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Covariant gravitational dynamics in 3+1+1 dimensions

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Abstract
We develop a 3+1+1 covariant formalism with cosmological and astrophysical applications. First we give the evolution and constraint equations both on the brane and off-brane in terms of 3-space covariant kinematical, gravito-electro-magnetic (Weyl) and matter variables. We discuss the junction conditions across the brane in terms of the new variables. Then we establish a *closure condition* for the equations on the brane. We also establish the connection of this formalism with isotropic and anisotropic cosmological brane-worlds. Finally we derive a new brane solution in the framework of our formalism: the *tidal charged Taub-NUT-(A)dS brane*, which obeys the closure condition.

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1. Introduction

In recent years the idea of exploring the possibility of the gravitational interaction acting in more than four dimensions (4d) attracted a lot of attention. In particular, one of the simplest such models arises from the curved generalization of the one-brane Randall–Sundrum model [1] (for a review see [2]), where gravity acts in five dimensions (5d); however, standard model fields are confined to the 4d brane. Nonstandard model fields can occur in the 5d spacetime. A $\mathcal{Z}_2$-symmetric embedding looks natural only if the brane is envisaged as a boundary; otherwise generic asymmetric embeddings should be allowed. The generic dynamics was given in [3, 4] in a 5d-covariant approach. Although promising at the level of allowing new degrees of freedom, which seem to be adjustable to explain for example dark matter [5]–[8], the complexity of the brane-world dynamics also represents a major impediment in obtaining simple exact solutions or in monitoring the evolution of perturbations. Therefore new approaches to gravitational dynamics in brane-worlds are worthy of study.

In [9] the 5d spacetime was foliated first by a family of 4d spacetimes and a second family of 4d space-like hypersurfaces (see figure 1 in [9]). Such a 5d spacetime is *double foliable*. This structure of the 5d spacetime allowed us to describe the gravitational degrees of freedom in terms of tensorial, vectorial and scalar quantities with respect to the 3-space emerging as

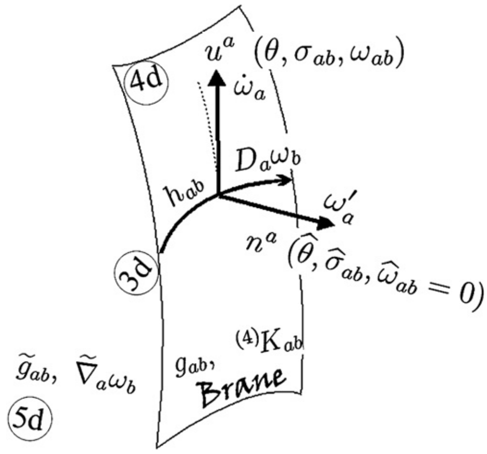

Figure 1. Elements of the 3+1+1 decomposition of the 5d spacetime geometry with metric $\widetilde{g}_{ab}$ and compatible connection $\widetilde{\nabla}$. The 4d brane with normal n^a has induced metric $g_{ab} = \widetilde{g}_{ab} - n_a n_b$ and extrinsic curvature $^{(4)}K_{ab} = g_a^c g_b^d \widetilde{\nabla}_c n_d$. The 4-velocity field u^a of observers in the brane defines local 3d orthogonal spatial patches, hypersurface-forming only when the vorticity $\omega_{ab} = 0$. The other kinematical characteristics of u^a are the expansion Θ and shear σ_{ab}. The expansion, shear and vorticity of n^a are $\widehat{\Theta}$, $\widehat{\sigma}_{ab}$ and $\widehat{\omega}_{ab}$, respectively, the latter vanishing on the brane. The temporal, off-brane and 3d covariant derivative are shown for the vorticity 1-form ω_a.

the intersection of the two hypersurfaces. They represent the gravitons, a gravi-photon, and a gravi-scalar and are given by quantities with well-defined geometrical meaning, namely the tensorial, vectorial and scalar projections of the spatial 4d metric and their canonically conjugated momenta: the extrinsic curvature (second fundamental form of the 3-spaces with respect to the temporal normal), normal fundamental form and normal fundamental scalar. The evolution equations for these variables were given in the manner of ADM both on the brane and outside it. An extension of this formalism towards a full Hamiltonian description was advanced in [10]. It has been shown that among all gravitational variables on the brane, only the momentum of the gravi-vector has a jump across the brane, related to the energy transport (heat flow) on the brane.

The formalism presented in [9, 10] however is not straightforward to be applied in brane cosmology. A formal difference would be the definition of the time derivative. In [9, 10] this was defined in the tradition of canonical gravity as a Lie-derivative taken along the temporal direction (which is not necessarily orthogonal to the 3-space) projected to the 3-space, whereas in cosmology by tradition the time derivative is defined as a covariant derivative along the normal to the 3-surfaces (this derivative happens to enter only in expressions projected onto the 3-space). Obviously this mismatch in the definitions of the time derivatives is not crucial, as the two definitions differ only in terms taken on the 3-space. However the formalism developed in [9, 10] relying on the double foliability of the 5d spacetime is unable to deal with the possible *vorticity of the word-lines of observers*.

In this paper we develop a formalism overcoming this inconvenience and derive the full set of evolution and constraint equations governing gravitational dynamics in a 3+1+1 covariant form. Provided the space-like normal n has vanishing vorticity on a hypersurface (the time evolution vector can still have vorticity), the formalism becomes suitable for describing gravitational dynamics on the brane (then n becomes the brane normal). In this sense the formalism is a generalization of the brane 3+1 covariant cosmology [11] (which in turn is a generalization of the general relativistic 3+1 covariant cosmology [12]–[15]).

This newly developed formalism also generalizes the s+1+1 covariant brane-world dynamics developed in [9]. Both the vector field n and the time evolution vector u are allowed to have vorticity in the present formalism. Though observational evidence suggests that the directly detectable 3+1 part of the universe is best described by a Friedmann brane with perfect fluid, when discussing cosmological perturbations, the vorticity of u should be allowed, similarly as in existing formalisms of covariant cosmology in both general relativity and brane-worlds. We can also argue for the need of keeping the vorticity of n. One reason would be that if it is vanishing at some initial instant, it should stay zero, and the formalism should be able to handle its 'evolution'. Secondly, and more important, the vorticity of n should not necessarily vanish at other locations than the brane, a gauge freedom worth to explore.

We note that a lower-dimensional 2+1+1 formalism was developed in [16, 17] and applied in the general relativistic covariant perturbations of Schwarzschild black holes and rotationally symmetric spacetime, and then for investigating spherically symmetric static solutions in $f(R)$ gravity theories in [18] .

In general relativity the important topic of cosmological perturbations, related to both the cosmic microwave background and structure formation, has a rich literature, from which (without claiming completeness) we mention [19]–[27], all based on the 3+1 covariant approach.

Brane-world cosmological perturbations are equally important; however, additional difficulties emerge due to the complexity of brane-world theory and the impediment to predict and perform observations on the brane. At a technical level, the latter is obstructed by the lack of closure of the perturbation equations on the brane. Despite impressive developments [28]–[40], many questions remain unanswered. Although we cannot overcome well-known difficulties, we expect our new formalism will provide the most convenient and complete tool-chest for approaching the problems. We also foresee the possibility of important applications for brane-world exact solutions.

We establish the generic 3+1+1 covariant formalism in section 2, by defining the kinematical quantities and the decomposition of the Weyl and energy–momentum tensors. We also relate the curvature and Ricci tensors and the three-dimensional (3d) scalar curvature to kinematic, non-local (Weyl) and matter-defined variables. In appendix A the commutation relations among all types of derivatives emerging in the formalism are given.

Section 3 contain the full 3+1+1 decomposed covariant gravitational dynamics and constraints, together with the available matter field evolutions. In appendix B we discuss the gauge freedom in the frame choice and give the transformations of all relevant quantities under infinitesimal frame transformations.

Taking into account that the brane is a 4d time-like hypersurface, in section 4 we discuss the decomposition of the Lanczos equation and of the sources of the effective Einstein equation. Then we specify the generic evolution and constraint equations on the brane, expressed in terms of quantities defined on the brane. These equations arise from combinations of the equations given in section 3, evaluated on the brane. Appendix C contains the brane equations for an asymmetric embedding. We continue section 4 by specializing to a symmetrically embedded brane, by taking into account the Lanczos equation. Then we conclude the section by deriving a *closure condition* for the equations on the brane.

Section 5 contains the derivation of the most important cosmological equations to a hypersurface (a generic brane), and then the discussion of the particular case of cosmological symmetries and perfect fluid source on the brane. As a test of our formalism we recover the

Friedmann, Raychaudhuri and energy balance equations and compare them with the relevant results of [4]. In subsection 5.2 and appendix D we also relate our formalism to anisotropic brane-world cosmologies.

Section 6 contains an astrophysical application of our formalism devoted to locally rotationally symmetric (LRS) spacetimes, at the end of which we recover a new brane solution with NUT charge. This solution obeys the previously derived closure condition.

Section 7 contains the concluding remarks.

Notations. Quantities defined on the 3-space orthogonal to both u^a and n^a carry no distinguishing mark and the 3d metric is denoted h_{ab}. Quantities defined on the brane carry the pre-index $^{(4)}$, the only exception being the 4d metric g_{ab}. Quantities defined on the full 5-space carry a distinguishing $\widetilde{}$ mark. Exceptionally, other quantities also carry the distinguishing $\widetilde{}$ mark. These are (a) the 3+1+1 decomposed components of the 5d energy–momentum tensor, which are defined on the 3-space orthogonal to both u^a and n^a, and (b) the kinematic and extrinsic curvature-type quantities related to another singled out spatial vector e^a in section 6. Calligraphic symbols denote 3+1+1 decomposed components of the 5d Weyl tensor. Whenever possible, identical symbols are used for quantities related to the temporal vector field u^a and the brane normal n^a, the latter distinguished by a $\widehat{}$ mark. We denote the average of a quantity f taken over the two sides of the brane by $\langle f\rangle$, and its jump by Δf. Angular brackets $\langle\rangle$ on abstract indices denote tensors which are projected in all indices with the metric h_{ab}, symmetrized and trace-free. Round brackets () and square brackets [] on indices denote the symmetric and antisymmetric parts, respectively.

2. The 3+1+1 covariant formalism

2.1. Decomposition of the metric

Let $u^a = \mathrm{d}x^a/\mathrm{d}\tau$ and $n^a = \mathrm{d}x^a/\mathrm{d}y$ be a time-like and a space-like vector field in the 5d spacetime, respectively, with τ and y the affine parameters of the respective non-null integral curves. They obey the normalization conditions $-u^a u_a = 1 = n^a n_a$ and the perpendicularity condition $u^a n_a = 0$. The 5d metric is decomposed as

$$\widetilde{g}_{ab} = n_a n_b + g_{ab}, \tag{1}$$

with

$$g_{ab} = -u_a u_b + h_{ab}, \tag{2}$$

the metric on the 4d temporal leaves y =const (with the brane at $y = 0$) and the spatial part h_{ab} of this metric obeying $u^a h_{ab} = n^a h_{ab} = 0$. We denote by ε_{abc} the volume element associated with h_{ab}. The 4-vector u^a represents the time-like velocity field of brane observers (see figure 1).

We employ three types of derivatives, all associated with projections of the 5d connection $\widetilde{\nabla}_i$. A dot and a prime denote covariant derivatives along the integral curves of u^a and n^a, respectively, while D is the 3d covariant derivative compatible with the metric h:

$$\dot{T}_{b..c} = u^a \widetilde{\nabla}_a T_{b..c}, \tag{3}$$

$$T'_{b..c} = n^a \widetilde{\nabla}_a T_{b..c}, \tag{4}$$

$$D_a T_{b..c} = h_a{}^d h_b{}^i..h_c{}^j \widetilde{\nabla}_d T_{i..j}. \tag{5}$$

Note that the D-derivative is the same as introduced in general relativity employing the corresponding projection of the ∇-derivative (∇ being the connection compatible with g_{ab}).

This is because both generate the covariant derivatives formed with the connection compatible with h_{ab}. Concerning the time derivative defined above, except for scalars, it differs from the corresponding general relativistic time derivative employed in the 3+1 covariant formalism (which is defined with ∇) in n^a and u^a terms. Nevertheless, when projected to the 3-manifold with $h_b{}^a$, the two definitions agree.

2.2. Kinematic quantities

We introduce the kinematic quantities through the decomposition of the 5d covariant derivative of u^a, n^a as

$$\widetilde{\nabla}_a u_b = -u_a A_b + \widehat{K} u_a n_b + K n_a n_b + n_a K_b + L_a n_b + K_{ab}, \tag{6a}$$
$$\widetilde{\nabla}_a n_b = n_a \widehat{A}_b + K n_a u_b + \widehat{K} u_a u_b - u_a \widehat{K}_b + L_a u_b + \widehat{K}_{ab}, \tag{6b}$$

where

$$\begin{aligned} A_a &= \dot{u}_{\langle a\rangle}, & \widehat{A}_a &= n'_{\langle a\rangle}, \\ K_a &= u'_{\langle a\rangle}, & \widehat{K}_a &= \dot{n}_{\langle a\rangle}, \\ K &= n^b u'_b, & \widehat{K} &= u^b \dot{n}_b, \\ L_a &= h_a{}^c n^d \widetilde{\nabla}_c u_d, \\ K_{ab} &= D_a u_b, & \widehat{K}_{ab} &= D_a n_b. \end{aligned} \tag{7}$$

As a rule, an overhat is used for kinematical quantities related to the vector field n^a in order to distinguish them from the similar kinematical quantities related to the vector field u^a. Here A_a is the acceleration. All scalars, vectors and tensors in the above decomposition are defined on the 3d manifold orthogonal to both n^a and u^a. The tensorial expressions K_{ab} and $\widehat{K}_{ab}$ can be further decomposed into (trace-, trace-free symmetric and antisymmetric) irreducible parts as follows:

$$K_{ab} = \frac{\Theta}{3} h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{8}$$

$$\widehat{K}_{ab} = \frac{\widehat{\Theta}}{3} h_{ab} + \widehat{\sigma}_{ab} + \widehat{\omega}_{ab}, \tag{9}$$

where we have defined the expansion, vorticity and shear of the vector fields u^a and n^a as

$$\begin{aligned} \Theta &= D^a u_a, \quad \omega_{ab} = D_{[a} u_{b]}, \quad \sigma_{ab} = D_{\langle a} u_{b\rangle}, \\ \widehat{\Theta} &= D^a n_a, \quad \widehat{\omega}_{ab} = D_{[a} n_{b]}, \quad \widehat{\sigma}_{ab} = D_{\langle a} n_{b\rangle}. \end{aligned} \tag{10}$$

The antisymmetric 3d tensors ω_{ab} and $\widehat{\omega}_{ab}$ can also be encoded in the vorticity vectors $\omega^a = \varepsilon^{abc}\omega_{bc}/2$ and $\widehat{\omega}^a = \varepsilon^{abc}\widehat{\omega}_{bc}/2$. When the vorticities of both vector fields n^a and u^a vanish $\omega_{ab} = \widehat{\omega}_{ab} = 0$, the tensorial expressions K_{ab} and $\widehat{K}_{ab}$ are symmetric and they represent the two extrinsic curvatures of a 3d hypersurface. The condition $\widehat{\omega}_{ab} = 0$ is also necessary in order to have a brane at $y = 0$, but not sufficient. Indeed the brane is a (3+1)-dimensional hypersurface which can exist only if the higher-dimensional vorticity of its normal $\widehat{\omega}_{ab} = \nabla_{[a} n_{b]}$ vanishes. This condition translates into $0 = g^c_{[a} g^d_{b]} \widetilde{\nabla}_c n_d = -u_{[a}\left(\widehat{K}_{b]} + L_{b]}\right) + \widehat{\omega}_{ab}$; therefore, (due to Frobenius' theorem) the necessary and sufficient conditions for the existence of the 3+1 brane are

$$\begin{aligned} &\widehat{\omega}_{ab}|_{y=0} = 0, \\ &L_a|_{y=0} = -\left.\widehat{K}_a\right|_{y=0}. \end{aligned} \tag{11}$$

In summary, the independent components of $\widetilde{\nabla}_a u_b$ are expressed by three scalars $(K, \widehat{K}, \Theta)$, four 3-vectors $(A_a, K_a, L_a, \omega_a)$ and a symmetric trace-free 3-tensor (σ_{ab}). The corresponding decomposition of $\widetilde{\nabla}_a n_b$ consists of the three scalars $(\widehat{K}, K, \widehat{\Theta})$, four 3-vectors $(\widehat{A}_a, \widehat{K}_a, L_a, \widehat{\omega}_a)$ and a symmetric trace-free 3-tensor $(\widehat{\sigma}_{ab})$. The irreducible decompositions of the covariant derivatives $\widetilde{\nabla}_a u_b$ and$\widetilde{\nabla}_a n_b$ have therefore 20 independent components each (due to the constraints $u^b \widetilde{\nabla}_a u_b = 0$ and $n^b \widetilde{\nabla}_a n_b = 0$).

2.3. *Gravito-electro-magnetic quantities*

The non-local gravitational properties of the 5d spacetime are carried by the 5d Weyl tensor, the principal directions of which lead to a classification scheme generalizing the general relativistic Petrov classification [41]. Here we are interested in the 3+1+1 decomposition[3] of $\widetilde{C}_{abcd}$, which can be given in terms of the quantities (with a total of 35 independent components):

$$\begin{aligned}
&\mathcal{E} = \widetilde{C}_{abcd} n^a u^b n^c u^d, && \\
&\mathcal{H}_k = \tfrac{1}{2}\varepsilon_k{}^{ab} \widetilde{C}_{abcd} u^c n^d, && \mathcal{F}_{kl} = \widetilde{C}_{abcd} h_{(k}{}^a u^b h_{l)}{}^c n^d, \\
&\widehat{\mathcal{E}}_k = \widetilde{C}_{abcd} h_k{}^a n^b u^c n^d, && \mathcal{E}_k = \widetilde{C}_{abcd} h_k{}^a u^b n^c u^d, \\
&\widehat{\mathcal{E}}_{kl} = \widetilde{C}_{abcd} h_{\langle k}{}^a n^b h_{l\rangle}{}^c n^d, && \mathcal{E}_{kl} = \widetilde{C}_{abcd} h_{\langle k}{}^a u^b h_{l\rangle}{}^c u^d, \\
&\widehat{\mathcal{H}}_{kl} = \tfrac{1}{2}\varepsilon_{(k}{}^{ab} h_{l)}{}^c \widetilde{C}_{abcd} n^d, && \mathcal{H}_{kl} = \tfrac{1}{2}\varepsilon_{(k}{}^{ab} h_{l)}{}^c \widetilde{C}_{abcd} u^d.
\end{aligned} \tag{12}$$

We note that all tensorial quantities defined above are trace-free (from the properties of the Weyl tensor), and further in the tensors $\mathcal{F}_{kl}$, $\mathcal{H}_{kl}$ and $\widehat{\mathcal{H}}_{kl}$ the brackets $\langle\rangle$ are equivalent with the round brackets (). The Weyl tensor in terms of the quantities defined in (12) is

$$\begin{aligned}
\widetilde{C}_{abcd} = & -2\left(\mathcal{E}_{d[a} h_{b]c} - \mathcal{E}_{c[a} h_{b]d}\right) + 2\left(\widehat{\mathcal{E}}_{d[a} h_{b]c} - \widehat{\mathcal{E}}_{c[a} h_{b]d}\right) - \frac{2}{3}\mathcal{E} h_{c[a} h_{b]d} \\
& + 2\left(\varepsilon_{cd}{}^i \widehat{\mathcal{H}}_{i[a} n_{b]} + \varepsilon_{ab}{}^i \widehat{\mathcal{H}}_{i[c} n_{d]}\right) - 2\left(\varepsilon_{cd}{}^i \mathcal{H}_{i[a} u_{b]} + \varepsilon_{ab}{}^i \mathcal{H}_{i[c} u_{d]}\right) \\
& + 2\left(n_c n_{[a} \widehat{\mathcal{E}}_{b]d} - n_d n_{[a} \widehat{\mathcal{E}}_{b]c}\right) + 2\left(u_c u_{[a} \mathcal{E}_{b]d} - u_d u_{[a} \mathcal{E}_{b]c}\right) \\
& - 2\left(u_c n_{[a} \mathcal{F}_{b]d} - u_d n_{[a} \mathcal{F}_{b]c} + n_c u_{[a} \mathcal{F}_{b]d} - n_d u_{[a} \mathcal{F}_{b]c}\right) \\
& - \left(n_{[c} u_{a]} \varepsilon_{bdk} + u_{[c} n_{b]} \varepsilon_{adk}\right) \mathcal{H}^k - \left(u_{[d} n_{a]} \varepsilon_{bck} + n_{[d} u_{b]} \varepsilon_{ack}\right) \mathcal{H}^k \\
& - 2\left(u_{[c} n_{d]} \varepsilon_{abk} + u_{[a} n_{b]} \varepsilon_{cdk}\right) \mathcal{H}^k - 2\left(\mathcal{E}_{[a} h_{b][c} n_{d]} + \mathcal{E}_{[c} h_{d][a} n_{b]}\right) \\
& - 2\left(\widehat{\mathcal{E}}_{[a} h_{b][c} u_{d]} + \widehat{\mathcal{E}}_{[c} h_{d][a} u_{b]}\right) + 4\mathcal{E} u_{[a} n_{b]} u_{[c} n_{d]} \\
& - 2\left(n_c u_d n_{[a} \widehat{\mathcal{E}}_{b]} - u_c n_d n_{[a} \widehat{\mathcal{E}}_{b]} + n_a u_b n_{[c} \widehat{\mathcal{E}}_{d]} - u_a n_b n_{[c} \widehat{\mathcal{E}}_{d]}\right) \\
& + 2\left(u_c n_d u_{[a} \mathcal{E}_{b]} - n_c u_d u_{[a} \mathcal{E}_{b]} + u_a n_b u_{[c} \mathcal{E}_{d]} - n_a u_b u_{[c} \mathcal{E}_{d]}\right) \\
& - \frac{2}{3}\mathcal{E}\left(n_d n_{[a} h_{b]c} - n_c n_{[a} h_{b]d}\right) + \frac{2}{3}\mathcal{E}\left(u_d u_{[a} h_{b]c} - u_c u_{[a} h_{b]d}\right).
\end{aligned} \tag{13}$$

This relation generalizes the general relativistic 3+1 covariant decomposition of the 4d Weyl tensor C_{abcd} (which has only ten independent components), where only two tensors $E_{kl} = C_{abcd} h_{\langle k}{}^a u^b h_{l\rangle}{}^c u^d$ and $H_{kl} = \frac{1}{2}\varepsilon_{(k}{}^{ab} h_{l)}{}^c C_{abcd} u^d$ appear, the electric and magnetic parts of the Weyl curvature. On the brane the relation between the set of variables $\mathcal{E}_{kl}$ and $\mathcal{H}_{kl}$ and the variables E_{kl} and H_{kl} is given by

$$\mathcal{E}_{ab} = E_{ab} + \frac{1}{2}\widehat{\mathcal{E}}_{ab} - \frac{1}{2}\left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\widehat{\sigma}_{ab} + \frac{1}{2}\widehat{\sigma}_{c\langle a}\widehat{\sigma}_{b\rangle}{}^c + \frac{1}{2}\widehat{K}_{\langle a}\widehat{K}_{b\rangle}, \tag{14}$$

$$\mathcal{H}_{ab} = H_{ab} - \varepsilon_{\langle a}{}^{cd}\widehat{\sigma}_{b\rangle c}\widehat{K}_d. \tag{15}$$

[3] The case of a generic $n + 1$ decomposition of the Weyl tensor was discussed in [42].

2.4. Decomposition of the energy–momentum tensor

The 5d gravitational dynamics is governed by the Einstein equation

$$\widetilde{G}_{ab} = -\widetilde{\Lambda}\widetilde{g}_{ab} + \widetilde{\kappa}^2[\widetilde{T}_{ab} + \tau_{ab}\delta\,(y)], \tag{16}$$

where $\widetilde{\kappa}^2$ denotes the 5d coupling constant. The sources of gravity are the 5d cosmological constant $\widetilde{\Lambda}$, the regular part of the 5d energy–momentum tensor $\widetilde{T}_{ab}$ and a distributional energy–momentum tensor with support on the brane:

$$\tau_{ab} = -\lambda g_{ab} + T_{ab}. \tag{17}$$

Here λ is the constant brane tension and T_{ab} describes standard model matter fields on the brane[4], decomposed with respect to a brane observer with 4-velocity u^a as

$$T_{ab} = \rho u_a u_b + q_{(a}u_{b)} + p h_{ab} + \pi_{ab}. \tag{18}$$

The quantities ρ, q_a, p and π_{ab} are the energy density, the energy current vector, the isotropic pressure and the symmetric trace-free anisotropic pressure tensor of the matter on the brane, respectively.

We decompose the regular part of the 5d energy–momentum tensor relative to u^a and n^a as

$$\widetilde{T}_{ab} = \widetilde{\rho}u_a u_b + 2\widetilde{q}_{(a}u_{b)} + 2\widetilde{q}u_{(a}n_{b)} + \widetilde{p}h_{ab} + \widetilde{\pi}n_a n_b + 2\widetilde{\pi}_{(a}n_{b)} + \widetilde{\pi}_{ab}, \tag{19}$$

where $\widetilde{\rho}$ is the relativistic energy density relative to u^a, $\widetilde{p}$ is the isotropic pressure, and $\widetilde{q}_a$ and $\widetilde{q}$ are the relativistic momentum densities on the 3-space and in the direction n^a. The quantities $\widetilde{\pi}$, $\widetilde{\pi}_a$ and the symmetric and trace-free tensor $\widetilde{\pi}_{ab}$ are related to the scalar, vectorial and tensorial components (with respect to the 3d space) of the 5d anisotropic pressure tensor, which is

$$\frac{3\,(\widetilde{\pi} - \widetilde{p})}{4}n_a n_b + 2\widetilde{\pi}_{(a}n_{b)} + \frac{\widetilde{p} - \widetilde{\pi}}{4}h_{ab} + \widetilde{\pi}_{ab}. \tag{20}$$

By employing the definitions given in this section we give the commutation relations among the temporal, off-brane and brane covariant derivatives in appendix A.

2.5. The Gauss equation and its contractions

We define the local 3d curvature tensor $\mathcal{R}_{abcd}$ of the space orthogonal to both u^a and n^a as

$$D_{[a}D_{b]}V_c - \omega_{ab}\dot{V}_{\langle c\rangle} + \widehat{\omega}_{ab}V'_{\langle c\rangle} = \frac{1}{2}\mathcal{R}_{abcd}V^d, \tag{21}$$

resulting in

$$\begin{aligned}\mathcal{R}_{abcd} = {} & h_a{}^i h_b{}^j h_c{}^k h_d{}^l \widetilde{R}_{ijkl} - (D_a u_c)\,(D_b u_d) + (D_a u_d)\,(D_b u_c) \\ & + (D_a n_c)\,(D_b n_d) - (D_a n_d)\,(D_b n_c)\,.\end{aligned} \tag{22}$$

This is a natural generalization of the definition used in general relativity [22, 44, 45]. By the definitions of the kinematical, gravito-electro-magnetic and matter variables, we have

$$\begin{aligned}\mathcal{R}_{abcd} = {} & \left[-\frac{2}{3}(\Theta^2 - \widehat{\Theta}^2) - \mathcal{E} + \widetilde{\Lambda} + \widetilde{\kappa}^2\,(\widetilde{\rho} + \widetilde{p} - \widetilde{\pi})\right]\frac{h_{c[a}h_{b]d}}{3} \\ & - 2\big(\mathcal{E}_{d[a}h_{b]c} - \mathcal{E}_{c[a}h_{b]d}\big) + 2\big(\widehat{\mathcal{E}}_{d[a}h_{b]c} - \widehat{\mathcal{E}}_{c[a}h_{b]d}\big) - \frac{2\widetilde{\kappa}^2}{3}\big(h_{d[a}\widetilde{\pi}_{b]c} - h_{c[a}\widetilde{\pi}_{b]d}\big)\end{aligned}$$

[4] For DGP/induced gravity-type models [43], T_{ab} should be replaced by $T_{ab} - (\gamma/\kappa^2)\,G_{ab}$, with G_{ab} the Einstein tensor constructed from the metric g_{ab} and γ the dimensionless induced gravity parameter. Randall–Sundrum-type brane-worlds are recovered for $\gamma \to 0$, on the $\varepsilon = -1$ branch.

$$
\begin{aligned}
&-\frac{2\Theta}{3}\left[\left(\sigma_{a[c}+\omega_{a[c}\right)h_{d]b}+\left(\sigma_{b[d}+\omega_{b[d}\right)h_{c]a}\right]-2\left(\sigma_{a[c}+\omega_{a[c}\right)\left(\sigma_{d]b}-\omega_{d]b}\right) \\
&+\frac{2\widehat{\Theta}}{3}\left[\left(\widehat{\sigma}_{a[c}+\widehat{\omega}_{a[c}\right)h_{d]b}+\left(\widehat{\sigma}_{b[d}+\widehat{\omega}_{b[d}\right)h_{c]a}\right]+2\left(\widehat{\sigma}_{a[c}+\widehat{\omega}_{a[c}\right)\left(\widehat{\sigma}_{d]b}-\widehat{\omega}_{d]b}\right),
\end{aligned}
\tag{23}
$$

and the local 3d Ricci tensor and Ricci scalar become

$$
\begin{aligned}
\mathcal{R}_{ac}=h^{bd}\mathcal{R}_{abcd}=&\left[-\frac{2}{3}(\Theta^2-\widehat{\Theta}^2)-2\mathcal{E}+\widetilde{\Lambda}+\widetilde{\kappa}^2\,(\widetilde{\rho}+\widetilde{p}-\widetilde{\pi})\right]\frac{h_{ac}}{3} \\
&+\mathcal{E}_{ac}-\widehat{\mathcal{E}}_{ac}+\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ac}-\frac{2\Theta}{3}\,(\sigma_{ac}+\omega_{ac})+\frac{2\widehat{\Theta}}{3}\,(\widehat{\sigma}_{ac}+\widehat{\omega}_{ac}) \\
&+\sigma_{ab}\sigma_c{}^b-\omega_b\omega^b h_{ac}+\omega_a\omega_c-2\sigma_{[a}{}^b\omega_{c]b} \\
&-\widehat{\sigma}_{ab}\widehat{\sigma}_c{}^b+\widehat{\omega}_b\widehat{\omega}^b h_{ac}-\widehat{\omega}_a\widehat{\omega}_c+2\widehat{\sigma}_{[a}{}^b\widehat{\omega}_{c]b},
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\mathcal{R}=h^{ac}\mathcal{R}_{ac}=&-2\mathcal{E}+\widetilde{\Lambda}+\widetilde{\kappa}^2\,(\widetilde{\rho}+\widetilde{p}-\widetilde{\pi})-\frac{2}{3}(\Theta^2-\widehat{\Theta}^2) \\
&-2\omega_a\omega^a+\sigma_{ab}\sigma^{ab}+2\widehat{\omega}_a\widehat{\omega}^a-\widehat{\sigma}_{ab}\widehat{\sigma}^{ab}.
\end{aligned}
\tag{25}
$$

Equation (25) can be referred to as a generalized Friedmann equation in the 5d spacetime, as will become evident in section 5. The general relativistic counterpart is presented in [44, 45].

3. 3+1+1 covariant dynamics

The *full set* of the 3+1+1 covariant evolution and constraint equations for the kinematic, gravito-electro-magnetic and matter variables are given by the projections of the Bianchi and Ricci identities for u^a and n^a. These equations hold as the main result of the paper, and they are presented in the following three subsections. In particular, the first subsection contains all Ricci identities; the second subsection contains twice contracted Bianchi identities, which by virtue of the Einstein equations describe evolutions for the energy density and currents, while the third subsection contains the rest of independent Bianchi identities. A related appendix B gives the transformation rules under a frame change for the totality of the kinematic, gravito-electro-magnetic and matter variables, to linear order accuracy.

3.1. Ricci identities

The Ricci identity for u^a gives the following independent equations:

$$
\begin{aligned}
0=&\dot{K}+\widehat{K}'+\widehat{A}_aA^a+K^2-\widehat{K}^2+L_aK^a-\widehat{K}_aK^a-L_a\widehat{K}^a \\
&+\mathcal{E}-\frac{\widetilde{\Lambda}}{6}+\frac{\widetilde{\kappa}^2}{6}\,(\widetilde{\rho}-\widetilde{\pi}+3\widetilde{p})\,,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
0=&\dot{\Theta}-D^aA_a+\frac{\Theta^2}{3}+\widehat{\Theta}\widehat{K}-A^aA_a-2\omega_a\omega^a+\sigma_{ab}\sigma^{ab}-\mathcal{E} \\
&+(\widehat{K}^aK_a+K^aL_a+L^a\widehat{K}_a)-\frac{\widetilde{\Lambda}}{2}+\frac{\widetilde{\kappa}^2}{2}\,(\widetilde{\rho}+\widetilde{\pi}+\widetilde{p})\,,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
0=&\dot{K}_{\langle a\rangle}-A'_{\langle a\rangle}+\left(K+\frac{\Theta}{3}\right)K_a+\left(K-\frac{\Theta}{3}\right)\widehat{K}_a+\mathcal{E}_a \\
&+\widehat{K}\,(\widehat{A}_a+A_a)+(\omega_{ba}+\sigma_{ba})\,(K^b-\widehat{K}^b)-\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_a,
\end{aligned}
\tag{28}
$$

$$0 = \dot{L}_{\langle a\rangle} + D_a\widehat{K} + \left(K + \frac{\Theta}{3}\right)L_a + \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)A_a + \left(K - \frac{\Theta}{3}\right)\widehat{K}_a$$
$$+ (\widehat{\omega}_{ab} + \widehat{\sigma}_{ab})\,A^b - (\omega_{ab} + \sigma_{ab})\,(\widehat{K}^b - L^b) + \mathcal{E}_a - \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_a, \tag{29}$$
$$0 = \dot{\omega}_{\langle a\rangle} - \frac{1}{2}\varepsilon_a{}^{cd}D_cA_d + \widehat{K}\widehat{\omega}_a + \frac{2\Theta}{3}\omega_a - \sigma_{ab}\omega^b + \frac{1}{2}\varepsilon_a{}^{cd}(K_c\widehat{K}_d + K_cL_d + \widehat{K}_cL_d), \tag{30}$$
$$0 = \dot{\sigma}_{\langle ab\rangle} - D_{\langle a}A_{b\rangle} + \frac{2\Theta}{3}\sigma_{ab} + \widehat{K}\widehat{\sigma}_{ab} - A_{\langle a}A_{b\rangle} + K_{\langle a}L_{b\rangle}$$
$$+ \widehat{K}_{\langle a}K_{b\rangle} + L_{\langle a}\widehat{K}_{b\rangle} + \omega_{\langle a}\omega_{b\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}{}^c + \mathcal{E}_{ab} - \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab}, \tag{31}$$
$$0 = \Theta' - D^aK_a - \left(K - \frac{\Theta}{3}\right)\widehat{\Theta} + (K^a + L^a)\widehat{A}_a - A^a(K_a - L_a)$$
$$- 2\widehat{\omega}_a\omega^a + \widehat{\sigma}_{ab}\sigma^{ab} - \widetilde{\kappa}^2\widetilde{q}, \tag{32}$$
$$0 = L'_{\langle a\rangle} - D_aK + \left(K - \frac{\Theta}{3}\right)\widehat{A}_a - \left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)L_a + \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)K_a$$
$$+ (\widehat{\sigma}_{ab} + \widehat{\omega}_{ab})\,(K^b + L^b) - (\omega_{ab} + \sigma_{ab})\,\widehat{A}^b - \widehat{\mathcal{E}}_a + \frac{\widetilde{\kappa}^2}{3}\widetilde{q}_a, \tag{33}$$
$$0 = \omega'_{\langle k\rangle} - \frac{1}{2}\varepsilon_k{}^{ab}D_aK_b - \frac{1}{2}\varepsilon_k{}^{ab}(A_b + \widehat{A}_b)(K_a - L_a) - \left(K - \frac{\Theta}{3}\right)\widehat{\omega}_k \tag{34}$$
$$+ \frac{\widehat{\Theta}}{3}\omega_k + \frac{1}{2}\varepsilon_k{}^{ab}\omega_a\widehat{\omega}_b - \frac{1}{2}\widehat{\sigma}_{ka}\omega^a - \frac{1}{2}\sigma_{ka}\widehat{\omega}^a + \frac{1}{2}\varepsilon_k{}^{ab}\sigma_{cb}\widehat{\sigma}_a{}^c - \frac{1}{2}\mathcal{H}_k, \tag{35}$$
$$0 = \sigma'_{\langle ab\rangle} - D_{\langle a}K_{b\rangle} - A_{\langle b}(K_{a\rangle} - L_{a\rangle}) + \widehat{A}_{\langle b}(K_{a\rangle} + L_{a\rangle}) + \frac{\widehat{\Theta}}{3}\sigma_{ab}$$
$$- \left(K - \frac{\Theta}{3}\right)\widehat{\sigma}_{ab} + \omega_{\langle a}\widehat{\omega}_{b\rangle} + \omega_{c\langle a}\widehat{\sigma}_{b\rangle}{}^c + \sigma_{\langle a}{}^d\widehat{\omega}_{b\rangle d} + \sigma_{c\langle a}\widehat{\sigma}_{b\rangle}{}^c + \mathcal{F}_{ab}, \tag{36}$$
$$0 = \varepsilon_k{}^{ab}D_aL_b + 2\left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\omega_k + 2\left(K - \frac{\Theta}{3}\right)\widehat{\omega}_k + \varepsilon_k{}^{ab}\omega_a\widehat{\omega}_b - \widehat{\sigma}_{ka}\omega^a + \sigma_{ka}\widehat{\omega}^a$$
$$- \varepsilon_k{}^{ab}\sigma_{cb}\widehat{\sigma}_a{}^c + \mathcal{H}_k, \tag{37}$$
$$0 = D^a\omega_a - A_a\omega^a + \widehat{\omega}^a\,(K_a - L_a)\,, \tag{38}$$
$$0 = D_{\langle c}\omega_{k\rangle} + \varepsilon_{ab\langle k}D^b\sigma_{c\rangle}{}^a + 2A_{\langle c}\omega_{k\rangle} - 2K_{\langle c}\widehat{\omega}_{k\rangle} - L_{\langle c}\widehat{\omega}_{k\rangle} + \varepsilon_{\langle k}{}^{ab}\widehat{\sigma}_{c\rangle b}L_a + \mathcal{H}_{kc}, \tag{39}$$
$$0 = D^b\sigma_{ab} + \varepsilon_a{}^{ck}D_c\omega_k - \frac{2}{3}D_a\Theta + \frac{2\widehat{\Theta}}{3}L_a - \varepsilon_a{}^{ck}L_c\widehat{\omega}_k$$
$$+ 2\varepsilon_a{}^{ck}\,(A_c\omega_k - K_c\widehat{\omega}_k) - \widehat{\sigma}_{ab}L^b + \widehat{\mathcal{E}}_a + \frac{2\widetilde{\kappa}^2}{3}\widetilde{q}_b. \tag{40}$$

The Ricci identity for n^a gives the following independent equations:

$$0 = \dot{\widehat{\Theta}} - D^a\widehat{K}_a + \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\Theta - (\widehat{K}^a - L^a)A_a + \widehat{A}^a(\widehat{K}_a + L_a) - 2\omega_a\widehat{\omega}^a + \widehat{\sigma}_{ab}\sigma^{ab} - \widetilde{\kappa}^2\widetilde{q}, \tag{41}$$
$$0 = \dot{\widehat{\omega}}_{\langle k\rangle} - \frac{1}{2}\varepsilon_k{}^{ab}D_a\widehat{K}_b + \frac{1}{2}\varepsilon_k{}^{ab}(\widehat{A}_b + A_b)(\widehat{K}_a + L_a) + \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\omega_k$$
$$+ \frac{\Theta}{3}\widehat{\omega}_k + \frac{1}{2}\varepsilon_k{}^{ab}\widehat{\omega}_a\omega_b - \frac{1}{2}\widehat{\sigma}_{ka}\omega^a - \frac{1}{2}\sigma_{ka}\widehat{\omega}^a + \frac{1}{2}\varepsilon_k{}^{ab}\widehat{\sigma}_{cb}\sigma_a{}^c + \frac{1}{2}\mathcal{H}_k, \tag{42}$$

$$0 = \dot{\widehat{\sigma}}_{\langle ab\rangle} - D_{\langle a}\widehat{K}_{b\rangle} + \widehat{A}_{\langle b}(\widehat{K}_{a\rangle} + L_{a\rangle}) - A_{\langle b}(\widehat{K}_{a\rangle} - L_{a\rangle}) + \frac{\Theta}{3}\widehat{\sigma}_{ab}$$
$$+ \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\sigma_{ab} + \omega_{\langle a}\widehat{\omega}_{b\rangle} - \omega_{c\langle a}\widehat{\sigma}_{b\rangle}{}^{c} - \sigma_{\langle a}{}^{d}\widehat{\omega}_{b\rangle d} + \sigma_{c\langle a}\widehat{\sigma}_{b\rangle}{}^{c} + \mathcal{F}_{ab}, \tag{43}$$

$$0 = \widehat{K}'_{\langle a\rangle} - \dot{\widehat{A}}_{\langle a\rangle} - \left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\widehat{K}_a - \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)K_a - K\left(A_a + \widehat{A}_a\right)$$
$$+ (\widehat{\omega}_{ba} + \widehat{\sigma}_{ba})\left(\widehat{K}^b - K^b\right) + \widehat{\mathcal{E}}_a - \frac{\widetilde{\kappa}^2}{3}\widetilde{q}_a, \tag{44}$$

$$0 = \widehat{\Theta}' - D^a\widehat{A}_a + \frac{\widehat{\Theta}^2}{3} - \Theta K + \widehat{A}^a\widehat{A}_a - 2\widehat{\omega}_a\widehat{\omega}^a + \widehat{\sigma}_{ab}\widehat{\sigma}^{ab} + \mathcal{E}$$
$$- (\widehat{K}^a K_a - \widehat{K}^a L_a - L^a K_a) + \frac{\widetilde{\Lambda}}{2} + \frac{\widetilde{\kappa}^2}{2}(\widetilde{\pi} + \widetilde{\rho} - \widetilde{p}), \tag{45}$$

$$0 = \widehat{\omega}'_{\langle a\rangle} - \frac{1}{2}\varepsilon_a{}^{cd}D_c\widehat{A}_d - K\omega_a + \frac{2\widehat{\Theta}}{3}\widehat{\omega}_a - \widehat{\sigma}_{ab}\widehat{\omega}^b - \frac{1}{2}\varepsilon_a{}^{cd}\left(K_c\widehat{K}_d + \widehat{K}_c L_d + K_c L_d\right), \tag{46}$$

$$0 = \widehat{\sigma}'_{\langle ab\rangle} - D_{\langle a}\widehat{A}_{b\rangle} + \frac{2\widehat{\Theta}}{3}\widehat{\sigma}_{ab} - K\sigma_{ab} + \widehat{A}_{\langle a}\widehat{A}_{b\rangle} + \widehat{K}_{\langle a}L_{b\rangle}$$
$$- \widehat{K}_{\langle a}K_{b\rangle} + L_{\langle a}K_{b\rangle} + \widehat{\omega}_{\langle a}\widehat{\omega}_{b\rangle} + \widehat{\sigma}_{c\langle a}\widehat{\sigma}_{b\rangle}{}^{c} + \widehat{\mathcal{E}}_{ab} + \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab}, \tag{47}$$

$$0 = D^a\widehat{\omega}_a + \widehat{A}_a\widehat{\omega}^a - \omega^a\left(\widehat{K}_a + L_a\right), \tag{48}$$

$$0 = D_{\langle c}\widehat{\omega}_{k\rangle} + \varepsilon_{ab\langle k}D^b\widehat{\sigma}_{c\rangle}{}^{a} - 2\widehat{A}_{\langle c}\widehat{\omega}_{k\rangle} + 2\widehat{K}_{\langle c}\omega_{k\rangle} - L_{\langle c}\omega_{k\rangle} + \varepsilon_{\langle k}{}^{ab}\sigma_{c\rangle b}L_a + \widehat{\mathcal{H}}_{kc}, \tag{49}$$

$$0 = D^b\widehat{\sigma}_{ab} + \varepsilon_a{}^{ck}D_c\widehat{\omega}_k - \frac{2}{3}D_a\widehat{\Theta} + \frac{2\Theta}{3}L_a - \varepsilon_a{}^{ck}L_c\omega_k$$
$$- 2\varepsilon_a{}^{ck}\left(\widehat{A}_c\widehat{\omega}_k - \widehat{K}_c\omega_k\right) - \sigma_{ab}L^b - \mathcal{E}_a - \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_a. \tag{50}$$

3.2. Conservations laws

The twice-contracted 5d Bianchi identities imply $\widetilde{\nabla}^a\widetilde{T}_{ab} = 0$, which can be decomposed into the projections taken with u, n and h, respectively:

$$0 = \dot{\widetilde{\rho}} + \widetilde{q}' + D^a\widetilde{q}_a + \widetilde{\rho}\,(K + \Theta) + K\widetilde{\pi} + \Theta\widetilde{p} + \widetilde{\pi}^{ab}\sigma_{ab}$$
$$- \widetilde{q}\left(2\widehat{K} - \widehat{\Theta}\right) + \widetilde{q}^a\left(2A_a - \widehat{A}_a\right) + \widetilde{\pi}^a\left(L_a + K_a\right), \tag{51}$$

$$0 = \dot{\widetilde{q}} + \widetilde{\pi}' + D^a\widetilde{\pi}_a + \widetilde{\pi}\left(\widehat{\Theta} - \widehat{K}\right) - \widehat{K}\widetilde{\rho} - \widehat{\Theta}\widetilde{p} - \widetilde{\pi}^{ab}\widehat{\sigma}_{ab}$$
$$+ \widetilde{q}\,(2K + \Theta) - \widetilde{\pi}^a\left(2\widehat{A}_a - A_a\right) - \widetilde{q}^a\left(\widehat{K}_a - L_a\right), \tag{52}$$

$$0 = \dot{\widetilde{q}}_{\langle k\rangle} + \widetilde{\pi}'_{\langle k\rangle} + D_k\widetilde{p} + D^a\widetilde{\pi}_{ak} + \frac{4\widehat{\Theta}}{3}\widetilde{\pi}_k + \frac{4\Theta}{3}\widetilde{q}_k - \widehat{K}\widetilde{\pi}_k + K\widetilde{q}_k + \widetilde{\rho}A_k + \widetilde{\pi}\widehat{A}_k - \widetilde{p}(\widehat{A}_k - A_k)$$
$$+ \widetilde{q}^a\omega_{ak} + \widetilde{q}^a\sigma_{ak} + \widetilde{\pi}^a\widehat{\omega}_{ak} + \widetilde{\pi}^a\widehat{\sigma}_{ak} + \widetilde{q}\left(\widehat{K}_k + K_k\right) - \widetilde{\pi}_{ak}\left(\widehat{A}^a - A^a\right). \tag{53}$$

The first of these equations is the continuity equation, as can be easily verified in the homogeneous, isotropic case ($\widetilde{q} = \widetilde{q}^a = \widetilde{\pi}^a = \widetilde{\pi}^{ab} = 0$ and $\Theta = 3H$) and for $K = 0$. The ensemble of the equations represent an incomplete set of evolution equations for the 5d matter (there are no evolution equations for $\widetilde{p}, \widetilde{\pi}, \widetilde{\pi}_a, \widetilde{\pi}_{ab}$).

3.3. Bianchi identities

The equations independent from equations (51)–(53) arising from the 5d Bianchi identities are:

$$
\begin{aligned}
0 = {} & \dot{\mathcal{E}} - D^a\widehat{\mathcal{E}}_a + \frac{4}{3}\Theta\mathcal{E} + \widehat{\mathcal{E}}_{ab}\sigma^{ab} - \mathcal{F}_{ab}\widehat{\sigma}^{ab} + 3\mathcal{H}_a\widehat{\omega}^a - \mathcal{E}_a(2\widehat{K}^a + L^a) - 2\widehat{\mathcal{E}}_a A^a \\
& - \frac{\widetilde{\kappa}^2}{2}(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p})^{\cdot} - \frac{2\widetilde{\kappa}^2}{3}D^a\widetilde{q}_a - \frac{2\widetilde{\kappa}^2}{3}\Theta\left(\widetilde{\rho} + \widetilde{p}\right) \\
& - \frac{2\widetilde{\kappa}^2}{3}\widehat{\Theta}\widetilde{q} - \frac{4\widetilde{\kappa}^2}{3}\widetilde{q}_a A^a - \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_a\left(2\widehat{K}^a + L^a\right) - \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab}\sigma^{ab},
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
0 = {} & \dot{\mathcal{E}}_{\langle k\rangle} + D^a\mathcal{F}_{ka} + \frac{1}{2}\varepsilon_k{}^{ab}D_a\mathcal{H}_b - \mathcal{E}_{ka}\left(\widehat{K}^a + L^a\right) - \widehat{\mathcal{E}}_{ka}L^a - \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\widehat{\mathcal{E}}_k + \frac{4}{3}\Theta\mathcal{E}_k \\
& + \frac{4\mathcal{E}}{3}\widehat{K}_k + \mathcal{F}_{ka}A^a - \frac{1}{2}\left(\sigma_{ka} + \omega_{ka}\right)\mathcal{E}^a - \left(\widehat{\sigma}_{ka} - 2\widehat{\omega}_{ka}\right)\widehat{\mathcal{E}}^a - 2\mathcal{H}_{ka}\widehat{\omega}^a - \widehat{\mathcal{H}}_{ka}\omega^a \\
& + \varepsilon_k{}^{ab}\widehat{\mathcal{H}}_{ca}\sigma_b{}^c - \frac{3}{2}\varepsilon_k{}^{ab}\mathcal{H}_a A_b + \frac{2\widetilde{\kappa}^2}{3}\dot{\widetilde{P}}_{\langle k\rangle} + \frac{2\widetilde{\kappa}^2}{3}D_k\widetilde{q} + \frac{2\widetilde{\kappa}^2}{3}\left(\widetilde{\pi} - \widetilde{p}\right)\widehat{K}_k \\
& + \frac{2\widetilde{\kappa}^2}{3}\left(\widetilde{\rho} + \widetilde{\pi}\right)L_k - \frac{2\widetilde{\kappa}^2}{3}\left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\widetilde{q}_k - \frac{2\widetilde{\kappa}^2}{3}\left(\widehat{\omega}_{ka} + \widehat{\sigma}_{ka}\right)\widetilde{q}^a \\
& - \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ka}\widehat{K}^a + \frac{2\widetilde{\kappa}^2}{3}\left(\omega_{ka} + \sigma_{ka}\right)\widetilde{\pi}^a + \frac{2\widetilde{\kappa}^2\Theta}{9}\widetilde{\pi}_k + \frac{2\widetilde{\kappa}^2\widetilde{q}}{3}A_k,
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
0 = {} & \dot{\mathcal{H}}_{\langle k\rangle} - \varepsilon_k{}^{ab}D_a\mathcal{E}_b - \widehat{\mathcal{H}}_{ka}A^a + \mathcal{H}_{ka}\widehat{K}^a - \mathcal{E}_{ka}\widehat{\omega}^a - \frac{8}{3}\mathcal{E}\widehat{\omega}_k + \mathcal{F}_{ka}\omega^a + \varepsilon_k{}^{ab}\mathcal{F}_{ac}\sigma_b{}^c + \Theta\mathcal{H}_k \\
& + \frac{3}{2}\left(\omega_{ka} - \sigma_{ka}\right)\mathcal{H}^a + \varepsilon_k{}^{ab}\left[\frac{3}{2}\left(\widehat{\mathcal{E}}_a\widehat{K}_b - A_a\mathcal{E}_b\right) - L_a\widehat{\mathcal{E}}_b\right] - \varepsilon_k{}^{ab}\mathcal{E}_{ac}\widehat{\sigma}_b{}^c + \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}D_a\widetilde{\pi}_b \\
& - \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}\widetilde{q}_a L_b + \frac{2\widetilde{\kappa}^2}{3}\widetilde{q}\omega_k + \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}\widetilde{\pi}_{ac}\widehat{\sigma}_b{}^c + \frac{2\widetilde{\kappa}^2}{3}\left(\widetilde{\pi} - \widetilde{p}\right)\widehat{\omega}_k + \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ka}\widehat{\omega}^a,
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
0 = {} & \dot{\widehat{\mathcal{E}}}_{\langle k\rangle} + \mathcal{E}'_{\langle k\rangle} - D_k\mathcal{E} - \mathcal{E}_{ka}\widehat{A}^a - \widehat{\mathcal{E}}_{ka}A^a + K\widehat{\mathcal{E}}_k - \widehat{K}\mathcal{E}_k + \frac{2}{3}(\Theta\widehat{\mathcal{E}}_k + \widehat{\Theta}\mathcal{E}_k) \\
& + 2(\omega_{ka} + \sigma_{ka})\widehat{\mathcal{E}}^a + 2(\widehat{\omega}_{ka} + \widehat{\sigma}_{ka})\mathcal{E}^a + \mathcal{F}_{ka}(K^a + \widehat{K}^a) + \frac{3}{2}\varepsilon_{kab}\mathcal{H}^a(\widehat{K}^b - K^b) \\
& - \frac{4}{3}\mathcal{E}(A_k - \widehat{A}_k) - \frac{\widetilde{\kappa}^2}{6}D_k\left(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p}\right) + \frac{\widetilde{\kappa}^2}{3}D^a\widetilde{\pi}_{ak} \\
& + \frac{2\widetilde{\kappa}^2}{9}(\widehat{\Theta}\widetilde{\pi}_k + \Theta\widetilde{q}_k) - \frac{\widetilde{\kappa}^2}{3}[(3\widehat{\omega}_{ka} + \widehat{\sigma}_{ka})\widetilde{\pi}^a + (3\omega_{ka} + \sigma_{ka})\widetilde{q}^a],
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
0 = {} & \dot{\mathcal{E}}_{\langle kj\rangle} - \mathcal{F}'_{\langle kj\rangle} - \varepsilon_{ab\langle k}D^a\mathcal{H}_{j\rangle}^{\;b} - \frac{1}{2}D_{\langle k}\widehat{\mathcal{E}}_{j\rangle} + \left(K - \frac{\Theta}{3}\right)\widehat{\mathcal{E}}_{kj} + \left(K + \Theta\right)\mathcal{E}_{kj} \\
& + \frac{2\mathcal{E}}{3}\sigma_{kj} + 2\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\mathcal{F}_{kj} - \widehat{\mathcal{E}}_{\langle k}\left(A_{j\rangle} - \frac{3}{2}\widehat{A}_{j\rangle}\right) + \frac{1}{2}\mathcal{E}_{\langle k}(4\widehat{K}_{j\rangle} - L_{j\rangle} - 3K_{k\rangle}) \\
& + \frac{3}{2}\mathcal{H}_{\langle k}\widehat{\omega}_{j\rangle} + 2\widehat{\mathcal{E}}_{a\langle k}\sigma_{j\rangle}{}^a + \frac{3}{2}\varepsilon_{\langle j}{}^{ab}\widehat{\sigma}_{k\rangle a}\mathcal{H}_b + \mathcal{E}_{\langle k}{}^a(\omega_{j\rangle a} - 3\sigma_{j\rangle a}) - \mathcal{F}_{\langle j}{}^a(\widehat{\omega}_{k\rangle a} - \widehat{\sigma}_{k\rangle a}) \\
& + \varepsilon_{\langle k}{}^{ab}\mathcal{H}_{j\rangle a}(2A_b - \widehat{A}_b) - \varepsilon_{\langle k}{}^{ab}\widehat{\mathcal{H}}_{j\rangle a}\left(K_b + L_b\right) + \frac{2\widetilde{\kappa}^2}{3}\dot{\widetilde{\pi}}_{\langle kj\rangle} + \frac{2\widetilde{\kappa}^2}{3}D_{\langle k}\widetilde{q}_{j\rangle} + \frac{2\widetilde{\kappa}^2}{9}\Theta\widetilde{\pi}_{jk}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle k}(3\widehat{K}_{j\rangle}+L_{j\rangle})+2\widetilde{\kappa}^2\widetilde{q}_{\langle k}A_{j\rangle}+\frac{2\widetilde{\kappa}^2}{3}\widetilde{q}\widehat{\sigma}_{jk}+\frac{2\widetilde{\kappa}^2}{3}\left(\widetilde{\rho}+\widetilde{p}\right)\sigma_{jk}\\
&+\frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}{}^{a}(\omega_{k\rangle a}+\sigma_{k\rangle a}),
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
0={}&\dot{\mathcal{F}}_{\langle kj\rangle}-\mathcal{E}'_{\langle kj\rangle}+D_{\langle k}\mathcal{E}_{j\rangle}+\left(\widehat{K}-\frac{\widehat{\Theta}}{3}\right)\mathcal{E}_{kj}+\widehat{K}\widehat{\mathcal{E}}_{kj}+\frac{4\mathcal{E}}{3}\widehat{\sigma}_{kj}+\left(2K+\frac{\Theta}{3}\right)\mathcal{F}_{kj}\\
&-\frac{3}{2}\mathcal{H}_{\langle k}\omega_{j\rangle}-\widehat{\mathcal{E}}_{\langle j}\left(\frac{3}{2}\widehat{K}_{k\rangle}-L_{k\rangle}-K_{k\rangle}\right)+\mathcal{F}_{\langle j}{}^{a}(\omega_{k\rangle a}+\sigma_{k\rangle a})-2\mathcal{E}_{\langle k}\left(\widehat{A}_{j\rangle}-\frac{3}{4}A_{j\rangle}\right)\\
&-\mathcal{E}_{\langle j}{}^{a}(\widehat{\omega}_{k\rangle a}+\widehat{\sigma}_{k\rangle a})+\varepsilon_{\langle k}{}^{ab}\mathcal{H}_{j\rangle a}(\widehat{K}_{b}-2K_{b})+\varepsilon_{\langle j}{}^{ab}\widehat{\mathcal{H}}_{k\rangle a}A_{b}+\frac{3}{2}\varepsilon_{\langle j}{}^{ab}\sigma_{k\rangle a}\mathcal{H}_{b}+\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}'_{\langle kj\rangle}\\
&-\frac{\widetilde{\kappa}^2}{3}D_{\langle k}\widetilde{\pi}_{j\rangle}+\frac{\widetilde{\kappa}^2}{9}\widehat{\Theta}\widetilde{\pi}_{kj}-\frac{\widetilde{\kappa}^2}{3}\left(\widetilde{\pi}-\widetilde{p}\right)\widehat{\sigma}_{kj}-\frac{\widetilde{\kappa}^2}{3}\widetilde{q}\sigma_{kj}\\
&+\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}{}^{a}(\widehat{\omega}_{k\rangle a}+\widehat{\sigma}_{k\rangle a})+\frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}\widehat{A}_{k\rangle}+\frac{\widetilde{\kappa}^2}{3}\widetilde{q}_{j}(2K_{k\rangle}-L_{k\rangle}),
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
0={}&\dot{\mathcal{H}}_{\langle kj\rangle}+\varepsilon_{ab\langle k}D^{a}\mathcal{E}_{j\rangle}{}^{b}+\widehat{K}\widehat{\mathcal{H}}_{kj}+\Theta\mathcal{H}_{kj}-\frac{3}{2}\mathcal{H}_{\langle k}\widehat{K}_{j\rangle}-\frac{3}{2}\widehat{\mathcal{E}}_{\langle j}\omega_{k\rangle}-(\omega_{a\langle k}+3\sigma_{a\langle k})\mathcal{H}_{j\rangle}^{a}\\
&+\varepsilon_{\langle k}{}^{ab}\mathcal{F}_{j\rangle a}(\widehat{K}_{b}+2L_{b})+\varepsilon_{\langle k}{}^{ab}(\widehat{\mathcal{E}}_{j\rangle a}-2\mathcal{E}_{j\rangle a})A_{b}-\frac{1}{2}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\widehat{\mathcal{E}}_{b}+\varepsilon_{\langle k}{}^{ab}\widehat{\sigma}_{j\rangle a}\mathcal{E}_{b}\\
&+3\mathcal{E}_{\langle j}\widehat{\omega}_{k\rangle}-\frac{\widetilde{\kappa}^2}{3}\varepsilon_{ab\langle k}D^{a}\widetilde{\pi}_{j\rangle}{}^{b}-\widetilde{\kappa}^2(\widetilde{q}_{\langle j}\omega_{k\rangle}+\widetilde{\pi}_{\langle j}\widehat{\omega}_{k\rangle})-\frac{\widetilde{\kappa}^2}{3}\varepsilon_{\langle k}{}^{ab}(\sigma_{j\rangle a}\widetilde{q}_{b}+\widehat{\sigma}_{j\rangle a}\widetilde{\pi}_{b}),
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
0={}&\dot{\widehat{\mathcal{E}}}_{\langle kj\rangle}-\mathcal{F}'_{\langle kj\rangle}-D_{\langle k}\widehat{\mathcal{E}}_{j\rangle}+\left(K+\frac{\Theta}{3}\right)\widehat{\mathcal{E}}_{kj}+K\mathcal{E}_{kj}+\frac{4\mathcal{E}}{3}\sigma_{kj}+\left(2\widehat{K}-\frac{\widehat{\Theta}}{3}\right)\mathcal{F}_{kj}-\frac{3}{2}\mathcal{H}_{\langle k}\widehat{\omega}_{j\rangle}\\
&-\mathcal{F}_{\langle j}{}^{a}(\widehat{\omega}_{k\rangle a}+\widehat{\sigma}_{k\rangle a})-\mathcal{E}_{\langle j}\left(\frac{3}{2}K_{k\rangle}+L_{k\rangle}-\widehat{K}_{k\rangle}\right)-2\widehat{\mathcal{E}}_{\langle k}(A_{j\rangle}-\frac{3}{4}\widehat{A}_{j\rangle})+\widehat{\mathcal{E}}_{\langle j}{}^{a}(\omega_{k\rangle a}+\sigma_{k\rangle a})\\
&-\varepsilon_{\langle k}{}^{ab}\widehat{\mathcal{H}}_{j\rangle a}(K_{b}-2\widehat{K}_{b})-\varepsilon_{\langle j}{}^{ab}\mathcal{H}_{k\rangle a}\widehat{A}_{b}+\frac{3}{2}\varepsilon_{\langle j}{}^{ab}\widehat{\sigma}_{k\rangle a}\mathcal{H}_{b}\\
&+\frac{\widetilde{\kappa}^2}{3}\dot{\widetilde{\pi}}_{\langle kj\rangle}+\frac{\widetilde{\kappa}^2}{3}D_{\langle k}\widetilde{q}_{j\rangle}+\frac{\widetilde{\kappa}^2}{3}\left(\widetilde{\rho}+\widetilde{p}\right)\sigma_{kj}\\
&+\frac{\widetilde{\kappa}^2\widetilde{q}}{3}\widehat{\sigma}_{kj}+\frac{\widetilde{\kappa}^2\Theta}{9}\widetilde{\pi}_{kj}+\frac{2\widetilde{\kappa}^2}{3}\widetilde{q}_{\langle j}A_{k\rangle}+\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}\left(2\widehat{K}_{k\rangle}+L_{k\rangle}\right)+\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}{}^{a}(\omega_{k\rangle a}+\sigma_{k\rangle a}),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
0={}&\dot{\widehat{\mathcal{H}}}_{\langle kj\rangle}+\varepsilon_{ab\langle k}D^{a}\mathcal{F}_{j\rangle}^{b}-\frac{1}{2}D_{\langle j}\mathcal{H}_{k\rangle}+\left(\widehat{K}+\frac{\widehat{\Theta}}{3}\right)\mathcal{H}_{kj}+\frac{2\Theta}{3}\widehat{\mathcal{H}}_{kj}-2\mathcal{H}_{a\langle k}\widehat{\sigma}_{j\rangle}{}^{a}\\
&+\widehat{\mathcal{H}}_{\langle j}{}^{a}(\omega_{k\rangle a}-\sigma_{k\rangle a})+\frac{3}{2}\mathcal{E}_{\langle j}\omega_{k\rangle}-3\widehat{\omega}_{\langle k}\widehat{\mathcal{E}}_{j\rangle}-\varepsilon_{\langle k}{}^{ab}\mathcal{E}_{j\rangle a}(\widehat{K}_{b}-L_{b})\\
&-\frac{3}{2}\mathcal{H}_{\langle k}A_{j\rangle}+\varepsilon_{\langle k}{}^{ab}\widehat{\mathcal{E}}_{j\rangle a}(2\widehat{K}_{b}+L_{b})+\frac{3}{2}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\mathcal{E}_{b}-\varepsilon_{\langle k}{}^{ab}\mathcal{F}_{j\rangle a}A_{b},
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
0={}&\mathcal{E}'+D^{a}\mathcal{E}_{a}+\frac{4}{3}\widehat{\Theta}\mathcal{E}-2\mathcal{E}_{a}\widehat{A}^{a}-\widehat{\mathcal{E}}_{a}(2K^{a}-L^{a})+3\omega_{a}\mathcal{H}^{a}+\mathcal{F}_{ab}\sigma^{ab}\\
&-\mathcal{E}_{ab}\widehat{\sigma}^{ab}-\frac{\widetilde{\kappa}^2}{2}\left(\widetilde{\rho}-\widetilde{\pi}+\widetilde{p}\right)'+\frac{2\widetilde{\kappa}^2}{3}D^{a}\widetilde{\pi}_{a}+\frac{2\widetilde{\kappa}^2\Theta}{3}\widetilde{q}\\
&-\frac{2\widetilde{\kappa}^2\widehat{\Theta}}{3}(\widetilde{p}-\widetilde{\pi})-\frac{4\widetilde{\kappa}^2}{3}\widetilde{\pi}_{a}\widehat{A}^{a}-\frac{2\widetilde{\kappa}^2}{3}\widetilde{q}_{a}(2K^{a}-L^{a})-\frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab}\widehat{\sigma}^{ab},
\end{aligned}
\tag{63}
$$

$$
0=\widehat{\mathcal{E}}'_{\langle k\rangle}-D^{a}\mathcal{F}_{ka}+\frac{1}{2}\varepsilon_{k}{}^{ab}D_{a}\mathcal{H}_{b}-\widehat{\mathcal{E}}_{ka}(K^{a}-L^{a})+\mathcal{E}_{ka}L^{a}-\frac{1}{2}(\widehat{\sigma}_{ka}+\widehat{\omega}_{ka})\widehat{\mathcal{E}}^{a}
$$

$$
\begin{aligned}
&-(\sigma_{ka}-2\omega_{ka})\,\mathcal{E}^{a}+2\widehat{\mathcal{H}}_{ka}\omega^{a}+\mathcal{H}_{ka}\widehat{\omega}^{a}+\mathcal{F}_{ka}\widehat{A}^{a}+\left(K-\frac{\Theta}{3}\right)\mathcal{E}_{k}-\frac{4\mathcal{E}}{3}K_{k}\\
&+\frac{4\widehat{\Theta}}{3}\widehat{\mathcal{E}}_{k}-\varepsilon_{k}{}^{ab}\mathcal{H}_{ca}\widehat{\sigma}_{b}{}^{c}+\frac{3}{2}\varepsilon_{k}{}^{ab}\mathcal{H}_{a}\widehat{A}_{b}+\frac{2\widetilde{\kappa}^{2}}{3}\widetilde{q}'_{\langle k\rangle}-\frac{2\widetilde{\kappa}^{2}}{3}D_{k}\widetilde{q}\\
&+\frac{2\widetilde{\kappa}^{2}}{3}\left(\widetilde{\rho}+\widetilde{p}\right)K_{k}-\frac{2\widetilde{\kappa}^{2}}{3}\left(\widetilde{\pi}+\widetilde{\rho}\right)L_{k}+\frac{2\widetilde{\kappa}^{2}}{3}\left(K-\frac{\Theta}{3}\right)\widetilde{\pi}_{k}+\frac{2\widetilde{\kappa}^{2}\widehat{\Theta}}{9}\widetilde{q}_{k}\\
&-\frac{2\widetilde{\kappa}^{2}}{3}\left(\omega_{ka}+\sigma_{ka}\right)\widetilde{\pi}^{a}+\frac{2\widetilde{\kappa}^{2}}{3}\left(\widehat{\omega}_{ka}+\widehat{\sigma}_{ka}\right)\widetilde{q}^{a}+\frac{2\widetilde{\kappa}^{2}\widetilde{q}}{3}\widehat{A}_{k}+\frac{2\widetilde{\kappa}^{2}}{3}\widetilde{\pi}_{ka}K^{a},
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
0={}&\mathcal{H}'_{\langle k\rangle}+\varepsilon_{k}{}^{ab}D_{a}\widehat{\mathcal{E}}_{b}+\mathcal{H}_{ka}\widehat{A}^{a}-\widehat{\mathcal{H}}_{ka}K^{a}+\widehat{\mathcal{E}}_{ka}\omega^{a}-\mathcal{F}_{ka}\widehat{\omega}^{a}\\
&+\varepsilon_{k}{}^{ab}\widehat{\mathcal{E}}_{ac}\sigma_{b}{}^{c}-\varepsilon_{k}{}^{ab}\mathcal{F}_{ac}\widehat{\sigma}_{b}{}^{c}+\frac{3}{2}(\widehat{\omega}_{ka}-\widehat{\sigma}_{ka})\mathcal{H}^{a}\\
&+\widehat{\Theta}\mathcal{H}_{k}-\frac{8}{3}\mathcal{E}\omega_{k}-\varepsilon_{k}{}^{ab}\left[\frac{3}{2}\widehat{A}_{a}\widehat{\mathcal{E}}_{b}-\frac{3}{2}\mathcal{E}_{a}K_{b}-L_{a}\mathcal{E}_{b}\right]-\frac{\widetilde{\kappa}^{2}}{3}\varepsilon_{k}{}^{ab}D_{a}\widetilde{q}_{b}+\frac{\widetilde{\kappa}^{2}}{3}\varepsilon_{k}{}^{ab}\widetilde{\pi}_{a}L_{b}\\
&-\frac{2\widetilde{\kappa}^{2}\widetilde{q}}{3}\widehat{\omega}_{k}+\frac{\widetilde{\kappa}^{2}}{3}\varepsilon_{k}{}^{ab}\widetilde{\pi}_{ac}\sigma_{b}{}^{c}-\frac{2\widetilde{\kappa}^{2}}{3}(\widetilde{\rho}+\widetilde{p})\omega_{k}+\frac{\widetilde{\kappa}^{2}}{3}\widetilde{\pi}_{ka}\omega^{a},
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
0={}&\mathcal{E}'_{\langle k\rangle}+D^{a}\widehat{\mathcal{E}}_{ka}-\frac{2}{3}D_{k}\mathcal{E}+\widehat{\Theta}\mathcal{E}_{k}\\
&+\frac{4\mathcal{E}}{3}\widehat{A}_{k}-\mathcal{E}_{ka}\widehat{A}^{a}-3\widehat{\mathcal{H}}_{ka}\widehat{\omega}^{a}+\left(K-\frac{2}{3}\Theta\right)\widehat{\mathcal{E}}_{k}+\mathcal{F}_{ka}(K^{a}-2L^{a})\\
&+\frac{1}{2}(3\widehat{\sigma}_{ka}+\widehat{\omega}_{ka})\mathcal{E}^{a}+(\sigma_{ka}+3\omega_{ka})\widehat{\mathcal{E}}^{a}-\frac{3}{2}\varepsilon_{k}{}^{ab}\mathcal{H}_{a}K_{b}+\varepsilon_{k}{}^{ab}\widehat{\mathcal{H}}_{ca}\widehat{\sigma}_{b}{}^{c}+\frac{2\widetilde{\kappa}^{2}}{3}\widetilde{\pi}'_{\langle k\rangle}+\frac{\widetilde{\kappa}^{2}}{3}D^{a}\widetilde{\pi}_{ka}\\
&-\frac{\widetilde{\kappa}^{2}}{3}D_{k}(\widetilde{\rho}+\widetilde{\pi}-\widetilde{p})+\frac{2\widetilde{\kappa}^{2}\widetilde{q}}{3}(K_{k}-2L_{k})+\frac{2\widetilde{\kappa}^{2}}{3}\left(K+\frac{\Theta}{3}\right)\widetilde{q}_{k}-\frac{2\widetilde{\kappa}^{2}}{3}\widetilde{\pi}_{ka}\widehat{A}^{a}\\
&+\frac{2\widetilde{\kappa}^{2}}{3}\left(\widetilde{\pi}-\widetilde{p}\right)\widehat{A}_{k}+\frac{2\widetilde{\kappa}^{2}}{3}\widehat{\Theta}\widetilde{\pi}_{k}+\frac{\widetilde{\kappa}^{2}}{3}[(\widehat{\omega}_{ka}+3\widehat{\sigma}_{ka})\,\widetilde{\pi}^{a}-(3\omega_{ka}+\sigma_{ka})\,\widetilde{q}^{a}],
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
0={}&\widehat{\mathcal{E}}'_{\langle kj\rangle}-\mathcal{E}'_{\langle kj\rangle}+\varepsilon_{ab\langle k}D^{a}\widehat{\mathcal{H}}_{j\rangle}{}^{b}+\frac{1}{2}D_{\langle k}\mathcal{E}_{j\rangle}+\frac{2\mathcal{E}}{3}\widehat{\sigma}_{jk}-\frac{2\widehat{\Theta}}{3}\mathcal{E}_{kj}+\widehat{\Theta}\widehat{\mathcal{E}}_{kj}-\mathcal{E}_{\langle k}\widehat{A}_{j\rangle}\\
&-3\mathcal{H}_{\langle k}\omega_{j\rangle}-\frac{1}{2}\widehat{\mathcal{E}}_{\langle k}(2K_{j\rangle}-L_{j\rangle})+2\mathcal{F}_{a\langle k}\sigma_{j\rangle}{}^{a}-\mathcal{E}_{\langle k}{}^{a}(\widehat{\omega}_{j\rangle a}-\widehat{\sigma}_{j\rangle a})+\widehat{\mathcal{E}}_{\langle k}{}^{a}(\widehat{\omega}_{j\rangle a}-3\widehat{\sigma}_{j\rangle a})\\
&-\frac{\Theta}{3}\mathcal{F}_{kj}+2\varepsilon_{\langle k}{}^{ab}\widehat{\mathcal{H}}_{j\rangle a}\widehat{A}_{b}-\varepsilon_{\langle k}{}^{ab}\mathcal{H}_{j\rangle a}\,(2K_{b}-L_{b})-\frac{\widetilde{\kappa}^{2}}{3}\widetilde{\pi}'_{\langle kj\rangle}-\frac{\widetilde{\kappa}^{2}}{3}\widetilde{q}_{\langle k}(4K_{j\rangle}-L_{j\rangle})\\
&+\frac{\widetilde{\kappa}^{2}}{3}D_{\langle k}\widetilde{\pi}_{j\rangle}-\frac{4\widetilde{\kappa}^{2}}{3}\widetilde{\pi}_{\langle k}\widehat{A}_{j\rangle}+\frac{\widetilde{\kappa}^{2}}{3}\widetilde{q}\sigma_{jk}+\frac{\widetilde{\kappa}^{2}}{3}(\widetilde{\pi}-\widetilde{p})\widehat{\sigma}_{jk}\\
&-\frac{\widetilde{\kappa}^{2}\widehat{\Theta}}{9}\widetilde{\pi}_{jk}-\frac{\widetilde{\kappa}^{2}}{3}\widetilde{\pi}_{\langle k}{}^{a}(\widehat{\omega}_{j\rangle a}+\widehat{\sigma}_{j\rangle a}),
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
0={}&\mathcal{H}'_{\langle kj\rangle}+\varepsilon_{ab\langle k}D^{a}\mathcal{F}_{j\rangle}{}^{b}+\frac{1}{2}D_{\langle j}\mathcal{H}_{k\rangle}-\left(K-\frac{\Theta}{3}\right)\widehat{\mathcal{H}}_{kj}+\frac{2\widehat{\Theta}}{3}\mathcal{H}_{kj}-\frac{3}{2}\mathcal{H}_{\langle k}\widehat{A}_{j\rangle}\\
&-\frac{3}{2}\widehat{\mathcal{E}}_{\langle j}\widehat{\omega}_{k\rangle}+3\omega_{\langle k}\mathcal{E}_{j\rangle}-2\widehat{\mathcal{H}}_{a\langle k}\sigma_{j\rangle}{}^{a}+\mathcal{H}_{\langle j}{}^{a}(\widehat{\omega}_{k\rangle a}-\widehat{\sigma}_{k\rangle a})+\varepsilon_{\langle k}{}^{ab}\widehat{\mathcal{E}}_{j\rangle a}\,(K_{b}+L_{b})\\
&-\varepsilon_{\langle k}{}^{ab}\mathcal{E}_{j\rangle a}\,(2K_{b}-L_{b})-\frac{3}{2}\varepsilon_{\langle k}{}^{ab}\widehat{\sigma}_{j\rangle a}\widehat{\mathcal{E}}_{b}+\varepsilon_{\langle k}{}^{ab}\mathcal{F}_{j\rangle a}\widehat{A}_{b},
\end{aligned}
\tag{68}
$$

$$
0=\widehat{\mathcal{H}}'_{\langle kj\rangle}+\varepsilon_{ab\langle k}D^{a}\widehat{\mathcal{E}}_{j\rangle}{}^{b}-\frac{3}{2}\mathcal{H}_{\langle k}K_{j\rangle}-K\mathcal{H}_{kj}+\widehat{\Theta}\widehat{\mathcal{H}}_{kj}-(\widehat{\omega}_{a\langle k}+3\widehat{\sigma}_{a\langle k})\widehat{\mathcal{H}}_{j\rangle}^{a}+\frac{3}{2}\mathcal{E}_{\langle j}\widehat{\omega}_{k\rangle}
$$

$$- \varepsilon_{\langle k}{}^{ab}(\mathcal{E}_{j\rangle a} - 2\widehat{\mathcal{E}}_{j\rangle a})\widehat{A}_b + \frac{1}{2}\varepsilon_{\langle k}{}^{ab}\widehat{\sigma}_{j\rangle a}\mathcal{E}_b - \varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\widehat{\mathcal{E}}_b - \varepsilon_{\langle k}{}^{ab}\mathcal{F}_{j\rangle a}(K_b - 2L_b)$$
$$- 3\widehat{\mathcal{E}}_{\langle j}\omega_{k\rangle} + \frac{\widetilde{\kappa}^2}{3}\varepsilon_{ab\langle k}D^a\widetilde{\pi}_{j\rangle}{}^b + \widetilde{\kappa}^2(\widetilde{q}_{\langle j}\omega_{k\rangle} + \widetilde{\pi}_{\langle j}\widehat{\omega}_{k\rangle}) + \frac{\widetilde{\kappa}^2}{3}\varepsilon_{\langle k}{}^{ab}(\sigma_{j\rangle a}\widetilde{q}_b + \widehat{\sigma}_{j\rangle a}\widetilde{\pi}_b), \quad (69)$$

$$0 = D^k\mathcal{H}_k - \widehat{\mathcal{H}}_{ab}\sigma^{ab} + \mathcal{H}_{ab}\widehat{\sigma}^{ab} - 3\mathcal{E}_a\omega^a - 3\widehat{\mathcal{E}}_a\widehat{\omega}^a, \quad (70)$$

$$0 = D^a\widehat{\mathcal{E}}_{ak} - D^a\mathcal{E}_{ak} - \frac{1}{3}D_k\mathcal{E} - \frac{\widehat{\Theta}}{3}\mathcal{E}_k - \frac{\Theta}{3}\widehat{\mathcal{E}}_k + 3\mathcal{H}_{ka}\omega^a - \varepsilon_k{}^{ab}\mathcal{H}_{ac}\sigma_b{}^c$$
$$+ \frac{1}{2}[(3\widehat{\omega}_{ka} + \widehat{\sigma}_{ka})\mathcal{E}^a + (3\omega_{ka} + \sigma_{ka})\widehat{\mathcal{E}}^a] - 3\widehat{\mathcal{H}}_{ka}\widehat{\omega}^a + \varepsilon_k{}^{ab}\widehat{\mathcal{H}}_{ac}\widehat{\sigma}_b{}^c - \frac{\widetilde{\kappa}^2}{3}D^a\widetilde{\pi}_{ak}$$
$$+ \frac{\widetilde{\kappa}^2}{6}D_k(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p}) - \frac{2\widetilde{\kappa}^2\Theta}{9}\widetilde{q}_k - \frac{2\widetilde{\kappa}^2\widehat{\Theta}}{9}\widetilde{\pi}_k + \widetilde{\kappa}^2\widetilde{q}^a\omega_{ka} + \frac{\widetilde{\kappa}^2}{3}\widetilde{q}^a\sigma_{ka}$$
$$+ \widetilde{\kappa}^2\widetilde{\pi}^a\widehat{\omega}_{ka} + \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}^a\widehat{\sigma}_{ka}, \quad (71)$$

$$0 = D^a\mathcal{H}_{ak} + \frac{1}{2}\varepsilon_k{}^{ab}D_a\widehat{\mathcal{E}}_b - \frac{\widehat{\Theta}}{3}\mathcal{H}_k - \frac{4\mathcal{E}}{3}\omega_k - \widehat{\mathcal{H}}_{ka}L^a - 2\mathcal{F}_{ka}\widehat{\omega}^a - (\widehat{\sigma}_{ka} - 2\widehat{\omega}_{ka})\mathcal{H}^a$$
$$-(\widehat{\mathcal{E}}_{ka} - 3\mathcal{E}_{ka})\omega^a - \varepsilon_k{}^{ab}(\mathcal{E}_{ac} - \widehat{\mathcal{E}}_{ac})\sigma_b{}^c - \frac{1}{2}\varepsilon_k{}^{ab}\mathcal{E}_aL_b + \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}D_a\widetilde{q}_b$$
$$- \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}\widetilde{\pi}_aL_b + \frac{2\widetilde{\kappa}^2}{3}\widetilde{q}\widehat{\omega}_k + \frac{2\widetilde{\kappa}^2}{3}(\widetilde{\rho} + \widetilde{p})\omega_k - \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ka}\omega^a - \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}\widetilde{\pi}_a{}^c\sigma_{bc}, \quad (72)$$

$$0 = D^a\widehat{\mathcal{H}}_{ak} - \frac{1}{2}\varepsilon_k{}^{ab}D_a\mathcal{E}_b - \frac{\Theta}{3}\mathcal{H}_k - \mathcal{H}_{ka}L^a + 2\mathcal{F}_{ka}\omega^a + (\mathcal{E}_{ka} - 3\widehat{\mathcal{E}}_{ka})\widehat{\omega}^a - \frac{4\mathcal{E}}{3}\widehat{\omega}_k$$
$$- (\sigma_{ka} - 2\omega_{ka})\mathcal{H}^a - \frac{1}{2}\varepsilon_k{}^{ab}L_a\widehat{\mathcal{E}}_b + \varepsilon_k{}^{ab}\widehat{\sigma}_b{}^c(\widehat{\mathcal{E}}_{ca} - \mathcal{E}_{ca}) - \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}D_a\widetilde{\pi}_b$$
$$- \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}L_a\widetilde{q}_b - \frac{2\widetilde{\kappa}^2\widetilde{q}}{3}\omega_k - \frac{2\widetilde{\kappa}^2}{3}(\widetilde{\pi} - \widetilde{p})\widehat{\omega}_k - \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ka}\widehat{\omega}^a + \frac{\widetilde{\kappa}^2}{3}\varepsilon_k{}^{ab}\widetilde{\pi}_{cb}\widehat{\sigma}_a{}^c. \quad (73)$$

4. 3+1 gravitational dynamics on the brane

In this section we consider distributional energy–momentum tensor sources on the brane, in addition to the regular energy–momentum tensor $\widetilde{T}_{ab}$. Such a distributional source comes together with a discontinuity in the extrinsic curvature, as related by the Lanczos equation.

4.1. The Lanczos equation

The extrinsic curvature of the brane is ${}^{(4)}K_{ab} = \nabla_{(c}n_{d)} = g^a_{(c}g^b_{d)}\widetilde{\nabla}_a n_b$, equal to the symmetrized version of the last four terms of equation (6*b*). Replacing $\widehat{K}_{ab}$ by expression (9), and specializing to the brane, cf equations (11), the extrinsic curvature is expressed as

$${}^{(4)}K_{ab} = \widehat{K}u_au_b - 2u_{(a}\widehat{K}_{b)} + \frac{\widehat{\Theta}}{3}h_{ab} + \widehat{\sigma}_{ab}. \quad (74)$$

As we approach the brane from left or right, the limiting values of the extrinsic curvature could be different, according to the embedding and 5d metric in the two regions. Therefore we introduce averages and differences of the extrinsic curvature.

The Lanczos equation [46, 47] relates the jump of the extrinsic curvature across the brane to the distributional matter layer:

$$\Delta{}^{(4)}K_{ab} = -\widetilde{\kappa}^2\left(\tau_{ab} - \frac{\tau}{3}g_{ab}\right). \quad (75)$$

The $u^a u^b$, $u^a h^b_c$, trace and trace-free parts of the $h^a_c h^b_d$ projections give

$$\Delta \widehat{K} = \frac{\widetilde{\kappa}^2}{3} (\lambda - 2\rho - 3p), \tag{76}$$

$$\Delta \widehat{K}_a = \widetilde{\kappa}^2 q_a, \tag{77}$$

$$\Delta \widehat{\Theta} = -\widetilde{\kappa}^2 (\lambda + \rho), \tag{78}$$

$$\Delta \widehat{\sigma}_{ab} = -\widetilde{\kappa}^2 \pi_{ab}. \tag{79}$$

The Lanczos equation is necessary in order to derive the gravitational dynamics on the brane, given by a scalar (the twice-contracted Gauss), a 4d vectorial (the Codazzi) and a 4d tensorial (the effective Einstein) equations [3]. The latter has been first derived in [48], later generalized to include bulk matter and asymmetric embedding contributions (equation (1) in [3]).

4.2. The 3+1 decomposition of the source terms of the effective Einstein equation

We give the 3+1 covariant decomposition of the source terms of the effective Einstein equation. This equation is [3]

$$G_{ab} = -\left(\Lambda - \frac{\widetilde{\kappa}^2 \langle \widetilde{\pi} \rangle}{2}\right) g_{ab} + \kappa^2 T_{ab} + \widetilde{\kappa}^4 S_{ab} - \left\langle {}^{(4)}\mathcal{E}_{ab} \right\rangle + \langle L_{ab} \rangle + \langle \mathcal{P}_{ab} \rangle . \tag{80}$$

The sources are the stress–energy tensor T_{ab} representing the standard model matter (decomposed in equation (18)); the source term S_{ab} quadratic in T_{ab} (dominant at high energies), $\langle \mathcal{P}_{ab} \rangle$ the pull-back to the brane of the bulk matter; $\langle L_{ab} \rangle$ a source term originating in the asymmetry of the embedding and ${}^{(4)}\mathcal{E}_{ab}$ the contribution of the electric part (relative to the vector n^a) of the 5d Weyl tensor. We have defined the 4d coupling constant κ^2 and the brane cosmological constant Λ as

$$6\kappa^2 = \widetilde{\kappa}^4 \lambda, \tag{81}$$

$$2\Lambda = \kappa^2 \lambda + \langle \widetilde{\Lambda} \rangle. \tag{82}$$

The quadratic source term is decomposed as

$$S_{ab} = \frac{1}{24}(2\rho^2 - 3\pi_{cd}\pi^{cd}) u_a u_b + \frac{1}{24}(2\rho^2 + 4p\rho - 4q_c q^c + \pi_{cd}\pi^{cd}) h_{ab} + \frac{1}{4} q_{\langle a} q_{b\rangle} + \frac{\rho}{3} q_{(a} u_{b)} - \frac{1}{2} q^c \pi_{c(a} u_{b)} - \frac{\rho + 3p}{12} \pi_{ab} - \frac{1}{4} \pi_{c\langle a} \pi_{b\rangle}{}^c. \tag{83}$$

Some of the numerical coefficients are corrected here with respect to the corresponding expression (7) given in [11].

The electric part of the 5d Weyl tensor expressed in terms of gravito-electro-magnetic quantities defined in section 2.3 is

$$\left\langle {}^{(4)}\mathcal{E}_{ab} \right\rangle = \langle \mathcal{E} \rangle \left(u_a u_b + \frac{1}{3} h_{ab} \right) - 2 \langle \widehat{\mathcal{E}}_{(a} \rangle u_{b)} + \langle \widehat{\mathcal{E}}_{ab} \rangle. \tag{84}$$

The asymmetry source term is decomposed as

$$\langle L_{ab} \rangle = \frac{1}{3} \left[\langle \widehat{\Theta} \rangle^2 - \frac{3}{2} \langle \widehat{\sigma}_{cd} \rangle \langle \widehat{\sigma}^{cd} \rangle \right] u_a u_b - u_{(a} \left[\frac{4}{3} \langle \widehat{\Theta} \rangle h_{b)c} - 2 \langle \widehat{\sigma}_{b)c} \rangle \right] \langle \widehat{K}^c \rangle$$

$$+ \frac{1}{9}\left[6\langle\widehat{\Theta}\rangle\langle\widehat{K}\rangle - 9\langle\widehat{K}^c\rangle\langle\widehat{K}_c\rangle - \langle\widehat{\Theta}\rangle^2 + \frac{9}{2}\langle\widehat{\sigma}_{cd}\rangle\langle\widehat{\sigma}^{cd}\rangle\right]h_{ab}$$
$$+ \langle\widehat{K}_a\rangle\langle\widehat{K}_b\rangle + \left\langle\frac{\widehat{\Theta}}{3} - \widehat{K}\right\rangle\langle\widehat{\sigma}_{ab}\rangle - \langle\widehat{\sigma}_a{}^c\rangle\langle\widehat{\sigma}_{bc}\rangle. \tag{85}$$

As the induced metric is continuous, the average of the trace $L = g^{ab}L_{ab}$ is the trace of the average:

$$\langle L\rangle = \langle\widehat{\sigma}_{cd}\rangle\,\langle\widehat{\sigma}^{cd}\rangle - 2\langle\widehat{K}_b\rangle\langle\widehat{K}^b\rangle - \frac{2}{3}\langle\widehat{\Theta}\rangle^2 + 2\langle\widehat{\Theta}\rangle\langle\widehat{K}\rangle. \tag{86}$$

For a symmetric embedding $\langle{}^{(4)}K_{ab}\rangle = 0$, therefore cf equation (74) $\langle\widehat{\Theta}\rangle = \langle\widehat{K}\rangle = 0 = \langle\widehat{K}_c\rangle = \langle\widehat{\sigma}_{cd}\rangle$, so that $\langle L_{ab}\rangle = 0$.

Finally,

$$\frac{6\mathcal{P}_{ab}}{\widetilde{\kappa}^2} = 3\,(\widetilde{\rho} + \widetilde{p})\,u_a u_b + 8\widetilde{q}_{(a}u_{b)} + (\widetilde{\rho} + \widetilde{p})\,h_{ab} + 4\widetilde{\pi}_{ab}. \tag{87}$$

4.3. Gravitational dynamics on the brane: generic embedding

In order to obtain the evolution and constraint equations on the brane, we select a subset of the Ricci and Bianchi equations given in subsections 3.1 and 3.3, by combining them in such a way that the off-brane derivatives of the kinematical and gravito-electro-magnetic quantities drop out. Additionally, the equations of this subset contain only quantities appearing in the effective Einstein equation and in the 4d theory.

First we express $\mathcal{H}_a$, $\mathcal{E}_a$, $\mathcal{F}_{ab}$ and $\widehat{\mathcal{H}}_{ab}$ from (37), (50), (43) and (49) respectively and we employ definitions (14), (15) in order to introduce E_{ab} and H_{ab} in place of $\mathcal{E}_{ab}$ and $\mathcal{H}_{ab}$. Inserting these into the system of equations given by the following equations: (41), (29), (54), (57)–(66), (27), (30), (31), (38), (39), (40), (58)–(61), (60), (71) and (72), evaluated at the brane, we obtain a system of equations to be referred to as the brane equations. These equations are either evolution or constraint equations on the brane and for a generic asymmetric embedding are presented in appendix C. The evolutions refer to the quantities $\widehat{\Theta}$, $\widehat{K}_a$, Θ, ω_a, σ_{ab}, $\mathcal{E}$, $\widehat{\mathcal{E}}_a$, E_{ab}, H_{ab}.

4.4. Gravitational evolution and constraint equations on a symmetrically embedded brane

In this subsection we restrict ourselves to *symmetrically embedded* branes. The Z_2-symmetric embedding arises when there is a perfect symmetry between the 5d spacetime regions on the two sides of the brane. In this case the extrinsic curvatures on the two sides of the brane are opposite. This is due to the fact that the normal vectors to the brane on its two sides are n^a and $-n^a$, respectively. Therefore $\Delta{}^{(4)}K_{ab} = 2{}^{(4)}K_{ab}$ and $\overline{K}_{ab} = 0$.

We present a system of evolution and constrain equations, which *hold on the brane and contain no off-brane derivatives*. We obtain these equations by specifying equations (C.1)–(C14) for a symmetrically embedded brane, and then replacing $\widehat{\Theta}$, $\widehat{K}_a$, $\widehat{K}$ and $\widehat{\sigma}_{ab}$ with the corresponding matter variables as given by the projections (76)–(79) of the Lanczos equation, again specified for a symmetrically embedded brane. Finally, we employ definitions (81) and (82), whenever possible.

For improved clarity we group all general relativistic contributions on the left hand side of the equations, keeping the brane-world contributions on the right-hand side:

$$\dot{\rho} + D^a q_a + (\rho + p)\,\Theta + 2q^a A_a + \pi_{ab}\sigma^{ab} = -2\widetilde{q}, \tag{88}$$

$$\dot{q}_{\langle a\rangle} + D_a p + D^b\pi_{ab} + \frac{4\Theta}{3}q_a + (\rho + p)\,A_a + \pi_{ab}A^b - \omega_{ab}q^b + \sigma_{ab}q^b = -2\widetilde{\pi}_a, \tag{89}$$

$$\begin{aligned}
0 = {} & \dot{\mathcal{E}} - D^a\widehat{\mathcal{E}}_a + \frac{4}{3}\Theta\mathcal{E} + \widehat{\mathcal{E}}_{ab}\sigma^{ab} - 2\widehat{\mathcal{E}}_a A^a + \frac{\widetilde{\kappa}^4}{4}\left[\pi^{ab}\pi^{\cdot}_{\langle ab\rangle} + \pi^{ab}D_a q_b + q^a D^b\pi_{ab} - \frac{2}{3}q^a D_a\rho\right. \\
& \left. + 2A_b q_a\pi^{ab} + \frac{\Theta}{3}\pi_{ab}\pi^{ab} + (\rho + p)\,\sigma_{ab}\pi^{ab} + \sigma_{ca}\pi_b{}^c\pi^{ab} + \frac{2\Theta}{3}q_a q^a - \sigma_{ab}q^a q^b\right] \\
& - \frac{\widetilde{\kappa}^2}{2}\left(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p}\right)^{\cdot} - \frac{2\widetilde{\kappa}^2}{3}D^a\widetilde{q}_a - \frac{2\widetilde{\kappa}^2}{3}\Theta\left(\widetilde{\rho} + \widetilde{p}\right) + 2\kappa^2\widetilde{q} \\
& + \frac{\widetilde{\kappa}^4}{3}\rho\widetilde{q} - \frac{4\widetilde{\kappa}^2}{3}\widetilde{q}_a A^a - \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab}\sigma^{ab},
\end{aligned} \tag{90}$$

$$\begin{aligned}
0 = {} & \dot{\widehat{\mathcal{E}}}_{\langle k\rangle} + \frac{4}{3}\Theta\widehat{\mathcal{E}}_k - \frac{1}{3}D_k\mathcal{E} - \frac{4\mathcal{E}}{3}A_k - D^a\widehat{\mathcal{E}}_{ka} - \widehat{\mathcal{E}}_{ka}A^a - (\omega_{ka} - \sigma_{ka})\,\widehat{\mathcal{E}}^a \\
& + \frac{\widetilde{\kappa}^4}{4}\left[-\dot{\pi}_{\langle ka\rangle}q^a - (\rho + p)\,D^b\pi_{kb} - q^a\sigma_{ck}\pi_a{}^c + \frac{2}{3}(\rho + p)\,D_k\rho - \frac{2\Theta}{3}(\rho + p)\,q_k\right. \\
& - 2q^a D_k q_a + q^a D_a q_k + \frac{1}{3}q_k D^a q_a - 2q^a A_{\langle k}q_{a\rangle} + \sigma_{ba}\pi_k{}^b q^a - \frac{2}{3}q_k\sigma_{ab}\pi^{ab} \\
& \left. + \pi^{ab}D_k\pi_{ab} - \pi_b{}^a D^b\pi_{ka} - \frac{1}{3}\pi_k{}^a D_a\rho - \varepsilon_{cab}q^a\omega^b\pi_k{}^c - \varepsilon_k{}^{ab}q_c\omega_a\pi_b{}^c\right] \\
& + \frac{\widetilde{\kappa}^2}{3}\left[-2\widetilde{\pi}'_{\langle k\rangle} + \frac{1}{2}D_k\left(\widetilde{\rho} + 3\widetilde{\pi} - 3\widetilde{p}\right) + \frac{\widetilde{\kappa}^2}{3}(2\lambda - \rho - 3p)\,\widetilde{\pi}_k - 2K\widetilde{q}_k\right. \\
& \left. - \frac{2}{3}\widetilde{q}(2\widehat{K}_k + K_k) + 2\widetilde{\pi}_{ka}\widehat{A}^a - 2\left(\widetilde{\pi} - \widetilde{p}\right)\widehat{A}_k + \frac{5\widetilde{\kappa}^2}{2}\pi_{ka}\widetilde{\pi}^a\right],
\end{aligned} \tag{91}$$

$$\begin{aligned}
& \dot{\Theta} - D^a A_a + \frac{\Theta^2}{3} - A^a A_a - 2\omega_a\omega^a + \sigma_{ab}\sigma^{ab} + \frac{\kappa^2}{2}(\rho + 3p) - \Lambda \\
& \quad = \mathcal{E} + \frac{\widetilde{\kappa}^4}{4}q^a q_a - \frac{\widetilde{\kappa}^4}{12}\rho\,(2\rho + 3p) - \frac{\widetilde{\kappa}^2}{2}\left(\widetilde{\rho} + \widetilde{\pi} + \widetilde{p}\right),
\end{aligned} \tag{92}$$

$$\begin{aligned}
& \dot{\sigma}_{\langle ab\rangle} - D_{\langle a}A_{b\rangle} + \frac{2\Theta}{3}\sigma_{ab} - A_{\langle a}A_{b\rangle} + \omega_{\langle a}\omega_{b\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}{}^c + E_{ab} - \frac{\kappa^2}{2}\pi_{ab} \\
& \quad = \frac{\widetilde{\kappa}^4}{8}q_{\langle a}q_{b\rangle} - \frac{\widetilde{\kappa}^4}{8}\pi_{c\langle a}\pi_{b\rangle}{}^c - \frac{\widetilde{\kappa}^4}{24}(\rho + 3p)\,\pi_{ab} - \frac{1}{2}\widehat{\mathcal{E}}_{ab} + \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab},
\end{aligned} \tag{93}$$

$$\dot{\omega}_{\langle a\rangle} - \frac{1}{2}\varepsilon_a{}^{cd}D_c A_d + \frac{2\Theta}{3}\omega_a - \sigma_{ab}\omega^b = 0, \tag{94}$$

$$D^b\sigma_{ab} - \frac{2}{3}D_a\Theta + \varepsilon_a{}^{ck}D_c\omega_k + 2\varepsilon_a{}^{ck}A_c\omega_k + \kappa^2 q_a = -\frac{\widetilde{\kappa}^4}{6}\rho q_a + \frac{\widetilde{\kappa}^4}{4}\pi_{ab}q^b - \widehat{\mathcal{E}}_a - \frac{2\widetilde{\kappa}^2}{3}\widetilde{q}_a, \tag{95}$$

$$D_{\langle c}\omega_{k\rangle} + \varepsilon_{ab\langle k}D^b\sigma_{c\rangle}{}^a + 2A_{\langle c}\omega_{k\rangle} + H_{ab} = 0, \tag{96}$$

$$D^a\omega_a - A_a\omega^a = 0, \tag{97}$$

$$\begin{aligned}
&\dot{E}_{\langle kj\rangle} - \varepsilon_{ab\langle k}D^aH_{j\rangle}{}^b + \Theta E_{kj} + E^a{}_{\langle k}(\omega_{j\rangle a} - 3\sigma_{j\rangle a}) + 2\varepsilon_{\langle k}{}^{ab}H_{j\rangle a}A_b \\
&\qquad + \frac{\kappa^2}{2}\left[\dot{\pi}_{\langle kj\rangle} + D_{\langle k}q_{j\rangle} + (\rho+p)\,\sigma_{kj} + 2q_{\langle k}A_{j\rangle} + \frac{\Theta}{3}\pi_{kj} + \pi_{\langle k}{}^a(\omega_{j\rangle a} + \sigma_{j\rangle a})\right] \\
&= \frac{1}{2}\dot{\widehat{\mathcal{E}}}_{\langle kj\rangle} - \frac{1}{2}D_{\langle k}\widehat{\mathcal{E}}_{j\rangle} + \frac{1}{2}\widehat{\mathcal{E}}_{\langle j}{}^a(\omega_{k\rangle a} + \sigma_{k\rangle a}) + \frac{\Theta}{6}\widehat{\mathcal{E}}_{kj} - \widehat{\mathcal{E}}_{\langle k}A_{j\rangle} + \frac{2\mathcal{E}}{3}\sigma_{kj} \\
&\quad + \frac{\widetilde{\kappa}^4}{24}\Big[(\rho+3p)\,\dot{\pi}_{\langle kj\rangle} + 6\pi_{\langle j}{}^a\dot{\pi}_{k\rangle a} + (\rho+3p)^{\cdot}\,\pi_{kj} \\
&\quad - 2q_{\langle k}D_{j\rangle}\,(\rho-3p) - 2\rho D_{\langle k}q_{j\rangle} + 3\pi_{\langle j}{}^aD_{k\rangle}q_a \\
&\quad + 6q_{\langle k}D^b\pi_{j\rangle b} + 3q_aD_{\langle k}\pi_{j\rangle}{}^a - 2\rho\,(\rho+p)\,\sigma_{kj} + 7\Theta q_{\langle k}q_{j\rangle} + 2\,(\rho+3p)\,q_{\langle k}A_{j\rangle} + \Theta\pi_{\langle j}{}^a\pi_{k\rangle a} \\
&\quad + \frac{\Theta}{3}\,(\rho+3p)\,\pi_{kj} + (\rho+3p)\,\pi_{\langle k}{}^a\left(\omega_{j\rangle a} + \sigma_{j\rangle a}\right) + 3q_{\langle k}\sigma_{j\rangle b}q^b + 3\pi_{\langle j}{}^a\omega_{k\rangle c}\pi_a{}^c + 3\sigma_{jk}q^aq_a \\
&\quad - 9q_{\langle k}\omega_{j\rangle a}q^a + 6q_{\langle k}\pi_{j\rangle a}A^a + 6\pi_{a\langle j}A_{k\rangle}q^a + 3\pi_c{}^a\pi_{\langle k}{}^c\sigma_{j\rangle a}\Big] \\
&\quad + \frac{\widetilde{\kappa}^2}{3}\left[\dot{\widetilde{\pi}}_{\langle kj\rangle} + D_{\langle k}\widetilde{q}_{j\rangle} - \frac{\widetilde{\kappa}^2}{2}\widetilde{\pi}_{\langle k}q_{j\rangle} + 4\widetilde{q}_{\langle k}A_{j\rangle} + (\widetilde{\rho} + \widetilde{p})\,\sigma_{jk} + \frac{\Theta}{3}\widetilde{\pi}_{jk} + \widetilde{\pi}_{\langle j}{}^a(\omega_{k\rangle a} + \sigma_{k\rangle a})\right],
\end{aligned} \tag{98}$$

$$\begin{aligned}
&\dot{H}_{\langle kj\rangle} + \varepsilon_{ab\langle k}D^aE_{j\rangle}{}^b + \Theta H_{kj} - 3\sigma_{a\langle k}H_{j\rangle}{}^a - \omega_{a\langle k}H_{j\rangle}{}^a - 2\varepsilon_{\langle k}{}^{ab}E_{j\rangle a}A_b \\
&\quad - \frac{\kappa^2}{2}\left(\varepsilon_{ab\langle k}D^a\pi_{j\rangle}{}^b + 3\omega_{\langle k}q_{j\rangle} + \varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}q_b\right) \\
&\quad = -\frac{1}{2}\varepsilon_{ab\langle k}D^a\widehat{\mathcal{E}}_{j\rangle}{}^b + \frac{1}{2}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\widehat{\mathcal{E}}_b + \frac{3}{2}\widehat{\mathcal{E}}_{\langle j}\omega_{k\rangle} \\
&\quad + \frac{\widetilde{\kappa}^4}{24}\Big[\varepsilon_{\langle k}{}^{cd}\pi_{j\rangle c}D_d\,(\rho+3p) - (\rho+3p)\,\varepsilon_{ab\langle k}D^a\pi_{j\rangle}{}^b \\
&\quad - 3\varepsilon_{ab\langle k}D^a\pi_{j\rangle c}\pi^{cb} - 3\varepsilon_{ab\langle k}\pi_{j\rangle c}D^a\pi^{cb} + 6\rho\omega_{\langle k}q_{j\rangle} + 2\rho\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}q_b \\
&\quad + 3\varepsilon_{ab\langle k}q_{j\rangle}D^aq^b + 3\varepsilon_{ab\langle k}q^bD^aq_{j\rangle} \\
&\quad + 3\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle b}\pi_a{}^cq_c - 9\pi_{\langle j}{}^a\omega_{k\rangle}q_a\Big] + \frac{\widetilde{\kappa}^2}{3}\left(\varepsilon_{ab\langle k}D^a\widetilde{\pi}_{j\rangle}{}^b + 3\widetilde{q}_{\langle j}\omega_{k\rangle} + \varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\widetilde{q}_b\right),
\end{aligned} \tag{99}$$

$$\begin{aligned}
&D^aE_{ak} - 3H_{ka}\omega^a + \varepsilon_k{}^{ab}H_{ac}\sigma_b{}^c + \frac{\kappa^2}{2}\left[-\frac{2}{3}D_k\rho + D^a\pi_{ak} + \frac{2}{3}\Theta q_k - \sigma_{kb}q^b - 3\varepsilon_k{}^{cd}q_c\omega_d\right] \\
&\quad = \frac{1}{2}D^a\widehat{\mathcal{E}}_{ak} - \frac{1}{3}D_k\mathcal{E} - \frac{\Theta}{3}\widehat{\mathcal{E}}_k + \frac{1}{2}\,(3\omega_{ka} + \sigma_{ka})\,\widehat{\mathcal{E}}^a + \frac{\widetilde{\kappa}^4}{24}\Big[\frac{4}{3}\rho D_k\rho + \pi_{ak}D^a\,(\rho+3p) \\
&\quad + (\rho+3p)\,D^a\pi_{ak} + 3\pi_a{}^cD^a\pi_{ck} - 4\pi^{ab}D_k\pi_{ab} + 3\pi_{kb}D^a\pi_a{}^b + 2q^aD_kq_a - 3q_aD^aq_k
\end{aligned}$$

$$
\begin{aligned}
&- 3q_k D^a q_a + 9\varepsilon_k{}^{ad} q_c \omega_a \pi_d{}^c - 3\pi_a{}^b q^a \sigma_{kb} - \frac{4}{3}\rho\Theta q_k \\
&\left. + 2\rho\sigma_{kb} q^b + 6\rho\varepsilon_k{}^{cd} q_c \omega_d + 2\Theta\pi_k{}^a q_a \right] \\
&- \frac{\widetilde{\kappa}^2}{3}\left[D^a \widetilde{\pi}_{ak} - \frac{1}{2} D_k \left(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p}\right) + \frac{2}{3}\Theta\widetilde{q}_k - \widetilde{q}^a \left(3\omega_{ka} + \sigma_{ka}\right) \right], \qquad (100)
\end{aligned}
$$

$$
\begin{aligned}
&D^a H_{ak} + 3E_{ka}\omega^a - \varepsilon_k{}^{ab} E_{ac}\sigma_b{}^c + \frac{\kappa^2}{2}\left[\varepsilon_k{}^{ab} D_a q_b + 2(\rho + p)\omega_k - \pi_k{}^c\omega_c - \varepsilon_k{}^{ab}\pi_{ac}\sigma_b{}^c\right] \\
&= -\frac{1}{2}\varepsilon_k{}^{ab} D_a \widehat{\mathcal{E}}_b + \frac{4\mathcal{E}}{3}\omega_k - \frac{1}{2}\widehat{\mathcal{E}}_{ak}\omega^a - \frac{1}{2}\varepsilon_k{}^{ab}\widehat{\mathcal{E}}_{ac}\sigma_b{}^c + \frac{\widetilde{\kappa}^4}{24}\left[3q^c\omega_c q_k - 2\varepsilon_k{}^{ab} q_b D_a \rho \right. \\
&- 2\rho\varepsilon_k{}^{ab} D_a q_b - 3\varepsilon_{abk} q^c D^b \pi_c{}^a - 3\varepsilon_k{}^{ab}\pi_a{}^c D_b q_c - 4\rho(\rho + p)\omega_k \\
&+ 3q_a q^a \omega_k - (\rho + 3p)\pi_k{}^c\omega_c \\
&\left. - 3\varepsilon_k{}^{ac}\sigma_{ab} q^b q_c + 3\pi_{ab}\pi^{ab}\omega_k - 3\pi_{ca}\pi_k{}^c\omega^a - (\rho + 3p)\varepsilon_k{}^{ab}\pi_{ac}\sigma_b{}^c - 3\varepsilon_k{}^{ab}\pi_{da}\pi_c{}^d\sigma_b{}^c \right] \\
&+ \frac{\widetilde{\kappa}^2}{3}\left[\widetilde{\pi}_{ka}\omega^a - \varepsilon_k{}^{ab} D_a \widetilde{q}_b - 2(\widetilde{\rho} + \widetilde{p})\omega_k + \varepsilon_k{}^{ab}\widetilde{\pi}_a{}^c\sigma_{bc} \right]. \qquad (101)
\end{aligned}
$$

For vanishing 5d matter these equations reduce to the corrected form of the equations (26)–(29) and appendix A of [11].

Equations (88) and (89) express the interchange of energy density and energy current between the brane and the 5d spacetime (due to the nonvanishing right-hand sides). In the absence of the 5d sources, these equations become evolution equations for the brane matter alone. Similar relations for the effective nonlocal energy density and effective nonlocal energy current are given by equations (90) and (91).

Even for the chosen $\mathcal{Z}_2$-symmetric embedding and with the simplifying assumption $\widetilde{T}_{ab} = 0$ (thus $0 = \widetilde{\rho} = \widetilde{q} = \widetilde{q}_a = \widetilde{\pi} = \widetilde{\pi}_a = \widetilde{\pi}_{ab} = \widetilde{p}$) the above equations do not close on the brane, due to the absence of evolution equations for $\widehat{\mathcal{E}}_{ab}$, π_{ab}, and p. Assumptions fixing π_{ab} (typically by kinetic considerations employing the Boltzmann equation) and p (by the choice of a continuity equation) are required already in general relativity for closing the system; however, on the brane (even with empty 5d spacetime) an additional assumption for $\widehat{\mathcal{E}}_{ab}$ is equally required [11]. From these considerations it is immediate to establish the closure in the special case $\widehat{\mathcal{E}}_{ab} = 0 = \pi_{ab}$, provided the equation of state is known.

As the closure is difficult to achieve even in the simple 5d vacuum case, in order to tackle realistic problems we need to consider the ensemble of all dynamical and constraint equations given in the preceding section.

4.5. Closure conditions

The general relativistic $3 + 1$ covariant formalism contains 10 gravito-electro-magnetic variables with 10 evolution equations (besides there are also 12 kinematic variables with 9 evolution equations and 15 constraints altogether). The $3 + 1 + 1$ brane-world formalism developed in this paper contains 35 gravito-electro-magnetic variables with 35 evolution equations (besides 35 kinematical variables with 28 evolution equations and 77 constraints altogether).

The subset of equations on the brane, given in subsection 4.4, has $10+9$ gravito-electro-magnetic variables with only $10+4$ evolution equations (there are also 12 kinematical variables with 9 evolution equations and 15 constraints altogether). The 9 new gravito-electro-magnetic variables are the quantities appearing on the right-hand side of equation (84); among them the last term $\widehat{\mathcal{E}}_{ab}$, representing 5 independent variables, has no evolution equation. It has been known that the system of equations is closed by the condition $\widehat{\mathcal{E}}_{ab}=0$ [11].

In this subsection we want to explore the extra information we have in the complete system of evolution equations derived. In particular, equation (61) contains the desired temporal evolution; however, it also contains terms not appearing in subsection 4.4. Remembering that on the brane $\widehat{\omega}_{ab}=0$ we could impose

$$\begin{aligned}\mathcal{F}'_{\langle kj\rangle} &= \left(2\widehat{K}-\frac{\widehat{\Theta}}{3}\right)\mathcal{F}_{kj}-\mathcal{F}_{\langle j}{}^{a}\widehat{\sigma}_{k\rangle a}+K\widehat{\mathcal{E}}_{kj}+K\mathcal{E}_{kj}-\mathcal{E}_{\langle j}\left(\frac{3}{2}K_{k\rangle}+L_{k\rangle}-\widehat{K}_{k\rangle}\right)\\&\quad+\frac{3}{2}\widehat{\mathcal{E}}_{\langle k}\widehat{A}_{j\rangle}-\varepsilon_{\langle k}{}^{ab}\widehat{\mathcal{H}}_{j\rangle a}(K_b-2\widehat{K}_b)-\varepsilon_{\langle j}{}^{ab}\mathcal{H}_{k\rangle a}\widehat{A}_b+\frac{3}{2}\varepsilon_{\langle j}{}^{ab}\widehat{\sigma}_{k\rangle a}\mathcal{H}_b,\end{aligned}\tag{102}$$

such that equation (61) becomes

$$0=\dot{\widehat{\mathcal{E}}}_{\langle kj\rangle}-D_{\langle k}\widehat{\mathcal{E}}_{j\rangle}+\frac{\Theta}{3}\widehat{\mathcal{E}}_{kj}+\frac{4\mathcal{E}}{3}\sigma_{kj}-2\widehat{\mathcal{E}}_{\langle k}A_{j\rangle}+\widehat{\mathcal{E}}_{\langle j}{}^{a}(\omega_{k\rangle a}+\sigma_{k\rangle a})+\mathcal{M}_{kj},\tag{103}$$

with the 5d matter contributions

$$\begin{aligned}\mathcal{M}_{kj} &= \frac{\widetilde{\kappa}^2}{3}\dot{\widetilde{\pi}}_{\langle kj\rangle}+\frac{\widetilde{\kappa}^2}{3}D_{\langle k}\widetilde{q}_{j\rangle}+\frac{\widetilde{\kappa}^2}{3}(\widetilde{\rho}+\widetilde{p})\sigma_{kj}+\frac{\widetilde{\kappa}^2\widetilde{q}}{3}\widehat{\sigma}_{kj}+\frac{\widetilde{\kappa}^2\Theta}{9}\widetilde{\pi}_{kj}\\&\quad+\frac{2\widetilde{\kappa}^2}{3}\widetilde{q}_{\langle j}A_{k\rangle}+\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}(2\widehat{K}_{k\rangle}+L_{k\rangle})+\frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}{}^{a}(\omega_{k\rangle a}+\sigma_{k\rangle a}).\end{aligned}\tag{104}$$

A particular solution of equation (102) would be

$$\mathcal{F}_{kj}=\widehat{\mathcal{H}}_{kj}=\mathcal{E}_j=\mathcal{H}_j=K=\widehat{A}_j=0.\tag{105}$$

None of the quantities above appear in the brane equations presented in subsection 4.4; thus, those equations are not altered by the choice (105), and the system becomes closed by equation (103)[5].

The 5d Schwarzschild–anti-de Sitter spacetime containing a Friedmann brane emerges as a special case of the spacetimes obeying equations (103) and (105), as they have $\mathcal{M}_{ab}=0=K$ and

$$\begin{aligned}&0=\mathcal{E}_a=\widehat{\mathcal{E}}_a=\mathcal{H}_a=L_a=K_a=\widehat{K}_a=\omega_a=\widehat{\omega}_a=A_a=\widehat{A}_a,\\&0=\mathcal{E}_{ab}=\mathcal{F}_{ab}=\mathcal{H}_{ab}=\widehat{\mathcal{E}}_{ab}=\widehat{\mathcal{H}}_{ab}=\sigma_{ab}=\widehat{\sigma}_{ab}.\end{aligned}\tag{106}$$

At the end of this section we emphasize that equation (103) closing the system of brane equations should allow for far more solutions than the trivial one for $\widehat{\mathcal{E}}_{ab}$. We will construct a specific example in section 6.

5. Cosmology

In this section we consider generic embeddings. By employing definitions (81) and (82), also conditions (11), arising from the existence of the brane, we can derive average and difference equations on the brane. We give here but the most relevant such dynamical equations. The rest of the equations are straightforward to derive, although the derivation may be lengthy.

[5] The no-go theorem for closure given in [32] does not apply here, as it was derived for perturbations of 5d anti-de Sitter spacetimes.

The average taken on the two sides of the brane of equation (25) reduces to the *generalized brane Friedmann equation*. We also rewrite $\langle\widehat{\Theta}^2\rangle = \langle\widehat{\Theta}\rangle^2 + (\Delta\widehat{\Theta})^2/4$ and use a similar relation for $\widehat{\sigma}_{ab}$. We take $\Delta\widehat{\Theta}$ from the Lanczos equations (76)–(79). Equation (25) then becomes

$$\frac{\mathcal{R}}{2} + \frac{\Theta^2}{3} - \Lambda - \kappa^2\rho\left(1 + \frac{\rho}{2\lambda}\right) - \frac{1}{2}\sigma_{ab}\sigma^{ab} + \omega_a\omega^a$$
$$= -\langle\mathcal{E}\rangle - \frac{\widetilde{\kappa}^4}{8}\pi_{ab}\pi^{ab} + \frac{\langle\widehat{\Theta}\rangle^2}{3} - \frac{1}{2}\langle\widehat{\sigma}_{ab}\rangle\langle\widehat{\sigma}^{ab}\rangle + \frac{\widetilde{\kappa}^2}{2}\langle\widetilde{\rho} + \widetilde{p} - \widetilde{\pi}\rangle. \tag{107}$$

The *generalized brane Raychaudhuri equation* is obtained from the average of equation (27), by employing the same sequence of simplifications as for the Friedmann equation. We obtain

$$\dot{\Theta} + \frac{\Theta^2}{3} + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - D^aA_a - A^aA_a + \frac{\kappa^2}{2}(\rho + 3p) - \Lambda$$
$$= \langle\mathcal{E}\rangle - \frac{\widetilde{\kappa}^4\rho}{12}(2\rho + 3p) + \frac{\widetilde{\kappa}^4}{4}q_aq^a - \langle\widehat{\Theta}\rangle\langle\widehat{K}\rangle + \langle\widehat{K}_a\rangle\langle\widehat{K}^a\rangle - \frac{\widetilde{\kappa}^2}{2}\langle\widetilde{\rho} + \widetilde{\pi} + \widetilde{p}\rangle. \tag{108}$$

Finally we give the *generalized brane energy-balance equation* from the jump across the brane of equation (41) by the same procedure:

$$\dot{\rho} + \Theta(\rho + p) + D_aq^a + 2A_aq^a + \pi_{ab}\sigma^{ab} = -\Delta\widetilde{q}. \tag{109}$$

These equations hold for arbitrary branes. Again, general relativistic contributions are on the left-hand sides, and brane-world contributions on the right-hand sides.

5.1. Friedmann brane with perfect fluid

In this case the conditions $\omega_a = 0 = \sigma_{ab} = \Delta\widehat{\sigma}_{ab}$ hold, arising from the particular geometry and matter source chosen; also $\mathcal{R} = 6k/a^2$, where a is the scale factor. Moreover $\Theta/3 = H \equiv \dot{a}/a$, where H is the Hubble parameter. The Friedmann, Raychaudhuri and energy-balance equations become

$$3\left(H^2 + \frac{k}{a^2}\right) = \Lambda + \kappa^2\rho\left(1 + \frac{\rho}{2\lambda}\right) - \langle\mathcal{E}\rangle$$
$$+ \frac{\langle\widehat{\Theta}\rangle^2}{3} - \frac{1}{2}\langle\widehat{\sigma}_{ab}\rangle\langle\widehat{\sigma}^{ab}\rangle + \frac{\widetilde{\kappa}^2}{2}\langle\widetilde{\rho} + \widetilde{p} - \widetilde{\pi}\rangle, \tag{110}$$

$$3(\dot{H} + H^2) + \frac{\kappa^2}{2}(\rho + 3p) - \Lambda = \langle\mathcal{E}\rangle - \frac{\widetilde{\kappa}^4\rho}{12}(2\rho + 3p) + \frac{\widetilde{\kappa}^4}{4}q_aq^a$$
$$-\langle\widehat{\Theta}\rangle\langle\widehat{K}\rangle + \langle\widehat{K}_a\rangle\langle\widehat{K}^a\rangle - \frac{\widetilde{\kappa}^2}{2}\langle\widetilde{\rho} + \widetilde{\pi} + \widetilde{p}\rangle, \tag{111}$$

$$\dot{\rho} + 3H(\rho + p) = -\Delta\widetilde{q}. \tag{112}$$

For symmetric embedding the equations simplify by $\langle\widehat{\Theta}\rangle = 0 = \langle\widehat{\sigma}_{ab}\rangle$. For generic asymmetric embedding equation (110) can be rewritten as

$$3\left(H^2 + \frac{k}{a^2}\right) = \Lambda + \kappa^2\rho\left(1 + \frac{\rho}{2\lambda}\right) + \kappa^2U - \frac{\langle L\rangle}{4} - \frac{\widetilde{\kappa}^2}{2}\langle\widetilde{\pi}\rangle, \tag{113}$$

where $\langle L\rangle$ is given by equation (86) and U is defined by

$$\kappa^2U = \frac{\langle\widehat{\Theta}\rangle^2}{6} + \frac{1}{2}\langle\widehat{\Theta}\rangle\langle\widehat{K}\rangle - \frac{1}{2}\langle\widehat{K}_b\rangle\langle\widehat{K}^b\rangle - \frac{1}{4}\langle\widehat{\sigma}_{ab}\rangle\langle\widehat{\sigma}^{ab}\rangle - \langle\mathcal{E}\rangle + \frac{\widetilde{\kappa}^2}{2}\langle\widetilde{\rho} + \widetilde{p}\rangle. \tag{114}$$

The quantity U is nothing but the *effective energy density* introduced in [4], encompassing the trace-free parts of the Weyl, embedding and 5d matter sources in the effective Einstein equation, given here in terms of the 3+1+1 kinematic, gravito-electro-magnetic and 5d matter variables.

5.2. Anisotropic brane-worlds

Full brane-world solutions with homogeneous, but anisotropic 5d spacetimes are also known.

In [49] a vacuum 5d static and anisotropic spacetime with cosmological constant admitting a moving Bianchi I brane was analyzed. The Z_2-symmetric junction conditions could be obeyed only by anisotropic stresses on the brane; hence, the brane cannot support a perfect fluid. Isotropy of the brane fluid could be achieved only for isotropic 5d spacetime and brane.

This setup was generalized in [50] by allowing for a non-static 5d spacetime. Assuming separability of the metric components, new 5d solutions combining the 4d Kasner solution with the static 5d solutions of [49] were obtained.

For the reader's convenience we give in appendix D the list of kinematical, gravito-electro-magnetic and matter variables for the five-dimensional models presented in [49]. Working out the respective quantities for other metrics would be a similar straightforward exercise.

6. Stationary vacuum spacetimes with local rotational symmetry

In this section we discuss an application of our formalism, by assuming vacuum in both 4d and 5d, but keeping the respective cosmological constants. The embedding of the brane is symmetric. Then the effective Einstein equation reduces to

$$G_{ab} = \Lambda g_{ab} - \mathcal{E}\left(u_a u_b + \frac{1}{3}h_{ab}\right) + 2\widehat{\mathcal{E}}_{(a}u_{b)} - \widehat{\mathcal{E}}_{ab}. \tag{115}$$

We are interested in stationary spacetimes; therefore, $\dot{f} = 0$ for any scalar field f. The stationarity implies a singled-out temporal Killing vector; therefore, the 3+1+1 decomposition of the gravitational dynamics developed in this paper turns out to be particularly useful for the study of gravitational dynamics on the brane (which defines the other singled-out direction).

We also specialize our treatment to spacetimes with local rotational symmetry (LRS) on the brane. Such a symmetry singles out an additional space-like unit vector field e^a, in the sense that there is a unique preferred spatial direction at each point that assigns the local axis of symmetry. Once such a special vector field is chosen, a further decomposition of the spatial quantities would lead to a generic 2+1+1+1 formalism. For this, the metric h_{ab} should be further decomposed as

$$h_{ab} = e_a e_b + q_{ab}, \tag{116}$$

where q_{ab} is the induced 2-metric on the surface perpendicular to both e^a and u^a, and lying in the brane.

In what follows we will see that these symmetries assure that the structure of the spacetime can be described solely in terms of scalars; thus, no vectors and tensorial quantities are needed. This is a powerful feature of the formalism. Furthermore, the symmetries assure that all scalars f depend only on the coordinate parametrizing the integral curves of the rotation axis field e^a. We denote this coordinate derivative as $f^\star \equiv e^a D_a f$ (a spatial covariant derivative along these integral curves).

6.1. Independent kinematic quantities related to the vector field e^a

6.1.1. Decomposition. For the purposes of the present application it is enough to give here the decomposition of the covariant brane derivative of the vector field e^a in terms of kinematical quantities and extrinsic curvature-type quantities analogous to the ones appearing in the decomposition of the vector fields u^a and n^a. We keep the familiar notations, with the remark that the quantities belonging to the decomposition of $\nabla_a e_b$ will carry a distinguishing $\widetilde{}$ mark. We thus have

$$\nabla_a e_b = \widetilde{L} u_a u_b - \widetilde{L}_a u_b - u_a \widetilde{K}_b + e_a \widetilde{A}_b + \frac{\widetilde{\Theta}}{2} q_{ab} + \widetilde{\sigma}_{ab} + \widetilde{\omega}_{ab}, \tag{117}$$

with

$$\begin{aligned} &\widetilde{L} = u^c u^d \widetilde{\nabla}_c e_d, \quad \widetilde{\Theta} = q^{ab} D_a e_b \ , \\ &\widetilde{L}_a = u^d h_a{}^c \widetilde{\nabla}_c e_d, \quad \widetilde{K}_b = u^c h_b{}^d \widetilde{\nabla}_c e_d, \quad \widetilde{A}_b = e^c q_b{}^d \widetilde{\nabla}_c e_d, \\ &\widetilde{\omega}_{ab} = q_{[a}{}^c q_{b]}{}^d \widetilde{\nabla}_c e_d, \quad \widetilde{\sigma}_{ab} = q_a{}^c q_b{}^d \widetilde{\nabla}_c e_d - \frac{\widetilde{\Theta}}{2} q_{ab}. \end{aligned} \tag{118}$$

We remark that $\widetilde{L}$ and $\widetilde{L}_a$ are not independent from the previously introduced sets of variables; they can be expressed in terms of projections of the kinematic quantities related to u^a:

$$\widetilde{L} = -e^a A_a \ , \quad \widetilde{L}_a = -e^d \left(\frac{\Theta}{3} h_{ad} + \sigma_{ad} + \omega_{ad} \right) \ . \tag{119}$$

By contrast, the quantities $\widetilde{\Theta}$, $\widetilde{K}_b$, $\widetilde{A}_b$, $\widetilde{\sigma}_{ab}$ and $\widetilde{\omega}_{ab}$ are independent of the rest of the kinematical and extrinsic curvature-type variables. Similar to ω_a and $\widehat{\omega}_a$, we can also define a rotation vector

$$\widetilde{\omega}_a = \frac{1}{2} \varepsilon_{abc} \widetilde{\omega}^{bc}. \tag{120}$$

6.1.2. LRS symmetry. The preferred space-like unit vector field e_a satisfies $u^a e_a = 0$, $e^a e_a = 1$. We employ here various results following from the LRS symmetry, following [51]. The symmetry and normalization implies

$$e^a D_a e_b = 0 \ , \quad \dot{e}_{\langle b \rangle} = 0 \ , \tag{121}$$

i.e. e_a is geodesic with respect to the connection compatible with h_{ab} and is Fermi propagated along the world line of a brane observer. The above equations and normalization further imply that equation (120) can be rewritten into the standard form

$$\widetilde{\omega}_a = \varepsilon_a{}^{bc} q_{[b}{}^e q_{c]}{}^d \widetilde{\nabla}_e e_d = \varepsilon_{abc} D^b e^c. \tag{122}$$

Due to LRS, all space-like vector fields must be proportional to e^a; thus

$$\begin{aligned} &A^a = A e^a, \quad \omega_a = \omega e_a, \quad \widetilde{\omega}_a = \widetilde{\omega} e_a. \\ &\widehat{\mathcal{E}}^a = \widehat{\mathcal{E}}^V e^a, \quad D^a \Theta = \Theta^\star e_a, \quad D_a \mathcal{E} = \mathcal{E}^\star e_a. \end{aligned} \tag{123}$$

The vector field e^a and the induced metric q_{ab} define a unique space-like trace-free symmetric tensor field e_{ab} as

$$e_{ab} = e_a e_b - \frac{q_{ab}}{2}, \tag{124}$$

satisfying

$$\begin{aligned} &u^a e_{ab} = 0, \quad e^a e_{ab} = e_b, \quad e^a{}_a = 0, \quad 2 e_{ac} e^c{}_b = e_{ab} + h_{ab}, \\ &e_{ab} e^{ab} = \frac{3}{2}, \quad D^b e_{ab} = \frac{3\widetilde{\Theta}}{2} e_a, \quad \dot{e}_{\langle ab \rangle} = 0. \end{aligned} \tag{125}$$

Again, due to LRS all 3d tracefree symmetric tensor fields are proportional to e_{ab}:

$$\sigma_{ab} = \frac{2\sigma}{\sqrt{3}} e_{ab} \ , \quad E_{ab} = \frac{2E}{\sqrt{3}} e_{ab} \ , \quad H_{ab} = \frac{2H}{\sqrt{3}} e_{ab} \ , \quad \widehat{\mathcal{E}}_{ab} = \frac{2\widehat{\mathcal{E}}}{\sqrt{3}} e_{ab} \ . \tag{126}$$

In equations (123) and (126) we have introduced suitably normalized scalars A, ω, $\widetilde{\omega}$, σ, $\widehat{\mathcal{E}}^V$, $\widehat{\mathcal{E}}$, E and H, replacing the vectorial and tensorial variables.

From LRS, by use of equations (121), (123), (126) and (125) it also follows that

$$\widetilde{L} = -A \ , \quad \widetilde{L}_a = -\left(\frac{\Theta}{3} + \frac{2\sigma}{\sqrt{3}}\right) e_a \ , \tag{127}$$

$$\widetilde{K}_a = \widetilde{A}_a = 0 \ , \quad \widetilde{\sigma}_{ab} = 0 \ . \tag{128}$$

Thus there are only two kinematic quantities related to e^a left, which are both non-trivial and independent from those introduced in section 2. These are $\widetilde{\Theta}$ and $\widetilde{\omega}$.

6.2. LRS class I type conditions

The general relativistic classification of the LRS models presented in [51] is recovered for $\widehat{\mathcal{E}}_a = 0 = \widehat{\mathcal{E}}_{ab}$ and $\mathcal{E} < 0$. For brane-worlds, when the above conditions do not hold even under the simplifying assumptions of this section, this classification should be refined; however, for the application we are interested in, we shall still assume the conditions $\widetilde{\omega} = \Theta = \sigma = 0$ characterizing the LRS class I of general relativity. From these conditions, equations (95), (122), (123) and $\varepsilon_{aij} e^i e^j = 0$ we also find $\widehat{\mathcal{E}}^V = 0$. Therefore, we have verified that equation (105) which closes the system of brane equations for the considered stationary vacuum spacetimes containing LRS class I type branes, trivally holds.

6.2.1. Dynamics. The evolution of the single kinematic quantity $\widetilde{\Theta}$ characterizing e^a can be inferred from the Ricci identity for e^a:

$$\widetilde{\Theta}^{\star} + \frac{\widetilde{\Theta}^2}{2} = -\frac{2E}{\sqrt{3}} + \frac{2\mathcal{E}}{3} + \frac{\widehat{\mathcal{E}}}{\sqrt{3}} - \frac{2\Lambda}{3}. \tag{129}$$

Other nontrivial brane equations are (91), (92), (93), (96), (97), (100) and (101). Under the assumptions of this section they simplify to

$$\mathcal{E}^{\star} + 4A\mathcal{E} + 2\sqrt{3}\left[\widehat{\mathcal{E}}^{\star} + \left(\frac{3\widetilde{\Theta}}{2} + A\right)\widehat{\mathcal{E}}\right] = 0, \tag{130}$$

$$A^{\star} + \left(\widetilde{\Theta} + A\right) A + 2\omega^2 + \Lambda = -\mathcal{E}, \tag{131}$$

$$A^{\star} + \left(A - \frac{\widetilde{\Theta}}{2}\right) A - \omega^2 - \sqrt{3}E = \frac{\sqrt{3}\widehat{\mathcal{E}}}{2}, \tag{132}$$

$$H + \left(2A - \widetilde{\Theta}\right)\frac{\sqrt{3}\omega}{2} = 0, \tag{133}$$

$$\omega^{\star} + \left(\widetilde{\Theta} - A\right)\omega = 0, \tag{134}$$

$$E^{\star} + \frac{3\widetilde{\Theta}}{2}E - 3\omega H = \frac{\widehat{\mathcal{E}}^{\star}}{2} + \frac{3\widetilde{\Theta}\widehat{\mathcal{E}}}{4} - \frac{\mathcal{E}^{\star}}{2\sqrt{3}}, \tag{135}$$

$$2H^{\star} + 3\widetilde{\Theta}H + 6E\omega = \frac{4\omega\mathcal{E}}{\sqrt{3}} - \omega\widehat{\mathcal{E}}. \tag{136}$$

Eliminating $A^{\star}$ from equations (131) and (132) we obtain the algebraic equation

$$0 = \frac{3\widetilde{\Theta}A}{2} + 3\omega^2 + \Lambda + \sqrt{3}E + \mathcal{E} + \frac{\sqrt{3}\widehat{\mathcal{E}}}{2}. \tag{137}$$

Also for any solution of the system (129)–(134), equations (135) and (136) are identically obeyed. This can be seen as follows. By taking the $\star$-derivative of equation (137) and employing equations (129)–(134) we obtain equation (135). Similarly, the $\star$ -derivative of equation (133), combined with equations (129)–(134), gives (136).

As we have six independent (two algebraic and four first-order differential) equations left for the seven variables ($\mathcal{E}$, $\widehat{\mathcal{E}}$, $\widetilde{\Theta}$, A, ω, E, H), we need to impose an additional ansatz, chosen as

$$\widehat{\mathcal{E}} = -\frac{2\mathcal{E}}{\sqrt{3}}. \tag{138}$$

This condition will considerably simplify the algebraic equation (137).

6.2.2. Discussion. The algebraic equations (133) and (137) give H and E in terms of the rest of variables. By equation (138) the system (129)–(131) and (134) reduces to

$$\widetilde{\Theta}^{\star} + \frac{\widetilde{\Theta}^2}{2} - \widetilde{\Theta}A - 2\omega^2 = 0, \tag{139}$$

$$\mathcal{E}^{\star} + 2\mathcal{E}\widetilde{\Theta} = 0, \tag{140}$$

$$A^{\star} + \left(\widetilde{\Theta} + A\right) A + 2\omega^2 + \Lambda = -\mathcal{E}, \tag{141}$$

$$\omega^{\star} + \left(\widetilde{\Theta} - A\right) \omega = 0. \tag{142}$$

From the newly arised two algebraic equations (140) and (142) we express

$$\widetilde{\Theta} = \left(\ln \mathcal{E}^{-1/2}\right)^{\star}, \tag{143}$$

$$A = \left(\ln \omega\mathcal{E}^{-1/2}\right)^{\star}. \tag{144}$$

In terms of the auxiliary variables

$$x = \ln \mathcal{E}^{-1/2}, \tag{145}$$

$$y = \ln \omega\mathcal{E}^{-1/2}, \tag{146}$$

the remaining equations (139) and (141) become

$$x^{\star\star} + \frac{(x^{\star})^2}{2} - x^{\star}y^{\star} = 2\mathrm{e}^{y-x}, \tag{147}$$

$$y^{\star\star} + (y^{\star})^2 + x^{\star}y^{\star} = -(2\mathrm{e}^{y-x} + \mathrm{e}^{-2x} + \Lambda). \tag{148}$$

They form a coupled second-order system, which would eventually give ω and $\mathcal{E}$ in full generality. The solution of this system is however not immediate; therefore, in what follows we will employ a metric ansatz in order to find a particular solution.

6.3. Taub-NUT-(A)dS solution with tidal charge

We take the following metric ansatz, compatible with the chosen symmetries:

$$\mathrm{d}s^2 = -\frac{f(r)}{g(r)}\left(\mathrm{d}t + \omega_{\mathrm{k}}\mathrm{d}\varphi\right)^2 + \frac{g(r)}{f(r)}\mathrm{d}r^2 + g(r)\left(\mathrm{d}\theta^2 + \Omega_{\mathrm{k}}^2\mathrm{d}\varphi^2\right), \tag{149}$$

where

$$\Omega_{\mathrm{k}} = \begin{cases} \sin\theta, & k = 1 \\ 1, & k = 0 \\ \sinh\theta, & k = -1 \end{cases}, \tag{150}$$

and $\omega_{\rm k}$ is another function of θ. The axis of LRS is given by

$$e^a = \frac{f^{1/2}}{g^{1/2}}\left(\frac{\partial}{\partial r}\right)^a . \tag{151}$$

Choosing the 1-form u_a as

$$u = -\frac{f^{1/2}}{g^{1/2}}\left(\mathrm{d}t + 2\omega_{\rm k}\mathrm{d}\varphi\right), \tag{152}$$

and employing $\widehat{\mathcal{E}}^V = 0$, equations (126), (138), (151), the electric part of the 5d Weyl tensor is found as

$$^{(4)}\mathcal{E}^t{}_t = {}^{(4)}\mathcal{E}^r{}_r = -{}^{(4)}\mathcal{E}^\theta{}_\theta = -{}^{(4)}\mathcal{E}^\varphi{}_\varphi = -\mathcal{E}\,, \quad {}^{(4)}\mathcal{E}^t{}_\varphi = -2\mathcal{E}\omega_{\rm k}\,. \tag{153}$$

Both from $\widehat{\mathcal{E}}_a \propto \widehat{\mathcal{E}}^V = 0$ and from H_{ab} being proportional to e_{ab} we find

$$\Omega_{\rm k}\frac{\mathrm{d}^2\omega_{\rm k}}{\mathrm{d}\theta^2} - \frac{\mathrm{d}\Omega_{\rm k}}{\mathrm{d}\theta}\frac{\mathrm{d}\omega_{\rm k}}{\mathrm{d}\theta} = 0. \tag{154}$$

Equivalently, by an integration,

$$\Omega_{\rm k}^{-1}\frac{\mathrm{d}\omega_{\rm k}}{\mathrm{d}\theta} = -2l \tag{155}$$

where l is constant. A second integration gives

$$\omega_{\rm k}(\theta) = \begin{cases} 2l\cos\theta + L, & k = 1 \\ -2l\theta + L, & k = 0 \\ -2l\cosh\theta + L, & k = -1 \end{cases}, \tag{156}$$

where L is another integration constant. Locally, this constant can be absorbed in a new time variable $t + L\varphi \to t$, which translates to the choice $L = 0$.

Direct computation, employing equation (156) shows that the metric ansatz is compatible with the chosen LRS class I conditions

$$\Theta = \sigma = a = \widehat{\mathcal{E}}^V = 0\,. \tag{157}$$

The nontrivial kinematic and Weyl quantities, by employing equations (149) and (156), are

$$\begin{aligned}
\widetilde{\Theta} &= \frac{f^{1/2}}{g^{3/2}}\frac{\mathrm{d}g}{\mathrm{d}r}, \quad A = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f}{g}\right)^{1/2}, \quad \omega = \frac{lf^{1/2}}{g^{3/2}},\\
\mathcal{E} &= \frac{f^{1/2}}{g^{5/4}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f^{1/2}}{g^{3/4}}\frac{\mathrm{d}g}{\mathrm{d}r}\right) - \frac{3l^2 f}{g^3} - \frac{k}{g} + \Lambda,\\
\frac{2\widehat{\mathcal{E}}}{\sqrt{3}} &= \frac{1}{3g}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} - \frac{2}{3g^2}\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\mathrm{d}g}{\mathrm{d}r} + \frac{f}{3g^3}\left(\frac{\mathrm{d}g}{\mathrm{d}r}\right)^2 + \frac{4l^2 f}{3g^3} + \frac{2k}{3g},\\
\frac{2E}{\sqrt{3}} &= \frac{1}{6g}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} - \frac{1}{3f^{1/2}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f^{3/2}}{g^2}\frac{\mathrm{d}g}{\mathrm{d}r}\right) - \frac{4l^2 f}{3g^3} - \frac{k}{3g},\\
\frac{2H}{\sqrt{3}} &= -l\frac{\mathrm{d}}{\mathrm{d}r}\left(fg^{-2}\right).
\end{aligned} \tag{158}$$

These quantities are constrained by equations (133), (137), (138), and (139)–(142). From here by straightforward algebra we find two independent equations for the metric functions $f(r)$ and $g(r)$:

$$g^{3/2}\frac{\mathrm{d}^2 g^{1/2}}{\mathrm{d}r^2} = l^2, \tag{159}$$

$$\frac{d^2 f}{dr^2} = 2k - 4\Lambda g. \tag{160}$$

By multiplying the first equation with dg/dr and integrating, we get

$$\left(\frac{dg}{dr}\right)^2 + 4l^2 = C_1 g, \tag{161}$$

with $C_1 > 0$ an integration constant. A further integration then gives

$$g(r) = \frac{C_1}{4}(r + C_2)^2 + \frac{4l^2}{C_1}, \tag{162}$$

with C_2 another integration constant. The constant C_2 can be absorbed into r by a redefinition of its origin; therefore, without restricting generality we choose $C_2 = 0$. Also the constant C_1 disappears in the following rescaling of the coordinates and parameters: $(2t/C_1^{1/2}, C_1^{1/2}r/2, 4l^2/C_1) \to (t, r, l^2)$. Formally this is the same as choosing $C_1 = 4$.

Equation (160) then gives the other metric function as

$$f(r) = C_4 + C_3 r + (k - 2l^2\Lambda)r^2 - \frac{\Lambda}{3}r^4, \tag{163}$$

with C_3 and C_4 emerging as integration constants. With the reparametrizations $C_3 = -2m$ and $C_4 = q - kl^2 + \Lambda l^4$ we find the brane solution given by equations (149), (150) and

$$\begin{aligned} f(r) &= k(r^2 - l^2) - 2mr + q - \Lambda\left(\frac{r^4}{3} + 2l^2r^2 - l^4\right), \\ g(r) &= r^2 + l^2, \\ \omega_k(\theta) &= \begin{cases} 2l\cos\theta, & k = 1 \\ -2l\theta, & k = 0 \\ -2l\cosh\theta, & k = -1 \end{cases}. \end{aligned} \tag{164}$$

This is quite similar to the charged-Taub-NUT-(A)dS solution of a general relativistic Einstein–Maxwell system with mass m, electric charge Q, NUT charge l, and cosmological constant Λ; however, the constant q replacing Q^2 is not restricted to positive values. A glance at the effective Einstein equation (115) shows that this constant could possibly arise only from the electric part of the 5d Weyl tensor, equation (153). Indeed from the fourth equation (158) we get

$$\mathcal{E} = -\frac{q}{(l^2 + r^2)^2}. \tag{165}$$

As q originates in the Weyl sector of the higher dimensional spacetime, the derived solution has the interpretation of a *Taub-NUT-(A)dS brane with tidal charge.*

7. Concluding remarks

In this paper we have developed a generic 3+1+1 covariant formalism for characterizing 5d gravitational dynamics on and outside a brane. The singled-out directions are the off-brane and temporal directions; thus, the 3-spaces are constant time sections of the brane. Generalizing previous approaches, like 3+1+1 with the extra requirement of double foliability [9, 10], 3+1 in general relativity [12]–[15] and in brane-worlds [11], finally 2+1+1 in general relativity [16, 17], we presented gravitational evolution and constraint equations in terms of kinematic, gravito-electro-magnetic and matter variables, defined as scalars, 3-vectors or 3-tensors. The number of variables being higher than for other lower dimensional formalisms, the 5d matter and especially the 5d Weyl tensor leads to a multitude of projections without counterpart in the mentioned approaches. Only the kinematical set of variables is similar. We have compared and checked our results with those presented in a recent work by Nzioki, Carloni, Goswami and Dunsby [18] on the 2+1+1 decomposition of $f(R)$ gravity and we give the correspondence between our and the notations of [18] in table 1.

Our generic formalism contains the complete set of dynamical equations in the 5d spacetime. All equations, with the exception of those in sections 4 and 6 are independent of the particular form of the dynamics on the brane. As such, they can be employed both to discuss DGP/induced gravity-type [43] branes (a project deferred for future work) and one-brane Randall–Sundrum-type branes. For the latter, we have given both the full set of equations on the brane and those containing off-brane evolutions.

The brane equations are more general than previously published results by the inclusion of arbitrary 5d sources. Although we have recovered the known fact that in the generic case this system does not close on the brane (except the trivial case $\widehat{\mathcal{E}}_{ab} = 0$), by deriving the complementary system of equations of the 5d dynamics we could establish a more generic *condition for closure*. This is given by either equations (102) or (105), which carry much richer possibilities than the previously known $\widehat{\mathcal{E}}_{ab} = 0$ case.

The initial value problem in general relativity is usually discussed in ADM-like variables (including modifications enhancing stability, like the use of variables with factored-out conformal factors [54]); therefore, a similar treatment would be possible to develop in the framework of the Hamiltonian approach presented in [9, 10]. A 3+1 covariant approach for discussing the evolution of cosmic microwave background anisotropies in the cold dark matter model was employed in [24]. The complete set of infinitesimal frame transformations given in the present paper may turn useful in the study of perturbations in a 3+1+1 setup.

We have decomposed the Lanczos equations and all source terms of the effective Einstein equation in terms of the 3+1+1 covariant variables. The ensemble of these results opens the possibility for applications, with both cosmological and other symmetries.

We have given generalized Friedmann, Raychaudhuri and energy balance equations for a generic brane, and by specifying for cosmological symmetries we have established correspondence with previous related work [4]. We have also established the correspondence with the anisotropic brane-world presented in [49].

We have also employed the 3+1+1 covariant formalism to discuss stationary spacetimes with local rotation symmetry of class I, imposed on the brane. We have shown that such spacetimes obey the closure condition presented here. The symmetries and metric ansatz (149) implemented in the 3+1+1 covariant formalism led to a simple decoupled system of second-order differential equations for the metric functions, the solution of which gave *a new exact solution of the effective Einstein equation, the tidal charged Taub-NUT-(A)dS brane*, given by equations (150) and (164).

In the spherically symmetric and rotating cases tidal charged brane solutions were already found [52, 53], which correspond to electrically charged general relativistic

Table 1. Comparison of the notations for the kinematic quantities with [18]

$n_a \to e_a$	$K \to \frac{1}{3}\Theta + \Sigma$	$\widehat{K} \to -A$
$A_a \to A_a$	$K_a \to \Sigma_a + \varepsilon_{ab}\Omega^b$	$\widehat{K}_a \to \alpha_a$
$\widehat{A}_a \to a_a$	$\Theta \to \Theta - \frac{3}{2}\Sigma$	$\widehat{\Theta} \to \phi$
$L_a \to \Sigma_a - \varepsilon_{ab}\Omega^b$	$\omega_{ab} \to \Omega\varepsilon_{ab}$	$\widehat{\omega}_{ab} \to \xi\varepsilon_{ab}$
$h_{ab} \to N_{ab}$	$\sigma_{ab} \to \Sigma_{ab}$	$\widehat{\sigma}_{ab} \to \zeta_{ab}$

Einstein–Maxwell solutions, when a formal identification of the electric charge squared with the tidal charge is carried on. Here we also found that replacing the electric charge squared in the electrically charged Taub-NUT-(A)dS solution of an Einstein–Maxwell system with a tidal charge originating in the Weyl curvature of the 5d space-time leads to a brane solution with the same symmetries. Unless the electric charge squared, the tidal charge however is allowed to have both positive and negative values, thus allowing to either weaken gravity, or contribute towards its confinement on the brane.

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Appendix A. Commutation relations

In this appendix we enlists some useful differential identities, obtained by computing the commutators among the derivatives $u^a\widetilde{\nabla}_a \equiv D/\mathrm{d}\tau$ (dot), $n^a\widetilde{\nabla}_a \equiv D/\mathrm{d}y$ (prime), and D_a (induced metric compatible covariant derivative) on scalars, brane-vectors and symmetric brane-tensors of second rank.

For a scalar field ϕ the following commutation relations hold:

$$n^a\widetilde{\nabla}_a(\dot{\phi}) - u^c\widetilde{\nabla}_c(\phi') = K\phi' + \widehat{K}\dot{\phi} + \left(K^a - \widehat{K}^a\right) D_a\phi, \tag{A.1}$$

$$D_a\phi' - h_a{}^i n^b\widetilde{\nabla}_b\left(D_i\phi\right) = -\left(K_a - L_a\right)\dot{\phi} + \widehat{A}_a\phi' + \frac{\widehat{\Theta}}{3}D_a\phi + \left(\widehat{\omega}_{ab} + \widehat{\sigma}_{ab}\right)D^b\phi, \tag{A.2}$$

$$D_a\dot{\phi} - h_a{}^i u^b\widetilde{\nabla}_b\left(D_i\phi\right) = -A_a\dot{\phi} + \left(\widehat{K}_a + L_a\right)\phi' + \frac{\Theta}{3}D_a\phi + \left(\omega_{ab} + \sigma_{ab}\right)D^b\phi, \tag{A.3}$$

$$D_{[a}D_{b]}\phi = \omega_{ab}\dot{\phi} - \widehat{\omega}_{ab}\phi'. \tag{A.4}$$

For a 3-vector field V^a the following commutation relations hold:

$$\begin{aligned} h_b{}^j n^a\widetilde{\nabla}_a(\dot{V}_{\langle j\rangle}) - h_b{}^j u^c\widetilde{\nabla}_c(V'_{\langle j\rangle}) = {} & -\varepsilon_{kab}\mathcal{H}^a V^b + KV'_{\langle b\rangle} + \widehat{K}\dot{V}_{\langle b\rangle} + \left(K^a - \widehat{K}^a\right)D_aV_b \\ & -A_aV^aK_b + \widehat{K}_aV^a\widehat{A}_b + K_aV^aA_b - \widehat{A}_aV^a\widehat{K}_b, \end{aligned} \tag{A.5}$$

$$\begin{aligned} h_c{}^i h_d{}^j u^a\widetilde{\nabla}_a\left(D_iV_j\right) - D_c\left(\dot{V}_{\langle d\rangle}\right) = {} & -\left(\widehat{K}_c + L_c\right)V'_{\langle d\rangle} + A_c\dot{V}_{\langle d\rangle} - \frac{\Theta}{3}D_cV_d \\ & + \frac{\widehat{\Theta}}{3}\widehat{K}_dV_c - \frac{\Theta}{3}A_dV_c - \left(\sigma_{ac} - \omega_{ac}\right)D^aV_d + \frac{1}{2}\widehat{\mathcal{E}}_aV^ah_{cd} \\ & - \frac{1}{2}\widehat{\mathcal{E}}_dV_c - \varepsilon_{dab}\mathcal{H}_c{}^bV^a + \frac{\widetilde{\kappa}^2}{3}\widetilde{q}_aV^ah_{cd} \\ & - \frac{\widetilde{\kappa}^2}{3}V_c\widetilde{q}_d - \left(\frac{\widehat{\Theta}}{3}h_{cd} + \widehat{\omega}_{cd} + \widehat{\sigma}_{cd}\right)\widehat{K}_aV^a + \widehat{K}_d\left(\widehat{\omega}_{ca} + \widehat{\sigma}_{ca}\right)V^a \\ & + \left(\frac{\Theta}{3}h_{cd} + \omega_{cd} + \sigma_{cd}\right)A_aV^a - A_d\left(\omega_{ca} + \sigma_{ca}\right)V^a, \end{aligned} \tag{A.6}$$

$$
\begin{aligned}
h_c{}^i h_d{}^j n^a \widetilde{\nabla}_a (D_i V_j) - D_c(V'_{\langle d\rangle}) = &-(L_c - K_c)\,\dot{V}_{\langle d\rangle} - \widehat{A}_c V'_{\langle d\rangle} - \frac{\widehat{\Theta}}{3} D_c V_d - \frac{\Theta}{3} K_d V_c + \frac{\widehat{\Theta}}{3}\widehat{A}_d V_c \\
&-(\widehat{\sigma}_{ac} - \widehat{\omega}_{ac}) D^a V_d - \frac{1}{2}\mathcal{E}_a V^a h_{cd} + \frac{1}{2}\mathcal{E}_d V_c - \varepsilon_{dab}\widehat{\mathcal{H}}_c{}^b V^a - \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_a V^a h_{cd} \\
&+ \frac{\widetilde{\kappa}^2}{3} V_c \widetilde{\pi}_d + \left(\frac{\Theta}{3} h_{cd} + \omega_{cd} + \sigma_{cd}\right) K_a V^a - K_d\,(\omega_{ca} + \sigma_{ca})\,V^a \\
&-\left(\frac{\widehat{\Theta}}{3} h_{cd} + \widehat{\omega}_{cd} + \widehat{\sigma}_{cd}\right)\widehat{A}_a V^a + \widehat{A}_d\,(\widehat{\omega}_{ca} + \widehat{\sigma}_{ca})\,V^a,
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
D_{[a} D_{b]} V_c = &\,\omega_{ab}\dot{V}_{\langle c\rangle} - \widehat{\omega}_{ab} V'_{\langle c\rangle} + h_{c[a}\mathcal{E}_{b]d} V^d + \mathcal{E}_{c[a} V_{b]} - h_{c[a}\widehat{\mathcal{E}}_{b]d} V^d - \widehat{\mathcal{E}}_{c[a} V_{b]} \\
&- \frac{1}{9}(\Theta^2 - \widehat{\Theta}^2) h_{c[a} V_{b]} - \frac{\Theta}{3}(\sigma_{c[a} - \omega_{c[a}) V_{b]} + \frac{\widehat{\Theta}}{3}(\widehat{\sigma}_{c[a} - \widehat{\omega}_{c[a}) V_{b]} \\
&- \frac{\Theta}{3} h_{c[a}(\sigma_{b]d} + \omega_{b]d}) V^d + \frac{\widehat{\Theta}}{3} h_{c[a}(\widehat{\sigma}_{b]d} + \widehat{\omega}_{b]d}) V^d - \frac{1}{3}\mathcal{E} h_{c[a} V_{b]} \\
&- (\sigma_{c[a} - \omega_{c[a})(\omega_{b]d} + \sigma_{b]d}) V^d + (\widehat{\sigma}_{c[a} - \widehat{\omega}_{c[a})(\widehat{\omega}_{b]d} + \widehat{\sigma}_{b]d}) V^d \\
&+ \frac{\widetilde{\Lambda}}{6} h_{c[a} V_{b]} + \frac{\widetilde{\kappa}^2}{6}\left(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p}\right) h_{c[a} V_{b]} + \frac{\widetilde{\kappa}^2}{3} h_{c[a}\widetilde{\pi}_{b]d} V^d + \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{c[a} V_{b]}.
\end{aligned}
\tag{A.8}
$$

For a symmetric trace-free 3-tensor field T_{ab} the following commutation relations hold:

$$
\begin{aligned}
h_{\langle c}{}^i h_{d\rangle}{}^j n^a \widetilde{\nabla}_a (\dot{T}_{\langle ij\rangle}) - h_{\langle c}{}^i h_{d\rangle}{}^j u^b \widetilde{\nabla}_b (T'_{\langle ij\rangle}) = &\, K T'_{\langle cd\rangle} + \widehat{K}\dot{T}_{\langle cd\rangle} + K^a D_a T_{cd} - \widehat{K}^a D_a T_{cd} \\
&-2K_{\langle c} T_{d\rangle a} A^a + 2\widehat{A}_{\langle c} T_{d\rangle a}\widehat{K}^a + 2A_{\langle c} T_{d\rangle a} K^a \\
&-2\widehat{K}_{\langle c} T_{d\rangle a}\widehat{A}^a - 2\varepsilon_{ab\langle d} T_{c\rangle}{}^a \mathcal{H}^b,
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
h_a{}^k h_{\langle b}{}^i h_{c\rangle}{}^j n^d \widetilde{\nabla}_d (D_k T_{ij}) - D_a(T'_{\langle bc\rangle}) = &-(L_a - K_a)\dot{T}_{\langle bc\rangle} - \widehat{A}_a T'_{\langle bc\rangle} - \frac{\widehat{\Theta}}{3} D_d T_{bc} + \mathcal{E}_{\langle b} T_{c\rangle a} \\
&-(\widehat{\omega}_{ad} + \widehat{\sigma}_{ad}) D^d T_{bc} - 2T_{\langle b}{}^d \varepsilon_{c\rangle di}\widehat{\mathcal{H}}_a{}^i + \frac{2\Theta}{3} h_{a\langle b} T_{c\rangle d} K^d \\
&- h_{a\langle b} T_{c\rangle}{}^d \mathcal{E}_d + \frac{2\widehat{\Theta}}{3}\widehat{A}_{\langle b} T_{c\rangle a} - \frac{2\widehat{\Theta}}{3} h_{a\langle b} T_{c\rangle d}\widehat{A}^d \\
&-2(\omega_{ad} + \sigma_{ad}) K_{\langle b} T_{c\rangle}{}^d + 2(\widehat{\omega}_{ad} + \widehat{\sigma}_{ad})\widehat{A}_{\langle b} T_{c\rangle}{}^d \\
&+2(\omega_{a\langle b} + \sigma_{a\langle b}) T_{c\rangle d} K^d - 2(\widehat{\omega}_{a\langle b} + \widehat{\sigma}_{a\langle b}) T_{c\rangle d}\widehat{A}^d \\
&- \frac{2\Theta}{3} K_{\langle b} T_{c\rangle a} - \frac{2\widetilde{\kappa}^2}{3} h_{a\langle b} T_{c\rangle}{}^d \widetilde{\pi}_d + \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle b} T_{c\rangle a},
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
h_a{}^k h_{\langle b}{}^i h_{c\rangle}{}^j u^d \widetilde{\nabla}_d (D_k T_{ij}) - D_a(\dot{T}_{\langle bc\rangle}) = &-(\widehat{K}_a + L_a) T'_{\langle bc\rangle} + A_a \dot{T}_{\langle bc\rangle} - \frac{\Theta}{3} D_d T_{bc} - \widehat{\mathcal{E}}_{\langle b} T_{c\rangle a} \\
&-(\omega_{ad} + \sigma_{ad}) D^d T_{bc} - 2T_{\langle b}{}^d \varepsilon_{c\rangle di}\mathcal{H}_a{}^i + h_{a\langle b} T_{c\rangle}{}^d \widehat{\mathcal{E}}_d \\
&- \frac{2\widehat{\Theta}}{3} h_{a\langle b} T_{c\rangle d}\widehat{K}^d + \frac{2\widehat{\Theta}}{3}\widehat{K}_{\langle b} T_{c\rangle a} - \frac{2\Theta}{3} A_{\langle b} T_{c\rangle a} \\
&+2(\widehat{\omega}_{ad} + \widehat{\sigma}_{ad})\widehat{K}_{\langle b} T_{c\rangle}{}^d - 2(\omega_{ad} + \sigma_{ad}) A_{\langle b} T_{c\rangle}{}^d \\
&-2(\widehat{\omega}_{a\langle b} + \widehat{\sigma}_{a\langle b}) T_{c\rangle d}\widehat{K}^d + 2(\omega_{a\langle b} + \sigma_{a\langle b}) T_{c\rangle d} A^d \\
&+ \frac{2\Theta}{3} h_{a\langle b} T_{c\rangle d} A^d + \frac{2\widetilde{\kappa}^2}{3} h_{a\langle b} T_{c\rangle}{}^d \widetilde{q}_d - \frac{2\widetilde{\kappa}^2}{3}\widetilde{q}_{\langle b} T_{c\rangle a}.
\end{aligned}
\tag{A.11}
$$

These results apply for arbitrary (not necessarily small) scalar, vector and tensor fields.

Appendix B. Infinitesimal frame transformations

An infinitesimal frame transformation from the diad (u^a, n^a) to the diad $(\overline{u}^a, \overline{n}^a)$ can be defined as a generalization of the corresponding general relativistic procedure [25] as:

$$\overline{u}_a = u_a + \upsilon_a + \nu n_a, \quad \text{with} \quad u^a \upsilon_a = n^a \upsilon_a = 0, \tag{B.1}$$

$$\overline{n}_a = n_a + l_a + m u_a, \quad \text{with} \quad u^a l_a = n^a l_a = 0, \tag{B.2}$$

where υ_a, l_a, ν, m are all $\mathcal{O}(1)$.[6] The new diad also obeys

$$\overline{u}^a \overline{u}_a = -1, \ \overline{n}^a \overline{n}_a = 1, \ \overline{u}^a \overline{n}_a = 0, \tag{B.3}$$

which implies

$$\nu = m. \tag{B.4}$$

For $\upsilon_a = l_a = 0$ the above parameters define an infinitesimal orthogonal transformation (this is a two-dimensional infinitesimal Lorentz boost). The parameters υ_a and l_a represent infinitesimal translations. These transformations represent gauge degrees of freedom, worth to explore in order to achieve particular tasks, for example to fulfil physical conditions, to conveniently close subsets of differential equations, etc.

The fundamental algebraic tensors h_{ab} and ε_{abc} change accordingly:

$$\overline{h}_{ab} = h_{ab} + 2u_{(a}\upsilon_{b)} - 2n_{(a}l_{b)}, \tag{B.5}$$

$$\overline{\varepsilon}_{abc} = \varepsilon_{abc} - \left(n_c \varepsilon_{abe} + 2n_{[a}\varepsilon_{b]ce}\right) l^e + \left(u_c \varepsilon_{abd} + 2u_{[a}\varepsilon_{b]cd}\right)\upsilon^d. \tag{B.6}$$

The new 3-metric obeys $\overline{h}_{ab}\overline{n}^a = \overline{h}_{ab}\overline{u}^a = 0$.

The covariant derivatives of the new diad vectors can be invariantly decomposed as

$$\widetilde{\nabla}_a \overline{u}_b = -\overline{u}_a \overline{A}_b + \overline{\widehat{K}}\,\overline{u}_a \overline{n}_b + \overline{K}\,\overline{n}_a \overline{n}_b + \overline{n}_a \overline{K}_b + \overline{L}_a \overline{n}_b + \frac{\overline{\Theta}}{3}\overline{h}_{ab} + \overline{\omega}_{ab} + \overline{\sigma}_{ab}, \tag{B.7}$$

$$\widetilde{\nabla}_a \overline{n}_b = \overline{n}_a \overline{\widehat{A}}_b + \overline{K}\,\overline{n}_a \overline{u}_b + \overline{\widehat{K}}\,\overline{u}_a \overline{u}_b - \overline{u}_a \overline{\widehat{K}}_b + \overline{L}_a \overline{u}_b + \frac{\overline{\widehat{\Theta}}}{3}\overline{h}_{ab} + \overline{\widehat{\omega}}_{ab} + \overline{\widehat{\sigma}}_{ab}, \tag{B.8}$$

which imply the following transformations for the kinematic quantities:

$$\overline{K} = K - \widehat{A}_a \upsilon^a + (K_a + L_a)\, l^a - m\widehat{K} + m', \tag{B.9}$$

$$\overline{\widehat{K}} = \widehat{K} - A_a l^a + (\widehat{K}_a - L_a)\upsilon^a - mK - \dot{m}, \tag{B.10}$$

$$\overline{\widehat{\Theta}} = \widehat{\Theta} + m\Theta + D^a l_a - l^a \widehat{A}_a - \upsilon^a (L_a - \widehat{K}_a), \tag{B.11}$$

$$\overline{\Theta} = \Theta + m\widehat{\Theta} + D^a \upsilon_a + \upsilon^a A_a - l^a (K_a + L_a), \tag{B.12}$$

$$\overline{A}_c = A_c + \upsilon^b A_b u_c + \widehat{K} l_c - l^b A_b n_c + \frac{\Theta}{3}\upsilon_c + (\omega_{ac} + \sigma_{ac})\,\upsilon^a + m(K_c + \widehat{K}_c) + \dot{\upsilon}_{\langle c\rangle}, \tag{B.13}$$

$$\overline{\widehat{A}}_c = \widehat{A}_c - l^b \widehat{A}_b n_c - K\upsilon_c + \upsilon^b \widehat{A}_b u_c + \frac{\widehat{\Theta}}{3} l_c + (\widehat{\omega}_{ac} + \widehat{\sigma}_{ac}) l^a + m(K_c + \widehat{K}_c) + l'_{\langle c\rangle}, \tag{B.14}$$

[6] The quantities denoted by $\mathcal{O}(1)$ all vanish for identical transformation. We also assume a first-order accuracy; thus, all quantities $\mathcal{O}(2) = \mathcal{O}(1)^2$ will be dropped.

$$\overline{K}_c = K_c - Kl_c + u_c\upsilon^b K_b - n_c l^b K_b + \frac{\Theta}{3}l_c + (\sigma_{ac} + \omega_{ac})\, l^a + m(A_c + \widehat{A}_c) + \upsilon'_{\langle c\rangle}, \tag{B.15}$$

$$\overline{\widehat{K}}_c = \widehat{K}_c + \widehat{K}\upsilon_c - n_c l^b \widehat{K}_b + u_c\upsilon^b\widehat{K}_b + \frac{\widehat{\Theta}}{3}\upsilon_c + (\widehat{\sigma}_{ac} + \widehat{\omega}_{ac})\upsilon^a + m(A_c + \widehat{A}_c) + \dot{l}_{\langle c\rangle}, \tag{B.16}$$

$$\overline{L}_c = L_c - \widehat{K}\upsilon_c - Kl_c + u_c\upsilon^a L_a - n_c l^a L_a - \frac{\widehat{\Theta}}{3}\upsilon_c + \frac{\Theta}{3}l_c + D_c m - (\widehat{\sigma}_{cb} + \widehat{\omega}_{cb})\upsilon^b + (\sigma_{ca} + \omega_{ca})l^a, \tag{B.17}$$

$$\overline{\widehat{\sigma}}_{cd} = \widehat{\sigma}_{cd} + m\sigma_{cd} + D_{\langle c}l_{d\rangle} + \upsilon_{\langle c}(\widehat{K}_{d\rangle} - L_{d\rangle}) - l_{\langle c}\widehat{A}_{d\rangle} + 2\upsilon^a\widehat{\sigma}_{a(d}u_{c)} - 2l^a\widehat{\sigma}_{a(d}n_{c)}, \tag{B.18}$$

$$\overline{\sigma}_{cd} = \sigma_{cd} + m\widehat{\sigma}_{cd} + D_{\langle c}\upsilon_{d\rangle} - l_{\langle c}(K_{d\rangle} + L_{d\rangle}) + \upsilon_{\langle c}A_{d\rangle} - 2l^a\sigma_{a(d}n_{c)} + 2\upsilon^a\sigma_{a(d}u_{c)}, \tag{B.19}$$

$$\overline{\widehat{\omega}}_{cd} = \widehat{\omega}_{cd} + m\omega_{cd} + D_{[c}l_{d]} + \upsilon_{[c}(\widehat{K}_{d]} - L_{d]}) - l_{[c}\widehat{A}_{d]} + 2\upsilon^a\widehat{\omega}_{a[d}u_{c]} - 2l^a\widehat{\omega}_{a[d}n_{c]}, \tag{B.20}$$

$$\overline{\omega}_{cd} = \omega_{cd} + m\widehat{\omega}_{cd} + D_{[c}\upsilon_{d]} - l_{[c}(K_{d]} + L_{d]}) + \upsilon_{[c}A_{d]} - 2l^a\omega_{a[d}n_{c]} + 2\upsilon^a\omega_{a[d}u_{c]}. \tag{B.21}$$

Similarly, the transformation laws for the gravito-electro-magnetic quantities are

$$\overline{\mathcal{E}} = \mathcal{E} + 2\mathcal{E}_a l^a + 2\widehat{\mathcal{E}}_a l^a, \tag{B.22}$$

$$\overline{\mathcal{E}}_k = \mathcal{E}_k - m\widehat{\mathcal{E}}_k - \frac{4}{3}\mathcal{E}l_k + \mathcal{E}_a\upsilon^a u_k - \mathcal{E}_a l^a n_k + \varepsilon_{kab}\mathcal{H}^a\upsilon^b + \mathcal{E}_{ka}l^a - \mathcal{F}_{ka}\upsilon^a, \tag{B.23}$$

$$\overline{\widehat{\mathcal{E}}}_k = \widehat{\mathcal{E}}_k - m\mathcal{E}_k + \frac{4}{3}\mathcal{E}\upsilon_k - \widehat{\mathcal{E}}_a l^a n_k + \widehat{\mathcal{E}}_a\upsilon^a u_k - \varepsilon_{kab}\mathcal{H}^a l^b + \widehat{\mathcal{E}}_{ka}\upsilon^a - \mathcal{F}_{ka}l^a, \tag{B.24}$$

$$\overline{\mathcal{H}}_k = \mathcal{H}_k - \frac{1}{2}\varepsilon_k{}^{ab}\widehat{\mathcal{E}}_a l_b - \frac{1}{2}\varepsilon_k{}^{ab}\mathcal{E}_a\upsilon_b - n_k\mathcal{H}_a l^a + u_k\mathcal{H}_a\upsilon^a - \varepsilon_k{}^{ab}\mathcal{E}_a\upsilon_b + \widehat{\mathcal{H}}_{ka}\upsilon^a - \mathcal{H}_{ka}l^a, \tag{B.25}$$

$$\overline{\mathcal{F}}_{kl} = \mathcal{F}_{kl} + 2u_{(k}\mathcal{F}_{l)a}\upsilon^a - 2n_{(k}\mathcal{F}_{l)a}l^a - \frac{3}{2}\mathcal{E}_{\langle k}\upsilon_{l\rangle} + \frac{3}{2}\widehat{\mathcal{E}}_{\langle k}l_{l\rangle} + m\widehat{\mathcal{E}}_{kl} + m\mathcal{E}_{kl} - \varepsilon_{ab(k}\widehat{\mathcal{H}}_{l)}{}^a\upsilon^b - \varepsilon_{ab(k}\mathcal{H}_{l)}{}^a l^b, \tag{B.26}$$

$$\overline{\widehat{\mathcal{E}}}_{kl} = \widehat{\mathcal{E}}_{kl} + 2m\mathcal{F}_{kl} + 2\upsilon_{\langle k}\widehat{\mathcal{E}}_{l\rangle} - \mathcal{E}_{\langle k}l_{l\rangle} + 2u_{(k}\widehat{\mathcal{E}}_{l)a}\upsilon^a - 2n_{(k}\widehat{\mathcal{E}}_{l)a}l^a - 2\varepsilon_{ab\langle k}\widehat{\mathcal{H}}_{l\rangle}{}^a l^b, \tag{B.27}$$

$$\overline{\mathcal{E}}_{kl} = \mathcal{E}_{kl} + 2m\mathcal{F}_{kl} - 2l_{\langle k}\mathcal{E}_{l\rangle} + \widehat{\mathcal{E}}_{\langle k}\upsilon_{l\rangle} + 2u_{\langle k}\mathcal{E}_{l\rangle a}\upsilon^a - 2n_{\langle k}\mathcal{E}_{l\rangle a}l^a - 2\varepsilon_{ab\langle k}\mathcal{H}_{l\rangle}{}^a\upsilon^b, \tag{B.28}$$

$$\overline{\mathcal{H}}_{kl} = \mathcal{H}_{kl} + m\widehat{\mathcal{H}}_{kl} + \frac{3}{2}\mathcal{H}_{\langle k}l_{l\rangle} - \varepsilon_{(k}{}^{ab}\mathcal{F}_{l)a}l_b + 2\varepsilon_{(k}{}^{ab}\mathcal{E}_{l)a}\upsilon_b - \varepsilon_{(k}{}^{ab}\widehat{\mathcal{E}}_{l)a}\upsilon_b - 2n_{(k}\mathcal{H}_{l)a}l^a + 2u_{(k}\mathcal{H}_{l)a}\upsilon^a, \tag{B.29}$$

$$\overline{\widehat{\mathcal{H}}}_{kl} = \widehat{\mathcal{H}}_{kl} + m\mathcal{H}_{kl} + \frac{3}{2}\mathcal{H}_{\langle k}\upsilon_{l\rangle} + \varepsilon_{(k}{}^{ab}\mathcal{F}_{l)a}\upsilon_b - 2\varepsilon_{(k}{}^{ab}\widehat{\mathcal{E}}_{l)a}l_b + \varepsilon_{(k}{}^{ab}\mathcal{E}_{l)a}l_b + 2u_{(k}\widehat{\mathcal{H}}_{l)a}\upsilon^a - 2n_{(k}\widehat{\mathcal{H}}_{l)a}l^a. \tag{B.30}$$

The matter variables transform as

$$\bar{\widetilde{\rho}} = \widetilde{\rho} - 2\upsilon^a\widetilde{q}_a - 2m\widetilde{q}, \tag{B.31}$$

$$\bar{\widetilde{\pi}} = \widetilde{\pi} + 2l^a\widetilde{\pi}_a - 2m\widetilde{q}, \tag{B.32}$$

$$\bar{\widetilde{p}} = \widetilde{p} - \frac{2}{3}\upsilon^a\widetilde{q}_a - \frac{2}{3}l^a\widetilde{\pi}_a, \tag{B.33}$$

$$\bar{\widetilde{q}} = \widetilde{q} + l^a\widetilde{q}_a - \upsilon^a\widetilde{\pi}_a - m(\widetilde{\rho} + \widetilde{\pi}), \tag{B.34}$$

$$\bar{\widetilde{q}}_a = \widetilde{q}_a - \widetilde{\rho}\upsilon_a - \widetilde{q}l_a - 2\widetilde{p}\upsilon_a - 2\widetilde{\pi}_{ab}\upsilon^b + u_a\upsilon^c\widetilde{q}_c - n_al^c\widetilde{q}_c - m\widetilde{\pi}_a, \tag{B.35}$$

$$\bar{\widetilde{\pi}}_a = \widetilde{\pi}_a - \widetilde{q}\upsilon_a + 2\widetilde{p}l_a - \widetilde{\pi}l_a + 2\widetilde{\pi}_{ab}l^b - n_al^c\widetilde{\pi}_c + u_a\upsilon^c\widetilde{\pi}_c - m\widetilde{q}_a, \tag{B.36}$$

$$\bar{\widetilde{\pi}}_{ab} = \pi_{ab} + 2\widetilde{p}\upsilon_{\langle a}u_{b\rangle} - 2\widetilde{p}l_{\langle a}n_{b\rangle} - 2\widetilde{q}_{\langle a}\upsilon_{b\rangle} + 2\pi_{\langle a}{}^d u_{b\rangle}\upsilon_d - 2\pi_{\langle a}{}^c n_{b\rangle}l_c - 2\widetilde{\pi}_{\langle a}l_{b\rangle}. \tag{B.37}$$

We have checked that in the particular case $l^a = 0 = m$, by applying the Lanczos equations (76)–(79) for eliminating the quantities $\widehat{K}$, $\widehat{\Theta}$, $L_a = -\widehat{K}_a$ and $\widehat{\sigma}_{ab}$, by suppressing the quantities related to the brane normal, in particular imposing $\widehat{\omega}_{ab} = 0$, we recover the linearized form of the transformation laws for the kinematical and dynamical quantities $(\Theta, \sigma_{ab}, \omega_{ab}, A_a)$ and (ρ, p, q_a, π_{ab}) of [25]. Similarly, by employing equations (14) and (15), we obtain the required transformations for the Weyl projections (E_{ab}, H_{ab}).

If we would like to apply the generic transformations derived in this appendix in a brane-world scenario, we have to impose $l^a = 0 = m$ on the brane (in order to preserve the vector field n^a at $y = 0$, which defines the brane); however, the derivatives along the off-brane direction (the derivatives denoted by prime) of these quantities can be different from zero even at $y = 0$.

Appendix C. Gravitational evolution and constraint equations on an asymmetrically embedded brane

The brane equations describing the gravitational dynamics are

$$0 = \dot{\widehat{\Theta}} - D^a\widehat{K}_a + \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\Theta - 2\widehat{K}^aA_a + \widehat{\sigma}_{ab}\sigma^{ab} - \widetilde{\kappa}^2\widetilde{q}, \tag{C.1}$$

$$\begin{aligned}0 = {} & \dot{\widehat{K}}_{\langle a\rangle} - D_a\left(\widehat{K} - \frac{2}{3}\widehat{\Theta}\right) - D^b\widehat{\sigma}_{ab} + \frac{4\Theta}{3}\widehat{K}_a - (\widehat{K} + \frac{\widehat{\Theta}}{3})A_a - \widehat{\sigma}_{ab}A^b - \omega_{ab}\widehat{K}^b \\ & + \sigma_{ab}\widehat{K}^b + \widetilde{\kappa}^2\widetilde{\pi}_a,\end{aligned} \tag{C.2}$$

$$\begin{aligned}0 = {} & \dot{\mathcal{E}} - D^a\widehat{\mathcal{E}}_a + \frac{4}{3}\Theta\mathcal{E} + \widehat{\mathcal{E}}_{ab}\sigma^{ab} - 2\widehat{\mathcal{E}}_aA^a + \widehat{\sigma}^{ab}\dot{\widehat{\sigma}}_{\langle ab\rangle} - \widehat{\sigma}^{ab}D_a\widehat{K}_b - \widehat{K}^aD^b\widehat{\sigma}_{ab} + \frac{2}{3}\widehat{K}^aD_a\widehat{\Theta} \\ & - 2A_b\widehat{K}_a\widehat{\sigma}^{ab} + \frac{\Theta}{3}\widehat{\sigma}_{ab}\widehat{\sigma}^{ab} + \left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\sigma_{ab}\widehat{\sigma}^{ab} + \sigma_{ca}\widehat{\sigma}_b{}^c\widehat{\sigma}^{ab} + \frac{2\Theta}{3}\widehat{K}_a\widehat{K}^a - \sigma_{ab}\widehat{K}^a\widehat{K}^b \\ & - \frac{\widetilde{\kappa}^2}{2}(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p})^{\cdot} - \frac{2\widetilde{\kappa}^2}{3}D^a\widetilde{q}_a - \frac{2\widetilde{\kappa}^2}{3}\Theta(\widetilde{\rho} + \widetilde{p}) - \frac{2\widetilde{\kappa}^2}{3}\widehat{\Theta}\widetilde{q} - \frac{4\widetilde{\kappa}^2}{3}\widetilde{q}_aA^a - \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab}\sigma^{ab},\end{aligned} \tag{C.3}$$

$$
\begin{aligned}
0 ={}& \dot{\widehat{\mathcal{E}}}_{\langle k\rangle} + \frac{4}{3}\Theta\widehat{\mathcal{E}}_k - \frac{1}{3}D_k\mathcal{E} - \frac{4\mathcal{E}}{3}A_k - D^a\widehat{\mathcal{E}}_{ka} - \widehat{\mathcal{E}}_{ka}A^a - (\omega_{ka}-\sigma_{ka})\widehat{\mathcal{E}}^a + \dot{\widehat{\sigma}}_{\langle ka\rangle}\widehat{K}^a \\
&- \left(\widehat{K}+\frac{\widehat{\Theta}}{3}\right) D^b\widehat{\sigma}_{kb} + \widehat{K}^a \sigma_{ck}\widehat{\sigma}_a{}^c + \frac{2}{3}\left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)D_k\widehat{\Theta} + \frac{2\Theta}{3}\left(\widehat{K}+\frac{\widehat{\Theta}}{3}\right)\widehat{K}_k \\
&- 2\widehat{K}^a D_k \widehat{K}_a + \widehat{K}^a D_a \widehat{K}_k + \frac{1}{3} \widehat{K}_k D^a \widehat{K}_a - 2\widehat{K}^a A_{\langle k}\widehat{K}_{a\rangle} - \sigma_{ba}\widehat{\sigma}_k{}^b \widehat{K}^a + \frac{2}{3} \widehat{K}_k \sigma_{ab}\widehat{\sigma}^{ab} \\
&+ \widehat{\sigma}^{ab} D_k\widehat{\sigma}_{ab} - \widehat{\sigma}_b{}^a D^b \widehat{\sigma}_{ka} - \frac{1}{3} \widehat{\sigma}_k{}^a D_a\widehat{\Theta} + \varepsilon_{cab}\widehat{K}^a \omega^b \widehat{\sigma}_k{}^c + \varepsilon_k{}^{ab}\widehat{K}_c \omega_a \widehat{\sigma}_b{}^c - \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}'_{\langle k\rangle} \\
&+ \frac{\widetilde{\kappa}^2}{6} D_k(\widetilde{\rho} + 3\widetilde{\pi} - 3\widetilde{p}) + \frac{2\widetilde{\kappa}^2}{3}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\widetilde{\pi}_k - \frac{2\widetilde{\kappa}^2}{3}K\widetilde{q}_k - \frac{2\widetilde{\kappa}^2}{3}\widetilde{q}(2\widehat{K}_k + K_k) \\
&+ \frac{2\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ka}\widehat{A}^a - \frac{2\widetilde{\kappa}^2}{3}(\widetilde{\pi} - \widetilde{p})\widehat{A}_k - \frac{5\widetilde{\kappa}^2}{3}\widehat{\sigma}_{ka}\widetilde{\pi}^a,
\end{aligned} \tag{C.4}
$$

$$
0 = \dot{\Theta} - D^aA_a + \frac{\Theta^2}{3} + \widehat{\Theta}\widehat{K} - A^aA_a - 2\omega_a\omega^a + \sigma_{ab}\sigma^{ab} - \widehat{K}^a\widehat{K}_a - \mathcal{E} - \frac{1}{2}\widetilde{\Lambda} + \frac{\widetilde{\kappa}^2}{2}(\widetilde{\rho}+\widetilde{\pi}+\widetilde{p}), \tag{C.5}
$$

$$
0 = \dot{\omega}_{\langle a\rangle} - \frac{1}{2} \varepsilon_a{}^{cd}D_cA_d + \frac{2\Theta}{3}\omega_a - \sigma_{ab}\omega^b, \tag{C.6}
$$

$$
\begin{aligned}
0 ={}& \dot{\sigma}_{\langle ab\rangle} - D_{\langle a}A_{b\rangle} + \frac{2\Theta}{3}\sigma_{ab} + \frac{1}{2}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\widehat{\sigma}_{ab} - A_{\langle a}A_{b\rangle} - \frac{1}{2}\widehat{K}_{\langle a}\widehat{K}_{b\rangle} \\
&+ \omega_{\langle a}\omega_{b\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}{}^c + \frac{1}{2}\widehat{\sigma}_{c\langle a}\widehat{\sigma}_{b\rangle}{}^c + E_{ab} + \frac{1}{2}\widehat{\mathcal{E}}_{ab} - \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{ab},
\end{aligned} \tag{C.7}
$$

$$
0 = D^a\omega_a - A_a\omega^a, \tag{C.8}
$$

$$
0 = D_{\langle c}\omega_{k\rangle} + \varepsilon_{ab\langle k}D^b\sigma_{c\rangle}{}^a + 2A_{\langle c}\omega_{k\rangle} + H_{ab}, \tag{C.9}
$$

$$
0 = D^b\sigma_{ab} - \frac{2}{3} D_a\Theta + \varepsilon_a{}^{ck}D_c\omega_k - \frac{2\widehat{\Theta}}{3}\widehat{K}_a + 2\varepsilon_a{}^{ck}A_c\omega_k + \widehat{\sigma}_{ab}\widehat{K}^b + \widehat{\mathcal{E}}_a + \frac{2\widetilde{\kappa}^2}{3}\widetilde{q}_a, \tag{C.10}
$$

$$
\begin{aligned}
0 ={}& \dot{E}_{\langle kj\rangle} - \frac{1}{2}\dot{\widehat{\mathcal{E}}}_{\langle kj\rangle} - \varepsilon_{ab\langle k}D^aH_{j\rangle}{}^b + \frac{1}{2} D_{\langle k}\widehat{\mathcal{E}}_{j\rangle} + \Theta E_{kj} - \frac{\Theta}{6}\widehat{\mathcal{E}}_{kj} + \widehat{\mathcal{E}}_{\langle k}A_{j\rangle} - \frac{2\mathcal{E}}{3}\sigma_{kj} \\
&- \frac{1}{2}\widehat{\mathcal{E}}_{\langle j}{}^a(\omega_{k\rangle a}+\sigma_{k\rangle a}) + E_{\langle k}{}^a(\omega_{j\rangle a} - 3\sigma_{j\rangle a}) + 2\varepsilon_{\langle k}{}^{ab}H_{j\rangle a}A_b - \frac{1}{2}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\dot{\widehat{\sigma}}_{\langle kj\rangle} \\
&- \widehat{\sigma}_{\langle j}{}^a\dot{\widehat{\sigma}}_{k\rangle a} - \frac{1}{2} \left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)^{\cdot}\widehat{\sigma}_{kj} + \widehat{K}_{\langle k}D_{j\rangle}(\widehat{K} - \widehat{\Theta}) - \frac{\widehat{\Theta}}{3}D_{\langle k}\widehat{K}_{j\rangle} + \frac{1}{2}\widehat{\sigma}_{\langle j}{}^aD_{k\rangle}\widehat{K}_a \\
&+ \widehat{K}_{\langle k}D^b\widehat{\sigma}_{j\rangle b} + \frac{1}{2}\widehat{K}_a D_{\langle k}\widehat{\sigma}_{j\rangle}{}^a + \frac{\widehat{\Theta}}{3}\left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\sigma_{kj} - \frac{7\Theta}{6}\widehat{K}_{\langle k}\widehat{K}_{j\rangle} + \left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\widehat{K}_{\langle k}A_{j\rangle} \\
&- \frac{\Theta}{6}\widehat{\sigma}_{\langle j}{}^a\widehat{\sigma}_{k\rangle a} - \frac{1}{2}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\widehat{\sigma}_{\langle k}{}^a(\omega_{j\rangle a}+\sigma_{j\rangle a}) - \frac{\Theta}{6}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\widehat{\sigma}_{kj} - \frac{1}{2}\widehat{K}_{\langle k}\sigma_{j\rangle b}\widehat{K}^b \\
&- \frac{1}{2}\widehat{\sigma}_{\langle j}{}^a\omega_{k\rangle c}\widehat{\sigma}_a{}^c - \frac{1}{2}\sigma_{jk}\widehat{K}^a\widehat{K}_a + \frac{3}{2}\widehat{K}_{\langle k}\omega_{j\rangle a}\widehat{K}^a + \widehat{K}_{\langle k}\widehat{\sigma}_{j\rangle a}A^a + \widehat{\sigma}_{a\langle j}A_{k\rangle}\widehat{K}^a \\
&- \frac{1}{2}\widehat{\sigma}_c{}^a\widehat{\sigma}_{\langle k}{}^c\sigma_{j\rangle a} - \frac{\widetilde{\kappa}^2}{3}\dot{\widetilde{\pi}}_{\langle kj\rangle} - \frac{\widetilde{\kappa}^2}{3}D_{\langle k}\widetilde{q}_{j\rangle} + \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle k}\widehat{K}_{j\rangle} - \frac{4\widetilde{\kappa}^2}{3}\widetilde{q}_{\langle k}A_{j\rangle} \\
&- \frac{\widetilde{\kappa}^2}{3}(\widetilde{\rho}+\widetilde{p})\sigma_{jk} - \frac{\widetilde{\kappa}^2}{9}\Theta\widetilde{\pi}_{jk} - \frac{\widetilde{\kappa}^2}{3}\widetilde{\pi}_{\langle j}{}^a(\omega_{k\rangle a}+\sigma_{k\rangle a}),
\end{aligned} \tag{C.11}
$$

$$
\begin{aligned}
0 = {} & \dot{H}_{\langle kj\rangle} + \varepsilon_{ab\langle k}D^{a}E_{j\rangle}{}^{b} + \frac{1}{2}\varepsilon_{ab\langle k}D^{a}\widehat{\mathcal{E}}_{j\rangle}{}^{b} + \Theta H_{kj} - 3\sigma_{a\langle k}H_{j\rangle}{}^{a} - \omega_{a\langle k}H_{j\rangle}{}^{a} \\
& - 2\varepsilon_{\langle k}{}^{ab}E_{j\rangle a}A_{b} - \frac{1}{2}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\widehat{\mathcal{E}}_{b} - \frac{3}{2}\widehat{\mathcal{E}}_{\langle j}\omega_{k\rangle} - \frac{1}{2}\varepsilon_{\langle k}{}^{cd}\widehat{\sigma}_{j\rangle c}D_{d}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right) \\
& + \frac{1}{2}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\varepsilon_{ab\langle k}D^{a}\widehat{\sigma}_{j\rangle}{}^{b} + \frac{1}{2}\varepsilon_{ab\langle k}D^{a}\widehat{\sigma}_{j\rangle c}\widehat{\sigma}^{cb} + \frac{1}{2}\varepsilon_{ab\langle k}\widehat{\sigma}_{j\rangle c}D^{a}\widehat{\sigma}^{cb} + \widehat{\Theta}\omega_{\langle k}\widehat{K}_{j\rangle} \\
& + \frac{\widehat{\Theta}}{3}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\widehat{K}_{b} - \frac{1}{2}\varepsilon_{ab\langle k}\widehat{K}_{j\rangle}D^{a}\widehat{K}^{b} - \frac{1}{2}\varepsilon_{ab\langle k}\widehat{K}^{b}D^{a}\widehat{K}_{j\rangle} + \frac{1}{2}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle b}\widehat{\sigma}_{a}{}^{c}\widehat{K}_{c} \\
& - \frac{3}{2}\widehat{\sigma}_{\langle j}{}^{a}\omega_{k\rangle}\widehat{K}_{a} - \frac{\widetilde{\kappa}^{2}}{3}\varepsilon_{ab\langle k}D^{a}\widetilde{\pi}^{b}_{j\rangle} - \widetilde{\kappa}^{2}\widetilde{q}_{\langle j}\omega_{k\rangle} - \frac{\widetilde{\kappa}^{2}}{3}\varepsilon_{\langle k}{}^{ab}\sigma_{j\rangle a}\widetilde{q}_{b},
\end{aligned} \tag{C.12}
$$

$$
\begin{aligned}
0 = {} & D^{a}E_{ak} - \frac{1}{2}D^{a}\widehat{\mathcal{E}}_{ak} + \frac{1}{3}D_{k}\mathcal{E} - 3H_{ka}\omega^{a} + \varepsilon_{k}{}^{ab}H_{ac}\sigma^{c}_{b} + \frac{\Theta}{3}\widehat{\mathcal{E}}_{k} - \frac{1}{2}(3\omega_{ka} + \sigma_{ka})\widehat{\mathcal{E}}^{a} \\
& - \frac{2\widehat{\Theta}}{9}D_{k}\widehat{\Theta} - \frac{1}{2}\widehat{\sigma}_{ak}D^{a}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right) - \frac{1}{2}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)D^{a}\widehat{\sigma}_{ak} - \frac{1}{2}\widehat{\sigma}_{a}{}^{c}D^{a}\widehat{\sigma}_{ck} + \frac{2}{3}\widehat{\sigma}^{ab}D_{k}\widehat{\sigma}_{ab} \\
& - \frac{1}{2}\widehat{\sigma}_{kb}D^{a}\widehat{\sigma}_{a}{}^{b} - \frac{1}{3}\widehat{K}^{a}D_{k}\widehat{K}_{a} + \frac{1}{2}\widehat{K}_{a}D^{a}\widehat{K}_{k} + \frac{1}{2}\widehat{K}_{k}D^{a}\widehat{K}_{a} + \frac{3}{2}\varepsilon_{k}{}^{ad}\widehat{K}_{c}\omega_{a}\widehat{\sigma}_{d}{}^{c} \\
& - \frac{1}{2}\widehat{\sigma}_{a}{}^{b}\widehat{K}^{a}\sigma_{kb} - \frac{2}{9}\widehat{\Theta}\Theta\widehat{K}_{k} + \frac{\widehat{\Theta}}{3}\sigma_{kb}\widehat{K}^{b} + \widehat{\Theta}\varepsilon_{k}{}^{cd}\widehat{K}_{c}\omega_{d} + \frac{\Theta}{3}\widehat{\sigma}_{k}{}^{a}\widehat{K}_{a} + \frac{\widetilde{\kappa}^{2}}{3}D^{a}\widetilde{\pi}_{ak} \\
& - \frac{\widetilde{\kappa}^{2}}{6}D_{k}(\widetilde{\rho} - \widetilde{\pi} + \widetilde{p}) + \frac{2\widetilde{\kappa}^{2}}{9}\Theta\widetilde{q}_{k} - \frac{\widetilde{\kappa}^{2}}{3}\widetilde{q}^{a}(3\omega_{ka} + \sigma_{ka}),
\end{aligned} \tag{C.13}
$$

$$
\begin{aligned}
0 = {} & D^{a}H_{ak} + \frac{1}{2}\varepsilon_{k}{}^{ab}D_{a}\widehat{\mathcal{E}}_{b} - \frac{4\mathcal{E}}{3}\omega_{k} + 3E_{ka}\omega^{a} - \varepsilon_{k}{}^{ab}E_{ac}\sigma_{b}{}^{c} + \frac{1}{2}\widehat{\mathcal{E}}_{ak}\omega^{a} + \frac{1}{2}\varepsilon_{k}{}^{ab}\widehat{\mathcal{E}}_{ac}\sigma_{b}{}^{c} \\
& - \frac{1}{2}\widehat{K}^{c}\omega_{c}\widehat{K}_{k} - \frac{1}{3}\varepsilon_{k}{}^{ab}\widehat{K}_{b}D_{a}\widehat{\Theta} - \frac{\widehat{\Theta}}{3}\varepsilon_{k}{}^{ab}D_{a}\widehat{K}_{b} - \frac{1}{2}\varepsilon_{abk}\widehat{K}^{c}D^{b}\widehat{\sigma}_{c}{}^{a} - \frac{1}{2}\varepsilon_{k}{}^{ab}\widehat{\sigma}_{a}{}^{c}D_{b}\widehat{K}_{c} \\
& + \frac{2\widehat{\Theta}}{3}\left(\widehat{K} + \frac{\widehat{\Theta}}{3}\right)\omega_{k} - \frac{1}{2}\widehat{K}_{a}\widehat{K}^{a}\omega_{k} + \frac{1}{2}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\widehat{\sigma}_{k}{}^{c}\omega_{c} + \frac{1}{2}\varepsilon_{k}{}^{ac}\sigma_{ab}\widehat{K}^{b}\widehat{K}_{c} \\
& - \frac{1}{2}\widehat{\sigma}_{ab}\widehat{\sigma}^{ab}\omega_{k} + \frac{1}{2}\widehat{\sigma}_{ca}\widehat{\sigma}_{k}{}^{c}\omega^{a} + \frac{1}{2}\left(\widehat{K} - \frac{\widehat{\Theta}}{3}\right)\varepsilon_{k}{}^{ab}\widehat{\sigma}_{ac}\sigma_{b}{}^{c} + \frac{1}{2}\varepsilon_{k}{}^{ab}\widehat{\sigma}_{da}\widehat{\sigma}_{c}{}^{d}\sigma_{b}{}^{c} \\
& - \frac{\widetilde{\kappa}^{2}}{3}\widetilde{\pi}_{ka}\omega^{a} + \frac{\widetilde{\kappa}^{2}}{3}\varepsilon_{k}{}^{ab}D_{a}\widetilde{q}_{b} + \frac{2\widetilde{\kappa}^{2}}{3}(\widetilde{\rho} + \widetilde{p})\omega_{k} - \frac{\widetilde{\kappa}^{2}}{3}\varepsilon_{k}{}^{ab}\widetilde{\pi}_{a}{}^{c}\sigma_{bc}.
\end{aligned} \tag{C.14}
$$

We note that this system of equations is valid on both sides of the brane.

Appendix D. Kinematical, gravito-electro-magnetic and matter variables for Bianchi I brane-worlds

In this appendix we rewrite the Bianchi I brane-world, presented in [49], in terms of our variables. The brane-world solution contains an unspecified function $V(y)$ of a Gauss normal coordinate y (which is however not related to the brane normal).

The kinematical quantities appearing on and outside the brane are presented in tables D1 and D2, respectively, while the gravito-electro-magnetic quantities on and outside the brane are given in tables D3 and D4. Tables D2 and D4 contain the quantities which were not computed in [49].

Table D1. Brane kinematical quantities (first column) for the brane-world [49] (second column; notations and equation numbers are from this reference)

Kinematic quantity	for the metric in [49]
A_a	0
Θ	Θ, as given by equation (63)
σ_{ab}	σ_{AB}, as given by equations (64) and (65)
ω_a	0

Table D2. Off-brane kinematical quantities (first column) for the brane-world [49] (second column; notations are from this reference)

Kinematic quantity	for the metric in [49]
$\widehat{K}$	$\frac{\varepsilon}{\sqrt{1-V^2}}\left(\frac{u'}{4u}+\frac{C_0}{u}+\frac{VV'}{1-V^2}\right)$
K	$-\frac{1}{\sqrt{1-V^2}}\left(\frac{Vu'}{4u}+\frac{VC_0}{u}+\frac{V'}{1-V^2}\right)$
$\widehat{\Theta}$	$\frac{\varepsilon}{\sqrt{1-V^2}}\left(\frac{3u'}{4u}-\frac{C_0}{u}\right)$
$\widehat{A}_a$	0
$\widehat{K}_a$	0
K_a	0
L_a	0
$\widehat{\omega}_a$	0
$\widehat{\sigma}_{ab}$	$\sum_{i=1}^{3}\widehat{\sigma}_i e_{ia}e_{ib}\ ,\quad \widehat{\sigma}_i=\frac{\varepsilon}{3u\sqrt{1-V^2}}(C_0+3C_i)$

Table D3. Brane gravito-electro-magnetic quantities (first column) for the brane-world [49] (second column; notations and equation numbers are from this reference)

Weyl quantity	for the metric in [49]
$\mathcal{E}$	$-\kappa^2\mathcal{U}$ given by equation (53)
$\widehat{\mathcal{E}}_a$	0
$\widehat{\mathcal{E}}_{ab}$	$-\kappa^2\mathcal{P}_{AB}$ given by equations (54) and (55)

Table D4. Off-brane gravito-electro-magnetic quantities (first column) for the brane-world [49] (second column; notations are from this reference)

Weyl quantity	for the metric in [49]
$\mathcal{E}_a$	0
$\mathcal{H}_a$	0
$\mathcal{E}_{ab}$	$\sum_{i=1}^{3}E_i e_{ia}e_{ib}\ ,\quad E_i=-\frac{1}{6u^2}\left[(C_0+3C_i)\,u'-\frac{3}{2}C-2\left(C_0^2+3C_i^2\right)\right]+\frac{1}{4u^2(1-V^2)}[(C_0+3C_i)\,u'-C+4C_i(C_0-C_i)]$
$\mathcal{F}_{ab}$	$\sum_{i=1}^{3}F_i e_{ia}e_{ib}\ ,\quad F_i=\frac{\varepsilon V}{(1-V^2)u^2}\left[\frac{C_0+3C_i}{4}u'+C_i\,(C_0-C_i)-\frac{C}{4}\right]$
$\mathcal{H}_{ab}$	0
$\widehat{\mathcal{H}}_{ab}$	0

The notations for the brane matter variables (after the straightforward change in the indices from lowercase to uppercase letters) are identical in this paper and in [49], with $q_a = 0$. There are no off-brane matter variables.

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