On-line hypergraph coloring*

J. Nagy-György[†] Cs. Imreh [‡]

Abstract

In this paper we investigate the online hypergraph coloring problem. In this online problem the algorithm receives the vertices of the hypergraph in some order $v_1, \ldots, v_n$ and it must color v_i by only looking at the subhypergraph $H_i = (V_i, E_i)$ where $V_i = \{v_1, \ldots, v_i\}$ and E_i contains the edges of the hypergraph which are subsets of V_i. We show that there exists no online hypergraph coloring algorithm with sublinear competitive ratio. Furthermore we investigate some particular classes of hypergraphs (k-uniform hypergraphs, hypergraphs with max degree k, hypergraphs with bounded matching number, projective planes), we analyse the online algorithm FF and give matching lower bounds for these classes.

keywords: online algorithms, combinatorial problems

1 Introduction

A coloring of a hypergraph is an assignment of positive integers to the vertices of the hypergraph so that every edge contains vertices having different colors. A k-coloring of a hypergraph is a coloring of it where the number of used colors is at most k. In the online hypergraph coloring problem the algorithm receives the vertices of the hypergraph in some order $v_1, \ldots, v_n$ and it must

*This research has been supported by the Hungarian National Foundation for Scientific Research, Grant F048587.

†Department of Mathematics, University of Szeged, Aradi Vértanúk tere 1, H-6720 Szeged, Hungary, email: Nagy-Gyorgy@math.u-szeged.hu

‡Department of Informatics, University of Szeged, Árpád tér 2, H-6720 Szeged, Hungary, email: cimreh@inf.u-szeged.hu, Fax: +36 62 546397

color v_i by only looking at the subhypergraph $H_i = (V_i, E_i)$ where $V_i = \{v_1, \ldots, v_i\}$ and E_i contains the edges of the hypergraph which are subsets of V_i.

Online hypergraph coloring is the generalization of online graph coloring. Online graph coloring is investigated in several papers, one can find many details on that problem in the survey [8]. Some results are proved about the natural online graph coloring algorithm First Fit (we denote it by FF). FF uses the smallest color for each vertex which does not make a monochromatic edge. In [6] it is shown that this algorithm is the best possible for the trees. Some modified models are also investigated. In [7] the relaxed model where the graph is known in advance is presented. This means that the decision maker knows the graph up to isomorphism but during the procedure it does not know which part of the graph is given. In [4] an extended model is investigated where it is allowed to reject the vertex from the graph, and the objective is the sum of the number of colors used to color the accepted vertices and the total rejection cost. This paper also contains results about the classical online graph coloring problem the case of the (k+1)-Claw free graphs and graphs with bounded degree are considered, and it is proved that algorithm FF is the best possible for them.

Concerning the online hypergraph coloring problem we are not aware of any results about this area. Some results from online graph coloring which belong to particular subclasses can be extended easily: the algorithms and their competitive analysis for the trees, the graphs with bounded degree, k-claw free graphs, graphs without induced C_3 and C_5 (see [4] and [9] for the results on graphs). On the other hand the results for the general graphs cannot be extended. In this paper we show that in contrast to the online graph coloring there is no online algorithm with sublinear competitive ratio for the general online hypergraph coloring problem. In the case of online graph coloring such algorithms exist, they are presented in [9] and [10]. Furtermore in the case of graphs much better results exist for the bipartite graphs (see [3] and [10] for details). For hypergraphs the hypergraph which proves the lower bound is 2-colorable.

We also investigate some particular hypergraph classes we give the performance of FF and we present matching lower bounds. We show that in the case of 2-colorable k-uniform hypergraphs no online algorithm exists which can color every such hypergraph with less colors than $\lceil n/(k-1) \rceil$ and we show that algorithm FF colors these hypergraphs with this much colors. We also consider 2-colorable hypergraphs with max degree k, we show that no online algorithm can color every such hypergraph with less color than $k+1$, and a simple online algorithm which colors these hypergraphs with this much colors is given. Finally, we consider the hypergraphs with matching number k, we show that FF colors these hypergraphs with $2k+1$ colors. As a consequence we obtain that this algorithm colors the projective planes with 3 colors. We show that this bound is the best possible, we prove that there exists no online algorithm which can color a projective plane with less then 3 colors. (We note that the projective planes of order $q > 2$ are 2-colorable).

The paper is organized as follows. In the next section the basic notations and definitions which are used in the paper are introduced, furthermore we give a more detailed description of the online hypergraph coloring problem. Later in Section 3 the main results are presented. Finally in Section 4 some open questions are introduced.

2 Notations

In this paper on hypergraph we mean the structure $H = (V, E)$ where V is the finite set of the hypergraph's vertices and $E \subseteq \rho(V)$ is the set of the edges where $\rho(V)$ is the set of the nonempty subsets of V. We suppose that each edge has at least two elements.

A coloring of a hypergraph is an assignment of positive integers (called colors) to the vertices of the hypergraph so that every edge contains vertices having different colors. For a hypergraph H the minimum number of colors which is enough to color the hypergraph is called the chromatic number of the hypergraph and denoted by $\chi(H)$.

An online hypergraph (defined first in [1]) is a structure $H^{<} = (H, <)$ where H is a hypergraph and $<$ is a linear ordering of its vertices. We call a vertex the first, second,..., and ending vertex of an edge according to the ordering $<$. An online hypergraph coloring algorithm colors the i-th vertex of the hypergraph by only looking at the subhypergraph $H_i = (V_i, E_i)$ where V_i contains the first i vertices and E_i contains the edges of the hypergraph which are subsets of V_i. A straightforward idea is to use the greedy algorithm FF to color online hypergraphs. FF uses the smallest color for each vertex which does not make a monochromatic edge.

Usually in the theory of online computation the efficiency of the online algorithms are measured by the competitive ratio (see [2] and [5]) where the online algorithm is compared to the optimal offline algorithm. To define this ratio in this case we need some more notations. We extend the notations which are used in the area of online graph coloring. For an online algorithm A and an online hypergraph $H^<$, the number of colors used by A to color $H^<$ is denoted by $\chi_A(H^<)$. For a hypergraph H, $\chi_A(H)$ denotes the maximum of the $\chi_A(H^<)$ values over all ordering $<$. The competitive ratio of an algorithm A on a class Γ of hypergraphs is $\max_{H\in\Gamma}\chi_A(H)/\chi(H)$.

We use the following notions from the theory of hypergraphs. A hypergraph is called k-uniform if each edge contains k vertices. The degree of a vertex is the number of edges which contains it, the maximal degree of a hypergraph is the maximum of the degrees of the vertices. By the matching number $\nu(H)$ of a hypergraph H we mean the maximal number of pairwise disjoint edges of H. Let us note that by the definitions it follows immediately that the matching number of the projective planes is 1.

3 Main results

3.1 2-colorable k-uniform hypergraphs

In this part we consider the case of 2-colorable k-uniform hypergraphs with $k \geq 3$. (We note that the 2-uniform hypergraphs are the graphs.) We prove the following result.

Theorem 1 *Let $k \geq 3$. For every online hypergraph coloring algorithm A there exists a 2-colorable k-uniform hypergraph H on n vertices with $\chi_A(H) \geq \lceil n/(k-1) \rceil$. If H is a k-uniform hypergraph then $\chi_{FF}(H) \leq \lceil n/(k-1) \rceil$.*

Proof. Let A be an arbitrary online hypergraph coloring algorithm, define the online hypergraph $H_{n,A}$ on n vertices as follows. Any vertex v_i is the ending vertex of the following edges: for each color c which is used by A for at least $k-1$ vertices we have and edge $e_{c,i}$ which contains the vertices colored with c by A and v_i.

Note that if $k-1$ vertices are colored with a color c by A, then each of the following vertices is contained in an edge together with these c-colored vertices, therefore no more vertex can obtain the color c. This means that each color is used at most $k-1$ times. Consequently we obtained that $\chi_A(H_{n,A}) \geq \lceil n/(k-1) \rceil$.

Now we show that this hypergraph is k-uniform and 2-colorable. Let us observe that each edge in $H_{n,A}$ contains k vertices, where the first $k-1$ are colored with the same color by A. Consider the following 2-coloring of $H_{n,A}$. Every vertex which is colored with a new color by A gets the color 1, the other vertices get the color 2. By the above observation on the first $k-1$ vertices of the edges it follows that the first vertex gets the offline color 1, the second vertex gets the color 2 (since $k \geq 3$ it has the same online color) thus there is no monochromatic edge in this coloring. Therefore we showed that the hypergraph is 2-colorable.

Now consider the algorithm FF. The graph is k-uniform therefore if less than $k-1$ vertices are colored by FF with some color c then FF does not use a further color for the next vertex. Therefore the number of colors used by FF is not more than $\lceil n/(k-1) \rceil$. □

The theorem shows that the competitive ratio of FF is $\lceil n/(k-1) \rceil/2$ on this class and no better algorithm can be defined for this class. Therefore we obtained the following result.

Corollary 2 *FF is an optimal online algorithm for the class of 2-colorable k-uniform hypergraphs.*

Remark: We must note that in this case not only FF but many reasonable coloring algorithms can color the hypergraph with $\lceil n/(k-1) \rceil$ colors. Any algorithm which uses at most $k-1$ times each color gives a coloring of the hypergraph.

Moreover this theorem also proves (with $k = 3$) that contrary to the case of the online graph coloring in the case of hypergraphs no online algorithm with sublinear competitive ratio exists.

Corollary 3 *For every online hypergraph coloring algorithm A there exists a 2-colorable hypergraph H on n vertices with $\chi_A(H) \geq n/2$.*

3.2 Hypergraphs with maximal degree k

In this part we consider the case of 2-colorable hypergraphs with maximal degree k. It is easy to see that any hypergraph with maximal degree k is $k+1$ colorable, algorithm FF colors them with $k+1$ colors. The following result shows that it is not possible to give a better algorithm even for 2-colorable hypergraphs from this class.

Theorem 4 *For every online hypergraph coloring algorithm A there exists a 2-colorable hypergraph H with maximal degree k such that $\chi_A(H) \geq k+1$.*

Proof. For an arbitrary online algorithm we use the following recursive algorithm to construct the hypergraph H. H_0 is a hypergraph which contains one vertex and no edge. Then H_0 is colored with one color by any online algorithm, and the maximal degree of H_0 is 0. Suppose that for every online algorithm A we have a hypergraph $H_k(A)$ which has maximal degree at most k and A uses at least $k+1$ colors for it. Let ONL be an arbitrary online algorithm. Let ONL' be the online algorithm which colors the vertices in the same way as ONL colors after the coloring of a disjoint copy of $H_k(ONL)$. We build $H_{k+1}(ONL)$ as follows. First we give $H_k(ONL)$ to the algorithm and then a disjoint hypergraph $H_k(ONL')$. We distinguish the following two cases. If ONL does not use the same $k+1$ colors for $H_k(ONL)$ and $H_k(ONL')$ then the sequence is finished, the hypergraph which we obtained is $H_{k+1}(ONL)$. Algorithm ONL uses at least $k+2$ colors for it, and the maximal degree is $k \leq k+1$. If ONL uses the same colors for $H_k(ONL)$ and $H_k(ONL')$ then we obtain H_{k+1} as follows. We give an extra vertex v at the end of the algorithm which closes the following vertices. For each color i we define one edge which contains the first vertex with color i from $H_k(ONL)$, the first vertex with color i from $H_k(ONL')$, and vertex v. Then ONL has to use a new color for vertex v, thus it uses $k+2$ colors. Considering the degrees of the vertices, v has degree $k+1$, the degree of the other vertices is increased by at most 1, therefore the maximum degree of $H_{k+1}(ONL)$ is at most $k+1$.

To prove the theorem we have to show that $H_k(ONL)$ is 2-colorable for each k and for each online algorithm ONL. We prove by induction that for each k and for each algorithm ONL there exists such a 2-coloring of $H_k(ONL)$ where each vertex which obtains a new color from algorithm ONL has color 1. For H_0 the statement is trivial. Now suppose that the statement is true for k, we prove it for $k+1$. Let ONL an arbitrary online algorithm. If $H_{k+1}(ONL)$ consist of the two disjoint hypergraphs $H_k(ONL)$ and $H_k(ONL')$ then the statement is trivial we can use the offline 2-colorings of $H_k(ONL)$ and $H_k(ONL')$, and the statement follows. Now suppose that $H_{k+1}(ONL)$ is given by the two hypergraphs $H_k(ONL)$ and $H_k(ONL')$ and one extra vertex v. Then we can give the following coloring. Color $H_k(ONL)$ by induction, color $H_k(ONL')$ also by induction but changing the colors, finally give color 1 to vertex v. By the induction hypothesis it follows that the edges which are included in $H_k(ONL)$ or in $H_k(ONL')$ are not monochromatic. The edges which intersect both $H_k(ONL)$ and $H_k(ONL')$ contains the first occurrence of the online color i in $H_k(ONL)$ (this get the offline color 1), the first occurrence of the online color i in $H_k(ONL')$ (this get the offline color 2), thus these edges are not monochromatic. Therefore we gave a 2-coloring. Furthermore the extra property of the coloring also remains valid, the first occurrences of the online colors used for $H_k(ONL)$ get offline color 1 by the induction hypothesis and the first occurrence of the further color (vertex v) again obtains the offline color 1. □

3.3 Hypergraphs with bounded matching number

Considering this class of hypergraphs FF can achieve the following performance.

Theorem 5 *For any hypergraph H algorithm FF gives a coloring of H with at most $2 \cdot \nu(H) + 1$ colors.*

Proof. Consider an arbitrary hypergraph H and suppose that FF used k colors to color it. For every $i \leq k/2$ consider a vertex v_i which obtains the color $2i$. By the definition of FF it follows that there exists an edge E_i whose last vertex is v_i and the other vertices obtain the color $2i - 1$ by FF. Then considering the edges E_i, $i = 1, \ldots, \lfloor k/2 \rfloor$ we obtain pairwise disjoint edges, thus $\nu(H) \geq \lfloor k/2 \rfloor$, and this proves the theorem. □

Since any two edges of the projective planes are intersecting (the matching number is 1) we also obtained the following result.

Corollary 6 *FF colors the projective planes with at most 3 colors.*

It is easy to see that the projective planes are two colorable if $q > 2$. (Consider three points which are not collinear. Color these points with color 1, the points of the lines which connect these points obtain the color 2, the other points of the plane gets color 1.) On the other hand as the following statement shows there exists no online algorithm which can use less colors than FF in this cases.

Theorem 7 *No online algorithm exists which can color a projective plane with less than 3 colors.*

Proof. Consider an arbitrary online algorithm A and a projective plane with order q. Give the points of the projective plane to the algorithm in the following order. First q^2 points arrive such that none of the lines are completed (the remaining $q+1$ points will give one line of the plane furthermore they will finish all the other lines). Now distinguish the following two cases.

If A uses more than two colors then the statement is obvious. Suppose that A uses at most two colors 1 and 2. Without loss of generality we can suppose that the number of points colored by 1 is not less than the number of points colored by 2.

First suppose that at least q points is colored by 2. Then the next point which arrives will finish a line which contains q points of color 1, and a line which contains q points of color 2. Since A cannot construct a monochromatic line it has to use a new color for this point and the statement of the theorem follows.

Now suppose that there are at most $q-1$ points of color 2. Then each of the $q+1$ uncolored points has a line through it with q points of color 1. So now on we cannot use color 1. Since the uncolored points are on one line, it is impossible to use only color 2. This completes the proof. $\square$

4 Open questions

The first open question is to find further special classes of hypergraphs, where some online algorithm can perform well. One of the interesting classes is the class of hypertrees (cycle free, connected hypergraphs). In the case of online trees it is proved (see [6] [8]) that FF has the best possible competitive ratio: $\log n$ for graphs on n vertices. The generalization of that algorithm also works well on hypertrees it uses $\log_k n$ colors for k-uniform hypertrees on n vertices, but it is not clear that there is no better online algorithm. We conjecture so that this is the best possible algorithm.

Another interesting question is to define in a different way the online hypergraph coloring problem. We could give more information to the algorithm. In one modified model the algorithm receives all the partial edges which contain the presented vertex at the arrival of each vertex. In this case our counterexample does not work, it is possible that in this modified model there exists a sublinear competitive algorithm. Further interesting model is to investigate the case of online coloring known hypergraphs extending the model defined in [7]. In this case our lower bounds are not true (except the bound on projective planes).

References

[1] N. Alon, U. Arad, Y. Azar, Independent Sets in Hypergraphs with Applications to Routing Via Fixed Paths, *Proc. of APPROX 99, LNCS 1671* 16–27, 1999

[2] A. Borodin, R. El-Yaniv, *Online Computation and Competitive Analysis*, Cambridge University Press, 1998.

[3] H.J. Broersma, A. Capponi and D. Paulusma On-Line Coloring of H-Free Bipartite Graphs, in *Proc. of 6th Italian Conference on Algorithms and Complexity, CIAC 2006, LNCS 3998* 284–295, 2006

[4] L. Epstein, A. Levin, G. J. Woeginger Graph coloring with rejection *Proc. of the ESA 2006, LNCS 4168*, 364–375, 2006.

[5] A. Fiat, and G. J. Woeginger (eds.) *Online algorithms: The State of the Art* Vol. 1442 of Lecture Notes in Computer Science, Springer-Verlag Berlin, Heidelberg, 1998

[6] A. Gyárfás, J. Lehel, On-line and first-fit colorings of graphs, *Journal of Graph Theory*, **12**, 217–227, 1988.

[7] M. M. Halldórsson: Online Coloring Known Graphs. Proc. of SODA 1999 917–918, 1999

[8] H. A. Kierstead, Coloring Graphs On-line, *Online algorithms: The State of the Art* (A. Fiat, and G. J. Woeginger (eds.)), Vol. 1442 of Lecture Notes in Computer Science, Springer-Verlag Berlin, Heidelberg, 281–305, 1998.

[9] H. A. Kierstead, On-line coloring k-colorable graphs, *Israel J. Math*, **105**, 93–104, 1998.

[10] L. Lovász, M. Saks, W. T. Trotter, An on-line graph coloring algorithm with sublinear performance ratio, *Discrete Mathematics*, **75**, 319–325, 1989.