ON THE NUMBER OF SLIM, SEMIMODULAR LATTICES

GÁBOR CZÉDLI, TAMÁS DÉKÁNY, LÁSZLÓ OZSVÁRT, NÓRA SZAKÁCS, AND BALÁZS UDVARI

Abstract. A lattice L is *slim* if it is finite and the set of its join-irreducible elements contains no three-element antichain. Slim, semimodular lattices were previously characterized by G. Czédli and E. T. Schmidt [10] as the duals of the lattices consisting of the intersections of the members of two composition series in a group. Our main result determines the number of (isomorphism classes of) these lattices of a given size in a recursive way. The corresponding planar Hasse diagrams, up to similarity, are also enumerated. We prove that the number of diagrams of slim, distributive lattices of a given length n is the n-th Catalan number. Besides lattice theory, the paper includes some combinatorial arguments on permutations and their inversions.

1. Introduction and target

The well-known concept of a composition series in a group goes back to Évariste Galois (1831), see J. J. Rotman [25, Thm. 5.9]. The Jordan-Hölder theorem, stating that any two composition series of a finite group have the same length, was also proved in the nineteenth century, see C. Jordan [17] and O. Hölder [16]. Let

$$\begin{aligned} \vec{H}: &\quad G = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_h = \{1\} \text{ and} \\ \vec{K}: &\quad G = K_0 \triangleright K_1 \triangleright \cdots \triangleright K_h = \{1\} \end{aligned} \tag{1.1}$$

be composition series of a group G. Consider the following structure:

$$\big(\{H_i \cap K_j : i, j \in \{0, \dots, h\}\}, \subseteq\big).$$

It is a lattice, a so-called *composition series lattice*. The study of these lattices led G. Grätzer and J.B. Nation [14] and G. Czédli and E. T. Schmidt [7] to recent generalizations of the Jordan-Hölder theorem. In order to give an abstract characterization of these lattices, G. Czédli and E. T. Schmidt [10] proved that composition series lattices are exactly the duals of slim, semimodular lattices, to be defined later. (See also [4] for a more direct approach to this result.)

Here we continue the investigations started by G. Czédli, L. Ozsvárt, and B. Udvari [4]. Our main goal is to determine the number $N_{\rm ssl}(n)$ of slim, semimodular lattices (equivalently, composition series lattices) of a given size n. Isomorphic lattices are, of course, counted only once. These lattices of a given length were previously enumerated in [4]; however, the present task is subtler. Since slim lattices are planar by G. Czédli and E. T. Schmidt [7, Lemma 2.2], we are also interested

Date: March 18, 2016.

Key words and phrases. Composition series, Jordan-Hölder theorem, counting lattices, number of inversions, number of permutations, slim lattice, planar lattice, semimodular lattice.

2010 *Mathematics Subject Classification.* 06C10.

This research was supported by the NFSR of Hungary (OTKA), grant numbers K77432 and K83219, and by TÁMOP-4.2.1/B-09/1/KONV-2010-0005 and TÁMOP-4.2.2/B-10/1-2010-0012.

in the number of their planar diagrams. (By a *diagram*, we always mean a Hasse diagram.) Due to the fact that we count *specific* lattices, we give a recursive description for $N_{\rm ssl}(n)$ that is far more efficient than the best known way to count *all* finite lattices of a given size n; see J. Heitzig and J. Reinhold [15] and the references therein. We also enumerate the planar diagrams of slim, semimodular lattices of size n, up to similarity to be defined later.

Outline. Section 2 belongs to Lattice Theory. After presenting the necessary concepts, it reduces the targeted problems to combinatorial problems on permutations. Section 3 belongs to Combinatorics. Theorem 3.2 determines the number of slim, semimodular lattices consisting of n elements. Proposition 3.3 gives the number of the planar diagrams of slim, semimodular lattices of size n such that similar diagrams are counted only once. The number of planar diagrams of slim, distributive lattices of a given length is proved to be a Catalan number in Proposition 3.4.

2. From slim, semimodular lattices to permutations

An overview of slim, semimodular lattices. All lattices occurring in this paper are assumed to be finite. The notation is taken from G. Grätzer [13].

The set of non-zero join-irreducible elements of a lattice L is denoted by $\mathrm{Ji}\, L$. If $\mathrm{Ji}\, L$ is a union of two chains (equivalently, if $\mathrm{Ji}\, L$ contains no three-element antichain), then L is called a *slim* lattice. Slim lattices are *planar* by G. Czédli and E. T. Schmidt [7, Lemma 2.2]. That is, they possess planar diagrams. Let D_1 and D_2 be planar lattice diagrams. A bijection $\xi\colon D_1 \to D_2$ is a *similarity map* if it is a lattice isomorphism and for all $x, y, z \in D_1$ such that $x \prec y$ and $x \prec z$, y is to the left of z if and only if $\xi(y)$ is to the left of $\xi(z)$. Following D. Kelly and I. Rival [18, p. 640], we say that D_1 and D_2 are *similar lattice diagrams* if there exists a similarity map $D_1 \to D_2$. We always consider and count planar diagrams up to similarity. Also, we consider only planar diagrams. A diagram is slim if it represents a slim lattice; other lattice properties apply for diagrams analogously. For example, a diagram is *semimodular* if so is the corresponding lattice L; that is, if for all $x, y, z \in L$ such that $x \preceq y$, the covering or equal relation $x \vee z \preceq y \vee z$ holds.

Let D be a planar diagram of a slim lattice L of length h. Note that L may have several non-similar diagrams since we can reflect D (or certain intervals of D) over a vertical axis. The left boundary chain of D is denoted by $\mathrm{BC}_\ell(D)$, while $\mathrm{BC}_{\rm r}(D)$ stands for its right boundary chain. These chains are maximal chains in L, and both are of length h by semimodularity. So we can write

$$\begin{aligned} &\mathrm{BC}_\ell(D) = \{0 = b_0 \prec b_1 \prec \dots \prec b_h\} \text{ and}\\ &\mathrm{BC}_{\rm r}(D) = \{0 = c_0 \prec c_1 \prec \dots \prec c_h\}. \end{aligned} \tag{2.1}$$

Note that, by G. Czédli and E. T. Schmidt [8, Lemma 6],

$$\mathrm{Ji}\, L = \mathrm{Ji}\, D \subseteq \mathrm{BC}_\ell(D) \cup \mathrm{BC}_{\rm r}(D). \tag{2.2}$$

The permutation of a slim, semimodular lattice. The present paper is based on the fundamental connection between planar, slim, semimodular diagrams and permutations. In this and the next subsections, we recall and develop the details of this connection in a way that fits [4], where the enumerative investigations of slim, semimodular lattices start. The following statement is a straightforward consequence of G. Czédli and E. T. Schmidt [9, Proof 4.7], combined with [8, Lemma

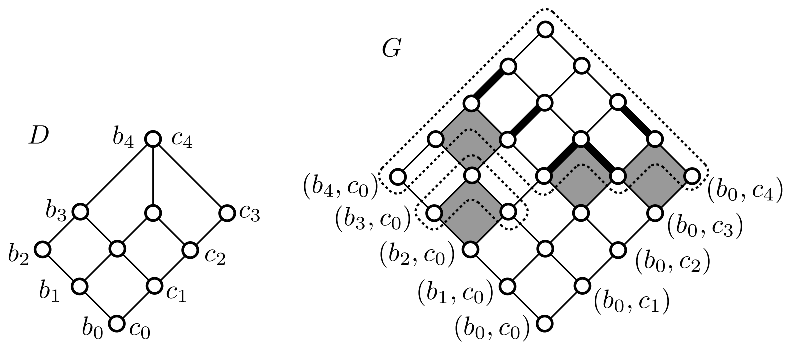

FIGURE 1. A diagram D and the corresponding grid diagram G

7]. (For a more general statement without semimodularity, the interested reader may want to see G. Czédli and G. Grätzer [3, Theorem 1-4.5].)

Lemma 2.1. *Assume that D and E are planar diagrams of a slim, semimodular lattice L. Then D is similar to E if and only if* $\mathrm{BC}_\ell(D) = \mathrm{BC}_\ell(E)$ *if and only if* $\mathrm{BC}_\mathrm{r}(D) = \mathrm{BC}_\mathrm{r}(E)$.

Next, with D as above, consider the diagram G of the (slim, distributive) lattice $\mathrm{BC}_\ell(D)\times\mathrm{BC}_\mathrm{r}(D)$ such that $\mathrm{BC}_\ell(D)\times\{0\}\subseteq\mathrm{BC}_\ell(G)$ and $\{0\}\times\mathrm{BC}_\mathrm{r}(D)\subseteq\mathrm{BC}_\mathrm{r}(G)$. Then G is determined up to similarity by Lemma 2.1, and it is called the *grid diagram* associated with D; see Figure 1. More generally, the diagram of the direct product of a chain $\{b_0 \prec b_1 \prec \cdots \prec b_m\}$ (to be placed on the bottom left boundary) and a chain $\{c_0 \prec c_1 \prec \cdots \prec c_n\}$ (to be placed at the bottom right boundary) is also called a *grid diagram of type* $m \times n$. For $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, let

$$\mathrm{cell}(i,j) = \{(b_{i-1},c_{j-1}),(b_i,c_{j-1}),(b_{i-1},c_j),(b_i,c_j)\};$$

this sublattice (and subdiagram) is called a 4-*cell*. The smallest join-congruence of G that collapses the top boundary $\{(b_i,c_{j-1}),(b_{i-1},c_j),(b_i,c_j)\}$ of this 4-cell is denoted by $\mathrm{con}_\vee(\mathrm{cell}(i,j))$. We recall the following statement from G. Czédli [2, Corollary 22]. For $(i,j) = (2,3)$, this statement is illustrated by Figure 1, where the non-singleton blocks are indicated by thick edges.

Lemma 2.2. *Let G be a grid diagram of type $m \times n$, and let $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. Denote* $\mathrm{con}_\vee(\mathrm{cell}(i,j))$ *by $\boldsymbol{\alpha}$. Then*

(i) *the $\boldsymbol{\alpha}$-block $(b_i,c_j)/\boldsymbol{\alpha}$ of (b_i,c_j) is $\{(b_i,c_{j-1}),(b_{i-1},c_j),(b_i,c_j)\}$;*
(ii) *$\{(b_s,c_{j-1}),(b_s,c_j)\}$ for $s > i$ and $\{(b_{i-1},c_t),(b_i,c_t)\}$ for $t > j$ are the two-element blocks of $\boldsymbol{\alpha}$;*
(iii) *the rest of $\boldsymbol{\alpha}$-blocks are singletons, and $\boldsymbol{\alpha}$ is cover-preserving.*

The following description of the join of join-congruences is borrowed from G. Czédli and E. T. Schmidt [5, Lemma 11].

Lemma 2.3. *Let $\boldsymbol{\beta}_i$, $i \in I$, be join-congruences of a join-semilattice F, and let $u, v \in F$. Then $(u,v) \in \bigvee_{i\in I}\boldsymbol{\beta}_i$ if and only if there is a $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$ and there are elements*

$$u = z_0 \le z_1 \le \cdots \le z_k = w_k \ge w_{k-1} \ge \cdots \ge w_0 = v$$

such that $\{(z_{j-1},z_j),(w_{j-1},w_j)\} \subseteq \bigcup_{i\in I}\boldsymbol{\beta}_i$ for $j \in \{1,\dots,k\}$.

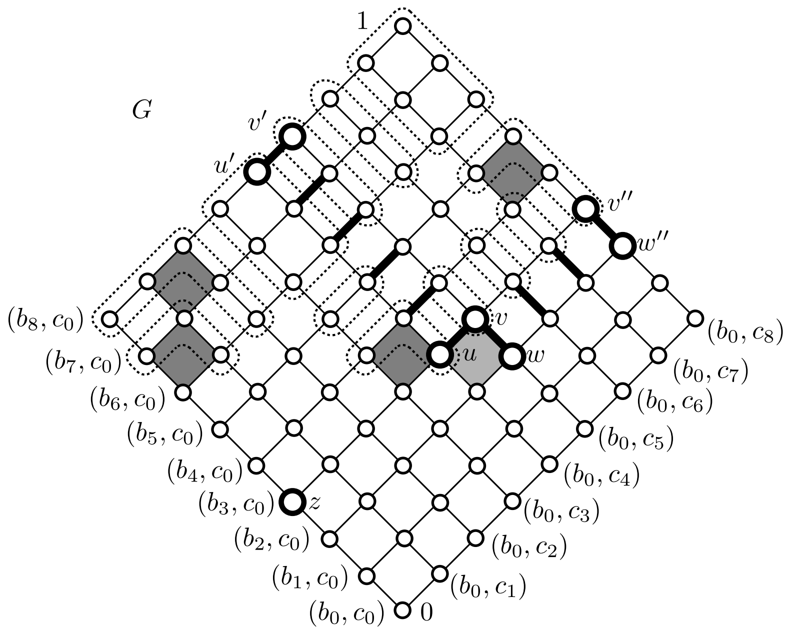

FIGURE 2. A grid

Combining Lemmas 2.2 and 2.3 we easily obtain the following corollary, which is implicit in G. Czédli [2].

Corollary 2.4. *Let G be a grid diagram of type $m \times n$, and let $k \in \mathbb{N} = \{1, 2, \dots\}$. Assume that $1 \leq i_1 < \cdots < i_k \leq m$ and that $j_1, \dots, j_k$ are pairwise distinct elements of $\{1, \dots n\}$. Consider the join-congruence $\boldsymbol{\beta} = \bigvee_{s=1}^{k} \mathrm{con}_\vee(\mathrm{cell}(i_s, j_s))$. Then $\boldsymbol{\beta}$ is cover-preserving, and it is described by the following rules.*

(i) $\big((b_i, c_j), (b_s, c_t)\big) \in \boldsymbol{\beta}$ *if and only if* $\{(b_i, c_j), (b_s, c_t)\} \subseteq (b_i \vee b_s, c_j \vee c_t)/\boldsymbol{\beta}$*;*
(ii) *for* $0 \leq r < s$, $\big((b_r, c_t), (b_s, c_t)\big) \in \boldsymbol{\beta}$ *if and only if for each* $x \in \{r+1, \dots, s\}$ *there is a (unique)* $p \in \{1, \dots, k\}$ *such that* $x = i_p$ *and* $j_p \leq t$*;*
(iii) *for* $0 \leq r < s$, $\big((b_t, c_r), (b_t, c_s)\big) \in \boldsymbol{\beta}$ *if and only if for each* $x \in \{r+1, \dots, s\}$ *there is a (unique)* $p \in \{1, \dots, k\}$ *such that* $x = j_p$ *and* $i_p \leq t$*.*

In Figure 1, this statement is illustrated for $m = n = 4$ and $k = 4$ so that the non-singleton blocks of $\boldsymbol{\beta}$ are indicated by dotted lines and the $\mathrm{cell}(i_s, j_s)$, $1 \leq s \leq 4$, are the grey cells, that is,

$$\begin{pmatrix} i_1 & \dots & i_4 \\ j_1 & \dots & j_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}. \tag{2.3}$$

Corollary 2.4 is also illustrated by Figure 2 for $m = n = 8$ and $k = 4$ where

$$\begin{pmatrix} i_1 & \dots & i_4 \\ j_1 & \dots & j_4 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 7 & 8 \\ 4 & 8 & 1 & 2 \end{pmatrix} \tag{2.4}$$

(only the dark grey 4-cells are considered, the light grey one should be disregarded).

Now, we consider the grid diagram G (of type $h \times h$) associated with D again. It follows from (2.2) that the map $\xi\colon G \to D$, $(x, y) \mapsto x \vee y$ is a surjective join-homomorphism. By G. Czédli and E. T. Schmidt [6, proof of Corollary 2], ξ is *cover-preserving*, that is, if $a, b \in G$ and $a \preceq b$, then $\xi(a) \preceq \xi(b)$. Thus its kernel, $\boldsymbol{\alpha} = \{(a, b) : \xi(a) = \xi(b)\}$ is a so-called *cover-preserving join-congruence* by definition, see [6]. If the $\boldsymbol{\alpha}$-block $(b_i, c_j)/\boldsymbol{\alpha}$ includes $\{(b_i, c_{j-1}), (b_{i-1}, c_j)\}$ but $(b_{i-1}, c_{j-1}) \notin (b_i, c_j)/\boldsymbol{\alpha}$, then $\operatorname{cell}(i, j)$ is called a *source cell* of $\boldsymbol{\alpha}$. In Figure 1, the source cells of $\boldsymbol{\alpha} = \operatorname{Ker} \xi$ are the grey ones. The set of these source cells is denoted by $\operatorname{SCells}(\boldsymbol{\alpha})$. With D, we associate a relation π_D (which turns out to be a permutation, see (2.3) for Figure 1) as follows:

$$\pi_D = \big\{(i, j) \in \{1, \ldots, h\}^2 : \operatorname{cell}(i, j) \in \operatorname{SCells}(\boldsymbol{\alpha})\big\}. \tag{2.5}$$

Modulo notational changes, the following lemma is included in G. Czédli and E. T. Schmidt [10]. Therefore, its "proof" below will only be a guide to [10]. Remember that similar diagrams are considered equal.

Lemma 2.5. *Let D be a slim, semimodular, planar diagram of length h, and let G, $\xi\colon G \to D$, $\boldsymbol{\alpha} = \operatorname{Ker} \xi$, and $\pi = \pi_D$ be as above.*

(i) *π is a permutation on $\{1, \ldots, h\}$.*
(ii) $\boldsymbol{\alpha} = \bigvee_{i=1}^{h} \operatorname{con}_\vee(\operatorname{cell}(i, \pi(i)))$.
(iii) *The mapping $D \mapsto \pi_D$ is a bijection from the set of slim, semimodular diagrams of length h to the set S_h of permutations acting on $\{1, \ldots, h\}$.*

Proof. Part (i) is the same as [10, Lemma 2.6]. Part (ii) follows from [10, Lemma 4.7], because $\bigvee_{i=1}^{h} \operatorname{con}_\vee(\operatorname{cell}(i, \pi(i))) = \bigvee_{\operatorname{cell}(i,j) \in \operatorname{SCells}(\boldsymbol{\alpha})} \operatorname{con}_\vee(\operatorname{cell}(i, j))$ by (2.5). Part (iii) is equivalent to the bijectivity of ψ_0, see [10, Definition 3.2(ii)] together with [10, Definition 2.5] and [10, Proposition 2.7], and ψ_0 is a bijection by [10, Theorem 3.3]. Note at this point that, by Lemma 2.1, similarity in our sense is equivalent to "boundary similarity", which is used in [10]. $\square$

Permutations determine the size. For a permutation $\sigma \in S_h$, the number $|\{(\sigma(i), \sigma(j)) : i < j \text{ and } \sigma(i) > \sigma(j)\}|$ of inversions of σ is denoted by $\operatorname{inv}(\sigma)$. The same notation applies for *partial permutations* (that is, bijections between two subsets of $\{1, \ldots, h\}$), only we have to stipulate that both $\sigma(i)$ and $\sigma(j)$ should be defined. For example, if σ is the partial permutation given in (2.4), then $\operatorname{inv}(\sigma) = 4$. The size $|D|$ of a diagram D is the number of elements of the lattice it determines. A crucial step of the paper is represented by the following statement.

Proposition 2.6. *With the assumptions of Corollary 2.4, let K be the lattice determined by G, and let τ denote the partial permutation $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{pmatrix}$. Then*

$$|K/\boldsymbol{\beta}| = (m + 1)(n + 1) + \operatorname{inv}(\tau) - k(m + n + 2) + \sum_{s=1}^{k} (i_s + j_s). \tag{2.6}$$

Proof. Corollary 2.4 gives a satisfactory understanding of $\boldsymbol{\beta} = \boldsymbol{\beta}_\tau$, which allows us to prove (2.6) by induction on k. For a first impression of the proof, the induction step will be preceded by an example.

The case $k = 0$ is obvious since then $\boldsymbol{\beta} = \boldsymbol{\beta}_\tau$ is the least join-congruence, $\operatorname{inv}(\tau) = 0$, and $K/\boldsymbol{\beta} \cong K$. Hence we assume that $k > 0$ and the lemma holds for all smaller values. We let

$$\sigma = \begin{pmatrix} i_2 & \dots & i_k \\ j_2 & \dots & j_k \end{pmatrix}.$$

Let $\{\mathrm{cell}(i_2, j_2), \dots, \mathrm{cell}(i_k, j_k)\}$ be denoted by $\mathrm{SCells}(\boldsymbol{\beta}_\sigma)$. It follows easily from Corollary 2.4 that $\mathrm{SCells}(\boldsymbol{\beta}_\sigma)$ is the set of source cells of $\boldsymbol{\beta}_\sigma$ in the earlier meaning. These source cells will also be called *dark grey cells*. Similarly, we have $\mathrm{SCells}(\beta) = \{\mathrm{cell}(i_1, j_1), \dots, \mathrm{cell}(i_k, j_k)\}$, and $\mathrm{cell}(i_1, j_1)$ is said to be the *light grey cell*.

Example. The situation for $k = 5$, $(i_1, j_1) = (2, 5)$,

$$\tau = \begin{pmatrix} 2 & 4 & 5 & 7 & 8 \\ 5 & 4 & 8 & 1 & 2 \end{pmatrix}, \text{ and, consequently, } \sigma = \begin{pmatrix} 4 & 5 & 7 & 8 \\ 4 & 8 & 1 & 2 \end{pmatrix}$$

is given by Figure 2. The cells of $\mathrm{SCells}(\boldsymbol{\beta}_\sigma)$ are depicted in dark grey, while $\mathrm{cell}(2, 5)$ is in light grey. The "action" of the light grey cell, that is $\mathrm{con}_\vee(\mathrm{cell}(i_1, j_1)) = \mathrm{con}_\vee(\mathrm{cell}(2, 5))$, is indicated by thick lines. (Note that σ is the partial permutation given in (2.4) but now the subscripts are shifted by 1.) At several places in the proof, we will reference Figure 2 to enlighten the argument with this example.

Now, returning to the proof, let

$$\boldsymbol{\beta}' = \bigvee_{s=2}^{k} \mathrm{con}_\vee(\mathrm{cell}(i_s, j_s)) = \bigvee_{s=2}^{k} \mathrm{con}_\vee(\mathrm{cell}(i_s, \sigma(i_s)));$$

its blocks are indicated by dotted lines in Figure 2. By the induction hypothesis, the number of $\boldsymbol{\beta}'$-blocks is

$$|K/\boldsymbol{\beta}'| = (m+1)(n+1) + \mathrm{inv}(\sigma) - (k-1)(m+n+2) + \sum_{s=2}^{k}(i_s + j_s). \tag{2.7}$$

Roughly saying, our job is to count how many $\boldsymbol{\beta}'$-blocks are glued together by the "action" of the light grey cell. Consider the following elements:

$$\begin{aligned} u &= (b_{i_1}, c_{j_1-1}), \quad & v &= (b_{i_1}, c_{j_1}), \quad & w &= (b_{i_1-1}, c_{j_1}), \quad & z &= (b_{i_1}, c_0), \\ u' &= (b_m, c_{j_1-1}), \quad & v' &= (b_m, c_{j_1}), \quad & v'' &= (b_{i_1}, c_n), \quad & w'' &= (b_{i_1-1}, c_n); \end{aligned} \tag{2.8}$$

Note that these elements are marked by enlarged circles in Figure 2. The restriction of $\boldsymbol{\beta}'$ to an interval I will be denoted by $\boldsymbol{\beta}'\rceil_I$, and $\boldsymbol{\omega}_I$ stands for the equality relation on I. Since no dark grey 4-cell occurs in the interval $[0, v'']$, Corollary 2.4 gives that $\boldsymbol{\beta}'\rceil_{[0,v'']} = \boldsymbol{\omega}_{[0,v'']}$. Similarly, there is no t such that $\big((b_t, c_{j_1-1}), (b_t, c_{j_1})\big) \in \boldsymbol{\beta}'$. Let $\boldsymbol{\gamma}_u$ be the join-congruence of $[z, u']$ defined by

$$\boldsymbol{\gamma}_u = \bigvee\big\{\mathrm{con}_\vee(\mathrm{cell}(i_s, j_s)) : 1 < s \le k,\ j_s < j_1\big\}; \tag{2.9}$$

it is the smallest join-congruence of $[z, u']$ that collapses the top boundaries of the dark grey 4-cells in $[z, u']$. We conclude that $\boldsymbol{\delta} = \boldsymbol{\omega}_{[0,v'']} \cup \boldsymbol{\omega}_{[u,v']} \cup \boldsymbol{\gamma}_u \cup [v, 1]^2$, which is clearly a join-congruence of K, includes $\boldsymbol{\beta}'$. Thus $\boldsymbol{\gamma}_u = \boldsymbol{\beta}'\rceil_{[z,u']}$. If (2.9) is understood in the interval $[z, v']$, then it defines a join-congruence $\boldsymbol{\gamma}_v$ of $[z, v']$, and we similarly obtain that $\boldsymbol{\gamma}_v = \boldsymbol{\beta}'\rceil_{[z,v']}$. The previous two equalities clearly yield that $\boldsymbol{\beta}'\rceil_{[u,u']} = \boldsymbol{\gamma}_u\rceil_{[u,u']}$ and $\boldsymbol{\beta}'\rceil_{[v,v']} = \boldsymbol{\gamma}_v\rceil_{[v,v']}$. Applying Corollary 2.4 to $[z, u']$ and to $[z, v']$, we obtain that $\boldsymbol{\beta}'\rceil_{[u,u']}$ partitions $[u, u']$ to $m + 1 - i_1 - q$ blocks and that $\boldsymbol{\beta}'\rceil_{[v,v']}$ partitions $[v, v']$ to $m + 1 - i_1 - q$ blocks, where

$$q = |\{s : 1 < s \le k,\ j_s < j_1\}|,$$

which is the number of dark grey 4-cells in $[z, u']$ (and also in $[z, v']$). We also obtain from Corollary 2.4 that the above-mentioned blocks are "positioned in parallel", that is, for $x, y \in [u, u']$, we have $(x, y) \in \boldsymbol{\beta}'$ if and only if $(x \vee v, y \vee v) \in \boldsymbol{\beta}'$.

We know that $\boldsymbol{\beta} = \boldsymbol{\beta}' \vee \mathrm{con}_\vee(\mathrm{cell}(i_1, j_1))$ in the lattice of join-congruences of K and also in the lattice of equivalences of K. The blocks of $\mathrm{con}_\vee(\mathrm{cell}(i_1, j_1))$ are given by Lemma 2.2; they are indicated by thick lines in Figure 2. Since $\boldsymbol{\beta}' \subseteq \boldsymbol{\delta}$, each element of $[w, w']$ belongs to a singleton $\boldsymbol{\beta}'$-block. There are $n + 1 - j_1$ such (singleton) $\boldsymbol{\beta}'$-blocks, and the northwest-southeast oriented thick edges merge them into other (not necessarily singleton) $\boldsymbol{\beta}'$-blocks. Similarly, the northeast-southwest oriented thick edges merge q $\boldsymbol{\beta}'$-blocks of $[z, u']$ to the respective blocks in $[v, v']$. Therefore,

$$|K/\boldsymbol{\beta}| = |K/\boldsymbol{\beta}'| - (m + 1 - i_1 - q) - (n + 1 - j_1). \tag{2.10}$$

Since q is the number of inversions with j_1, we have that $q = \mathrm{inv}(\tau) - \mathrm{inv}(\sigma)$. Combining this equation with (2.7) and (2.10) we obtain the desired (2.6). $\square$

Proposition 2.7. *Let D be a slim, semimodular, planar diagram, and let π be the permutation associated with D in* (2.5). *Then $|D| = h + 1 + \mathrm{inv}(\pi)$.*

Proof. Let L be the lattice determined by D. It follows from Lemma 2.5 and the Homomorphism Theorem that $|D| = |L| = |G/\boldsymbol{\alpha}|$. Hence Proposition 2.6 applies, and the substitution $(m, n, k, \sigma) := (h, h, h, \pi)$ clearly turns the right side of (2.6) into $h + 1 + \mathrm{inv}(\pi)$. $\square$

Permutations corresponding to slim, distributive lattices. For a planar diagram D of a slim, semimodular lattice L, let $\mathrm{PrInt}(D)$ denote the set of prime intervals of L, that is, the set of edges of D. Let $[a, b], [c, d] \in \mathrm{PrInt}(D)$. These two prime intervals are *consecutive* if they are opposite sides of a 4-cell. We say that we go from $[a, b]$ to $[c, d]$ *upwards* if d is the top element of this 4-cell. Otherwise, if c is the bottom element of the 4-cell, we go *downwards*. The transitive reflexive closure of the relation

$$\big\{\big([a, b], [c, d]\big) : [a, b] \text{ and } [c, d] \text{ are consecutive}\big\}$$

is called *prime projectivity*. It is an equivalence relation on $\mathrm{PrInt}(D)$, and its blocks are called *trajectories*. If there are no $[a, b], [c_1, d_1], [c_2, d_2] \in \mathrm{PrInt}(D)$ such that $[c_1, d_1] \neq [c_2, d_2]$, $[a, b]$ and $[c_i, d_i]$ are consecutive for $i \in \{1, 2\}$, and either the trajectory T containing $[a, b]$ goes upwards from $[a, b]$ to $[c_1, d_1]$ and $[c_2, d_2]$, or it goes downwards to $[c_1, d_1]$ and $[c_2, d_2]$, then we say that the *trajectories of D do not branch out*. A trajectory that does not branch out can be visualized by its *strip*, which is the set of 4-cells determined by consecutive edges of the trajectory. For example, the strip from $[g_B, g'_B]$ to $[h_B, h'_B]$ in Figure 3 is depicted in grey. If only some consecutive edges of a trajectory are taken, then they determine a *strip section*. We recall the following statement from G. Czédli and E. T. Schmidt [7, Lemma 2.8].

Lemma 2.8. *The trajectories of D do not branch out. Each trajectory starts at a unique prime interval of* $\mathrm{BC}_\ell(D)$*, and it goes to the right. First it goes upwards* (*possibly in zero steps*)*, then it goes downwards* (*possibly in zero steps*)*, and finally it reaches a unique prime interval of* $\mathrm{BC}_\mathrm{r}(D)$*. In particular, once it is going down, there is no further turn.*

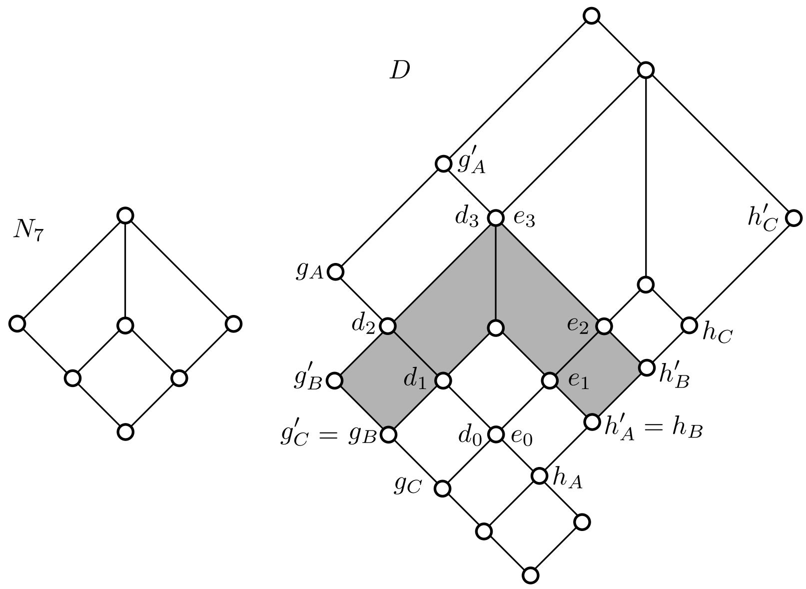

FIGURE 3. N_7 and a trajectory

Assume that D is a slim, semimodular diagram with boundary chains (2.1). By Lemma 2.8, for each $i \in \{1, \dots, h\}$ there is a unique $j \in \{1, \dots, h\}$ such that the trajectory starting at $[b_{i-1}, b_i]$ arrives at $[c_{j-1}, c_j]$. This defines a map $\hat{\pi}_D \colon \{1, \dots, h\} \to \{1, \dots, h\}$, $i \mapsto j$. For example, $\hat{\pi}_D$ for Figure 3 is

$$\hat{\pi}_D = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 7 & 6 & 4 & 1 & 8 & 3 & 5 \end{pmatrix}. \tag{2.11}$$

This gives an alternative way to associate a permutation with D using the following statement from G. Czédli and E. T. Schmidt [10, Proposition 2.7].

Lemma 2.9. *For any planar, slim, semimodular diagram D, $\hat{\pi}_D$ equals π_D defined in* (2.5).

Let $\pi \in S_h$. We say that the permutation π *contains the 321 pattern* if there are $i < j < k \in \{1, \dots, h\}$ such that $\pi(i) > \pi(j) > \pi(k)$. For general background on permutation patterns, which we do not need here, see M. Bóna [1, Theorem 2.3]. The distributivity of D is characterized by the following statement.

Proposition 2.10. *Let D be a slim, semimodular diagram, and let $\pi = \pi_D$ denote the permutation associated with it. Then D is distributive if and only if π does not contain the 321 pattern.*

Proof. The idea of the proof is simple: the distributivity of a slim, semimodular lattice is characterized by the lack of cover-preserving N_7 sublattices, and

$$\pi_{N_7} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Furthermore, a trajectory can change its direction from going upwards to going downwards only at a cover-preserving N_7 sublattice. Below, we turn this pictorial idea into a rigorous proof.

In virtue of Lemma 2.9, we work with $\pi = \hat{\pi}_D$. In order to prove the necessity part of Proposition 2.10, we assume that D is not distributive. We obtain from G. Czédli and E. T. Schmidt [8, Lemma 15] that D contains, as a cover-preserving sublattice, a copy of N_7, given in Figure 3. Let $\{d_0 \prec d_1 \prec d_2 \prec d_3\}$ and $\{e_0 \prec e_1 \prec e_2 \prec e_3\}$ be the left and right boundary chains, respectively, of a subdiagram of D representing N_7, see Figure 3. Let A, B, C denote the trajectories containing $[e_0, e_1]$, $[e_1, e_2]$, and $[e_2, e_3]$, respectively. The corresponding strip sections, starting at these edges and going to the right, are denoted by A^*, B^*, and C^*, respectively. Let us denote the last members of these trajectories by $[h_A, h'_A], [h_B, h'_B], [h_C, h'_C] \in \mathrm{PrInt}(\mathrm{BC_r}(D))$, respectively. We claim that

$$h_A \prec h'_A \leq h_B \prec h'_B \leq h_C \prec h'_C. \tag{2.12}$$

Suppose for a contradiction that $h'_A \not\leq h_B$. Then $h'_B \leq h_A$ since $[h_A, h'_A] \neq [h_B, h'_B]$ by Lemma 2.8, and h'_B and h_A belong to the chain $\mathrm{BC_r}(D)$. Thus A^* must cross B^* at a 4-cells such that A^* crosses this 4-cell upwards (that is, to the northeast). But this is impossible by Lemma 2.8 since A and thus A^* went downwards previously at $[e_0, e_1]$. A similar contradiction is obtained from $h'_B \not\leq h_C$ since B^* goes downwards through $[e_1, e_2]$, and thus it cannot cross a square upwards later. This proves (2.12).

Next, let $[g_A, g'_A], [g_B, g'_B], [g_C, g'_C] \in \mathrm{PrInt}(\mathrm{BC}_\ell(D))$ denote the first edges of A, B, C, respectively. Since $[d_0, d_1] \in C$, $[d_1, d_2] \in B$, and $[d_2, d_3] \in A$, the left-right dual of the argument leading to (2.12) yields that

$$g_C \prec g'_C \leq g_B \prec g'_B \leq g_A \prec g'_A. \tag{2.13}$$

Therefore, in virtue of Lemma 2.9, (2.12) together with (2.13) yields a 321 pattern in π.

Now, to prove the sufficiency part, assume that D is distributive. Then it is dually slim by G. Czédli and E. T. Schmidt [8, Lemma 16]. Hence, by the dual of [8, Lemma 16], no element of D has more than two lower covers. Thus each trajectory goes (entirely) either upwards, or downwards; that is, a trajectory cannot make a turn. Suppose for a contradiction that π contains a 321 pattern. Then, like previously, we have trajectories A, B, C such that (2.12) and (2.13) hold. Any two of the corresponding strips must cross at a 4-cell since their starting edges are in the opposite order as their ending edges are. Therefore any two of the three strips go to different directions, which is impossible since there are only two directions: upwards and downwards. This contradiction completes the proof. $\square$

Permutations with the same lattice. To accomplish our goal, we have to know when two permutations determine the same slim, semimodular lattice. Below, we recall the necessary information and notation from G. Czédli and E. T. Schmidt [10] and G. Czédli, L. Ozsvárt, and B. Udvari [4]. (The interested reader may also want to see the overview on slim semimodular lattices in G. Grätzer [3].) Assume that $1 \leq u \leq v \leq h$ and $\pi \in S_h$. If $I = [u, v] = \{i \in \mathbb{N} : u \leq i \leq v\}$ is nonempty and $[1, u-1]$, I, and $[v+1, h]$ are closed with respect to π, then I is called a *section* of π. Sections minimal with respect to set inclusion are called *segments*. Let $\mathrm{Seg}(\pi)$ denote the set of all segments of π. For example, if $\pi = \hat{\pi}_D$ from (2.11), then π has only one segment, $\{1, \ldots, 8\}$. Another example is

$$\begin{gathered} \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 1 & 2 & 6 & 5 & 7 & 9 & 10 & 8 \end{pmatrix} \\ \text{with } \mathrm{Seg}(\pi) = \big\{\{1,2,3,4\}, \{5,6\}, \{7\}, \{8,9,10\}\big\}. \end{gathered} \tag{2.14}$$

For a subset A of $\{1,\dots,h\}$, let $\pi\rceil_A$ denote the restriction of π to A. The set of $A\to A$ permutations is denoted by S_A. Notice that $\mathrm{Seg}(\tau)$ also makes sense for $\tau\in S_A$ since the natural order of $\{1,\dots,h\}$ is automatically restricted to A. If $A\in\mathrm{Seg}(\pi)$, then $\pi\rceil_A\in S_A$ and $\pi\rceil_{\{1,\dots,h\}-A}\in S_{\{1,\dots,h\}-A}$. The unique $I_1\in\mathrm{Seg}(\pi)$ with $1\in I_1$ is the *initial segment* of π. We adopt the following terminology:

$$\begin{aligned}&\mathrm{head}(\pi)=\pi\rceil_{I_1}\in S_{I_1}\text{ is the \textit{head} of }\pi,\\ &\mathrm{body}(\pi)=\pi\rceil_{\{1,\dots,h\}-I_1}\in S_{\{1,\dots,h\}-I_1}\text{ is the \textit{body} of }\pi.\end{aligned}\tag{2.15}$$

Note that $\mathrm{body}(\pi)$ can be the empty permutation acting on $\varnothing$. Clearly, the pair $(\mathrm{head}(\pi),\mathrm{body}(\pi))$ determines π; however, the two components of the pair $(\mathrm{head}(\pi),\mathrm{body}(\pi))$ are not arbitrary. We say that $\pi\in S_h$ is *irreducible*, if its initial segment is $\{1,\dots,h\}$. Note that π is irreducible if and only if $\mathrm{head}(\pi)=\pi$ or, equivalently, if and only if $\mathrm{body}(\pi)=\varnothing$. Note also that the largest element in the initial segment of π need not belong to the π-orbit of 1. In particular, as it is exemplified by the restriction of π in (2.14) to $\{1,2,3,4\}$, if $\sigma\in S_h$ is an involution, then its irreducibility does not imply $\sigma(1)=h$. Clearly, if $I=[1,u]$ is a *nonempty initial interval* of $\{1,\dots,h\}$, that is, if $1\le u\le h$, and, in addition, $\sigma\in S_I$, and $\tau\in S_{\{1,\dots,h\}-I}$, then (σ,τ) coincides with $(\mathrm{head}(\pi),\mathrm{body}(\pi))$ for some $\pi\in S_h$ if and only if $\sigma\in S_I$ is irreducible. For $\pi\in S_h$, the *degree* of π is h. We define the *block* $[\pi]^\sim$ of π by induction on the degree of π as follows. If π is irreducible, then we let $[\pi]^\sim=\{\pi,\pi^{-1}\}$. Otherwise, let $[\pi]^\sim=\bigl\{\sigma:\mathrm{head}(\sigma)\in\{\mathrm{head}(\pi),\mathrm{head}(\pi)^{-1}\}\text{ and }\mathrm{body}(\sigma)\in[\mathrm{body}(\pi)]^\sim\bigr\}$. For example, if π is taken from (2.14), then $[\pi]^\sim$ consists of four permutations. Note that

$$\begin{aligned}&S_h/{\sim}=\{[\pi]^\sim:\pi\in S_h\}\text{ and, for every }\pi\in S_h,\\ &[\pi]^\sim=\{\sigma:[\mathrm{head}(\sigma)]^\sim=[\mathrm{head}(\pi)]^\sim\text{ and }[\mathrm{body}(\sigma)]^\sim=[\mathrm{body}(\pi)]^\sim\}.\end{aligned}\tag{2.16}$$

is the partition on S_h associated with the so-called "sectionally inverse or equal" relation introduced in G. Czédli and E. T. Schmidt [10]. It is well-known from H. A. Rothe [24], see also D. E. Knuth [19] or one can prove it easily, that $\mathrm{inv}(\sigma)=\mathrm{inv}(\sigma^{-1})$. This implies that $\mathrm{inv}(\sigma)=\mathrm{inv}(\pi)$ for every $\sigma\in[\pi]^\sim$. Hence we can define $\mathrm{inv}([\pi]^\sim)$ by the equation $\mathrm{inv}([\pi]^\sim)=\mathrm{inv}(\pi)$. While part (iii) of Lemma 2.5 deals with diagrams, now we recall its lattice version from [10].

Lemma 2.11 (G. Czédli and E. T. Schmidt [10, Theorem 3.3])**.** *Let D and E be slim, semimodular, planar diagrams. Then D and E determine isomorphic lattices if and only if $[\pi_D]^\sim=[\pi_E]^\sim$.*

3. Counting

Slim, semimodular lattices. We introduce the following notation.

$$\begin{aligned}&P(h,k)=\{\pi\in S_h:\mathrm{inv}(\pi)=k\}, &&\widehat{P}(h,k)=\{\pi\in P(h,k):\pi\text{ is irreducible}\},\\ &I(h,k)=\{\pi\in P(h,k):\pi^2=\mathrm{id}\}, &&P^\sim(h,k)=\{[\pi]^\sim:\pi\in P(h,k)\},\\ &P^\sim_{s,t}(h,k)=\{[\pi]^\sim:\pi\in P(h,k),\ \mathrm{head}(\pi)\in S_s,\text{ and }\mathrm{inv}(\mathrm{head}(\pi))=t\},\\ &\widehat{I}(h,k)=\{\pi\in I(h,k):\pi\text{ is irreducible}\}.\end{aligned}$$

Here P and I comes from "permutation" and "involution". Their parameters denote the length of permutations and the number of inversions, while $^\sim$ and $\widehat{\ }$ stand for blocks and irreducibility, respectively. The sizes of these sets are denoted by the corresponding lower case letters; for example, $p^{\sim}(h,k) = |P^{\sim}(h,k)|$. (Note that, as opposed to us, the literature denotes $|P(h,k)|$ usually by $I_h(k)$ rather than $p(h,k)$.) The binary function p is well-studied. Let

$$G_h(x) = \sum_{j=0}^{\binom{h}{2}} p(h,j)x^j \tag{3.1}$$

denote its generating function. We recall the following result of O. Rodriguez [23] and Muir [20], see also D. E. Knuth [19, p. 15], or M. Bóna [1, Theorem 2.3].

Lemma 3.1. $G_h(x) = \prod_{j=1}^{h} \sum_{t=0}^{j-1} x^t = \prod_{j=1}^{h} \frac{1-x^j}{1-x}$.

We mention that for the generating function $G_h^{\mathrm{inv}}(x) = \sum_{j=0}^{\binom{h}{2}} i(h,j)x^j$ of i, W. M. B. Dukes [11, Proposition 2.8] gives the following recursive description:

$$G_0^{\mathrm{inv}}(x) = G_1^{\mathrm{inv}}(x) = 1, \quad G_h^{\mathrm{inv}}(x) = G_{h-1}^{\mathrm{inv}}(x) + \frac{x(1-x^{2(h-1)})}{1-x^2} \cdot G_{h-2}^{\mathrm{inv}}(x). \tag{3.2}$$

We will not use (3.2) since it is easier to compute $i(h,k)$ by (3.6), see later. We are now in the position to formulate the following theorem.

Theorem 3.2. *The number $N_{\mathrm{ssl}}(n)$ of slim, semimodular lattices of size n is determined by Lemma 3.1 together with the following* (*recursive*) *formulas*

$$N_{\mathrm{ssl}}(n) = \sum_{h=0}^{n-1} p^{\sim}(h, n-h-1), \tag{3.3}$$

$$p^{\sim}(h,k) = \frac{1}{2} \cdot \sum_{s=1}^{h} \sum_{t=0}^{k} \big(\widehat{p}(s,t) + \widehat{i}(s,t)\big) \cdot p^{\sim}(h-s, k-t), \tag{3.4}$$

$$\widehat{p}(h,k) = p(h,k) - \sum_{s=1}^{h-1} \sum_{t=0}^{k} \widehat{p}(s,t) \cdot p(h-s, k-t), \tag{3.5}$$

$$i(h,k) = i(h-1,k) + \sum_{s=2}^{h} i(h-2, k-2s+3), \tag{3.6}$$

$$\widehat{i}(h,k) = i(h,k) - \sum_{s=1}^{h-1} \sum_{t=0}^{k} \widehat{i}(s,t) \cdot i(h-s, k-t) \tag{3.7}$$

for $n, h \in \mathbb{N}$ and $k \in \mathbb{N}_0$, and with the initial values

$$p^{\sim}(h,0) = p(h,0) = i(h,0) = 1 = \widehat{p}(1,0) = \widehat{i}(1,0) \ \textit{for}\ h \in \mathbb{N}_0,$$

$$p^{\sim}(h,k) = p(h,k) = \widehat{p}(h,k) = i(h,k) = \widehat{i}(h,k) = 0 \ \textit{if}\ k > \binom{h}{2} \ \textit{or}\ \{h,k\} \not\subseteq \mathbb{N}_0,$$

$$\widehat{p}(h,0) = \widehat{i}(h,0) = 0, \ \textit{if}\ h > 1.$$

Notice that $\binom{h}{k} = 0$ if $k > h$. Clearly, together with the initial values, (3.6) determines the function i, (3.7) gives the function $\widehat{i}$, we can evaluate the function p based on Lemma 3.1 and (3.1), then (3.5) determines the function $\widehat{p}$, (3.4) yields $p^{\sim}$, and, finally, (3.3) yields $N_{\mathrm{ssl}}(n)$.

Proof of Theorem 3.2. By Lemma 2.11, we have to count the blocks $[\pi]^{\sim}$ that give rise to n-element lattices. The initial values are obvious.

If $\pi \in S_h$, then $\operatorname{inv}([\pi]^{\sim}) = \operatorname{inv}(\pi)$ equals $n-h-1$ by Proposition 2.7. This implies (3.3).

Next, $[\operatorname{head}(\pi)]^{\sim}$ is a singleton if $\operatorname{head}(\pi)^2 = \mathrm{id}$, and it is two-element otherwise. Thus, by (2.16), the number of blocks $[\pi]^{\sim} \in P^{\sim}(h,k)$ with $\operatorname{head}(\pi)^2 = \mathrm{id}$ is

$$\sum_{s=1}^{h}\sum_{t=0}^{k} \widehat{i}(s,t)\cdot p^{\sim}(h-s,k-t). \tag{3.8}$$

Similarly, the number of blocks $[\pi]^{\sim} \in P^{\sim}(h,k)$ with $\operatorname{head}(\pi)^2 \neq \mathrm{id}$ is

$$\sum_{s=1}^{h}\sum_{t=0}^{k} \frac{1}{2}\cdot\bigl(\widehat{p}(s,t)-\widehat{i}(s,t)\bigr)\cdot p^{\sim}(h-s,k-t). \tag{3.9}$$

Forming the sum of (3.8) and (3.9), we obtain (3.4).

The subtrahend on the right of (3.5) is the number of the reducible members of $P(h,k)$. This implies (3.5).

For $\pi \in I(h,k)$, let $s = \pi(1)$. There are exactly $i(h-1,k)$ many such π with $s = 1$; this gives the first summand in (3.6). Next, assume that $s > 1$, and note that $\pi(s) = 1$ since $\pi^2 = \mathrm{id}$. Then, in the second row of the matrix

$$\begin{pmatrix} 1 & 2 & \dots & s-1 & s & s+1 & \dots & h \\ s & \pi(2) & \dots & \pi(s-1) & 1 & \pi(s+1) & \dots & \pi(h) \end{pmatrix},$$

there are $s-1$ inversions of the form $(x,1)$, $s-2$ inversions of the form (s,y) with $y \neq 1$, and we also have the inversions of $\sigma = \pi\rceil_{\{1,\dots,h\}-\{1,s\}}$. Therefore, σ has $k-(s-1+s-2)$ inversions, whence σ can be selected in $i(h-2,k-2s+3)$ ways. This explains the second part of (3.6), completing the proof of equation (3.6).

Finally, the argument for (3.7) is essentially the same as that for (3.5) since the subtrahend in (3.7) is the number of reducible members of $I(h,k)$. $\square$

Slim, semimodular diagrams. Due to Lemma 2.5(iii), the first part of the previous proof for (3.3) clearly yields the following statement. Based on Lemma 3.1, it gives an effective way to count the diagrams in question.

Proposition 3.3. *Up to similarity, the number $N_{\mathrm{ssd}}(n)$ of planar, slim, semimodular lattice diagrams with n elements is*

$$N_{\mathrm{ssd}}(n) = \sum_{h=0}^{n-1} p(h,n-h-1).$$

Proof. If $\pi \in S_h$ determines an n-element diagram, then $\operatorname{inv}(\pi)$ equals $n-h-1$ by Proposition 2.7. This together with Lemma 2.5(iii) implies our statement. $\square$

Slim distributive diagrams. As opposed to the previous statement, we are going to enumerate these diagrams of a given length rather than a given size. Let $C_h = (h+1)^{-1}\cdot\binom{2h}{h}$ denote the h-th *Catalan number*, see, for example, M. Bóna [1].

Proposition 3.4. *Up to similarity, the number of planar, slim, distributive lattice diagrams of length h is C_h.*

Proof. By Lemma 2.5(iii) and Proposition 2.10, we need the number of permutations in S_h that do not contain the pattern 321. This number is C_h by M. Bóna [1, Corollary 4.7]. $\square$

3.1. **Calculations with Computer Algebra.** It follows easily from Theorem 3.2 that $N_{\rm ssl}(1) = 1$, $N_{\rm ssl}(2) = 1$, $N_{\rm ssl}(3) = 1$,$N_{\rm ssl}(4) = 2$, $N_{\rm ssl}(5) = 3$, $N_{\rm ssl}(6) = 5$, $N_{\rm ssl}(7) = 9$, $N_{\rm ssl}(8) = 16$, $N_{\rm ssl}(9) = 29$, and these values can easily be checked by listing the corresponding lattices. One can use computer algebra to obtain, say, $N_{\rm ssl}(20) = 33\,701$, $N_{\rm ssl}(30) = 25\,051\,415$, and $N_{\rm ssl}(40) = 19\,057\,278\,911$. In a desktop computer with 3GHz Intel® Core$^{\rm TM}$2 Duo Processor E8400 and 3.25 GB of RAM from 2008, one can compute

$$N_{\rm ssl}(50) = 14\,546\,017\,036\,127$$

in about three hours.

To indicate that semimodularity together with slimness is a strong assumption, we conclude the paper with the following comparison. While we computed $N_{\rm ssl}(18) = 9070$ with our computer described above in four seconds, it took about six days and a parallel algorithm using fifty 450 MHz processors of a Cray T3e computer to count *all* 18-element lattices, see J. Heitzig and J. Reinhold [15].

References

[1] M. Bóna, Combinatorics of permutations. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2004. xiv+383 pp. ISBN: 1-58488-434-7.
[2] G. Czédli, The matrix of a slim semimodular lattice, Order, **29** (2012), 85–103.
[3] G. Czédli, G. Grätzer, Planar Semimodular Lattices and Their Diagrams, in G. Grätzer, F. Wehrung, Lattice Theory: Special Topics and Applications, Birkhäuser, Basel 2014.
[4] G. Czédli, L. Ozsvárt, B. Udvari, How many ways can two composition series intersect? Discrete Mathematics, **312** (2012), 3523–3536.
[5] G. Czédli, E. T. Schmidt, Some results on semimodular lattices, Contributions to General Algebra 19 (Proc. Olomouc Conf. 2010), Johannes Hein verlag, Klagenfurt (2010), pp. 45–56.
[6] G. Czédli, E. T. Schmidt, How to derive finite semimodular lattices from distributive lattices?, Acta Mathematica Hungarica, **121** (2008), 277–282.
[7] G. Czédli, E. T. Schmidt, The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices, Algebra Universalis **66** (2011), 69–79.
[8] G. Czédli, E. T. Schmidt, Slim semimodular lattices. I. A visual approach, Order **29** (2012), 481–497.
[9] G. Czédli, E. T. Schmidt, Slim semimodular lattices. II. A description by patchwork systems, Order **30** (2013), 689–721.
[10] G. Czédli, E. T. Schmidt, Composition series in groups and the structure of slim semimodular lattices. Acta Sci. Math. (Szeged) **79**, 369-390 (2013).
[11] W. M. B. Dukes, Permutation statistics on involutions. European Journal of Combinatorics 28 (2007), 186–198.
[12] M. Erné, J. Heitzig, J. Reinhold, On the number of distributive lattices, Electron. J. Combin. 9 (2002), no. 1, Research Paper 24, 23 pp.
[13] G. Grätzer, Lattice Theory: Foundation, Birkhäuser Verlag, Basel, 2011.
[14] G. Grätzer, J. B. Nation, A new look at the Jordan-Hölder theorem for semimodular lattices, Algebra Universalis (in press).
[15] J. Heitzig, J. Reinhold, Counting finite lattices, Algebra Universalis 48 (2002), 43–53.
[16] O. Hölder, Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen, Math. Ann. 34 (1889), 26–56.
[17] C. Jordan, Traité des substitutions et des équations algebraique, Gauthier-Villars, Paris, 1870.
[18] Kelly, D., Rival, I.: Planar lattices. Canad. J. Math. **27** (1975), 636–665.
[19] D. E. Knuth, The Art of Computer Programming III. Sorting and Searching, 2nd ed., Addison-Wesley, Reading, MA, 1998.

[20] Muir, On a simple term of a determinant, Proc. Royal S. Edinborough **21** (1898-9), 441-477.
[21] J. B. Nation, Notes on Lattice Theory, `http://www.math.hawaii.edu/~jb/books.html`
[22] M. M. Pawar, B. N. Waphare, Enumeration of nonisomorphic lattices with equal number of elements and edges, Indian J. Math. 45 (2003), 315–323.
[23] O. Rodriguez, Note sur les inversion, ou dérangements produits dans les permutations, J. de Math. **4**, 1839, pp. 236–240.
[24] H. A. Rothe, Über Permutationen, in Beziehung auf die Stellen ihrer Elemente. Anwendung der daraus abgeleiteten Satze auf das Eliminationsproblem, in Sammlung combinatorisch-analytischer Abhandlungen, edited by K. F. Hindenburg, 2 (Leipzig: 1800), Bey G. Fleischer dem jüngern, 263–305.
[25] J. J. Rotman, An Introduction to the Theory of Groups, 4th ed., Springer Verlag, New York, 1995.
[26] R. Schmidt, Subgroup Lattices of Groups, de Gruyter Expositions in Mathematics, vol. 14, Walter de Gruyter & Co., Berlin, 1994.
[27] R. Schmidt, Planar subgroup lattices. Algebra Universalis 55 (2006), 3–12.
[28] M. Stern, Semimodular Lattices. Theory and Applications, Encyclopedia of Mathematics and its Applications. 73, Cambridge University Press, 1999.

E-mail address: `czedli@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~czedli/`

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: `tdekany@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~tdekany/`

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: `ozsvartl@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~ozsvartl/`

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: `szakacsn@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~szakacs/`

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: `udvarib@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~udvarib/`

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720