On the angle sum of lines

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Abstract. What is the maximum of the sum of the pairwise (non-obtuse) angles formed by \( n \) lines in the Euclidean 3-space? This question was posed by Fejes Tóth in (Acta Math Acad Sci Hung 10:13–19, 1959). Fejes Tóth solved the problem for \( n \leq 6 \), and proved the asymptotic upper bound \( n^2 \pi/5 \) as \( n \to \infty \). He conjectured that the maximum is asymptotically equal to \( n^2 \pi/6 \) as \( n \to \infty \). The main result of this paper is an upper bound on the sum of the angles of \( n \) lines in the Euclidean 3-space that is asymptotically equal to \( 3n^2 \pi/16 \) as \( n \to \infty \).

Mathematics Subject Classification. 52C35.

Keywords. Angle sum of lines, Upper bound.

1. Introduction. Consider \( n \) lines in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) which all pass through the origin \( o \). What is the maximum \( S(n, d) \) of the sum of the pairwise (non-obtuse) angles formed by the lines? This question was raised by Fejes Tóth in [3] for \( d = 3 \). For general \( d \), the problem is formulated, for example, in [5].

The conjectured maximum of the angle sum is attained by the following configuration: Let \( n = k \cdot d + m \) (\( 0 \leq m < d \)), and denote by \( x_1, \ldots, x_d \) the axes of a Cartesian coordinate system in \( \mathbb{R}^d \). Take \( k + 1 \) copies of each one of the axes \( x_1, \ldots, x_m \), and take \( k \) copies of each one of the axes \( x_{m+1}, \ldots, x_d \). The sum of the pairwise angles in this configuration is

\[
\frac{d(d-1)k^2}{2} + mk(d-1) + \frac{m(m-1)}{2} \pi.
\]

Fejes Tóth stated this conjecture only for \( d = 3 \), however, it is quite natural to extend it to any \( d \) (see [5]). To the best of our knowledge, this problem is unsolved for \( d \geq 3 \).

In the case \( d = 3 \), Fejes Tóth [3] proved the conjecture for \( n \leq 6 \). He determined \( S(n, 3) \) for \( n \leq 5 \) by direct calculation, and he obtained \( S(6, 3) \).