

# ON THE GRAPH CONDITION REGARDING THE $F$ -INVERSE COVER PROBLEM

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ABSTRACT. In [1], Auinger and Szendrei have shown that every finite inverse monoid has an  $F$ -inverse cover if and only if each finite graph admits a locally finite group variety with a certain property. We study this property and prove that the class of graphs for which a given group variety has the required property is closed downwards in the minor ordering, and can therefore be described by forbidden minors. We find these forbidden minors for all varieties of Abelian groups, thus describing the graphs for which such a group variety satisfies the above mentioned condition.

## 1. INTRODUCTION

An *inverse monoid* is a monoid  $M$  with the property that for each  $a \in M$  there exists a unique element  $a^{-1} \in M$  (the inverse of  $a$ ) such that  $a = aa^{-1}a$  and  $a^{-1} = a^{-1}aa^{-1}$ . Every inverse monoid may be embedded in a suitable symmetric inverse monoid  $I_V$ . Here  $I_V$  is the monoid of all partial injective maps from  $V$  to  $V$  (i.e. bijections between subsets of the set  $V$ ) with respect to the usual composition of partial maps.

We refer the reader to the books by Lawson [8] or Petrich [10] for the basics of the theory of inverse monoids. In particular, the *natural partial order* on an inverse monoid  $M$  is defined as follows:  $a \leq b$  if  $a = eb$  for some idempotent  $e \in M$ . In  $I_V$ , this partial order is the one defined by the restriction of partial maps. We also recall that inverse monoids, like groups, form a variety of algebras of type  $(2, 1, 0)$ , and free inverse monoids exist on any set. The free inverse monoid on the set  $X$  is, like the free group, obtained as a factor of the free monoid with the involution  $^{-1}$ , denoted by  $FMI(X)$  (see [8] for details).

It is well known that each inverse monoid admits a smallest group congruence which is usually denoted by  $\sigma$ . An inverse monoid is  *$F$ -inverse* if each  $\sigma$ -class has a greatest element with respect to the natural partial order.

The notion of an  $F$ -inverse monoid is among the most important ones in the theory of inverse semigroups, for example, free inverse monoids are  $F$ -inverse [8, 10]. Moreover, they play an important role in the theory of partial actions of groups, see Kellendonk and Lawson [6], and in this context they implicitly occur in Dehornoy [2, 3]. In Kaarli and Márki [5], they occur

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in the context of universal algebra. Even in analysis they are useful: see Nica [9], Khoshkam and Skandalis [7] and Steinberg [12] for their role in the context of  $C^*$ -algebras.

An  $F$ -inverse monoid  $F$  is an  $F$ -inverse cover of the inverse monoid  $M$  if there exists an idempotent separating surjective homomorphism from  $F$  to  $M$ . It is well known that every inverse monoid has an  $F$ -inverse cover. The proof is quite simple and constructive, the inverse cover it yields is an idempotent pure factor of a free inverse monoid, and therefore is always infinite. The question of whether finite inverse monoids admit a finite  $F$ -inverse cover was first proposed by Henckell and Rhodes [4], and has become one of the biggest open problems regarding finite inverse semigroups since.

In [1], Auinger and Szendrei have translated the  $F$ -inverse cover problem to the language of group varieties and graphs. They have proven that the question is equivalent to whether there is, for every finite graph, a locally finite group variety for which a certain condition is satisfied. These results are summarized in Section 2, and Section 3 contains general observations regarding this condition, including the fact that the class of graphs for which a given group variety has the required property is closed downwards in the minor ordering, and can therefore be described by forbidden minors. Section 4 contains our main result, which, using forbidden minors, describes the graphs for which there is a variety of Abelian groups satisfying the required condition. Not surprisingly, it turns out that these graphs consist of a quite narrow segment of all finite graphs.

## 2. PRELIMINARIES

We define graphs in this paper to be finite and directed. The set of vertices of a graph  $\Gamma$  is denoted by  $V(\Gamma)$ , the set of edges by  $E(\Gamma)$ , and the initial and terminal vertices of an edge  $e$  are denoted by  $\iota e$  and  $\tau e$  respectively. We also say that  $e$  is a  $(\iota e, \tau e)$ -edge. We consider  $\Gamma$  as the union of its vertices and edges, the union and intersection of subgraphs of  $\Gamma$  are meant in this sense. Connectedness of graphs will, however, be regarded in an undirected sense throughout the paper, that is, we call a digraph *connected* (*two-edge-connected*) if the underlying undirected graph is connected (two-edge-connected). Recall that an undirected graph is called *two-edge-connected* if it is connected and remains connected whenever an edge is removed.

A *path* in  $\Gamma$  is a sequence  $e_1 \cdots e_n$  of consecutive edges which, by definition, means that  $\tau e_i = \iota e_{i+1}$ ,  $i = 1, \dots, n-1$ , or an empty path around an arbitrary vertex. There is an evident notion of initial and terminal vertices of paths, also denoted by  $\iota$  and  $\tau$  respectively: if  $p = e_1 \cdots e_n$  for some  $n \in \mathbb{N}$ , then  $\iota p = \iota e_1$ ,  $\tau p = \tau e_n$ , and if  $p$  is an empty path around  $v$ , then  $\iota p = \tau p = v$ . We do, however, also need to consider paths in a more general, undirected sense. We therefore introduce the graph  $\bar{\Gamma}$ , for which  $V(\bar{\Gamma}) = V(\Gamma)$  and  $E(\bar{\Gamma}) = E(\Gamma) \cup (E(\Gamma))^{-1}$ , where  $E(\Gamma) \cap (E(\Gamma))^{-1} = \emptyset$ ,

and for an edge  $e \in E(\Gamma)$ ,  $e^{-1}$  is a  $(\tau e, \iota e)$ -edge. We will often consider paths in  $\bar{\Gamma}$ . In this context, we can also define the inverse of a path  $p$  in  $\bar{\Gamma}$ , denoted by  $p^{-1}$ , to be the  $(\tau p, \iota p)$ -path traversing the edges of  $p$  in the opposite direction. Note that a path in  $\bar{\Gamma}$  can be regarded as a word in the free monoid  $FMI(E(\Gamma))$  with involution, where, of course, the empty paths correspond to the empty word, and the word corresponding to the path  $p$  is of course the inverse of the words corresponding to  $p^{-1}$ . For a path  $p$  in  $\bar{\Gamma}$ , the *graph spanned by  $p$* , denoted by  $\langle p \rangle$ , is the subgraph of  $\Gamma$  consisting of the vertices  $p$  traverses and edges  $e \in E(\Gamma)$  for which  $e$  or  $e^{-1}$  occurs in  $p$ .

An *inverse category* is a category in which each arrow  $x$  admits a unique arrow  $x^{-1}$  satisfying  $x = xx^{-1}x$  and  $x^{-1} = x^{-1}xx^{-1}$ . The natural partial order on inverse categories is defined the same way as for inverse monoids, that is,  $x \leq y$  if  $x = ey$  for some idempotent arrow  $e$ .

We summarize some definitions and results of [1] necessary to formulate our results. Let  $\mathbf{V}$  be a variety of inverse monoids and let  $X$  be an alphabet. For words  $u, v \in FMI(X)$ , we put  $u \equiv_{\mathbf{V}} v$  if the identity  $u = v$  holds in  $\mathbf{V}$ . It is well known that  $\equiv_{\mathbf{V}}$  is a fully invariant congruence on  $FMI(X)$ , and  $F_{\mathbf{V}}(X) = FMI(X) / \equiv_{\mathbf{V}}$  is the relatively free inverse monoid in  $\mathbf{V}$  on  $X$ . We denote by  $[u]_{\mathbf{V}}$  the  $\equiv_{\mathbf{V}}$ -class of  $u$ , that is, the value of  $u$  in  $F_{\mathbf{V}}(X)$ .

Let  $\Gamma$  be a graph and  $\mathbf{V}$  be a variety of inverse monoids. By  $F_{g\mathbf{V}}(\Gamma)$ , we denote the *free  $g\mathbf{V}$ -category* on  $\Gamma$ : its set of vertices is  $V(\Gamma)$ , its set of  $(i, j)$ -arrows is

$$F_{g\mathbf{V}}(\Gamma)(i, j) = \{(i, [p]_{\mathbf{V}}, j) : p \text{ is an } (i, j)\text{-path in } \bar{\Gamma}\},$$

and the product of two consecutive arrows is defined by

$$(i, [p]_{\mathbf{V}}, j)(j, [q]_{\mathbf{V}}, k) = (i, [p]_{\mathbf{V}}[q]_{\mathbf{V}}, k) = (i, [pq]_{\mathbf{V}}, k).$$

The inverse of an arrow is given by

$$(i, [p]_{\mathbf{V}}, j)^{-1} = (j, [p]_{\mathbf{V}}^{-1}, i) = (j, [p^{-1}]_{\mathbf{V}}, i).$$

An important case is  $\mathbf{V} = \mathbf{Sl}$ , the variety of semilattices, in which case  $[p]_{\mathbf{Sl}}$  can be identified with the subgraph  $\langle p \rangle$  of  $\Gamma$  spanned by  $p$ . The  $(i, j)$ -arrows of  $F_{g\mathbf{Sl}}(\Gamma)$  are therefore precisely the triples  $(i, \Delta, j)$ , where  $\Delta$  is a connected subgraph containing  $i$  and  $j$ . The natural partial order on  $F_{g\mathbf{Sl}}(\Gamma)$  is conveniently described as

$$(i, \Delta_1, j) \leq (k, \Delta_2, l) \text{ if and only if } i = k, j = l \text{ and } \Delta_1 \supseteq \Delta_2.$$

A *dual premorphism*  $\psi: C \rightarrow D$  between inverse categories is a graph homomorphism satisfying  $(x\psi)^{-1} = x^{-1}\psi$  and  $(xy)\psi \geq x\psi \cdot y\psi$ . According to [1], every finite inverse monoid admits a finite  $F$ -inverse cover if and only if, for every (finite) connected graph  $\Gamma$ , there exist a locally finite group variety  $\mathbf{U}$  and a dual premorphism  $\psi: F_{g\mathbf{U}}(\Gamma) \rightarrow F_{g\mathbf{Sl}}(\Gamma)$  with  $\psi|_{\Gamma} = \text{id}_{\Gamma}$ .

Now fix a connected graph  $\Gamma$  and a group variety  $\mathbf{U}$ . We assign to each arrow  $x$  of  $F_{g\mathbf{U}}(\Gamma)$  two sequences of finite subgraphs of  $\Gamma$  as follows: let

$$(2.1) \quad C_0(x) = \bigcap \{ \langle p \rangle : (\iota p, [p]_{\mathbf{U}}, \tau p) = x \},$$

and let  $P_0(x)$  be the connected component of  $C_0(x)$  containing  $\iota x$ . If  $C_n(x), P_n(x)$  are already defined for all  $x$ , then put

$$C_{n+1}(x) = \bigcap \{P_n(x_1) \cup \cdots \cup P_n(x_k) : k \in \mathbb{N}, x_1 \cdots x_k = x\},$$

and again,  $P_{n+1}(x)$  is the connected component of  $C_{n+1}(x)$  containing  $\iota x$ .

It is easy to see that

$$C_0(x) \supseteq P_0(x) \supseteq \cdots \supseteq C_n(x) \supseteq P_n(x) \supseteq C_{n+1}(x) \supseteq P_{n+1}(x) \supseteq \cdots$$

for all  $x$  and  $n$ . We define  $P(x)$  to be  $\bigcap_{n=0}^{\infty} P_n(x)$ , which is a connected subgraph of  $\Gamma$  containing  $\iota x$ . According to [1, Lemma 3.1], there exists a dual premorphism  $\psi: F_{g\mathbf{U}}(\Gamma) \rightarrow F_{g\mathbf{S1}}(\Gamma)$  with  $\psi|_{\Gamma} = \text{id}_{\Gamma}$  if and only if  $\tau x \in P(x)$  for all  $x$ , and in this case, the assignment  $x \mapsto (\iota x, P(x), \tau x)$  gives such a dual premorphism. If  $\tau x \notin P(x)$  for some  $x = (\iota p, [p]_{\mathbf{U}}, \tau p)$ , then we call  $p$  a *breaking path* over  $\mathbf{U}$ .

In [1],  $C_0(x)$  is incorrectly defined to be the graph spanned by the  $\mathbf{U}$ -content of  $x$  together with  $\iota x$ . From the proof of [1, Lemma 3.1] (see the inclusion  $\mu(x\psi) \subseteq C_0(x)$ ), it is clear that the definition of  $C_0(x)$  needed is the one in (2.1). The following proposition states that in the cases crucial for the main result [1, Theorem 5.1], i.e., where  $\Gamma$  is the Cayley graph of a finite group, these two definitions are equivalent in the sense that  $P_0(x)$ , and so the sequence  $P_n(x)$  does not depend on which definition we use. For our later convenience, let  $\hat{C}_0(x)$  denote the graph which is the union of the  $\mathbf{U}$ -content of  $x$  and  $\iota x$ .

**Lemma 2.1.** *If  $\Gamma$  is two-edge-connected, then for any arrow  $x$  of  $F_{g\mathbf{U}}(\Gamma)$ , the subgraphs  $C_0(x)$  and  $\hat{C}_0(x)$  can only differ in isolated vertices (distinct from  $\iota x$  and  $\tau x$ ).*

**Proof.** Let  $x$  be an arrow of  $F_{g\mathbf{U}}(\Gamma)$ . It is clear that  $\hat{C}_0(x) \subseteq C_0(x)$ . For the converse, put  $x = (\iota p, [p]_{\mathbf{U}}, \tau p)$ , and suppose  $e$  is an edge of  $\langle p \rangle$  such that  $e \notin \hat{C}_0(x)$ . Let  $s_e$  be a  $(\iota e, \tau e)$ -path in  $\bar{\Gamma}$  not containing  $e$  — such a path exists since  $\Gamma$  is two-edge-connected. Let  $p_{e \rightarrow s_e}$  be the path obtained from  $p$  by replacing all occurrences of  $e$  by  $s_e$ . Then  $p \equiv_{\mathbf{U}} p_{e \rightarrow s_e}$ , and  $e \notin \langle p_{e \rightarrow s_e} \rangle$ , hence  $e \notin C_0(x)$ , which completes the proof.  $\square$

**Remark 2.2.** We remark that the condition of  $\Gamma$  being two-edge-connected is necessary in Lemma 2.1, that is, when  $\Gamma$  is not two-edge-connected, the subgraphs  $C_0(x)$  and  $\hat{C}_0(x)$  can in fact be different. Put, for example,  $\mathbf{U} = \mathbf{Ab}$ , the variety of Abelian groups, and let  $e$  be an edge of  $\Gamma$  for which  $\Gamma \setminus \{e\}$  is disconnected. Let  $p = ese^{-1}$  be a path in  $\bar{\Gamma}$ , where  $s \not\equiv_{\mathbf{Ab}} 1$  and  $e, e^{-1}$  do not occur in  $s$ . Then the subgraph spanned by the  $\mathbf{Ab}$ -content of  $p$  does not contain  $e$ , whereas any path  $p'$  which is coterminal with and  $\mathbf{Ab}$ -equivalent to  $p$  must contain the edge  $e$ , as there is no other  $(\iota e, \tau e)$ -path in  $\bar{\Gamma}$ .

For a group variety  $\mathbf{U}$ , we say that a graph  $\Gamma$  satisfies property  $(S_{\mathbf{U}})$ , or  $\Gamma$  is  $(S_{\mathbf{U}})$  for short, if  $\tau x \in P(x)$  holds for any arrow  $x$  of  $F_{g\mathbf{U}}(\Gamma)$ . By [1],

each finite inverse monoid has a finite  $F$ -inverse cover if and only if each finite connected graph is  $(S_{\mathbf{U}})$  for some locally finite group variety  $\mathbf{U}$ . This property  $(S_{\mathbf{U}})$  for finite connected graphs is our topic for the remaining part of the paper.

We recall that by [1, Lemmas 4.1 and 4.2], the following holds.

**Lemma 2.3.** *If a graph  $\Gamma$  is  $(S_{\mathbf{U}})$  for some group variety  $\mathbf{U}$ , then so is any redirection of  $\Gamma$ , and any subgraph of  $\Gamma$ .*

However, we remark that the lemma following these observations in [1], namely Lemma 4.3 is false. It states that if a simple graph  $\Gamma$  is  $(S_{\mathbf{U}})$ , then so is any graph obtained from  $\Gamma$  by adding parallel edges (where both “simple” and “parallel” are meant in the undirected sense). Our main result Theorem 4.1 yields counterexamples.

**Lemma 2.4.** *If  $\mathbf{U}$  and  $\mathbf{V}$  are group varieties for which  $\mathbf{U} \subseteq \mathbf{V}$ , then  $(S_{\mathbf{U}})$  implies  $(S_{\mathbf{V}})$ .*

**Proof.** Suppose  $\Gamma$  is  $(S_{\mathbf{U}})$ , let  $p$  be any path in  $\bar{\Gamma}$ . Put  $x^{\mathbf{U}} = (\iota p, [p]_{\mathbf{U}}, \tau p) \in F_{g\mathbf{U}}(\Gamma)$ , and similarly let  $x^{\mathbf{V}} = (\iota p, [p]_{\mathbf{V}}, \tau p) \in F_{g\mathbf{V}}(\Gamma)$ . Since  $\mathbf{U} \subseteq \mathbf{V}$ , we have  $C_0(x^{\mathbf{U}}) \subseteq C_0(x^{\mathbf{V}})$ . Also, since  $q \equiv_{\mathbf{V}} q_1 \cdots q_n$  implies  $q \equiv_{\mathbf{U}} q_1 \cdots q_n$ , we obtain  $P_n(x^{\mathbf{U}}) \subseteq P_n(x^{\mathbf{V}})$  by induction. Since  $\tau p \in P_n(x^{\mathbf{U}})$  by assumption, this yields  $\tau p \in P_n(x^{\mathbf{V}})$ , that is,  $\Gamma$  is  $(S_{\mathbf{V}})$ .  $\square$

### 3. FORBIDDEN MINORS

In this section, we prove that, given a group variety  $\mathbf{U}$ , the class of graphs satisfying  $(S_{\mathbf{U}})$  can be described by forbidden minors.

Let  $\Gamma$  be a graph and let  $e$  be a  $(u, v)$ -edge of  $\Gamma$  such that  $u \neq v$ . The operation which removes  $e$  and simultaneously merges  $u$  and  $v$  to one vertex is called *edge-contraction*. We call  $\Delta$  a *minor* of  $\Gamma$  if it can be obtained from  $\Gamma$  by edge-contraction, omitting vertices and edges, and redirecting edges.

**Proposition 3.1.** *Suppose  $\Gamma$  and  $\Delta$  are connected graphs such that  $\Delta$  is a minor of  $\Gamma$ . Then, if  $\Delta$  is non- $(S_{\mathbf{U}})$ , so is  $\Gamma$ .*

**Proof.** By Lemma 2.3, adding edges and vertices to, or redirecting some edges of a graph does not change the fact that it is non- $(S_{\mathbf{U}})$ . Therefore let us suppose that  $\Delta$  is obtained from  $\Gamma$  by contracting an edge  $e$  for which  $\iota e \neq \tau e$ . Let  $x_1, \dots, x_n$  be the edges of  $\Gamma$  having  $\iota e$  as their terminal vertex. For a path  $p$  in  $\bar{\Delta}$ , let  $p_{+e}$  denote the path in  $\bar{\Gamma}$  obtained by replacing all occurrences of  $x_j$  ( $j = 1, \dots, n$ ) by  $x_j e$  (and all occurrences of  $x_j^{-1}$  by  $e^{-1} x_j^{-1}$ ). Similarly, for a subgraph  $\Delta'$  of  $\Delta$ , let  $\Delta'_{+e}$  denote the subgraph of  $\Gamma$  obtained from  $\Delta'$  by taking its preimage under the edge-contraction containing the edge  $e$  if  $\Delta'$  contains some  $x_j$  ( $j = 1, \dots, n$ ), and its preimage without  $e$  otherwise. Obviously, we have  $\langle p_{+e} \rangle = \langle p \rangle_{+e}$  for any path  $p$  in  $\bar{\Delta}$ .

Note that if  $p$  is a path in  $\bar{\Delta}$  traversing the edges  $f_1, \dots, f_k$ , then  $p_{+e}$ , considered as a word in  $FMI(\{e, f_1, \dots, f_k\})$ , is obtained from the word

$p$  by substituting  $(x_j e)$  for  $x_j$  ( $j = 1, \dots, n$ ), and leaving the other edges unchanged. Putting  $x = (\iota p, [p]_{\mathbf{U}}, \tau p)$  and  $x_{+e} = (\iota p_{+e}, [p_{+e}]_{\mathbf{U}}, \tau p_{+e})$ , this implies  $(C_0(x))_{+e} \supseteq C_0(x_{+e})$  for any path  $p$  is  $\overline{\Delta}$ . Moreover, we also see that, for any paths  $q, q_1, \dots, q_k$  in  $\overline{\Delta}$ , we have  $q \equiv_{\mathbf{U}} q_1 \cdots q_k$  if and only if  $q_{+e} \equiv_{\mathbf{U}} (q_1)_{+e} \cdots (q_k)_{+e}$ . Using that for any subgraph  $\Delta' \subseteq \Delta$ , the connected components of  $\Delta'$  and  $\Delta'_{+e}$  are in one-one correspondence, an induction shows that  $(P_n(x))_{+e} \supseteq P_n(x_{+e})$  for every  $n$ . In particular,  $P_n(x)$  contains  $\tau p$  if and only if  $(P_n(x))_{+e}$  contains  $\tau p_{+e}$ . Therefore if  $p$  is a breaking path in  $\Delta$  over  $\mathbf{U}$ , then  $\tau p_{+e} \notin (P_n(x))_{+e}$  and hence  $\tau p_{+e} \notin P_n(x_{+e})$ , that is,  $p_{+e}$  is a breaking path in  $\Gamma$  over  $\mathbf{U}$ , which proves our statement.  $\square$

By the previous proposition, the class of all graphs containing a breaking path over  $\mathbf{U}$  (that is, of all non- $(S_{\mathbf{U}})$  graphs) is closed upwards in the minor ordering, hence, it is determined by its minimal elements. According to the theorem of Robertson and Seymour [11], there is no infinite anti-chain in the minor ordering, that is, the set of minimal non- $(S_{\mathbf{U}})$  graphs must be finite.

These observations are summarized in the following theorem:

**Theorem 3.2.** *For any group variety  $\mathbf{U}$ , there exist a finite set of connected graphs  $\Gamma_1, \dots, \Gamma_n$  such that the graphs containing a breaking path over  $\mathbf{U}$  are exactly those having one of  $\Gamma_1, \dots, \Gamma_n$  as a minor.*

By Lemma 2.4, if  $\mathbf{U}$  and  $\mathbf{V}$  are group varieties with  $\mathbf{U} \subseteq \mathbf{V}$ , the forbidden minors for  $\mathbf{U}$  are smaller (in the minor ordering) than the ones for  $\mathbf{V}$ .

The next statement contains simple observations regarding the nature of forbidden minors.

**Proposition 3.3.** *For any group variety  $\mathbf{U}$ , the set of minimal non- $(S_{\mathbf{U}})$  graphs are two-edge-connected graphs without loops.*

**Proof.** We show that if  $\Gamma$  is a non- $(S_{\mathbf{U}})$  graph which has loops or is not two-edge-connected, then there exists a graph below  $\Gamma$  in the minor ordering which is also non- $(S_{\mathbf{U}})$ . Indeed, suppose that  $\Gamma$  has a loop  $e$ , and take  $\Gamma \setminus \{e\}$ . For a path  $p$  in  $\overline{\Gamma}$ , let  $p_{-e}$  denote the corresponding path in  $\overline{\Gamma}$  obtained by omitting all occurrences of  $e$ , and for an arrow  $x = (\iota p, [p]_{\mathbf{U}}, \tau p) \in F_{g\mathbf{U}}(\Gamma)$ , put  $x_{-e} = (\iota p, [p_{-e}]_{\mathbf{U}}, \tau p) \in F_{g\mathbf{U}}(\Gamma \setminus \{e\})$ . Then it is easy to see by induction that  $C_n(x_{-e}) \subseteq C_n(x) \setminus \{e\}$  and  $P_n(x_{-e}) \subseteq P_n(x) \setminus \{e\}$  for every  $x$  and  $n$ , and hence  $\tau p \in P_n(x_{-e})$  implies  $\tau p \in P_n(x) \setminus \{e\}$ .

Now suppose  $\Gamma$  is not two-edge-connected, that is, there is a  $(u, v)$ -edge  $e$  of  $\Gamma$  for which  $\Gamma \setminus \{e\}$  is disconnected. Then let  $\Gamma_{u=v}$  denote the graph which we obtain from  $\Gamma$  by contracting  $e$ . For a path  $p$  in  $\overline{\Gamma}$ , let  $p_{u=v}$  denote the path in  $\overline{\Gamma_{u=v}}$  which we obtain by omitting all occurrences of  $e$  from  $p$ , and for an arrow  $x = (\iota p, [p]_{\mathbf{U}}, \tau p) \in F_{g\mathbf{U}}(\Gamma)$ , put  $x_{u=v} = (\iota p_{u=v}, [p_{u=v}]_{\mathbf{U}}, \tau p_{u=v}) \in F_{g\mathbf{U}}(\Gamma_{u=v})$ . Observe that for coterminal paths  $s, t$  in  $\overline{\Gamma}$ ,  $s \equiv_{\mathbf{U}} t$  implies  $s_{u=v} \equiv_{\mathbf{U}} t_{u=v}$ . This, by induction yields  $C_n(x_{u=v}) \subseteq C_n(x)_{u=v}$  and  $P_n(x_{u=v}) \subseteq P_n(x)_{u=v}$  for all  $n$ , and hence  $\tau p \in P_n(x_{u=v})$  implies  $\tau p \in P_n(x)_{u=v}$ .  $\square$

## 4. MAIN RESULT

In this section, we describe the forbidden minors (in the sense of the previous section) for all non-trivial varieties of Abelian groups. Denote by  $\mathbf{Ab}$  the variety of all Abelian groups.

**Theorem 4.1.** *A connected graph contains a breaking path over  $\mathbf{Ab}$  if and only if its minors contain at least one of the graphs in Figure 1.*

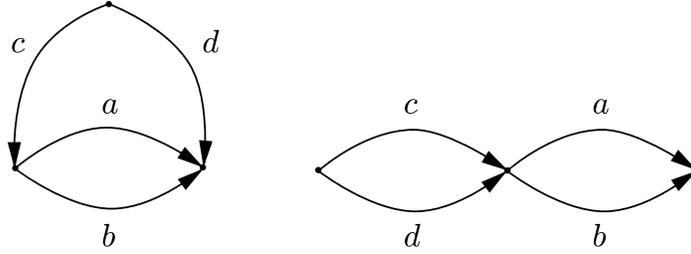


FIGURE 1

**Proof.** First, suppose  $\Gamma$  is a finite graph which does not have either graph in Figure 1 as a minor. Then  $\Gamma$  is either a cycle of length  $n$  for some  $n \in \mathbb{N}_0$  with possibly some trees and loops attached, or a graph with at most 2 vertices. According to Proposition 3.3,  $\Gamma$  contains a breaking path if and only if its greatest two-edge-connected minor does, which, in the former case is the cycle  $\Gamma_n$  of length  $n$ , and in the latter case is a two-edge-connected graph on at most 2 vertices. It is easy to see that both in cycles  $\Gamma_n$  or graphs on at most 2 vertices, for any path  $p$ , the  $\mathbf{Ab}$ -content  $\hat{C}_0(x)$  with  $x = (\iota p, [p]_{\mathbf{Ab}}, \tau p)$  is connected, therefore by Lemma 2.1, these graphs do not contain a breaking path over  $\mathbf{Ab}$ .

For the converse part, we prove that both graphs in Figure 1 contain a breaking path over  $\mathbf{Ab}$  — namely, the path  $a$ . For brevity, denote  $\iota a, \tau a$  and  $\iota c$  by  $u, v$  and  $w$  respectively, and put  $x = (u, [a]_{\mathbf{Ab}}, v)$ . Since both graphs are two-edge-connected, Lemma 2.1 implies that  $C_0(x)$  and the  $\mathbf{Ab}$ -content  $\hat{C}_0(x) = \langle a \rangle$  are (almost) the same, that is,  $P_0(x) = \langle a \rangle$  in both cases. Now put  $x_1 = (u, [c^{-1}]_{\mathbf{Ab}}, w)$ ,  $x_2 = (w, [cab^{-1}c^{-1}]_{\mathbf{Ab}}, w)$ ,  $x_3 = (w, [cb]_{\mathbf{Ab}}, v)$ , and note that  $x = x_1 x_2 x_3$ , that is,  $C_1(x) \subseteq P_0(x) \cap (P_0(x_1) \cup P_0(x_2) \cup P_0(x_3))$ . Again, using  $\hat{C}_0$  and Lemma 2.1, we obtain that  $\hat{C}_0(x_1) = \langle c \rangle$ ,  $\hat{C}_0(x_2) = \{w\} \cup \langle ab^{-1} \rangle$ ,  $\hat{C}_0(x_3) = \langle cb \rangle$ , and so  $P_0(x_1) \cup P_0(x_2) \cup P_0(x_3) = \langle c \rangle \cup \{w\} \cup \langle cb \rangle = \langle cb \rangle$  for both graphs in Figure 1. Therefore  $C_1(x) \subseteq \langle a \rangle \cap \langle cb \rangle = \{u, v\}$  and so  $v \notin P_1(x) \subseteq \{u\}$ . Hence  $a$  is, indeed, a breaking path over  $\mathbf{Ab}$  in both graphs.  $\square$

**Corollary 4.2.** *For any non-trivial variety  $\mathbf{U}$  of Abelian groups, a connected graph contains a breaking path over  $\mathbf{U}$  if and only if its minors contain at least one of the graphs in Figure 1.*

**Proof.** The statement is proven in Theorem 4.1 if  $\mathbf{U} = \mathbf{Ab}$ . Now let  $\mathbf{U}$  be a proper subvariety of  $\mathbf{Ab}$ . Then  $\mathbf{U}$  is the variety of Abelian groups of exponent  $n$  for some positive integer  $n \geq 2$ . By Lemma 2.4, the forbidden minors for  $\mathbf{U}$  must be minors of one of the forbidden minors of  $\mathbf{Ab}$ , that is, by Proposition 3.3, they are either the same, or the only forbidden minor is the cycle  $\Gamma_2$  of length two. However, it is clear that  $\Gamma_2$  contains no breaking path over  $\mathbf{U}$  for the same reason as in the case of  $\mathbf{Ab}$ , which proves our statement.  $\square$

**Remark 4.3.** For the variety  $\mathbf{1}$  of trivial groups, a finite connected graph is  $(S_1)$  if and only if it is a tree with some loops attached. That is, even the smallest two-edge-connected graph in the minor ordering, the cycle of length two contains a breaking path over  $\mathbf{1}$ .

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ON THE GRAPH CONDITION REGARDING THE  $F$ -INVERSE COVER PROBLEM 9

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