg condition

$$\sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\mathbf{Z}_k^{(n)}\| > \theta\}} | \mathcal{F}_{k-1}) \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty \quad (5.12)$$

for all $\theta > 0$ and $T > 0$. We have $\mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\mathbf{Z}_k^{(n)}\| > \theta\}} | \mathcal{F}_{k-1}) \leq \theta^{-2} \mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^4 | \mathcal{F}_{k-1})$ and

$$\|\mathbf{Z}_k^{(n)}\|^4 \leq 2(n^{-2} + n^{-6} X_{k-1}^4) M_k^4.$$

Hence, for all $\theta > 0$ and $T > 0$, we have

$$\sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E} (\|\mathbf{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\mathbf{Z}_k^{(n)}\| > \theta\}}) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

since $\mathbb{E}(M_k^4) = O(1)$ and $\mathbb{E}(M_k^4 X_{k-1}^4) \leq \sqrt{\mathbb{E}(M_k^8) \mathbb{E}(X_{k-1}^8)} = O(k^4)$ by Corollary B.5. This yields (5.12). \square

A Appendix: SDE for CBI processes

One can rewrite the SDE (1.1) in a form which does not contain integrals with respect to non-compensated Poisson random measures (see, SDE (2.6)), and then one can perform a linear transformation in order to remove randomness from the drift as follows, see Lemma 4.1 in Barczy et al. Barczy et al (2016+). This form is very useful for handling M_k , $k \in \mathbb{N}$.

Lemma A.1 *Let (c, a, b, ν, μ) be a set of admissible parameters such that the moment condition (2.3) holds. Let $(X_t)_{t \in \mathbb{R}_+}$ be a pathwise unique \mathbb{R}_+ -valued strong solution to the SDE (1.1) such that $\mathbb{E}(X_0) < \infty$. Then*

$$\begin{aligned} X_t &= e^{B(t-s)} X_s + \int_s^t e^{B(t-u)} A \, du + \int_s^t e^{B(t-u)} \sqrt{2cX_u} \, dW_u \\ &\quad + \int_s^t \int_0^\infty \int_0^\infty e^{B(t-u)} z \mathbb{1}_{\{v \leq X_{s-}\}} \tilde{N}(du, dz, dv) + \int_s^t \int_0^\infty e^{B(t-u)} z \tilde{M}(du, dz) \end{aligned}$$

for all $s, t \in \mathbb{R}_+$, with $s \leq t$. Consequently,

$$\begin{aligned} M_k &= \int_{k-1}^k e^{B(k-u)} \sqrt{2cX_u} \, dW_u + \int_{k-1}^k \int_0^\infty \int_0^\infty e^{B(k-u)} z \mathbb{1}_{\{v \leq X_{s-}\}} \tilde{N}(du, dz, dv) \\ &\quad + \int_{k-1}^k \int_0^\infty e^{B(k-u)} z \tilde{M}(du, dz), \quad k \in \mathbb{N}. \end{aligned}$$

Proof The last statement follows from (3.3), since $A \int_{k-1}^k e^{B(k-u)} \, du = A \int_0^1 e^{B(1-u)} \, du = \mathcal{A}$. \square

Note that the formulas for $(X_t)_{t \in \mathbb{R}_+}$ and $(M_k)_{k \in \mathbb{N}}$ in Lemma A.1 can be found as the first displayed formula in the proof of Lemma 2.1 in Huang et al. Huang et al (2011), and formulas (1.5) and (1.7) in Li and Ma Li and Ma (2015), respectively.

Lemma A.2 *Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $X_0 = 0$, $a \neq 0$ or $\nu \neq 0$, and $B = 0$ (hence it is critical). Suppose that $C = 0$ and the moment conditions (2.7) hold with $q = 2$. Then*

$$M_k = \int_{k-1}^k \int_0^\infty z \widetilde{M}(du, dz), \quad k \in \mathbb{N}.$$

and the sequence $(M_k)_{k \in \mathbb{N}}$ consists of independent and identically distributed random vectors.

Proof The assumption $C = 0$ implies $c = 0$ and $\mu = 0$ (see, Remark 2.7), thus, by Lemma A.1, we obtain the formula for M_k , $k \in \mathbb{N}$.

A Poisson point process admits independent increments, hence M_k , $k \in \mathbb{N}$, are independent.

For each $k \in \mathbb{N}$, the Laplace transform of the random variable M_k has the form

$$\begin{aligned} \mathbb{E}(e^{-\theta M_k}) &= \exp \left\{ - \int_{k-1}^k \int_0^\infty (1 - e^{-\theta r}) ds \nu(dr) \right\} \\ &= \exp \left\{ - \int_0^1 \int_0^\infty (1 - e^{-\theta r}) du \nu(dr) \right\} = \mathbb{E}(e^{-\theta M_1}) \end{aligned}$$

for all $\theta \in \mathbb{R}_+$, see, i.e., Kyprianou (Kyprianou 2014, page 44), hence M_k , $k \in \mathbb{N}$, are identically distributed. \square

B Appendix: On moments of CBI processes

In the proof of Theorem 3.1, good bounds for moments of the random variables $(M_k)_{k \in \mathbb{Z}_+}$ and $(X_k)_{k \in \mathbb{Z}_+}$ are extensively used. The following estimates are proved in Barczy and Pap (Barczy and Pap 2016+, Lemmas B.2 and B.3).

Lemma B.1 *Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $\mathbb{E}(X_0^q) < \infty$ and the moment conditions (2.7) hold with some $q \in \mathbb{N}$. Suppose that $B = 0$ (hence the process is critical). Then*

$$\sup_{t \in \mathbb{R}_+} \frac{\mathbb{E}(X_t^q)}{(1+t)^q} < \infty. \quad (\text{B.1})$$

In particular, $\mathbb{E}(X_t^q) = O(t^q)$ as $t \rightarrow \infty$ in the sense that $\limsup_{t \rightarrow \infty} t^{-q} \mathbb{E}(X_t^q) < \infty$.

Lemma B.2 *Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $\mathbb{E}(X_0^q) < \infty$ and the moment conditions (2.7) hold, where $q = 2p$ with some $p \in \mathbb{N}$. Suppose that $B = 0$ (hence the process is critical). Then, for the martingale differences $M_n = X_n - \mathbb{E}(X_n | X_{n-1})$, $n \in \mathbb{N}$, we have $\mathbb{E}(M_n^{2p}) = O(n^p)$ as $n \rightarrow \infty$ that is, $\sup_{n \in \mathbb{N}} n^{-p} \mathbb{E}(M_n^{2p}) < \infty$.*

We have $\text{Var}(M_k | \mathcal{F}_{k-1}) = \text{Var}(X_k | X_{k-1})$ and $\text{Var}(X_k | X_{k-1} = x) = \text{Var}(X_1 | X_0 = x)$ for all $x \in \mathbb{R}_+$, since $(X_t)_{t \in \mathbb{R}_+}$ is a time-homogeneous Markov process. Hence Proposition 4.8 in Barczy et al. Barczy et al (2016+) implies the following formula for $\text{Var}(M_k | \mathcal{F}_{k-1})$.

Proposition B.3 Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $\mathbb{E}(X_0^2) < \infty$ and the moment conditions (2.7) hold with $q = 2$. Then for all $k \in \mathbb{N}$, we have

$$\text{Var}(M_k | \mathcal{F}_{k-1}) = V X_{k-1} + V_0,$$

where

$$\begin{aligned} V &:= C \int_0^1 e^{B(1+u)} du, \\ V_0 &:= \int_0^\infty z^2 \nu(dz) \int_0^1 e^{2Bu} du + AC \int_0^1 \left(\int_0^{1-u} e^{Bv} dv \right) e^{2Bu} du. \end{aligned}$$

Note that $V_0 = \text{Var}(X_1 | X_0 = 0)$. Moreover, if $B = 0$, i.e., in the critical case, we have $V = C$.

Proposition B.4 Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $\mathbb{E}(X_0^q) < \infty$ and the moment conditions (2.7) hold with some $q \in \mathbb{N}$. Then for all $j \in \{1, \dots, q\}$, there exists a polynomial $P_j : \mathbb{R} \rightarrow \mathbb{R}$ having degree at most $\lfloor j/2 \rfloor$, such that

$$\mathbb{E}(M_k^j | \mathcal{F}_{k-1}) = P_j(X_{k-1}), \quad k \in \mathbb{N}. \quad (\text{B.2})$$

The coefficients of the polynomial P_j depends on c, a, b, ν, μ .

Proof We have

$$\mathbb{E}(M_k^j | \mathcal{F}_{k-1}) = \mathbb{E}[(X_k - \mathbb{E}(X_k | X_{k-1}))^j | X_{k-1}]$$

and

$$\mathbb{E}[(X_k - \mathbb{E}(X_k | X_{k-1}))^j | X_{k-1} = x] = \mathbb{E}[(X_1 - \mathbb{E}(X_1 | X_0 = x))^j | X_0 = x]$$

for all $x \in \mathbb{R}_+$, since $(X_t)_{t \in \mathbb{R}_+}$ is a time-homogeneous Markov process. Replacing w by e^{Bt} in the formula for $\mathbb{E}[(we^{-Bt}(Y_t - \mathbb{E}(Y_t))^k)]$ in the proof of Barczy et al. (Barczy et al 2016+, Theorem 4.5), and then using the law of total probability, one obtains

$$\begin{aligned} \mathbb{E}[(X_t - \mathbb{E}(X_t))^j] &= j(j-1)c \int_0^t e^{jB(t-s)} \mathbb{E}[(X_s - \mathbb{E}(X_s))^{j-2} X_s] ds \\ &\quad + \sum_{\ell=0}^{j-2} \binom{j}{\ell} \int_0^\infty z^{j-\ell} \mu(dz) \int_0^t e^{jB(t-s)} \mathbb{E}[(X_s - \mathbb{E}(X_s))^\ell X_s] ds \quad (\text{B.3}) \\ &\quad + \sum_{\ell=0}^{j-2} \binom{j}{\ell} \int_0^\infty z^{j-\ell} \nu(dz) \int_0^t e^{jB(t-s)} \mathbb{E}[(X_s - \mathbb{E}(X_s))^\ell] ds \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $j \in \{1, \dots, q\}$, and hence, for each $t \in \mathbb{R}_+$ and $j \in \{1, \dots, q\}$, there exists a polynomial $P_{t,j} : \mathbb{R} \rightarrow \mathbb{R}$ having degree at most $\lfloor j/2 \rfloor$, such that

$$\mathbb{E}[(X_t - \mathbb{E}(X_t))^j] = \mathbb{E}[P_{t,j}(X_0)],$$

where the coefficients of the polynomial $P_{t,j}$ depends on c, a, b, ν, μ , which clearly implies the statement with $P_j := P_{1,j}$. \square

Corollary B.5 Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $X_0 = 0$, $a \neq 0$ or $\nu \neq 0$, and $B = 0$ (hence the process is critical). Suppose that the moment conditions (2.7) hold with some $q \in \mathbb{N}$. Then

$$\mathbb{E}(X_k^i) = O(k^i), \quad \mathbb{E}(M_k^{2j}) = O(k^j)$$

for $i, j \in \mathbb{Z}_+$ with $i \leq q$ and $2j \leq q$.
 If, in addition, $C = 0$, then

$$\mathbb{E}(|M_k|^i) = O(1)$$

for $i \in \mathbb{Z}_+$ with $i \leq q$.

Proof The first and second statements follow from Lemmas B.1 and B.2, respectively.

If $C = 0$, then, by Lemma A.2, M_k , $k \in \mathbb{N}$, are independent and identically distributed, thus

$$\mathbb{E}(|M_k|^i) = \mathbb{E}(|M_1|^i) = O(1)$$

for $i \in \mathbb{Z}_+$ with $i \leq q$. \square

Corollary B.6 Let $(X_t)_{t \in \mathbb{R}_+}$ be a CBI process with parameters (c, a, b, ν, μ) such that $X_0 = 0$, $a \neq 0$ or $\nu \neq 0$, and $B = 0$ (hence the process is critical). Suppose that the moment conditions (2.7) hold with some $\ell \in \mathbb{N}$. Then

(i) for all $i \in \mathbb{Z}_+$ with $i \leq \lfloor \ell/2 \rfloor$, and for all $\theta > i + 1$, we have

$$n^{-\theta} \sum_{k=1}^n X_k^i \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty, \quad (\text{B.4})$$

(ii) for all $i \in \mathbb{Z}_+$ with $i \leq \ell$, for all $T > 0$, and for all $\theta > i + \frac{i}{\ell}$, we have

$$n^{-\theta} \sup_{t \in [0, T]} X_{[nt]}^i \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty, \quad (\text{B.5})$$

(iii) for all $i \in \mathbb{Z}_+$ with $i \leq \lfloor \ell/4 \rfloor$, for all $T > 0$, and for all $\theta > i + \frac{1}{2}$, we have

$$n^{-\theta} \sup_{t \in [0, T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} [X_k^i - \mathbb{E}(X_k^i | \mathcal{F}_{k-1})] \right| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \rightarrow \infty. \quad (\text{B.6})$$

Proof The statements can be derived exactly as in Barczy et al. (Barczy et al 2014, Corollary 9.2 of arXiv version). \square

C Appendix: CLS estimators

Lemma C.1 If $(X_t)_{t \in \mathbb{R}_+}$ is a CBI process with parameters (c, a, b, ν, μ) such that $B = 0$ (hence it is critical), $\mathbb{E}(X_0) < \infty$, and the moment condition (2.3) holds, then $\mathbb{P}(H_n) \rightarrow 1$ as $n \rightarrow \infty$, and hence, the probability of the existence of a unique CLS estimator $(\hat{\varrho}_n, \hat{A}_n)$ converges to 1 as $n \rightarrow \infty$, and this CLS estimator has the form given in (3.5) on the event H_n .

Proof First, note that for all $n \in \mathbb{N}$,

$$\begin{aligned} \Omega \setminus H_n &= \left\{ \omega \in \Omega : \sum_{k=1}^n X_{k-1}^2(\omega) - \frac{1}{n} \left(\sum_{i=1}^n X_{i-1}(\omega) \right)^2 = 0 \right\} \\ &= \left\{ \omega \in \Omega : \sum_{k=1}^n \left(X_{k-1}(\omega) - \frac{1}{n} \sum_{i=1}^n X_{i-1}(\omega) \right)^2 = 0 \right\} \\ &= \left\{ \omega \in \Omega : X_{k-1}(\omega) = \frac{1}{n} \sum_{i=1}^n X_{i-1}(\omega), k \in \{1, \dots, n\} \right\} \\ &= \{ \omega \in \Omega : 0 = X_0(\omega) = X_1(\omega) = \dots = X_{n-1}(\omega) \} \\ &= \left\{ \omega \in \Omega : \frac{1}{n^2} \sum_{i=1}^n X_{i-1}(\omega) = 0 \right\}, \end{aligned}$$

where we used that $X_0 = 0$ and $X_k \geq 0$, $k \in \mathbb{Z}_+$.

By continuous mapping theorem, we obtain

$$\frac{1}{n^2} \sum_{k=1}^n X_k \xrightarrow{\mathcal{D}} \int_0^1 \mathcal{Y}_t dt \quad \text{as } n \rightarrow \infty, \quad (\text{C.1})$$

see, e.g., the method of the proof of Proposition 3.1 in Barczy et al. Barczy et al (2010).

By the proof of Theorem 3.4, we have $\mathbb{P}(\int_0^1 \mathcal{Y}_t dt > 0) = 1$. Thus the distribution function of $\int_0^1 \mathcal{Y}_t dt$ is continuous at 0, and hence, by (C.1),

$$\mathbb{P}(H_n) = \mathbb{P}\left(\sum_{i=1}^n X_{i-1} > 0\right) = \mathbb{P}\left(\frac{1}{(n-1)^2} \sum_{i=1}^n X_{i-1} > 0\right) \rightarrow \mathbb{P}\left(\int_0^1 \mathcal{Y}_t dt > 0\right) = 1$$

as $n \rightarrow \infty$. \square

D Appendix: a version of the continuous mapping theorem

The following version of continuous mapping theorem can be found for example in Kallenberg (Kallenberg 2002, Theorem 4.27).

Lemma D.1 (Kallenberg) *Let (S, d_S) and (T, d_T) be metric spaces and $(\xi_n)_{n \in \mathbb{N}}$, ξ be random elements with values in S such that $\xi_n \xrightarrow{\mathcal{D}} \xi$ as $n \rightarrow \infty$. Let $f : S \rightarrow T$ and $f_n : S \rightarrow T$, $n \in \mathbb{N}$, be measurable mappings and $C \in \mathcal{B}(S)$ such that $\mathbb{P}(\xi \in C) = 1$ and $\lim_{n \rightarrow \infty} d_T(f_n(\xi_n), f(\xi)) = 0$ if $\lim_{n \rightarrow \infty} d_S(\xi_n, \xi) = 0$ and $\xi \in C$. Then $f_n(\xi_n) \xrightarrow{\mathcal{D}} f(\xi)$ as $n \rightarrow \infty$.*

E Appendix: convergence of random step processes

We recall a result about convergence of random step processes towards a diffusion process, see Ispány and Pap Ispány and Pap (2010). This result is used for the proof of convergence (4.1).

Theorem E.1 *Let $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times r}$ be a continuous function. Assume that uniqueness in the sense of probability law holds for the SDE*

$$d\mathcal{U}_t = \gamma(t, \mathcal{U}_t) d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad (\text{E.1})$$

with initial value $\mathcal{U}_0 = \mathbf{u}_0$ for all $\mathbf{u}_0 \in \mathbb{R}^d$, where $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is an r -dimensional standard Wiener process. Let $(\mathcal{U}_t)_{t \in \mathbb{R}_+}$ be a solution of (E.1) with initial value $\mathcal{U}_0 = \mathbf{0} \in \mathbb{R}^d$.

For each $n \in \mathbb{N}$, let $(\mathbf{U}_k^{(n)})_{k \in \mathbb{N}}$ be a sequence of d -dimensional martingale differences with respect to a filtration $(\mathcal{F}_k^{(n)})_{k \in \mathbb{Z}_+}$, that is, $\mathbb{E}(\mathbf{U}_k^{(n)} | \mathcal{F}_{k-1}^{(n)}) = 0$, $n \in \mathbb{N}$, $k \in \mathbb{N}$. Let

$$\mathcal{U}_t^{(n)} := \sum_{k=1}^{\lfloor nt \rfloor} \mathbf{U}_k^{(n)}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Suppose that $\mathbb{E}(\|\mathbf{U}_k^{(n)}\|^2) < \infty$ for all $n, k \in \mathbb{N}$. Suppose that for each $T > 0$,

$$(i) \sup_{t \in [0, T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \text{Var}(\mathbf{U}_k^{(n)} | \mathcal{F}_{k-1}^{(n)}) - \int_0^t \gamma(s, \mathcal{U}_s^{(n)}) \gamma(s, \mathcal{U}_s^{(n)})^\top ds \right\| \xrightarrow{\mathbb{P}} 0,$$

$$(ii) \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}(\|\mathbf{U}_k^{(n)}\|^2 \mathbb{1}_{\{\|\mathbf{U}_k^{(n)}\| > \theta\}} | \mathcal{F}_{k-1}^{(n)}) \xrightarrow{\mathbb{P}} 0 \text{ for all } \theta > 0,$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability. Then $\mathcal{U}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{U}$ as $n \rightarrow \infty$.

Note that in (i) of Theorem E.1, $\|\cdot\|$ denotes a matrix norm, while in (ii) it denotes a vector norm.

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