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Strong Chang's Conjecture and the tree property at ω_2 ☆

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ABSTRACT

We prove that a strong version of Chang's Conjecture together with $2^\omega = \omega_2$ implies there are no ω_2-Aronszajn trees.

1. Introduction

Given a regular cardinal κ, we say that κ has the *tree property* if every tree T of height κ and levels of size $< \kappa$, T has a cofinal branch, and it is usually denoted by $\mathrm{TP}(\kappa)$. Trees of height κ with levels of size $< \kappa$ with no cofinal branches are usually called *κ-Aronszajn*.

We list some historical results involving the Tree Property for different regular cardinals. König's Lemma gives some sufficient conditions for a tree to have a cofinal branch. He proved in [5] that $\mathrm{TP}(\omega)$ holds. However, Aronszajn showed that we cannot generalize König's Lemma for trees of height ω_1 by constructing an ω_1-Aronszajn tree (see [2]). Considering trees of height ω_2 with levels of size at most $\aleph_1$, it turns out to be independent from the usual axioms of Set Theory. We recall also the result by Silver, where if $\mathrm{TP}(\omega_2)$ holds, then $\aleph_2$ is weakly compact in L (Theorem 5.9 in [6]). On the other hand, Mitchell proved that if κ is a weakly compact, then there is a generic extension where $\kappa = \omega_2 = 2^\omega$ and $\mathrm{TP}(\omega_2)$ holds (see [6]). In particular, $\mathrm{TP}(\omega_2)$ is equiconsistent with the existence of a weakly compact cardinal.

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In these notes we work with a strong version of Chang's Conjecture (see Definition 3.1 of the present notes and also see Theorem 1.3 in [9] for an earlier reference) here denoted by CC^*. On one hand, Todorčević and Torres-Pérez proved that under a stronger version of CC^*, the negation of CH implies there are no special ω_2-Aronszajn trees (see [15]). On the other hand, Sakai and Velickovic proved that under SSR, a strengthening of CC^* (see [1]), the negation of CH together with $\mathrm{MA}_{\omega_1}(\mathrm{Cohen})$ implies the strong tree property at ω_2 and so in particular it implies $\mathrm{TP}(\omega_2)$ (see [8]).

We prove in these notes that CC^* and the negation of CH imply $\mathrm{TP}(\omega_2)$. Observe that by a result of Todorčević (see [14]), CC^* implies $2^\omega \leq \omega_2$, so under CC^*, $\neg\mathrm{CH}$ is equivalent to $2^\omega = \omega_2$.

We make a remark for the necessity of $\neg\mathrm{MA}_{\omega_1}(\mathrm{Cohen})$ in [8]:

Theorem 1.1 *(Folklore). Assume that there exists a strongly compact cardinal. Then there exists a forcing extension in which* $\mathrm{SSR} + \neg\mathrm{MA}_{\omega_1}(\mathrm{Cohen}) + \neg\mathrm{CH}$ *holds.*

The following fact is used:

Fact 1.1. *(Shelah [10], Chapter XIII, 1.6 and 1.10) Assume that κ is a strongly compact cardinal. Let $(P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa)$ be a revised countable support iteration of semi-proper posets of size $< \kappa$ such that $\kappa = \omega_2$ in V^{P_κ}. Then* SSR *holds in $V^{\mathbb{P}_\kappa}$.*

Assume that κ is strongly compact in V. Let $(P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa)$ be the countable support iteration of random forcing. Here recall that a revised countable support iteration coincides with a countable support iteration for proper posets. Note also that $\kappa = \omega_2$ in V^{P_κ}. Hence SSR holds in $V^{\mathbb{P}_\kappa}$ by the above fact. Moreover, $\mathrm{MA}_{\omega_1}(\mathrm{Cohen})$ fails in V^{P_κ} as adding random reals makes $\mathbf{non}(\mathcal{B})$ into ω_1.

2. Preliminaries and basic definitions

Given a limit ordinal γ, a subset $A \subseteq \gamma$ is *unbounded in* γ if $\sup(A) = \gamma$. A is *closed* in γ if for every limit ordinal $\beta < \gamma$, if $A \cap \beta$ is unbounded in β, then $\beta \in A$. A set $A \subseteq \gamma$ is often called a *club set in* γ if it is closed and unbounded in γ. A set $S \subseteq \gamma$ is stationary, if $S \cap A \neq \emptyset$ for every A club in γ.

The following result involving stationary sets is known as *Fodor's Lemma* or the *Pressing Down Lemma for ordinals.*

Lemma 2.1. *(Fodor [3]) Let κ be a regular uncountable cardinal. Then for every $S \subseteq \kappa$ stationary, and for every $f : S \to \kappa$ such that $f(\alpha) < \alpha$ for every $\alpha \in S$, there is $\xi < \kappa$ such that $f^{-1}(\{\xi\})$ is stationary.*

We use a general version of a stationary set, originally by Jech, but in these notes we use an equivalent version due to Kueker (see for example, Theorem 8.28 in [4]). Given an infinite set A, we denote by $[A]^{<\omega}$ the collection of finite subsets of A. Similarly, let $[A]^\omega$ denote the collection of all subsets of A of size ω. We say that a set $S \subseteq [A]^\omega$ is *stationary in* $[A]^\omega$ if for every function $F : [A]^{<\omega} \to A$, there is $X \in [A]^\omega$ such that $F(e) \in X$ for every $e \in [X]^{<\omega}$.

The following lemma is the generalized version of the Pressing Down Lemma (see Theorem 8.24 in [4]).

Lemma 2.2 *(Jech). For every stationary set $S \subseteq [A]^\omega$ and for every function $f : S \to A$ such that $f(X) \in X$ for every $X \in S$, there is $a \in A$ such that $f^{-1}(\{a\})$ is stationary.*

The couple $\langle T, <_T \rangle$ is a *tree* whenever $<_T$ is a partial order of T, and for every $t \in T$, the set $\{s \in T : s <_T t\}$ is well-ordered by $<_T$. Some times we may just write the tree T, assuming there is an implicit order. We denote by $\mathrm{pred}_T(t)$ the set of all the $<_T$-predecessors of t in T, and by $\mathrm{ht}_T(t) = \mathrm{o.t.}(\mathrm{pred}_T(t))$.

We will denote by $T_\xi = \{t \in T : \mathrm{ht}_T(t) = \xi\}$. Often we will just drop off the subindex T if the context is clear.

For $A, B \subseteq T$ we denote by $A \perp B$ if for every $s \in A$ and every $t \in B$, s and t are not comparable. Similarly, for $s, t \in T$ and $A \subseteq T$, let $s \perp t$ and $s \perp A$ iff $\{s\} \perp \{t\}$ and $\{t\} \perp A$ respectively.

Given an ordinal $\lambda \geq \omega_2$, we recall the *Weak Reflection Principle for* λ, WRP(λ).

Definition 2.1. WRP(λ) is the following statement: For any stationary subset S of $[\lambda]^\omega$, there is $X \subset \lambda$ such that

(1) $|X| = \omega_1$,
(2) $\omega_1 \subseteq X$ and $S \cap [X]^\omega$ is a stationary subset of $[X]^\omega$.

Todorčević showed the following (see Lemma 6 in [14]):

Lemma 2.3 *(Todorčević).* CC* *implies* WRP(ω_2)*.*

3. Main Theorem

In this section we prove our main result.

Theorem 3.1. *Under* CC**,* $\neg$CH *is equivalent to the tree property at* ω_2*.*

We follow very closely the proof of Theorem 2.2 in [15]. It is a classical result of Specker that TP(ω_2) implies $\neg$CH (see [11]).

Given two sets M^*, M we will denote by $M^* \sqsupseteq M$ iff $M^* \supseteq M$ and $M^* \cap \omega_1 = M \cap \omega_1$. Consider the following strong version of Chang's Conjecture:

Definition 3.1 *(*CC**).* There are arbitrarily large uncountable regular cardinals θ such that for every well-ordering $<$ of H_θ, and every countable elementary submodel $M \prec \langle H_\theta; \in, <\rangle$, and every ordinal $\eta < \omega_2$, there exists an elementary countable submodel $M^* \prec \langle H_\theta; \in, <\rangle$ such that $M^* \sqsupseteq M$ and $(M^* \cap \omega_2) \setminus \eta \neq \emptyset$.

We will need the following Proposition for the proof of Lemma 3.1, namely in Claim 3.1.

Proposition 3.1. *Let T be a κ-Aronszajn tree (κ a regular cardinal). Given a regular cardinal $\mu < \kappa$, consider a family of collection of nodes $\langle A_\xi : \xi \in X\rangle$ such that X contains a stationary set consisting of ordinals of cofinality at least μ, $A_\xi \subseteq T_\xi$ and $|A_\xi| < \mu$ for every $\xi \in X$. Then for every λ large enough such that $\{\kappa, T, X, \langle A_\xi : \xi \in X\rangle, \ldots\} \subset H_\lambda$ and for every elementary submodel $N \prec \langle H_\lambda; \in, <, \kappa, T, X, \langle A_\xi : \xi \in X\rangle, \ldots\rangle$ such that $A_\xi \subseteq N$ for every $\xi \in X \cap N$, then for every $t \in T$ of height at least* $\sup(N \cap \kappa)$ *there are unboundedly many (in* $\sup(N \cap \kappa)$*) $\xi \in X \cap N$ such that every $s \in A_\xi$ is incomparable with t.*

Proof. Suppose otherwise, and take $t \in T$ of height at least $\sup(N \cap \kappa)$ and $\alpha \in N$ such that for all $\xi \in X \cap N \setminus \alpha$, there is a node $t_\xi \in A_\xi$ such that $t_\xi \leq_T t$. Without loss of generality, we can suppose that X is a stationary set consisting of ordinals greater than α and of cofinality at least μ.

Since $|A_\xi| < \mu$ for any $\xi \in X$, there is an ordinal $\beta_\xi < \xi$ such that for any $s, s' \in A_\xi$, $s = s' \leftrightarrow s \restriction \beta_\xi = s' \restriction \beta_\xi$. By elementarity and using Fodor's Lemma, we can find $\beta \in N \cap X$ and a stationary set $S \in N$ such that for any $\xi \in S$, $s = s' \leftrightarrow s \restriction \beta = s' \restriction \beta$ for any $s, s' \in A_\xi$.

Then for every $s \in A_\beta$, we can define a function $f_s : S \to T$ such that $f_s(\xi)$ is the unique $s_\xi \in A_\xi$ such that $s_\xi > s$. Since $A_\beta \subseteq N$, in particular $s = t \restriction \beta \in N$, and therefore f_s is defined in N. However, by our initial assumption, $f_s(\xi) = t_\xi$ for every $\xi \in S \cap N$, and so f_s defines in N a cofinal branch of T, contradiction. $\square$

Let T be an ω_2-Aronszajn tree. In order to simplify the proof, without loss of generality, we suppose that $T \subseteq \omega_2$ and let $e : \omega_2 \times \omega_1 \to T$ be a bijective function such that $e(\delta, \xi) \in T_\delta$ for every $(\delta, \xi) \in \omega_2 \times \omega_1$. Let θ be sufficiently large such that T, e and all relevant parameters are members of H_θ.

Lemma 3.1. *Assume* CC* *and that T is an ω_2-Aronszajn tree. For every $M \prec H_\theta$ countable, and for every $\eta_0, \eta_1 \in \omega_2$, we can find $M_0, M_1 \prec H_\theta$ countable such that:*

(1) $M \cap \omega_1 = M_0 \cap \omega_1 = M_1 \cap \omega_1$,
(2) $M_0 \cap \omega_2 \setminus \eta_0 \neq \emptyset$ *and* $M_1 \cap \omega_2 \setminus \eta_1 \neq \emptyset$,
(3) $\exists \delta_0 \in (M_0 \cap \omega_2)$ *and* $\delta_1 \in (M_1 \cap \omega_2)$ *such that* $(M_0 \cap T_{\delta_0}) \perp (M_1 \cap T_{\delta_1})$.

Proof. Fix $\lambda > \theta$ sufficiently large such that CC* holds in H_λ and M, η_0, η_1 and all relevant parameters are in H_λ. Let $N \prec H_\lambda$ such that if $\gamma = \sup(N \cap \omega_2)$, then $\mathrm{cof}(\gamma) = \omega_1$.

Fix M_1 witnessing CC* for M and γ.

We need the following Claim:

Claim 3.1. *For every $t \in T$ of height at least γ, there is $M^* \sqsupseteq M$ with $M^* \in N$ and $\beta \in M^* \cap \omega_2$ such that $t \perp T_\beta \cap M^*$.*

Proof. Assume otherwise, and take $t \in T$ of height at least γ such that for every $M^* \in N$ with $M^* \sqsupseteq M$, for each $\beta \in M^* \cap \omega_2$, there is an $s_\beta \in (T_\beta \cap M^*)$ such that $s_\beta < t$.

We work inside N in this paragraph. Using that CC* holds in N, build a sequence of models $\langle M_\eta : \eta \in \omega_2\rangle$ such that $M_\eta \sqsupseteq M$ and $M_\eta \cap \omega_2 \setminus \eta \neq \emptyset$ for every $\eta \in \omega_2$. Let β_ξ be the minimum $\beta \in \omega_2 \setminus \xi$ such that there is $\eta \in \omega_2$ such that $\beta_\xi = \min(M_\eta \cap \omega_2 \setminus \eta)$. Let η_ξ be the minimum $\eta \in \omega_2$ such that $\beta_\xi = \min(M_\eta \cap \omega_2 \setminus \eta)$. Define $\langle A_\xi : \xi \in \omega_2\rangle$ by setting A_ξ to be the set of nodes r in T_ξ with $r \leq s$ for some $s \in M_{\eta_\xi} \cap T_{\beta_\xi}$. Remark that since M_{η_ξ} is countable, so is A_ξ.

By Proposition 3.1, there are unboundedly many $\xi \in N \cap \omega_2$ such that $t \perp A_\xi$, so choose one of such ξ's. Then there is $s \in M_{\eta_\xi} \cap T_{\beta_\xi}$ such that $s <_T t$. Thus there is $r \in A_\xi$ such that $r \leq_T s <_T t$, contradicting that r and t are incomparable. $\square$

Let $\{t_n : n \in \omega\}$ be an enumeration of $M_1 \cap T \setminus \gamma$. Using Claim 3.1, build a $\subseteq$-increasing sequence $\langle M_n^0 : n \in \omega\rangle$ of countable elementary submodels of H_θ such that for every $n \in \omega$, $M_n^0 \in N$ and $M_n^0 \sqsupseteq M$, and such that there is $\beta \in M_n^0 \cap \omega_2$ with $t_n \perp M_n^0 \cap T_\beta$. Let M_0 be an end-extension of $\bigcup_{n<\omega} M_n^0$ derived from CC* and η_0. Let $\delta_0 = \min(M_0 \cap \omega_2 \setminus \eta_0)$ and $\delta_1 = \min(M_1 \cap \omega_2 \setminus \gamma)$. We claim it suffices.

Take $s \in T_{\delta_0} \cap M_0$ and $t \in T_{\delta_1} \cap M_1$. In particular, there is $n \in \omega$ and $\beta \in M_n^0 \cap \omega_2$ such that $t = t_n$ and $t \perp T_\beta \cap M_n^0$. Since $\beta \in M_n^0 \subseteq M_0$, we have $s\restriction_\beta \in M_0$. Moreover, since the enumeration function $e \in M_n^0 \subseteq M_0$ and $M_n^0 \cap \omega_1 = M_0 \cap \omega_1$, we have $T_\beta \cap M_0 = T_\beta \cap M_n^0$ and so $s\restriction_\beta \in M_n^0$. Therefore $s\restriction_\beta$ is not comparable with t, and so neither are s and t.

This finishes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Assume* CC*. *Let T be an ω_2-Aronszajn tree. If the set*

$$S_T = \{A \in [\omega_2]^\omega : \forall t \in T(\mathrm{pred}(t) \cap A \textit{ is bounded in } \sup(A))\}$$

is nonstationary, then CH *holds.*

Proof. Let $f : [\omega_2]^{<\omega} \to \omega_2$ such that the set C_f of closure points of f (i.e. $X \in C_f$ iff for every $e \in [X]^{<\omega}$, $f(e) \in X$) is disjoint with S_T. We can suppose that $T \subseteq \omega_2$ and $e : \omega_1 \times \omega_2 \to T$ is a bijection such that $e(\delta, \beta) \in T_\delta$. Let λ be sufficiently large such that T, S_T, f, e and all relevant parameters are members of H_λ.

Using previous lemma, build a binary tree $\langle M_\sigma \rangle_{\sigma \in 2^{<\omega}}$ of countable elementary submodels of H_λ with the property that for every $\sigma \in 2^{<\omega}$

(1) $M_\sigma \cap \omega_1 = M_{\sigma\frown 0} \cap \omega_1 = M_{\sigma\frown 1} \cap \omega_1$,
(2) $M_\sigma \cap \omega_2 \subsetneq M_{\sigma\frown 0} \cap \omega_2$ and $M_\sigma \cap \omega_2 \subsetneq M_{\sigma\frown 1} \cap \omega_2$,
(3) there exists $\delta_0 \in (M_{\sigma\frown 0} \cap \omega_2)$ and $\delta_1 \in (M_{\sigma\frown 1} \cap \omega_2)$ such that $T_{\delta_0} \cap M_{\sigma\frown 0} \perp T_{\delta_0} \cap M_{\sigma\frown 1}$,
(4) for every $r \in 2^\omega$, if $M_r = \bigcup_{n\in\omega} M_{r\restriction n}$, then for every $r, r' \in 2^\omega$, $\sup(M_r \cap \omega_2) = \sup(M_{r'} \cap \omega_2)$.

Let δ be the common supremum of every $M_r \cap \omega_2$, $r \in 2^\omega$. Then for every $r \in 2^\omega$, there is $t_r \in T_\delta \cap M_r$ such that for every $\mathrm{pred}(t_r) \cap M_r$ is unbounded in δ.

Claim 3.2. *The application $r \mapsto t_r$ is an injection from 2^ω to T_δ (and so* CH *does hold).*

Proof. Let $r_0, r_1 \in 2^\omega$ with $r_0 \neq r_1$ and denote by t_i the node t_{r_i} for $i \in \{0, 1\}$. We will find two predecessors of t_0 and t_1 that are incomparable.

Let $n \in \omega$ such that $r_0\restriction_n = r_1\restriction_n = \sigma$, and $r_0\restriction_{n+1} \neq r_1\restriction_{n+1}$. Without loss of generality suppose $r_i(n) = i$ for $i \in \{0, 1\}$.

Since $(M_{r_i} \cap \omega_2) \notin S_T$, we can find $s_i <_T t_i$ with $s_i \in M_{r_i\restriction_{m_i}}$ for some $m_i > n$. By the construction of our binary tree, we can take $\delta_0 \in M_{r_0\restriction_{n+1}}$ and $\delta_1 \in M_{r_1\restriction_{n+1}}$ such that $T_{\delta_0} \cap M_{r_0\restriction_{n+1}} \perp T_{\delta_1} \cap M_{r_1\restriction_{n+1}}$. However, observe that for $i \in \{0, 1\}$, $\delta_i \in M_{r_i\restriction_{n+1}} \subseteq M_{r_1}$, and so $t_i\restriction_{\delta_i} \in M_{r_i\restriction_{n+1}}$. Therefore, $t_0\restriction_{\delta_0}$ and $t_1\restriction_{\delta_1}$ are incomparable, and so $t_0 \neq t_1$. □

This finishes the proof of Lemma 3.2. □

We are now ready to finish the proof of our Theorem. From the previous lemma we know that the set S_T is stationary in $[\omega_2]^{\omega_0}$. Let $S'_T = S_T \cap C_e$, where C_e is the club of all countable subsets of ω_2 closed under the level enumeration function e of T.

We now use that CC* implies WRP(ω_2) (Lemma 2.3). Take $X \subseteq \omega_2$ of size $\aleph_1$ such that $\omega_1 \subseteq X$ and where $S'_T \cap [X]^\omega$ is stationary. Take $t \in T$ of height at least $\sup(X)$.

From the definition of S_T, for every $A \in S'_T \cap [X]^\omega$ we can choose $\beta_A \in A$ such that if $s \in \mathrm{pred}(T) \cap A$, then $s < \beta_A$. By the Pressing Down Lemma, there is a stationary set $S \subseteq S'_T \cap [X]^\omega$ and a β such that $\beta_A = \beta$ for all $A \in S$. Let $\xi \in \omega_1$ such that $e(\beta, \xi) = t\restriction_\beta$. Observe that S is in particular cofinal in $[X]^\omega$ so $\bigcup S = X$. Since $\omega_1 \subseteq X$, pick $A \in S$ such that $\xi \in A$. Therefore, $e(\beta, \xi) \in A \cap \mathrm{pred}(t)$, and so $e(\beta, \xi) < \beta$. But this is a contradiction, since in general $e(\beta, \xi) \geq \beta$ for any $\beta \in \omega_2$. This ends the proof of our Theorem.

4. Some final remarks

We mention some related previous results. R. Strullu proved that the Map Reflection Principle, introduced by Moore in [7], together with MA_{ω_1} implies TP(ω_2) (see [12]). Also it is implicit in B. Velickovic and H. Sakai's results ([8]) that WRP(ω_2) + MA_{ω_1}(Cohen) implies TP(ω_2).

We remark that the results in [15] were in the context of Rado's Conjecture (RC), which is the following statement in Todorčević's equivalent version:

Definition 4.1 *(*RC*).* Every tree T of height $\aleph_1$ is special, i.e., the countable union of antichains if and only if every subtree of T of size $\aleph_1$ is also special.

Todorčević proved via a large cardinal that RC is consistent, and showed it is independent from ZFC. In particular, RC is not compatible with MA_{ω_1} (see final remarks in [13]).

As we have mentioned, in [15], it was proved that Rado's Conjecture together with the negation of the Continuum implies there are no special ω_2-Aronszajn trees. One natural question was which extra condition we could add to Rado's Conjecture to obtain that there are no ω_2-Aronszajn trees at all. Since Rado's Conjecture is consistent with both CH and ¬CH, and CH implies $\neg\mathrm{TP}(\omega_2)$, we needed at least to add the condition ¬CH to RC if we wanted to obtain $\mathrm{TP}(\omega_2)$. However, as we have mentioned, RC is not consistent with MA_{ω_1}, so we could not have similar results as the one cited above.

Todorčević proved in [14] that RC implies CC^*. Therefore, a consequence of the result in the present paper is that the condition ¬CH was not only needed, but also sufficient to add to RC to get $\mathrm{TP}(\omega_2)$.

Corollary 4.1. RC *and* ¬CH *imply* $\mathrm{TP}(\omega_2)$.

As we have mentioned, Todorčević proved in [14] that CC^* implies $\mathrm{WRP}(\omega_2)$. The following question is still open.

Question 4.1. *Do* $\mathrm{WRP}(\omega_2)$ *and* ¬CH *imply together* $\mathrm{TP}(\omega_2)$*?*

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