COVERING THE SPHERE BY EQUAL ZONES

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Abstract. A zone of half-width w on the unit sphere S^2 in Euclidean 3-space is a spherical segment of spherical width $2w$ which is symmetric to o. L. Fejes Tóth raised the following question in [5]: what is the minimal w_n such that one can cover S^2 with n zones of width $2w_n$? This question can be considered as a spherical relative of the famous plank problem of Tarski. We prove lower bounds for the minimum width w_n for all $n \geq 5$.

1. Introduction

Let S^2 denote the unit sphere in 3-dimensional Euclidean space $\mathbb{R}^3$ centred at the origin o. The spherical distance $d_s(x, y)$ of two points $x, y \in S^2$ is defined as the length of a (shorter) geodesic arc connecting x and y on S^2, or equivalently, the central angle $\angle xoy$ spanned by x and y. Following L. Fejes Tóth [5], a *zone* Z of half-width w in S^2 is the parallel domain of radius w of a great circle C, that is,

$$Z(C, w) := \{x \in S^2 \mid d_s(x, C) \leq w\}.$$

We call C the central great circle of Z. In this paper, we investigate the following problem.

Problem 1 (L. Fejes Tóth [5])**.** *For a given n, find the smallest number w_n such that one can cover S^2 with n zones of half-width w_n. Find also the optimal configurations of zones that realize the optimal coverings.*

We note that in the same paper [5] L. Fejes Tóth also asked the analogous question with not necessarily congruent zones, and conjectured that the sum of the widths of the zones that can cover S^2 is always at least π. Furthermore, L. Fejes Tóth [5] posed the question: what is the minimum of the sum of the widths of n (not necessarily congruent) zones that can cover a spherically convex disc on S^2? These questions are similar to the classical plank problem of Tarski, see for example Bezdek [1] for a recent survey on this topic.

L. Fejes Tóth formulated the following conjecture:

Conjecture 1 (L. Fejes Tóth [5])**.** *For $n \geq 1$, $w_n = \pi/(2n)$.*

It is clear that $w_n \leq \pi/(2n)$ since n zones of half-width $\pi/(2n)$, whose central great circles all pass through a pair of antipodal points of S^2 and which are distributed evenly, cover S^2. On the other hand, as the zones must cover S^2, the sum of their areas must be at least (actually, greater than) 4π, that is, $w_n > \arcsin(1/n)$.

Rosta [13] proved that $w_3 = \pi/6$, and that the unique optimal configuration consists of three zones whose central great circles pass through two antipodal points of S^2 and are distributed evenly. Linhart [9] showed that $w_4 = \pi/8$, and the unique optimal configuration is similar to the one for $n = 3$. To the best of our knowledge, no further results about this problem have been achieved to date and thus L. Fejes Tóth's conjecture remains open.

The paper is organized as follows. In Section 2, we determine the area of the intersection of two congruent zones as a function of their half-widths and the angle of their central great circles under some suitable restrictions. In Section 3, we use the currently known best upper bounds for the maximum of the minimal pairwise spherical distances of n points in S^2 to estimate from above the contribution of a zone in an optimal covering. Adding up these estimated contributions, we obtain a lower bound for w_n, which is the main result of our paper, and it is stated in Theorem 1. Finally, we calculate the numerical values of the established lower bound for some specific n.

2. Intersection of two zones

We start with the following simple observation. Consider two zones Z_1 and Z_2 of half-width w whose central great circles make an angle α. If $\alpha \geq 2w$, then the intersection of Z_1 and Z_2 is the union of two disjoint congruent spherical domains. These domains are symmetric to each other with respect to o, and they resemble to a rhombus which is bounded by four small circular arcs of equal (spherical) length. If $\alpha \leq 2w$, then the intersection is a connected, band-like domain. Let $2F(w, \alpha)$ denote the area of $Z_1 \cap Z_2$.

Lemma 1. *Let $0 \leq w \leq \pi/4$ and $2w \leq \alpha \leq \pi/2$. Then*

$$F(w, \alpha) = 2\pi + 4 \sin w \arcsin\left(\frac{1 - \cos\alpha}{\cot w \sin\alpha}\right) + 4 \sin w \arcsin\left(\frac{1 + \cos\alpha}{\cot w \sin\alpha}\right) - 2 \arccos\left(\frac{\cos\alpha - \sin^2 w}{\cos^2 w}\right) - 2 \arccos\left(\frac{-\cos\alpha - \sin^2 w}{\cos^2 w}\right). \tag{1}$$

Moreover, $F(w, \alpha)$ is a monotonically decreasing function of α in the interval $[0, \pi/2]$.

Proof. First, we prove (1). Let Z_1 be the zone of half-width w whose central great circle C_1 is the intersection of the xy-plane with S^2. Let c_1 and c_3 denote the small circles which bound Z_1 such that c_1 is contained in the closed half-space $z \geq 0$.

Let Z_2 be the zone of half-width w whose central great circle C_2 is the intersection of S^2 with the plane which contains the y-axis and which makes an angle α with the xy-plane as shown in Figure 1. Let c_2 and c_4 be the small circles bounding Z_2, cf. Figure 1.

The intersection $Z_1 \cap Z_2$ is the union of two connected components R_1 and R_2. Assume that R_1 is contained in the closed half-space $y \leq 0$. Let c_i', $i = 1, \ldots, 4$ denote the arc of c_i that bounds R_1. Observe that the c_i' are of equal length; we denote their common arc length by $l(w, \alpha)$. The radii of $c_1, \ldots, c_4$ are all equal to $\cos w$.

Assume that the boundary ∂R_1 of R_1 is oriented such that the small circular arcs follow each other in the cyclic order c_1', c_2', c_3', c_4'. For $i \in \{1, \ldots, 4\}$, let $\varphi_i(w, \alpha)$ denote the turning angle of ∂R_1 at the intersection point of c_i' and c_{i+1}' with the convention that $c_5 = c_1$. Notice that the signed geodesic curvature of ∂R_1 (in its smooth points) is equal to $-\tan w$.

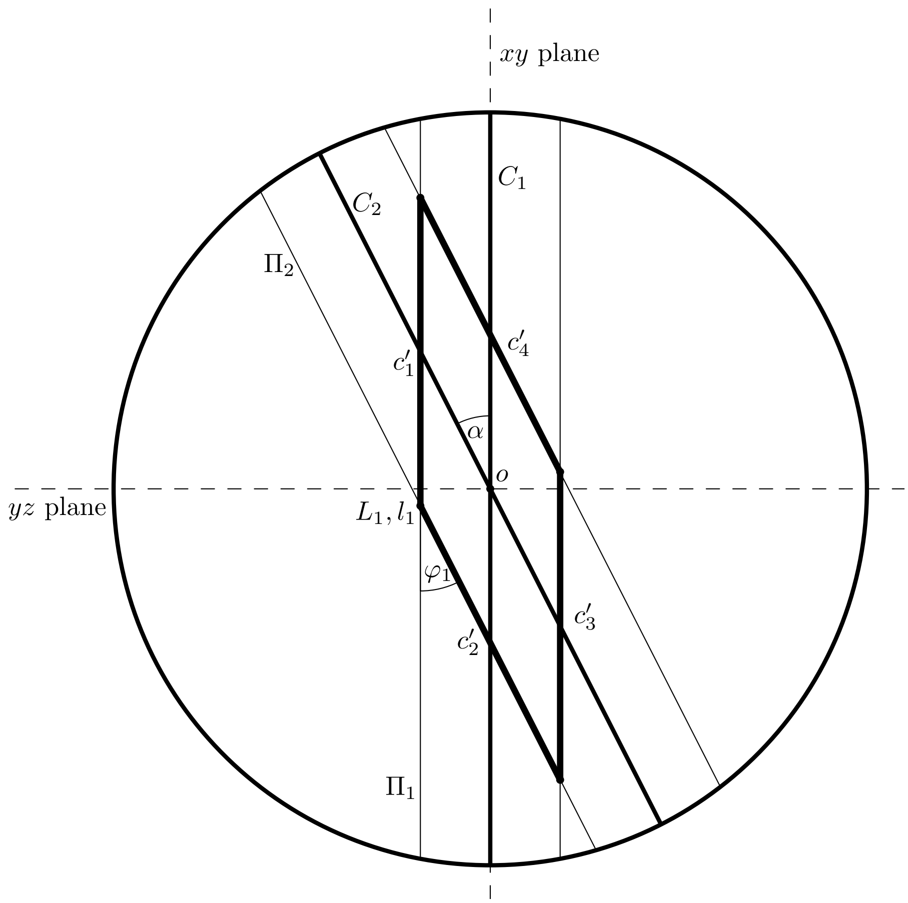

FIGURE 1. Orthogonal projection onto the xz plane

By the Gauss–Bonnet Theorem it holds that

$$F(w,\alpha) = 2\pi + 4\tan w \cdot l(w,\alpha) - \sum_{i=1}^{4} \varphi_i(w,\alpha).$$

Next, we calculate the $\varphi_i(w,\alpha)$. Note that $\varphi_i(w,\alpha) = \varphi_{i+2}(w,\alpha)$ for $i = 1, 2$.

Let Π_1 be the plane whose normal vector is $u_1 = (0,0,1)$ and contains the point $(0,0,\sin w)$. Let Π_2 be the plane which we get by rotating Π_1 around the y-axis by angle α so its normal vector is $u_2 = (-\sin\alpha, 0, \cos\alpha)$, see Figure 1. Note that $S^2 \cap \Pi_1 = c_1$ and $S^2 \cap \Pi_2 = c_2$.

$$\Pi_1 : \quad z = \sin w$$
$$\Pi_2 : \quad -x\sin\alpha + z\cos\alpha = \sin w$$

Now let $L_1=\Pi_1\cap\Pi_2$ and $L_1\cap S^2=\{l_1,l_1'\}$, such that l_1 has negative y-coordinate. Then

$$l_1=\left(\sin w(\cot\alpha-\csc\alpha),-\sqrt{1-\sin^2 w(1+(\cot\alpha-\csc\alpha)^2)},\sin w\right).$$

Let Π be the plane that is tangent to S^2 in l_1, and let $E_1=\Pi_1\cap\Pi$ and $E_2=\Pi_2\cap\Pi$. Then φ_1 is one of the angles made by E_1 and E_2. Let $v_1=l_1\times u_1$ and $v_2=l_1\times u_2$. Then v_1 and v_2 are vectors parallel to E_1 and E_2, respectively, such that their orientations agree with that of ∂R_1.

$$\begin{aligned}v_1&=\left(-\sqrt{1-\sin^2 w(1+(\cot\alpha-\csc\alpha)^2)},-\sin w(\cot\alpha-\csc\alpha),0\right)\\ v_2&=\Big(-\cos\alpha\sqrt{1-\sin^2 w(1+(\cot\alpha-\csc\alpha)^2)},-\cos\alpha\sin w(\cot\alpha-\csc\alpha)-\\ &\quad-\sin\alpha\sin w,-\sin\alpha\sqrt{1-\sin^2 w(1+(\cot\alpha-\csc\alpha)^2)}\Big).\end{aligned}$$

We only need to calculate the lengths of v_1 and v_2 and their scalar product. By routine calculations we obtain

$$\varphi_1=\arccos\frac{\langle v_1,v_2\rangle}{|v_1||v_2|}=\arccos\left(\frac{\cos\alpha-\sin^2 w}{\cos^2 w}\right).$$

The angle φ_2 can be evaluated similarly; one only needs to write $\pi-\alpha$ in place of α in the above calculations. Then

$$\varphi_2=\arccos\left(\frac{-\cos\alpha-\sin^2 w}{\cos^2 w}\right).$$

To finish the calculation, we need to find $l(w,\alpha)$. Let $l_i:=c_i'\cap c_{i+1}'$ for $i=1,2,3,4$ with $c_5'=c_1'$. Let d_i, $i=1,\dots,4$ be the absolute value of the y-coordinate of l_i. Simple trigonometry shows that

$$d_1=\frac{1-\cos\alpha}{\cot w\sin\alpha},$$

and

$$d_4=\frac{1+\cos\alpha}{\cot w\sin\alpha}.$$

Then the length of c_1' is equal to the following

$$\begin{aligned}l(w,\alpha)&=\cos w\arcsin d_1+\cos w\arcsin d_4\\ &=\cos w\arcsin\left(\frac{1-\cos\alpha}{\cot w\sin\alpha}\right)+\cos w\arcsin\left(\frac{1+\cos\alpha}{\cot w\sin\alpha}\right).\end{aligned}\tag{2}$$

In summary,

$$\begin{aligned}F(w,\alpha)=\ &2\pi+4\sin(w)\arcsin\left(\tan(w)(\csc(\alpha)+\cot(\alpha))\right)\\ &+4\sin(w)\arcsin\left(\tan(w)(\csc(\alpha)-\cot(\alpha))\right)\\ &-2\arccos\left(\frac{\cos(\alpha)-\sin^2(w)}{\cos^2(w)}\right)\\ &-2\arccos\left(\frac{\cos(\alpha)+\sin^2(w)}{-\cos^2(w)}\right)\end{aligned}\tag{3}$$

Finally, we prove that F is monotonically decreasing in α. This is obvious in the interval $[0,2w]$.

Let $\varepsilon > 0$ be sufficiently small with $\alpha + \varepsilon \leq \pi/2$. Consider the spherical "rhombus" R_1^* which is obtained as the intersection of Z_1 and another zone Z_2^* of half-width w whose central great circle C_2^* is the intersection of S^2 with the plane which contains the y-axis and which makes an angle $\alpha + \varepsilon$ with the xy-plane, similarly as for Z_2 above. Let F_1 be the area of $R_1 \setminus R_1^*$ and F_1^* be the area of $R_1^* \setminus R_1$. For the monotonicity of $F(w, \alpha)$ in α, we only need to show that $F_1 > F_1^*$.

The region $R_1 \setminus R_1^*$ consists of two disjoint congruent connected domains (in fact, two triangular regions bounded by arcs of small circles). Note that one such region, say P, is fully contained in the positive hemisphere of S^2 $(z \geq 0)$, and the other region is contained in the negative hemisphere $(z \leq 0)$. Similarly, let Q be the one of the two connected, congruent and disjoint regions whose union is $R_1^* \setminus R_1$ and which has a common (boundary) point with P. Let $q = P \cap Q$, then q has positive z-coordinate. It easily follows from the position of q that the arc $c_2 \cap Q$ is longer than $c_2 \cap P$, and, similarly, $c_2^* \cap Q$ is longer than $c_2^* \cap P$, so the area of Q is larger than the area of P, which completes the proof of the Lemma. □

Remark 1. Let Z_1 and Z_2 be two zones of half-width $w \in (0, \pi/4]$ which make an angle α. Then it is clear that the area of $Z_1 \cup Z_2$ is a monotonically increasing function of α for $\alpha \in [0, 2w]$.

3. A lower bound for w_n

For an integer $n \geq 3$, let d_n denote the maximum of the minimal pairwise (spherical) distances of n points on the unit sphere S^2. Finding d_n is a long-standing problem of discrete geometry which goes back to the Dutch botanist Tammes (1930) (see [15]). As of now, the exact value of d_n is only known in the following cases.

n	d_n	
3	$2\pi/3$	L. Fejes Tóth [7]
4	1.91063	L. Fejes Tóth [7]
5	$\pi/2$	Schütte, van der Waerden [14]
6	$\pi/2$	L. Fejes Tóth [7]
7	1.35908	Schütte, van der Waerden [14]
8	1.30653	Schütte, van der Waerden [14]
9	1.23096	Schütte, van der Waerden [14]
10	1.15448	Danzer [4]
11	1.10715	Danzer [4]
12	1.10715	L. Fejes Tóth [7]
13	0.99722	Musin, Tarasov [11]
14	0.97164	Musin, Tarasov [10]
24	0.76255	Robinson [12]

Table 1. Known values of d_n

Alternate proofs were given by Hárs [8] for the case $n = 10$, and by Böröczky [2] for the case $n = 11$.

For $n \geq 3$, L. Fejes Tóth (see [6]) proved the following upper estimate

$$d_n \leq \tilde{\delta}_n := \arccos\left(\frac{\cot^2\left(\frac{n}{n-2}\frac{\pi}{6}\right) - 1}{2}\right), \tag{4}$$

where equality holds exactly in the cases $n = 3, 4, 6, 12$ (see table above). Moreover, $\lim_{n\to\infty} \tilde{\delta}_n/d_n = 1$, that is, $\tilde{\delta}_n$ provides an exact asymptotic upper bound for d_n as $n \to \infty$.

Robinson [12] improved the upper estimate (4) of L. Fejes Tóth as follows. Assume that the pairwise distances between the n points on the sphere are all at least a where $0 < a < \arctan 2$. Let $\Delta_1(a)$ denote the area of an equilateral spherical triangle with side lengths a, and $\Delta_2(a)$ denote the area of a spherical triangle with two sides of length a making an angle of $2\pi - 4\alpha$. Let δ_n be the unique solution of the equation $4n\Delta_1(a) + (2n - 12)\Delta_2(a) - 12\pi = 0$. Then (cf. [12]) $d_n \leq \delta_n \leq \tilde{\delta}_n$ for $n \geq 13$.

Let $d_n^* := \min\{\pi/2, d_n\}$ for $n \geq 2$, and let

$$\delta_n^* := \begin{cases} d_n^* & \text{for } 3 \leq n \leq 14 \text{ and } n = 24, \\ \delta_n & \text{otherwise.} \end{cases} \tag{5}$$

We will also need a lower bound on d_n for our argument. We note that, for example, van der Waerden [16] proved a non-trivial lower bound on d_n, however, for our purposes the following simpler bound is sufficient. Set $\varrho_n := \arccos(1-2/n)$, and consider a maximal (saturated) set of points $p_1, \dots, p_m$ on the unit sphere S^2, such that their pairwise spherical distances are at least ϱ_n. By maximality it follows that the spherical circular discs (spherical caps) of radius ϱ_n centered at $p_1, \dots, p_m$ cover S^2. As the (spherical) area of such a cap is $4\pi/n$, we obtain that $m \cdot 4\pi/n \geq 4\pi$, that is, $m \geq n$, which implies that $\varrho_n := \arccos(1 - 2/n) \leq d_n$. As $x \leq \arccos(1 - x^2/2)$ for $0 \leq x \leq 1$, the following inequality is immediate

$$\frac{2}{\sqrt{n}} \leq d_n^* \leq \delta_n^*. \tag{6}$$

For $0 \leq \alpha \leq \pi/2$ and $n \geq 3$ we introduce $f(w, \alpha) = 4\pi \sin w - 2F(w, \alpha)$ and

$$G(w, n) = 4\pi \sin w + \sum_{i=2}^{n} f(w, \delta_{2i}^*).$$

Lemma 2. *For a fixed $n \geq 3$, the function $G(w, n)$ is continuous and monotonically increasing in w in the interval $[0, \delta_{2n}^*/3]$. Furthermore, $G(0, n) = 0$ and $G(\delta_{2n}^*/3, n) \geq 4\pi$.*

Proof. The continuity of G and that $G(0, n) = 0$ are obvious. First we show that the function $f(w, \alpha)$ is monotonically increasing in w for $0 \leq w \leq \alpha/3$. This clearly implies that $G(w, n)$ is also monotonically increasing in the interval stated in the lemma. As $n \geq 3$, we may and do assume that $w \leq \delta_6^*/3 = \pi/6$.

Note that $f(w, \alpha)$ is the area of a zone of half-width w minus the area of its intersection with a second zone of half-width w whose central great circle makes an angle α with the central great circle of the first zone. With the same notations as in the proof of Lemma 1, it is clear that for sufficiently small $\Delta w > 0$, the quantity $f(w + \Delta w, \alpha) - f(w, \alpha)$ is (roughly) proportional to $2l(c_1) - 4l(c_1') - 4l(c_2') =$

$2(l(c_1) - 4l(c_1'))$. Notice that, for a fixed $w \in [0, \pi/4]$, the function $l(c_1') = l(w, \alpha)$ is monotonically decreasing in α for $\alpha \in [2w, \pi/2]$. Thus, using $3w \leq \alpha$,

$$l(c_1) - 4l(c_1') \geq l(c_1) - 4l(w, 3w) =$$
$$= 4\cos w\left(\frac{\pi}{2} - \arcsin\left(\frac{1-\cos(3w)}{\cot w \sin(3w)}\right) - \arcsin\left(\frac{1+\cos(3w)}{\cot w \sin(3w)}\right)\right).$$

One can check that if $w \in (0, \pi/6]$, then both arguments in the above arcsin functions take on values in $[0, 2/3]$. By the monotonicity and convexity of arcsin, we obtain that

$$\arcsin\left(\frac{1-\cos(3w)}{\cot w \sin(3w)}\right) + \arcsin\left(\frac{1+\cos(3w)}{\cot w \sin(3w)}\right) \leq \arcsin(2/3)\frac{3\tan w}{\sin(3w)}$$
$$\leq \arcsin(2/3)\frac{3\tan(\pi/6)}{\sin(\pi/2)} = \frac{2\sqrt{3}}{3} < \frac{\pi}{2},$$

which shows the monotonicity of $G(w, n)$.

Finally, we show that $G(\delta_{2n}^*/3, n) \geq 4\pi$. For $n \leq 24$, this statement can be checked by direct calculation, thus we may assume $n \geq 25$. Using the definitions of G and f, and Lemma 1, we obtain that

$$G\left(\frac{\delta_{2n}^*}{3}, n\right) = n \cdot 4\pi \sin\frac{\delta_{2n}^*}{3} - 2 \cdot \sum_{i=2}^{n} F\left(\frac{\delta_{2n}^*}{3}, \delta_{2i}^*\right)$$
$$\geq 4n\pi \sin\frac{\delta_{2n}^*}{3} - 2\sum_{i=2}^{n} F\left(\frac{\delta_{2n}^*}{3}, \delta_{2n}^*\right)$$
$$= 4n\pi \sin\frac{\delta_{2n}^*}{3} - 2(n-1)F\left(\frac{\delta_{2n}^*}{3}, \delta_{2n}^*\right)$$
$$\geq 4n\pi \sin\frac{\delta_{2n}^*}{3} - 2(n-1)F\left(\frac{\delta_{2n}^*}{3}, \frac{2\delta_{2n}^*}{3}\right). \tag{7}$$

Note that $\delta_{2n}^* = \delta_{2n}$ for $n \geq 25$. Elementary trigonometry yields that

$$F\left(\frac{\alpha}{2}, \alpha\right) = 4\sin\frac{\alpha}{2}\arcsin\left(\tan^2\frac{\alpha}{2}\right) + 2\pi\sin\frac{\alpha}{2} - 2\arccos\left(1 - 2\tan^2\frac{\alpha}{2}\right).$$

Thus (7) is equal to

$$4\pi\sin\frac{\delta_{2n}}{3} + 4(n-1)\left(\arccos\left(1 - 2\tan^2\frac{\delta_{2n}}{3}\right) - 2\sin\frac{\delta_{2n}}{3}\arcsin\left(\tan^2\frac{\delta_{2n}}{3}\right)\right).$$

As $n \geq 25$, we have that $0 < \delta_{2n} < 0.75$. Using that $\cos x \geq 1 - x^2/2$ for $x \in [0, \pi/2]$, we obtain that

$$\arccos\left(1 - 2\tan^2\frac{\delta_{2n}}{3}\right) \geq 2\tan\frac{\delta_{2n}}{3}.$$

Similarly, as for $0 < x < 0.16$ we have that $x < 1.01\sin x$, we obtain that

$$2\sin\frac{\delta_{2n}}{3}\arcsin\left(\tan^2\frac{\delta_{2n}}{3}\right) < 2.02\tan^3\frac{\delta_{2n}}{3}.$$

Finally, using that $x - 1.01x^3 > x - 1.01 \cdot 0.4^2 \cdot x > 0.8x$ for $0 < x < 0.4$, we obtain that (7) can be estimated from below as follows

$$G\left(\frac{\delta_{2n}^*}{3}, n\right) \geq 6.4(n-1)\tan\frac{\delta_{2n}}{3} > 2.1(n-1)\delta_{2n}.$$

By (6) we know that $\delta_{2n} > \sqrt{2}/\sqrt{n}$, and thus the proof of Lemma 2 is complete. $\square$

Now, we are ready to state our main theorem.

Theorem 1. *For $n \geq 3$, let w_n^* denote the unique solution of the equation $G(w,n) = 4\pi$ in the interval $[0, \delta_{2n}^*/3]$. Then* $\arcsin(1/n) < w_n^* \leq w_n$.

Proof. Let $Z_i(w_n, C_i), i = 1, \ldots, n$ be zones that form a (minimal with respect to w) covering of S^2. For $i \in \{1, \ldots, n\}$, let p_i be one of the poles of C_i and let $p_{n+i} = -p_i$. Then there exist two points $p_{i_1}, p_{j_1} \in \{p_1, \ldots, p_{2n}\}$ with $i_1 < j_1$ and $j_1 \neq n + i_1$ (that is, p_{i_1} and p_{j_1} are poles of two different great circles) such that $d_s(p_{i_1}, p_{j_1}) \leq d_{2n}^*$. Observe that the area of the part of Z_{i_1} that is not covered by any Z_k with $i_1 \neq k$ is at most $f(w, \delta_{2n}^*)$ by Lemma 1, inequality (6) and Remark 1. Now, remove Z_{i_1} from the covering and repeat the argument for the remaining zones. Note that in the last step of the process, there is only one zone left Z_{i_n}, so the area of the part of Z_{i_n} not covered by any other zone is $4\pi \sin w$.

If for $k = 1, \ldots, n$ we add the areas of Z_{i_k} not covered by any Z_{i_l} for $l > k$, we obtain the function $G(w, n)$. Since $Z_1, \ldots, Z_n$ cover S^2, therefore $G(w, n) \geq 4\pi$, which shows that $w_n^* \leq w_n$. It is also clear form the argument that $\arcsin(1/n) < w_n^*$. This finishes the proof of Theorem 1. $\square$

4. Concluding remarks

Remark 2. Instead of Robinson's bound δ_n, one may use the original bound $\tilde{\delta}_n$ of L. Fejes Tóth, and prove Theorem 1, obtaining a lower bound $\tilde{w}_n^*$ for w_n. Clearly, this bound is slightly weaker than w_n^*, that is, $\tilde{w}_n^* \leq w_n^* \leq w_n$. However, we note that, thanks to the explicit formula (4), $\tilde{w}_n^*$ can be computed more easily than w_n^*. The difference between w_n^* and $\tilde{w}_n^*$ is shown in Table 2 for some specific values of n.

We also mention that for certain values of n Robinson's upper bound has been improved, see for example Böröczky and Szabó [3] for the cases $n = 15, 16, 17$. These stronger upper bounds, if included in the calculations, would provide only a very small improvement on w_n^*, so we decided to use only the known solutions of the Tammes problem and Robinson's general upper bound.

Remark 3. We note that the analogous question to Problem 1 can be raised in higher dimensions as well. A zone $Z = Z(C, w)$ of half-width w on the unit sphere S^{d-1} of the d-dimensional Euclidean space $\mathbb{R}^d$ is the parallel domain of radius w a great sphere C. What is the mininal $w(d, n)$ such that one can cover S^{d-1} with n zones of half-width $w(d, n)$, and what configurations realize the optimal coverings? We do not wish to formulate a conjecture about this problem, instead, we note the following simple fact. For $d \geq 4$, $w(d, 3) = \pi/6$. One can see this the following way. Let $Z_i = Z(C_i, w)$, $i = 1, 2, 3$ be three zones that cover S^{d-1}. Assume that $C_i = S^{d-1} \cap H_i$ for $i = 1, 2, 3$ where H_i is a hyperplane. Let $L = \cap_i H_i$. Then L is a linear subspace of $\mathbb{R}^d$, and $\dim L \geq d - 3$. Let $L^\perp$ denote the linear subspace of $\mathbb{R}^d$ which is the orthogonal complement of L. Clearly, $L^\perp \cap S^{d-1} = S^j$, where $j \leq 2$. If $\dim L^\perp = 1$, then $w = \pi/2$. So we may assume that $\dim L^\perp = 2$ or 3. Notice that the zones $Z_i, i = 1, 2, 3$ cover S^{d-1} if and only if the zones $Z_i' = Z_i \cap (L^\perp \cap S^{d-1}), i = 1, 2, 3$ cover $L^\perp \cap S^{d-1} = S^j$. We note also that the half-widths of $Z_i', i = 1, 2, 3$ are all equal to w. Now, if $j = 1$, then it is clear that

n	$\arcsin(1/n)$	$\tilde{w}_n^*$	w_n^*	$\pi/(2n)$
5	0.20135	0.22983	0.22983	0.31415
6	0.16744	0.18732	0.18732	0.26179
7	0.14334	0.15824	0.15824	0.22439
8	0.12532	0.13692	0.13692	0.19634
9	0.11134	0.12063	0.12067	0.17453
10	0.10016	0.10782	0.10787	0.15707
11	0.09103	0.09748	0.09753	0.14279
12	0.08343	0.08895	0.08899	0.13089
13	0.07699	0.08179	0.08183	0.12083
14	0.07148	0.07569	0.07573	0.11219
15	0.06671	0.07044	0.07048	0.10471
16	0.06254	0.06587	0.06591	0.09817
17	0.05885	0.06185	0.06189	0.09239
18	0.05558	0.05830	0.05833	0.08726
19	0.05265	0.05513	0.05516	0.08267
20	0.05002	0.05229	0.05232	0.07853
21	0.04763	0.04972	0.04975	0.07479
22	0.04547	0.04740	0.04743	0.07139
23	0.04349	0.04528	0.04531	0.06829
24	0.04167	0.04335	0.04337	0.06544
25	0.04001	0.04157	0.04159	0.06283
50	0.02000	0.02050	0.02051	0.03141
100	0.01000	0.01016	0.01017	0.01570

TABLE 2. Bounds for w_n

$w \geq \pi/6$ by elementary geometry, and if $j = 2$, then by Rosta's result [13], it holds that $w \geq w_3 = \pi/6$. Finally, in both cases, $w = \pi/6$ suffices to cover S^{d-1}.

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