Gravitational waveforms for black hole binaries with unequal masses

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Abstract. We derived a post-Newtonian (PN) inspiral only gravitational waveform for unequal mass, spinning black hole binaries. Towards the end of the inspiral the larger spin dominates over the orbital angular momentum (while the smaller spin is negligible), hence the name Spin-Dominated Waveforms (SDW). Such systems are common sources for future gravitational wave detectors and during the inspiral the largest amplitude waves are emitted exactly in its last part. The SDW waveforms emerge as a double expansion in the PN parameter and the ratio of the orbital angular momentum to the dominant spin.

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1. Introduction

Gravitational wave detectors like the Advanced LIGO (aLIGO), or the planned Einstein Telescope (ET), LAGRANGE and eLISA (NGO) space missions will measure gravitational waves from black hole binaries of various total masses m. For astrophysical black hole binaries (with total mass m a few ten times of the mass of the sun $M_\odot$), the comparable mass and the unequal mass case are both likely. For supermassive black hole binaries (total mass is between 10^6 $M_\odot \div 10^{10}$ $M_\odot$) the typical mass ratio ν is between 0.3 and 0.03 [4], [5].

For unequal masses the mass ratio can stand as a second small parameter. The purpose of this paper is to give an approximation for the gravitational waveforms in the small mass ratio regime.

2. Spin-dominated waveforms

It was shown in Ref. [4], that for rapidly spinning black hole binaries, the smaller spin is of order ν^2 compared to the dominant spin S_1, thus it can be neglected to first order in ν. Furthermore the ratio of the orbital angular momentum L_N and S_1 was also given [4]

$$\frac{S_1}{L_N} \approx \varepsilon^{1/2} \nu^{-1} \chi_1 \ , \tag{1}$$

where $\varepsilon = Gm/c^2 r \approx v^2/c^2$ (with r the orbital separation and v the orbital velocity of the reduced mass particle $\mu = m_1 m_2/m$, G the gravitational constant, c the speed of light) is the post-Newtonian (PN) parameter and $\chi_1 \in [0, 1]$ is the dimensionless spin. For maximally spinning black holes $\chi_1 = 1$.

As the PN parameter increases throughout the inspiral, the relation (1) shows, that S_1 will dominate over L_N at the end of the inspiral (thus the approximated waveforms are called Spin-Dominated Waveforms, SDW). This condition at the technical level is included in the smallness of the parameter $\xi = \varepsilon^{-1/2} \nu$.

PN waveforms were previously calculated to 1.5 PN order [6], [1], and to 2 PN order in Ref. [3]. In order to approximate the waveforms in the small mass ratio regime, we expand the waveforms in both parameters ε and ξ. The waveforms have the following structure [9]:

$$
\begin{aligned}
h_{+\atop\times} \ = \ & \frac{2G^2 m^2 \varepsilon^{1/2} \xi}{c^4 D r} \left[h_{+\atop\times}^0 + \beta_1 h_{+\atop\times}^{0\beta} + \varepsilon^{1/2} \left(h_{+\atop\times}^{0.5} + \beta_1 h_{+\atop\times}^{0.5\beta} - 2\xi h_{+\atop\times}^0 \right) \right. \\
& + \varepsilon \left(h_{+\atop\times}^1 - 4\xi h_{+\atop\times}^{0.5} + \beta_1 h_{+\atop\times}^{1\beta} + h_{+\atop\times}^{1SO} + \beta_1 h_{+\atop\times}^{1\beta SO} \right) \\
& \left. + \varepsilon^{3/2} \left(h_{+\atop\times}^{1.5} + h_{+\atop\times}^{1.5SO} + h_{+\atop\times}^{1.5tail} \right) \right] \ ,
\end{aligned}
\tag{2a}
$$

D being the luminosity distance to the source. The terms are of different ε and ξ orders, as indicated in Table 1,and are given in detail in Ref. [9]. The angle β_1 span by $\mathbf{J}$ and $\mathbf{S}_1$ is of order ξ too [9].

Table 1. SDW contributions of different ξ and ε orders. The SO terms contain the dominant spin.

	ε^0	$\varepsilon^{1/2}$	ε^1	$\varepsilon^{3/2}$
ξ^0	$h_{+\atop\times}^0$	$h_{+\atop\times}^{0.5}$	$h_{+\atop\times}^1, h_{+\atop\times}^{1SO}$	$h_{+\atop\times}^{1.5}, h_{+\atop\times}^{1.5SO}, h_{+\atop\times}^{1.5tail}$
ξ^1	$h_{+\atop\times}^{0\beta}$	$h_{+\atop\times}^{0.5\beta}$	$h_{+\atop\times}^{1\beta}, h_{+\atop\times}^{1\beta SO}$	

3. Limits of validity

We impose the smallness condition $\xi \leq 0.1$. This defines a lower limit of the PN parameter $\varepsilon_1 = Gm/c^2 r_1 = 100\nu^2$, implying an upper limit for the mass ratio, $\nu_{\max} = 0.0316 \approx 1 : 32$. The upper limit for ε is defined by the end of the inspiral (chosen here as $\varepsilon_2 = 0.1$ [7]).

From the expression $m = c^3 \varepsilon^{3/2} (\pi G f)^{-1}$ including the gravitational wave frequency f, also the leading order radiative orbital angular frequency evolution [8] an integration leads to the time Δt during the binary evolves from ε_1 to ε_2

$$\Delta t = \frac{5Gm}{2^8 c^3} \frac{(1+\nu)^2}{\nu} \left(\varepsilon_1^{-4} - \varepsilon_2^{-4} \right) . \tag{3}$$

Δt is shown as function of m and ν on Fig 1. Even with the SDW approximation holding, the lower sensitivity bound ($f_{\min} = 10^{-4}$ for eLISA [2]) of the instrument may impose a larger value of the PN parameter, as the lower validity bound $\varepsilon_{f_{\min}}$. Hence Δt is calculated from $\max(\varepsilon_1, \varepsilon_{f_{\min}})$ to ε_2.

A lower limit for the mass ratio comes from the assumption that the second compact object has at least the mass of a neutron star ($1.4\ M_\odot$). The total mass is bounded from above by the lower frequency bound of
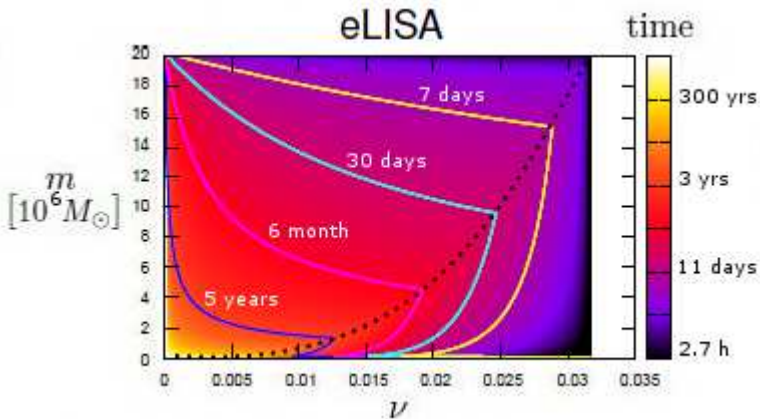

Figure 1. The time interval Δt until which the SDWs can be detected by eLISA (NGO) as function of the total mass m and mass ratio ν. Δt either begins at the lower bound of the sensitivity range of eLISA ($\varepsilon_{f\,\min}$), or when the SDW approximation begins to hold (ε_1), and ends at the end of the inspiral (chosen here as $\varepsilon_2 = 0.1$). The color code is logarithmic.

the detector (for eLISA $m = 2 \times 10^7 M_\odot$, hence the minimal mass ratio for the eLISA detector is $\nu_{\min} = 7 \times 10^{-8}$).

4. Concluding Remarks

For unequal mass ratios the larger spin dominates over the orbital angular momentum at the end of the inspiral. We have quantified this by the introduction of a second small parameter ξ and computed the respective waveforms as a series expansion in both this and the PN parameter. A comparison between the general waveforms of Ref. [1] and the SDWs showed that the SDWs are approximately 80% shorter, due to the smaller parameter space and the second expansion in ξ. We expect the SDWs to be useful tools in gravitational wave detection.

References

[1] Arun, K.G., Buonanno, A., Faye, G. and Ochsner, E., "Higher-order spin effects in the amplitude and phase of gravitational waveforms emitted by inspiraling compact binaries: Ready-to-use gravitational waveforms", *Phys.Rev.D*, **79**, 104023, (2009). [0810.5336].
[2] Bintruy, P., Boh, A., Caprini, C. and Dufaux, J., "Cosmological Backgrounds of Gravitational Waves and eLISA/NGO: Phase Transitions, Cosmic Strings and Other Sources", (2012). [1201.0983].
[3] Buonanno, A., Faye, G. and Hinderer, T., "Spin effects on gravitational waves from inspiraling compact binaries at second post-Newtonian order", (2012). [1209.6349].
[4] Gergely, L. Á. and Biermann, P. L., "The spin-flip phenomenon in supermassive black hole binary mergers", *Astrophys. J.*, **697**, 1621–1633, (2009). [0704.1968].
[5] Gergely, L. Á., Biermann, P. L. and Caramete, L. I., "Supermassive black hole spin-flip during the inspiral", *Class. Quantum Grav.*, **27**, 194009, (2010). [1005.2287].
[6] Kidder, L. E., "Coalescing binary systems of compact objects to $(\text{post})^{5/2}$-Newtonian order. V. Spin Effects", *Phys.Rev. D*, **52**, 821–847, (1995). [9506022].
[7] Levin, J., McWilliams, S. T. and Contreras, H., "Inspiral of Generic Black Hole Binaries: Spin, Precession, and Eccentricity", *Class. Quantum Grav.*, **28**, 175001, (2011). [1009.2533].
[8] Mikóczi, B., Vasúth, M. and Gergely, L. Á., "Self-interaction spin effects in inspiralling compact binaries", *Phys. Rev. D*, **71**, 124043, (2005). [0504538].
[9] Tápai, M., Keresztes Z. and Gergely, L. Á., "Spin-dominated waveforms for unequal mass compact binaries", (2012). [1209.1722].