On the Breiman conjecture

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Abstract

Let $Y_1, Y_2, \ldots$ be positive, nondegenerate, i.i.d. G random variables, and independently let $X_1, X_2, \ldots$ be i.i.d. F random variables. In this note we show that whenever $\sum X_i Y_i / \sum Y_i$ converges in distribution to nondegenerate limit for some $F \in \mathcal{F}$, in a specified class of distributions $\mathcal{F}$, then G necessarily belongs to the domain of attraction of a stable law with index less than 1. The class $\mathcal{F}$ contains those nondegenerate X with a finite second moment and those X in the domain of attraction of a stable law with index $1 < \alpha < 2$.

1 Introduction and results

Let $Y, Y_1, \ldots$ be positive, nondegenerate, i.i.d. random variables with distribution function [df] G, and independently let $X, X_1, \ldots$ be i.i.d. nondegenerate random variables with df F. Let ϕ_X denote the characteristic function [cf] of X. We shall use the notation $Y \in D(\beta)$ to mean that Y is in the domain of attraction of a stable law of index $0 < \beta < 1$, and $Y \in D(0)$ will denote that $1 - G$ is slowly varying at infinity. Furthermore $\mathcal{RV}_\infty(\rho)$ will signify the class of positive measurable functions regularly varying at infinity with index ρ, and $\mathcal{RV}_0(\rho)$ the class of positive measurable functions regularly varying at zero with index ρ. In particular, using this notation $Y \in D(\beta)$, with $0 \le \beta < 1$, if and only if $\overline{G} := 1 - G \in \mathcal{RV}_\infty(-\beta)$.

For each integer $n \ge 1$ set

$$T_n = \sum_{i=1}^n X_i Y_i / \sum_{i=1}^n Y_i. \tag{1}$$

Notice that $\mathbb{E}|X| < \infty$ implies that T_n is stochastically bounded. Theorem 4 of Breiman [2] says that T_n converges in distribution along the full sequence $\{n\}$ for *every* X with finite expectation, and with at least one limit law being nondegenerate if and only if

$$Y \in D(\beta), \text{ with } 0 \le \beta < 1. \tag{2}$$

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Let $\mathcal{X}$ denote the class of nondegenerate random variables X with $\mathbb{E}|X| < \infty$ and let $\mathcal{X}_0$ denote those $X \in \mathcal{X}$ such that $\mathbb{E}X = 0$. At the end of his paper Breiman conjectured that if for *some* $X \in \mathcal{X}$, T_n converges in distribution to some nondegenerate random variable T, written

$$T_n \to_d T, \text{ as } n \to \infty, \text{ with } T \text{ nondegenerate}, \tag{3}$$

then (2) holds. By Proposition 2 (in the case $\beta = 0$) and Theorem 3 (in the case $0 < \beta < 1$) of [2], for any $X \in \mathcal{X}$, (2) implies (3), in which case T, in the case $0 < \beta < 1$, has a distribution related to the arcsine law. Using this fact, we see that his conjecture can restated to be: for any $X \in \mathcal{X}$, (2) is equivalent to (3).

It has proved to be surprisingly challenging to resolve. Mason and Zinn [8] partially verified Breiman's conjecture. They established that whenever X is nondegenerate and satisfies $\mathbb{E}|X|^p < \infty$ for some $p > 2$, then (2) is equivalent to (3). In this note we further extend this result.

Theorem *Assume that for some $X \in \mathcal{X}_0$, $1 < \alpha \le 2$, positive slowly varying function L at zero and $c > 0$,*

$$\frac{-\log\left(\mathfrak{Re}\, \phi_X(t)\right)}{|t|^{\alpha} L\left(|t|\right)} \to c, \text{ as } t \to 0. \tag{4}$$

Whenever (3) holds then $Y \in D(\beta)$ for some $\beta \in [0, 1)$.

Let $\mathcal{F}$ denote the class of random variables that satisfy the conditions of the theorem. Applying our theorem in combination with Proposition 2 and Theorem 3 of [2] we get the following corollary.

Corollary *Whenever $X - \mathbb{E}X \in \mathcal{F}$, (2) is equivalent to (3).*

Remark 1 It can be inferred from Theorem 8.1.10 of Bingham et al. [1] (see also Theorem 1 and 5 of Pitman [9]) that for $X \in \mathcal{X}_0$, (4) holds for some $1 < \alpha < 2$, positive slowly varying function L at zero and $c > 0$ if and only if X satisfies $\mathbb{P}\left\{|X| > x\right\} \sim L(1/x)x^{-\alpha}c\Gamma(\alpha)\frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)$. Note that a random variable $X \in \mathcal{X}_0$ in the domain of attraction of a stable law of index $1 < \alpha < 2$ satisfies (4). For $\alpha = 2$ there is no simple condition equivalent to (4). By Theorem 5 of Pitman [9] for $\alpha = 2$ condition (4) implies that

$$\frac{1 - \mathfrak{Re}\, \phi_X(t)}{t^2} \sim \int_0^{t^{-1}} u\mathbb{P}\{|X| > u\}\mathrm{d}u, \quad \text{as } t \downarrow 0. \tag{5}$$

Also a random variable $X \in \mathcal{X}_0$ with variance $0 < \sigma^2 < \infty$ fulfills (4) with $\alpha = 2$, $L = 1$ and $c = \sigma^2/2$. Theorem 3 in [9] states that $\mathbb{P}\{|X| > x\} \in \mathcal{RV}_\infty(-2)$ implies (5), from which, combined with Proposition 1.5.9a [1], condition (4) follows.

Remark 2 Consult Kevei and Mason [7] for a fairly exhaustive study of the asymptotic distributions of T_n along subsequences, along with revelations of their unexpected properties.

The theorem follows from the two propositions below. First we need more notation. For any $\alpha \in (1, 2]$ define for $n \ge 1$

$$S_n(\alpha) = \frac{\sum_{i=1}^n Y_i^\alpha}{\left(\sum_{i=1}^n Y_i\right)^\alpha}. \tag{6}$$

Proposition 1 *Assume that the assumptions of the theorem hold. Then for some $0 < \gamma \le 1$*

$$\mathbb{E}S_n(\alpha) \to \gamma, \text{ as } n \to \infty. \tag{7}$$

The next proposition is interesting in its own right. It is an extension of Theorem 5.3 by Fuchs et al. [4], where $\alpha = 2$ (see also Proposition 3 of [8]).

Proposition 2 *If (7) holds with some $\gamma \in (0,1]$ then $Y \in D(\beta)$, for some $\beta \in [0,1)$, where $-\beta \in (-1,0]$ is the unique solution of*

$$\mathrm{Beta}(\alpha-1, 1-\beta) = \frac{\Gamma(\alpha-1)\Gamma(1-\beta)}{\Gamma(\alpha-\beta)} = \frac{1}{\gamma(\alpha-1)}.$$

In particular, $Y \in D(0)$ for $\gamma = 1$.

Conversely, if $Y \in D(\beta)$, $0 \le \beta < 1$, then (7) holds with

$$\gamma = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1-\beta)} = \frac{1}{(\alpha-1)\mathrm{Beta}(\alpha-1, 1-\beta)}.$$

2 Proofs

Set for each $n \ge 1$, $R_i = Y_i / \sum_{l=1}^{n} Y_l$, for $i = 1, \dots, n$. For notational ease we drop the dependence of R_i on $n \ge 1$. Consider the sequence of strictly decreasing continuous functions $\{\varphi_n\}_{n\ge1}$ on $[1,\infty)$ defined by $\varphi_n(y) = \mathbb{E}\left(\sum_{i=1}^{n} R_i^y\right)$, $y \in [1,\infty)$. Note that each function φ_n satisfies $\varphi_n(1) = 1$. By a diagonal selection procedure for each subsequence of $\{n\}_{n\ge1}$ there is a further subsequence $\{n_k\}_{k\ge1}$ and a right continuous nonincreasing function ψ such that φ_{n_k} converges to ψ at each continuity point of ψ.

Lemma 1 *Each such function ψ is continuous on $(1,\infty)$.*

Proof Choose any subsequence $\{n_k\}_{k\ge1}$ and a right continuous nonincreasing function ψ such that φ_{n_k} converges to ψ at each continuity point of ψ in $(1,\infty)$. Select any $x > 1$ and continuity points $x_1, x_2 \in (1,\infty)$ of ψ such that $1 < x_1 < x < x_2 < \infty$. Set $\rho_1 = x_1 - 1$ and $\rho_2 = x_2 - 1$. Since $\rho_2/\rho_1 > 1$ we get by Hölder's inequality

$$\sum_{i=1}^{n_k} R_i^{x_1} = \sum_{i=1}^{n_k} R_i^{\rho_1} R_i \le \left(\sum_{i=1}^{n_k} R_i^{\rho_2} R_i\right)^{\rho_1/\rho_2} = \left(\sum_{i=1}^{n_k} R_i^{x_2}\right)^{\rho_1/\rho_2}.$$

Thus by taking expectations and using Jensen's inequality we get $\varphi_{n_k}(x_1) \le (\varphi_{n_k}(x_2))^{\rho_1/\rho_2}$. Letting $n_k \to \infty$, we have $\psi(x_1) \le (\psi(x_2))^{\rho_1/\rho_2}$. Since $x_1 < x$ and $x_2 > x$ can be chosen arbitrarily close to x we conclude by right continuity of ψ at x that $\psi(x-) = \psi(x+) = \psi(x)$. □

Proof of Proposition 1 For a complex z, we use the notation for the principal branch of the logarithm, $Log(z) = \log|z| + \imath \arg z$, where $-\pi < \arg z \le \pi$, i.e. $z = |z| \exp(\imath \arg z)$. We see that for all t

$$\mathbb{E}\exp(\imath tT_n) = \mathbb{E}\left(\prod_{j=1}^{n}\phi_X(tR_j)\right)$$
$$= \mathbb{E}\left(\prod_{j=1}^{n}\exp\left(Log\phi_X(tR_j)\right)\right).$$

Since $\mathbb{E}X = 0$ we have $\mathfrak{Re}\,\phi_X(u) = 1 - o_+(u)$, where $o_+(u) \geq 0$, and $o_+(u)/u \to 0$ as $u \to 0$; and $\mathfrak{Im}\,\phi_X(u) = o(u)$. This when combined with

$$(\arctan\theta)' = \frac{1}{1+\theta^2}$$

gives as $u \to 0$,

$$\arg\phi_X(u) = \arctan\left(\frac{\mathfrak{Im}\,\phi_X(u)}{\mathfrak{Re}\,\phi_X(u)}\right) = o(u).$$

Note that for all $|u| > 0$ sufficiently small so that $\mathfrak{Re}\,\phi_X(u) > 0$

$$Log\phi_X(u) = Log(\mathfrak{Re}\,\phi_X(u) + \imath\mathfrak{Im}\,\phi_X(u)) = \log\mathfrak{Re}\,\phi_X(u) + Log\left(1 + \imath\frac{\mathfrak{Im}\,\phi_X(u)}{\mathfrak{Re}\,\phi_X(u)}\right),$$

where for the second term

$$\mathfrak{Re}\,Log\left(1 + \imath\frac{\mathfrak{Im}\,\phi_X(u)}{\mathfrak{Re}\,\phi_X(u)}\right) = \frac{1}{2}\left(\frac{\mathfrak{Im}\,\phi_X(u)}{\mathfrak{Re}\,\phi_X(u)}\right)^2(1 + o(u)),\ \text{as } u \to 0.$$

Thus for every $\varepsilon > 0$ for all $|t| > 0$ sufficiently small and independent of $n \geq 1$ and $R_1, \ldots, R_n$

$$1 - \varepsilon^2 t^2 \leq \cos(\varepsilon t) \leq \mathfrak{Re}\left(\exp\left\{\sum_{j=1}^{n} Log\left(1 + \imath\frac{\mathfrak{Im}\,\phi_X(tR_j)}{\mathfrak{Re}\,\phi_X(tR_j)}\right)\right\}\right) \leq e^{2^{-1}\varepsilon t^2} \leq 1 + \varepsilon t^2.$$

Thus we obtain

$$\mathbb{E}\exp\left\{\sum_{j=1}^{n}\log\mathfrak{Re}\,\phi_X(tR_j)\right\}\left(1 - \varepsilon^2 t^2\right) \leq \mathbb{E}\left(\mathfrak{Re}\,\exp(\imath tT_n)\right)$$
$$= \mathfrak{Re}\,\mathbb{E}\exp(\imath tT_n)$$
$$\leq \mathbb{E}\exp\left\{\sum_{j=1}^{n}\log\mathfrak{Re}\,\phi_X(tR_j)\right\}(1 + \varepsilon t^2).$$

We shall show (4) implies that (7) holds for some $0 < \gamma \leq 1$. Now using (4) we get for any $0 < \delta < c$ and all $|t|$ small enough independent of $n \geq 1$,

$$-\varepsilon t^2 + \log\mathbb{E}\exp\left(-(c+\delta)|t|^\alpha\left(\sum_{i=1}^{n} R_i^\alpha L(|t|R_i)\right)\right) \leq \log\left(\mathfrak{Re}\,\mathbb{E}\exp(\imath tT_n)\right)$$
$$\leq \varepsilon t^2 + \log\mathbb{E}\exp\left(-(c-\delta)|t|^\alpha\left(\sum_{i=1}^{n} R_i^\alpha L(|t|R_i)\right)\right).$$

Next since $\log s/(1-s) \to -1$ as $s \nearrow 1$, for all $|t|$ small enough independent of $n \geq 1$ and $R_1, \ldots, R_n$, (keeping in mind that $\sum_{i=1}^{n} R_i = 1$ and $1 < \alpha \leq 2$)

$$\begin{aligned}
&\log \mathbb{E} \exp \left(-(c+\delta) |t|^{\alpha} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right) \right) \\
&\geq -\left(1+\frac{\delta}{2}\right) \mathbb{E} \left(1 - \exp \left(-(c+\delta) |t|^{\alpha} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right) \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
&\log \mathbb{E} \exp \left(-(c-\delta) |t|^{\alpha} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right) \right) \\
&\leq -\left(1-\frac{\delta}{2}\right) \mathbb{E} \left(1 - \exp \left(-(c-\delta) |t|^{\alpha} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right) \right) \right).
\end{aligned}$$

Further since $(1-\exp(-y))/y \to 1$ as $y \searrow 0$, for all $|t|$ small enough independent of $n \geq 1$,

$$\begin{aligned}
&-\left(1+\frac{\delta}{2}\right) \mathbb{E} \left(1 - \exp \left(-(c+\delta) |t|^{\alpha} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right) \right) \right) \\
&\geq -(1+\delta)(c+\delta) |t|^{\alpha} \mathbb{E} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right)
\end{aligned}$$

and

$$\begin{aligned}
&-\left(1-\frac{\delta}{2}\right) \mathbb{E} \left(1 - \exp \left(-(c-\delta) |t|^{\alpha} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right) \right) \right) \\
&\leq -(1-\delta)(c-\delta) |t|^{\alpha} \mathbb{E} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right).
\end{aligned}$$

Therefore for all $|t|$ small enough independent of n,

$$\begin{aligned}
&-\varepsilon t^2 - (1+\delta)(c+\delta) |t|^{\alpha} \mathbb{E} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right) \\
&\leq \log \left(\mathfrak{Re}\, \mathbb{E} \exp (\imath t T_n) \right) \\
&\leq \varepsilon t^2 - (1-\delta)(c-\delta) |t|^{\alpha} \mathbb{E} \left(\sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) \right).
\end{aligned}$$

By the Potter's bound, Theorem 1.5.6 (i) in [1], for all $A > 1$ and $1 < \alpha_1 < \alpha < \alpha_2$, for all $t > 0$ small enough independent of $n \geq 1$,

$$A^{-1} \sum_{i=1}^{n} R_i^{\alpha_2} \leq \sum_{i=1}^{n} R_i^{\alpha} L(|t| R_i) / L(|t|) \leq A \sum_{i=1}^{n} R_i^{\alpha_1}. \tag{8}$$

We see now that for all $n \geq 1$ and $0 < 4\varepsilon < c$, appropriate $1 < \alpha_1 < \alpha < \alpha_2$ and all $|t|$ small enough independent of n,

$$\begin{aligned}
&-\varepsilon t^2 - (1+\varepsilon)(c+2\varepsilon)|t|^\alpha L(|t|)\,\mathbb{E}S_n(\alpha_2) \\
&= -\varepsilon t^2 - (1+\varepsilon)(c+2\varepsilon)|t|^\alpha L(|t|)\,\mathbb{E}\left(\sum_{i=1}^n R_i^{\alpha_2}\right) \\
&\leq \log\left(\mathfrak{Re}\,\mathbb{E}\exp(\imath t T_n)\right) \\
&\leq \varepsilon t^2 - (1-\varepsilon)(c-2\varepsilon)|t|^\alpha L(|t|)\,\mathbb{E}\left(\sum_{i=1}^n R_i^{\alpha_1}\right) \\
&= \varepsilon t^2 - (1-\varepsilon)(c-2\varepsilon)|t|^\alpha L(|t|)\,\mathbb{E}S_n(\alpha_1).
\end{aligned}$$

Choose any subsequence $\{n_k\}_{k\geq 1}$ and a right continuous nonincreasing function ψ such that φ_{n_k} converges to ψ at each continuity point of ψ, which by Lemma 1 above is all $(1,\infty)$. We see that $\mathbb{E}S_{n_k}(\alpha) \to \psi(\alpha)$, $\mathbb{E}S_{n_k}(\alpha_1) \to \psi(\alpha_1)$ and $\mathbb{E}S_{n_k}(\alpha_2) \to \psi(\alpha_2)$, where necessarily $0 < \psi(\alpha_2) \leq \psi(\alpha) \leq \psi(\alpha_1) \leq 1$. We see that for all $|t|$ sufficiently small independent of the subsequence $n_k \geq 1$,

$$\begin{aligned}
-\varepsilon t^2 - (1+\varepsilon)(c+3\varepsilon)|t|^\alpha L(|t|)\psi(\alpha_2) &\leq \log\left(\mathfrak{Re}\,\mathbb{E}\exp(\imath t T)\right) \\
&\leq \varepsilon t^2 - (1-\varepsilon)(c-3\varepsilon)|t|^\alpha L(|t|)\psi(\alpha_1),
\end{aligned} \tag{9}$$

where T is the nondegenerate limit in (3). Note that if $\psi(\alpha_1) = 0$ then because of monotonicity $\psi(\alpha_2) = 0$, so we would have $\lim_{t\to 0} t^{-2}\mathbb{E}[1-\cos(tT)] = 0$, which by an easy argument based on a classical probability inequality (see Lemma 1, p. 268 of Chow and Teicher [3]), implies that $\mathbb{P}\{T = 0\} = 1$, contrary to our assumptions. Therefore $\psi(\alpha_1) > 0$.

From (9) we obtain $|t|$ sufficiently small independent of the subsequence $n_k \geq 1$,

$$\begin{aligned}
-\varepsilon - (1+\varepsilon)(c+3\varepsilon)\psi(\alpha_2) &\leq \log\left(\mathfrak{Re}\,\mathbb{E}\exp(\imath t T_{n_k})\right)/\left(|t|^\alpha L(|t|)\right) \\
&\leq \varepsilon - (1-\varepsilon)(c-3\varepsilon)\psi(\alpha_1),
\end{aligned}$$

where for $\alpha = 2$ we use that $\liminf_{t\searrow 0} L(t) > 0$; see Remark 1. Since $0 < 4\varepsilon < c$ can be made arbitrarily small and $0 \leq \psi(\alpha_1) - \psi(\alpha_2)$ can be made as close to zero as desired, by letting $n_k \to \infty$, we get that for all $|t|$ sufficiently small

$$-\varepsilon - (1+\varepsilon)(c+4\varepsilon)\psi(\alpha) \leq \log\left(\mathfrak{Re}\,\mathbb{E}\exp(\imath t T)\right)/\left(|t|^\alpha L(|t|)\right) \leq \varepsilon - (1-\varepsilon)(c-4\varepsilon)\psi(\alpha),$$

which can happen only if $\psi(\alpha)$ does not depend on $\{n_k\}$. Thus (7) holds for some $0 < \gamma \leq 1$, namely $\gamma = \psi(\alpha)$. □

Proof of Proposition 2 To begin with, we note that whenever (7) holds, necessarily $\mathbb{E}Y = \infty$. To see this, write $D_n^{(1)} = \max_{1\leq i\leq n} Y_i/\left(\sum_{i=1}^n Y_i\right)$ and observe that

$$\begin{aligned}
\left(D_n^{(1)}\right)^\alpha &= \max_{1\leq i\leq n} \frac{Y_i^\alpha}{\left(\sum_{i=1}^n Y_i\right)^\alpha} \leq S_n(\alpha) \\
&\leq \max_{1\leq i\leq n} \frac{Y_i^{\alpha-1}}{\left(\sum_{i=1}^n Y_i\right)^{\alpha-1}} = \left(D_n^{(1)}\right)^{\alpha-1}.
\end{aligned}$$

From these inequalities it is easy to prove that $\mathbb{E}S_n(\alpha) \to 0$, $n \to \infty$, if and only if

$$D_n^{(1)} \to_P 0, \; n \to \infty. \tag{10}$$

Proposition 1 of Breiman [2] says that (10) holds if and only there exists a sequence of positive constants B_n converging to infinity such that

$$\sum_{i=1}^{n} Y_i / B_n \to_P 1, \; n \to \infty. \tag{11}$$

Since $\mathbb{E}Y < \infty$ obviously implies (11), it readily follows that $\mathbb{E}S_n(\alpha) \to 0$, $n \to \infty$, and thus (7) cannot hold.

We shall first prove the first part of Proposition 2. Following similar steps as in [8] we have that

$$\begin{aligned}
\mathbb{E}\frac{\sum_{i=1}^n Y_i^\alpha}{(\sum_{i=1}^n Y_i)^\alpha} &= n\mathbb{E}\frac{Y_1^\alpha}{(\sum_{i=1}^n Y_i)^\alpha} \\
&= \frac{n}{\Gamma(\alpha)}\mathbb{E}\int_0^\infty Y_1^\alpha e^{-t\sum_{i=1}^n Y_i} t^{\alpha-1}\mathrm{d}t \\
&= \frac{n}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathbb{E}\left(e^{-tY_1}Y_1^\alpha\right)(\mathbb{E}e^{-tY_1})^{n-1}\mathrm{d}t \\
&=: \frac{n}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\phi_\alpha(t)\phi_0(t)^{n-1}\mathrm{d}t.
\end{aligned}$$

Next, assuming (7) and arguing as in the proof of Theorem 3 in [2] we get

$$s\int_0^\infty t^{\alpha-1}\phi_\alpha(t)e^{s\log\phi_0(t)}\mathrm{d}t \to \gamma\Gamma(\alpha), \quad s \to \infty, \tag{12}$$

where $0 < \gamma \leq 1$. For $y \geq 0$, let $q(y)$ denote the inverse of $-\log\varphi_0(t)$. Changing the variables to $y = -\log\phi_0(t)$ and $t = q(y)$, we get from (12) that

$$s\int_0^\infty (q(y))^{\alpha-1}\,\phi_\alpha\left(q(y)\right)\exp\left(-sy\right)\mathrm{d}q(y) \to \gamma\Gamma(\alpha), \text{ as } s \to \infty.$$

By Karamata's Tauberian theorem, see Theorem 1.7.1′ on page 38 of [1], we conclude that

$$v^{-1}\int_0^v (q(x))^{\alpha-1}\,\phi_\alpha\left(q(x)\right)\mathrm{d}q(x) \to \gamma\Gamma(\alpha), \text{ as } v \searrow 0,$$

which, in turn, by the change of variable $y = q(x)$ gives

$$\frac{\int_0^t y^{\alpha-1}\phi_\alpha(y)\mathrm{d}y}{-\log\phi_0(t)} \to \gamma\Gamma(\alpha), \text{ as } t \searrow 0.$$

Now using that $-\log\phi_0(t) \sim 1 - \phi_0(t)$ as $t \to 0$, we end up with

$$\lim_{t\to 0}\frac{\int_0^t y^{\alpha-1}\phi_\alpha(y)\mathrm{d}y}{1-\phi_0(t)} = \gamma\Gamma(\alpha). \tag{13}$$

Since $\phi_\alpha(y) = \int_0^\infty e^{-uy} u^\alpha G(\mathrm{d}u)$, by Fubini's theorem

$$\begin{aligned}
\int_0^t y^{\alpha-1}\phi_\alpha(y)\mathrm{d}y &= \int_0^\infty u^\alpha G(\mathrm{d}u) \int_0^t y^{\alpha-1} e^{-uy}\mathrm{d}y \\
&= \int_0^\infty G(\mathrm{d}u) \int_0^{ut} z^{\alpha-1} e^{-z}\mathrm{d}z \\
&= \int_0^\infty \overline{G}(z/t) z^{\alpha-1} e^{-z}\mathrm{d}z \\
&= t^\alpha \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut}\mathrm{d}u.
\end{aligned}$$

A partial integration gives

$$1 - \phi_0(t) = t \int_0^\infty \overline{G}(u) e^{-ut}\mathrm{d}u.$$

So (13) reads

$$t^{\alpha-1} \frac{\int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut}\mathrm{d}u}{\int_0^\infty \overline{G}(u) e^{-ut}\mathrm{d}u} \to \gamma\Gamma(\alpha), \text{ as } t \searrow 0, \tag{14}$$

with $0 < \gamma \le 1$. Let us define the function for $t > 0$

$$f(t) = \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut}\mathrm{d}u. \tag{15}$$

Clearly, f is monotone decreasing and since $\mathbb{E}Y = \infty$, $\lim_{t\to 0} f(t) = \infty$. We shall show that f is regularly varying at 0, which by Lemma 3 of Pitman [9], implies that $\overline{G}$ is regularly varying at infinity. We use the identity

$$u^{1-\alpha} e^{-ut} = \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} e^{-(y+t)u}\mathrm{d}y,$$

which holds for $u > 0$ and $\alpha \in (1, 2]$. (This is the *Weyl-transform*, or *Weyl-fractional integral* of the function e^{-ut}.) This identity combined with Fubini's theorem (everything is nonnegative) gives

$$\begin{aligned}
\frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} f(y+t)\mathrm{d}y &= \int_0^\infty \overline{G}(u) u^{\alpha-1}\mathrm{d}u \frac{1}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} e^{-(y+t)u}\mathrm{d}y \\
&= \int_0^\infty \overline{G}(u) e^{-ut}\mathrm{d}u.
\end{aligned}$$

So we can rewrite (14) as

$$\lim_{t\searrow 0} \frac{t^{\alpha-1} f(t)}{\int_0^\infty y^{\alpha-2} f(t+y)\mathrm{d}y} = \frac{\gamma\Gamma(\alpha)}{\Gamma(\alpha-1)} = \gamma(\alpha-1). \tag{16}$$

A change of variable gives

$$\int_0^\infty y^{\alpha-2} f(t+y)\mathrm{d}y = t^{\alpha-1} \int_1^\infty (u-1)^{\alpha-2} f(ut)\mathrm{d}u,$$

and so we have

$$\lim_{t\searrow 0}\int_1^\infty (u-1)^{\alpha-2}\frac{f(ut)}{f(t)}\mathrm{d}u = [\gamma(\alpha-1)]^{-1}. \tag{17}$$

We can rewrite f as

$$f(t) = \int_0^\infty \overline{G}(u)u^{\alpha-1}e^{-ut}\mathrm{d}u = t^{-\alpha}\int_0^\infty \overline{G}(u/t)u^{\alpha-1}e^{-u}\mathrm{d}u,$$

from which we see that the function

$$g(t) = \int_0^\infty \overline{G}(u/t)u^{\alpha-1}e^{-u}\mathrm{d}u = t^\alpha f(t)$$

is bounded and nondecreasing. Substituting g into (17) we obtain

$$\lim_{t\to 0+}\int_1^\infty (u-1)^{\alpha-2}u^{-\alpha}\frac{g(ut)}{g(t)}\mathrm{d}u = [\gamma(\alpha-1)]^{-1}. \tag{18}$$

Write $g_\infty(x) = g(x^{-1})$, $x>0$. Then (18) has the form

$$\int_1^\infty (u-1)^{\alpha-2}u^{-\alpha}\frac{g_\infty(x/u)}{g_\infty(x)}\mathrm{d}u = \frac{k\overset{M}{*}g_\infty(x)}{g_\infty(x)} \to [\gamma(\alpha-1)]^{-1}, \quad \text{as } x\to\infty, \tag{19}$$

where

$$k(u) = \begin{cases}(u-1)^{\alpha-2}u^{-\alpha+1}, & u>1,\\ 0, & 0<u\le 1,\end{cases}$$

and

$$k\overset{M}{*}h(x) = \int_0^\infty h(x/u)k(u)/u\mathrm{d}u$$

is the *Mellin-convolution* of h and k. Note that the *Mellin-transform* of k,

$$\begin{aligned}\widetilde{k}\,(z) &= \int_1^\infty (u-1)^{\alpha-2}\,u^{-\alpha-z}\mathrm{d}u = \int_0^1 (1-v)^{\alpha-2}\,v^z\mathrm{d}v\\ &= \frac{\Gamma\,(\alpha-1)\,\Gamma\,(1+z)}{\Gamma\,(\alpha+z)} = \mathrm{Beta}(\alpha-1, 1+z)\end{aligned}$$

is convergent for $z>-1$. We apply a version of the Drasin-Shea theorem (Theorem 5.2.3 on page 273 of [1]). To do this we must verify the following conditions:

1. $\widetilde{k}$ has a maximal convergent strip $a<\mathfrak{Re}\,z<b$ such that $a<0$ and $b>0$, $\widetilde{k}\,(a+)=\infty$ and $\widetilde{k}\,(b-)=\infty$ if $b<\infty$. Our $\widetilde{k}$ satisfies this condition with $a=-1$ and $b=\infty$.

2. Our function of interest

$$g_\infty(x) = g(x^{-1}) = \int_0^\infty \overline{G}(ux)u^{\alpha-1}e^{-u}\mathrm{d}u,\ x>0,$$

is certainly positive and locally bounded.

3. Also our function g_∞ is of bounded decrease, since for $\lambda > 1$

$$\frac{g_\infty(\lambda x)}{g_\infty(x)} = \lambda^{-\alpha} \frac{(\lambda x)^\alpha g(1/(\lambda x))}{x^\alpha g(1/x)} = \lambda^{-\alpha} \frac{f(1/(\lambda x))}{f(1/x)} \geq \lambda^{-\alpha},$$

so its lower Matuszewska index is at least $-\alpha$.

Therefore by Theorem 5.2.3 of [1], whenever,

$$\frac{k \overset{M}{*} g_\infty(x)}{g_\infty(x)} \to c, \quad \text{as } x \to \infty, \tag{20}$$

then $\widetilde{k}(\rho) = c$ for some $\rho \in (-1, \infty)$. (In our case by (19), $c = [\gamma(\alpha-1)]^{-1}$.) Moreover, since $\widetilde{k}\,(z)$ is strictly decreasing on $(-1, \infty)$ and $\widetilde{k}\,(0) = \frac{1}{\alpha-1}$, for any $0 < \gamma \leq 1$ the solution ρ to $\widetilde{k}(\rho) = [\gamma(\alpha-1)]^{-1}$ must lie in $(-1, 0]$. Theorem 5.2.3 of [1] also says that $g_\infty(x)$ is regularly varying at infinity with index $0 \geq \rho > -1$.

Next since $g_\infty(x) = g(x^{-1}) = x^{-\alpha} f(x^{-1}) \in \mathcal{RV}_\infty(\rho)$, where $\widetilde{k}(\rho) = c$, $g \in \mathcal{RV}_0(-\rho)$, which implies that $f \in \mathcal{RV}_0(-\rho - \alpha)$. Recalling that

$$f(t) = \int_0^\infty \overline{G}(u) u^{\alpha-1} e^{-ut} \mathrm{d}u,$$

the Karamata Tauberian theorem now gives that

$$\int_0^x \overline{G}(u) u^{\alpha-1} \mathrm{d}u \in \mathcal{RV}_\infty(\alpha + \rho).$$

Thus by Lemma 3 of Pitman [9], $\overline{G}(u) \in \mathcal{RV}_\infty(\rho)$.

This says that $Y \in D(\beta)$, where $\rho = -\beta \in (-1, 0]$ and β is the unique solution of

$$\mathrm{Beta}(\alpha - 1, 1 - \beta) = \frac{\Gamma(\alpha-1)\Gamma(1-\beta)}{\Gamma(\alpha-\beta)} = \frac{1}{\gamma(\alpha-1)}.$$

We now turn to the proof of the second part of Proposition 2. First consider the case $\beta = 0$. Let $0 \leq D_n^{(n)} \leq \cdots \leq D_n^{(1)}$ denote the order statistics of $Y_1/\left(\sum_{i=1}^n Y_i\right), \ldots, Y_n/\left(\sum_{i=1}^n Y_i\right)$. We see that

$$\mathbb{E}\left(D_n^{(1)}\right)^\alpha \leq \mathbb{E} S_n(\alpha) = \sum_{i=1}^n \mathbb{E}\left(D_n^{(i)}\right)^\alpha \leq \mathbb{E}\left(D_n^{(1)}\right)^{\alpha-1} \leq 1.$$

Now $D_n^{(1)} \to_P 1$ if and only if $Y \in D(0)$. (See Theorem 1 of Haeusler and Mason [5] and their references.) Thus if $Y \in D(0)$ then (7) holds with $\gamma = 1$.

Now assume that $Y \in D(\beta)$, $0 < \beta < 1$. In this case, there exists a sequence of positive constants $\{a_n\}_{n\geq 1}$, such that $a_n^{-1} \sum_{i=1}^n Y_i \to_d U$, where U is a β-stable random variable, with characteristic function

$$\mathbb{E} e^{\imath t U} = \exp\left\{\beta \int_0^\infty (e^{\imath t u} - 1) u^{-\beta-1} u\right\}.$$

Moreover, $Y^\alpha \in D(\beta/\alpha)$, and it is easy to check that $a_n^{-\alpha}\sum_{i=1}^n Y_i^\alpha \to_d V$, where V is a β/α-stable random variable, with cf

$$\mathbb{E}e^{\imath tV} = \exp\left\{\frac{\beta}{\alpha}\int_0^\infty (e^{\imath tu}-1)u^{-\beta/\alpha-1}u\right\}.$$

Since

$$\lim_{n\to\infty} n\mathbb{P}\{Y > a_n u, Y^\alpha > a_n^\alpha v\} = \lim_{n\to\infty} n\overline{G}(a_n(u \vee v^{1/\alpha})) = u^{-\beta} \wedge v^{-\beta/\alpha} =: \Pi((u,\infty)\times(v,\infty)),$$

for $u, v \geq 0$, $u+v > 0$, using Corollary 15.16 of Kallenberg [6] one can show that the joint convergence also holds, and the limiting bivariate Lévy measure is Π. That is

$$\left(a_n^{-1}\sum_{i=1}^n Y_i, a_n^{-\alpha}\sum_{i=1}^n Y_i^\alpha\right) \to_d (U,V),$$

where the limiting bivariate random vector has cf

$$\mathbb{E}e^{\imath(sU+tV)} = \exp\left\{\int_{[0,\infty)^2}\left(e^{\imath(su+tv)}-1\right)\Pi(u,v)\right\} = \exp\left\{\beta\int_0^\infty\left(e^{\imath(su+tu^\alpha)}-1\right)u^{-\beta-1}u\right\}.$$

Since $\mathbb{P}\{U>0\} = \mathbb{P}\{V>0\} = 1$, we obtain

$$S_n(\alpha) \to_d \frac{V}{U^\alpha}.$$

Thus since $S_n(\alpha) \leq 1$ for all $n \geq 1$,

$$\mathbb{E}S_n(\alpha) \to \mathbb{E}\left(\frac{V}{U^\alpha}\right) =: \gamma \leq 1.$$

Clearly $\mathbb{P}\{U<\infty\} = 1$, which implies that $0 < \mathbb{E}\left(\frac{V}{U^\alpha}\right) \leq 1$, and thus by the first part of Proposition 2,

$$0 < \gamma = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1-\beta)} < 1.$$

□

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