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On the convexity of a hitting distribution for discrete random walks

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Abstract. We examine the convexity of the hitting distribution of the real axis for symmetric random walks on $\mathbb{Z}^2$. We prove that for a random walk starting at $(0,h)$, the hitting distribution is convex on $[h-2,\infty)\cap\mathbb{Z}$ if $h\geq 2$. We also show an analogous fact for higher-dimensional discrete random walks. This paper extends the results of a recent paper [NT].

1. Introduction

Let $\mathbb{Z}$ be the set of the integers and $\mathbb{Z}^2$ the integer lattice on the plane. We will consider (discrete) random and non-random walks on $\mathbb{Z}^2$ with four possible (unit) steps: $\leftarrow$, $\rightarrow$, $\uparrow$ and $\downarrow$. (In a symmetric random walk each step is equally likely.) By the *length* of a (finite) walk we mean the number of its steps. We will mostly work with special walks. We say that a $(k_1,h)\rightsquigarrow(k_2,0)$ walk is *positive*, if it stays strictly above the x-axis before its last step. ($P\rightsquigarrow Q$ indicates a walk with starting point P and endpoint Q.)

We denote by $p_{k_2}^{(k_1,h)}$ the probability that a symmetric random walk on $\mathbb{Z}^2$, started from the point (k_1,h), first hits the x-axis at the point $(k_2,0)$. We will use the shorter form $p_k^h := p_k^{(0,h)}$, too. In [NT] it has been proved that the sequence $\{p_k^1\}_{k=0}^\infty$ is convex, that is, $p_k^1\leq\frac{1}{2}(p_{k-1}^1+p_{k+1}^1)$ for all $k\in\mathbb{N}=\{1,2,\dots\}$. Improving the technique used there, we obtain a simple, transparent convexity result also in the case $h\geq 2$. The problem was suggested by V. Totik (personal communication).

Theorem 1. *The sequence $\{p_k^h\}_{k=h-2}^\infty$ is convex for all $h\geq 2$.*

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For the case $h = 1$, the proof given in [NT] relies on the fact that the number of positive $(0,1) \rightsquigarrow (k,0)$ walks of arbitrary fixed length starting with an up step is not more than the number of different walks of the same type and length starting with a left or right step. This can be shown by giving an injective length-preserving map from the set of walks starting with an up step into the set of walks starting with a left or right step. Similarly, in order to prove Theorem 1, it is sufficient to give an injective length-preserving map from the set of positive $(0,h) \rightsquigarrow (k,0)$ walks starting with an up or down step into the set of walks of the same type starting with a left or right step in the case of $k \geq h-1$. Before stating this formally, let us introduce a notation and make a remark. Let $\mathcal{W}_{k_2}^{(k_1,h)}$ be the set of positive walks from (k_1,h) to $(k_2,0)$, and $\mathcal{W}_k^h := \mathcal{W}_k^{(0,h)}$. The walks in $\mathcal{W}_k^h$ starting with an up, down, left or right step can be identified with the walks in $\mathcal{W}_k^{h+1}$, $\mathcal{W}_k^{h-1}$, $\mathcal{W}_k^{(-1,h)}$, and $\mathcal{W}_k^{(1,h)}$, respectively, by omitting the first step. With these notations and conventions, the main lemma of this paper can be stated as follows.

Lemma 2. *For integers h,k such that $h \geq 2$ and $k \geq h-1$, there exists a length-preserving injection of $\mathcal{W}_k^{h+1} \cup \mathcal{W}_k^{h-1}$ into $\mathcal{W}_k^{(-1,h)} \cup \mathcal{W}_k^{(1,h)}$.*

We prove this lemma in the next section and see why it implies our main theorem. Then we will discuss some open problems and possible extensions of Theorem 1 in the last section. We investigate the tightness of the bound $h-2$, and sketch the proof of a higher-dimensional analogue of the theorem. We prefer purely combinatorial arguments throughout the paper.

2. Proof of Theorem 1

First we show that Theorem 1 is implied by Lemma 2. Theorem 1 claims that

$$p_k^h \leq \frac{1}{2}\left(p_{k-1}^h + p_{k+1}^h\right) \tag{1}$$

holds for all $h \geq 2$ and $k \geq h-1$. Conditioning on the first step, we clearly have

$$p_k^h = \frac{1}{4}\left(p_k^{h+1} + p_k^{h-1} + p_k^{(-1,h)} + p_k^{(1,h)}\right),$$

thus, using the obvious facts $p_k^{(-1,h)} = p_{k+1}^h$ and $p_k^{(1,h)} = p_{k-1}^h$, inequality (1) is equivalent to

$$p_k^{h+1} + p_k^{h-1} \leq p_k^{(-1,h)} + p_k^{(1,h)}. \tag{2}$$

Since for $h > 0$,

$$p_{k_2}^{(k_1,h)} = \sum_{W \in \mathcal{W}_{k_2}^{(k_1,h)}} \left(\frac{1}{4}\right)^{|W|} \tag{3}$$

where $|W|$ denotes the length of W, Lemma 2 indeed implies (2) for the required h and k values, and so Theorem 1.

Proof of Lemma 2. We give such an injection ϕ. Pick an arbitrary walk $W \in \mathcal{W}_k^{h+1} \cup \mathcal{W}_k^{h-1}$. Let P be the lattice point where W first hits a diagonal or a side of the square with vertices $(h,0)$, $(h,2h)$, $(-h,2h)$ and $(-h,0)$. (Such a P obviously exists.) P divides W into two parts, let W_- be the subwalk preceding (the first visit of) P, and let W_+ be the rest of W.

If $W \in \mathcal{W}_k^{h-1}$, then, as $k \geq h-1$, P lies on a diagonal. Let the image $\phi(W)$ be the walk obtained from W by reflecting W_- across the diagonal containing P (if $P=(0,h)$ then choose $y=x+h$), and leaving W_+ unchanged, see Figure 1. Clearly, $|\phi(W)|=|W|$, and since the reflected part of W stays within the square, $\phi(W)$ does not hit the x-axis in a forbidden point, so $\phi(W) \in \mathcal{W}_k^{(-1,h)} \cup \mathcal{W}_k^{(1,h)}$ is also immediate.

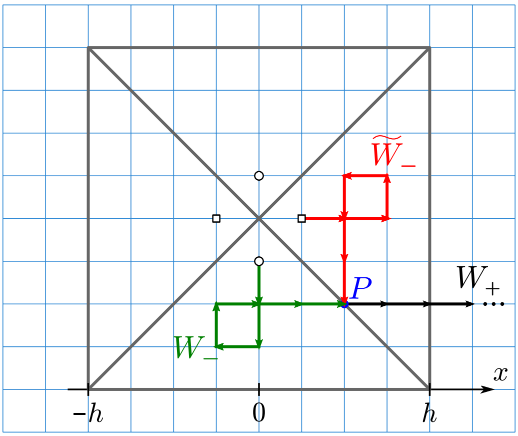

Figure 1. The case when P lies on a diagonal

If $W \in \mathcal{W}_k^{h+1}$, then P lies on one of the lines $y=x+h$, $y=-x+h$, and $y=2h$. If P lies on a diagonal, then follow the reflecting method introduced at the case of $W \in \mathcal{W}_k^{h-1}$. Now suppose that $P=(m,2h)$ (here $m \in \{-(h-2),\dots,(h-2)\}$). We will prove in the next lemma that there exists a length-preserving injection $\psi\colon \mathcal{W}_k^{(m,2h)} \to \mathcal{W}_k^{(h,h+m)} \cup \mathcal{W}_k^{(-h,h-m)}$. Note that $W_+ \in \mathcal{W}_k^{(m,2h)}$, and if we reflect W_- across the diagonal $y=x+h$ or $y=-x+h$, then the obtained walk $\widetilde{W}_-$ ends at $(h,h+m)$ or $(-h,h-m)$, respectively, denoted by P' and P'' on Figure 2. So $\phi(W)$ can be defined to be the concatenation of $\widetilde{W}_-$ and $\psi(W_+)$, where $\widetilde{W}_-$ is obtained by the reflection of the above which sends P, the endpoint of W_-, to the starting point of $\psi(W_+)$.

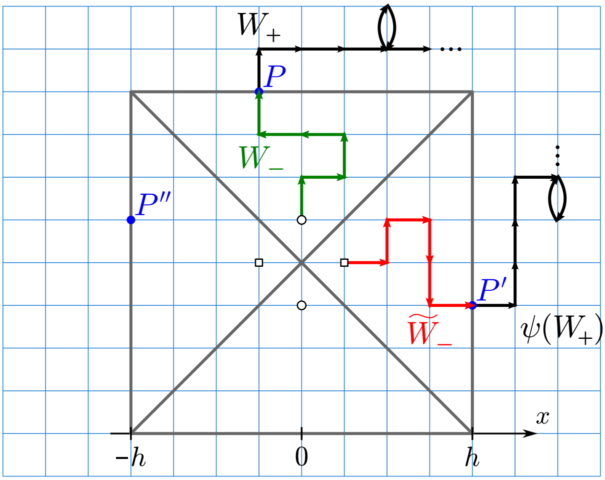

Figure 2. The case when P lies on the top side of the square

It is straightforward to check that ϕ has the required properties. The injectivity follows from the facts that the conversions of W_- and W_+ are clearly injective and that, for any walk $W' \in \mathcal{W}_k^{(-1,h)} \cup \mathcal{W}_k^{(1,h)}$, the only possible point where the two converted parts can be glued together is the point where W' first hits a diagonal or side of the same square as above. ∎

Now we establish the lemma used in the proof. It generalizes a result of [NT], the existence of the injective length-preserving map mentioned in Section 1, which corresponds to the case $h = 1$, $m = 0$. In fact, the idea of the proof of the base case is adapted to the general setting.

Lemma 3. *For integers h, k, m such that $h \geq 1$ and $-h < m < h$, there exists a length-preserving injection of $\mathcal{W}_k^{(m,2h)}$ into $\mathcal{W}_k^{(h,h+m)} \cup \mathcal{W}_k^{(-h,h-m)}$. (Note that there is no condition on k.)*

Proof. We give such an injection ψ. Pick an arbitrary walk $W \in \mathcal{W}_k^{(m,2h)}$. Let $\rightarrow_t$, $\uparrow_t$, and $\downarrow_t$ be the number of right, up, and down steps, respectively, among the first t steps of W. Let t_0 be the smallest natural number which is a solution of one of the following equations:

$$\rightarrow_t - \uparrow_t = h - m, \tag{4}$$

$$\downarrow_t - \rightarrow_t = h + m. \tag{5}$$

Note that at $t = 0$ the left-hand side is less than the right-hand side at both equations, but summing up the two equations, we get $\downarrow_t - \uparrow_t = 2h$, what happens to be true after the last step. Taking into account that $\rightarrow_t - \uparrow_t$ and $\downarrow_t - \rightarrow_t$ change at most one in each step, we conclude that such a t_0 exists, and is strictly less than the length of W. We note that t_0 cannot be a solution of both equations, since in Case 1 (that is when t_0 satisfies (4)) the t_0^{th} step is a right step, while in Case 2 (that is when t_0 satisfies (5)) it must be a down step.

In Case 1, we define $\psi(W)$ by the following method. It starts from the point $(h, h + m)$. We get the first t_0 steps of $\psi(W)$ from the first t_0 steps of W by interchanging the right and up steps (and leaving the rest unchanged), and we get the last $|W| - t_0$ steps of $\psi(W)$ by keeping these steps of W. It can be easily seen that the first (t_0-step) sections of W and $\psi(W)$ end at the same point. The two walks from here are identical. To show that $\psi(W)$ is in $\mathcal{W}_k^{(h,h+m)}$, we have to see that $\psi(W)$ does not meet the x-axis before the last step. It is obvious that there is no problem with the last $((|W| - t_0)$-step) part of $\psi(W)$, we have to check the first section. For any $t \leq t_0$, we have $\downarrow_t - \rightarrow_t < h + m$ for W, and hence we have $\downarrow_t - \uparrow_t < h + m$ in case of $t \leq t_0$ for $\psi(W)$. This means exactly that the walk remains above the x-axis.

In Case 2, we define $\psi(W)$ in a similar way, but we start $\psi(W)$ from $(-h, h-m)$, and in the first (t_0-step) section we will interchange the down and right steps. Now $\psi(W) \in \mathcal{W}_k^{(-h,h-m)}$.

The given map ψ is clearly length-preserving, and it is easy to see that it is also injective. This is left to the reader. ■

We note that, as can be immediately deduced from result (1.4) of [GKS], the number of l-step walks of $\mathcal{W}_{k_2}^{(k_1,h)}$ has the closed form, with the notation $k = k_2 - k_1$,

$$\binom{l-1}{(l+k-h)/2}\binom{l-1}{(l+k+h-2)/2} - \binom{l-1}{(l+k-h-2)/2}\binom{l-1}{(l+k+h)/2},$$

from which another proof of Lemma 2 can be obtained, as the required inequalities can be verified by an elementary (but a bit tedious) calculation. (In the above formula, the binomial coefficient $\binom{l-1}{r}$ is defined to be 0, if $r \notin \{0, \dots, l-1\}$.)

3. Further results and problems

After scaling by h^{-1}, a random walk on $\mathbb{Z}^2$ starting from $(0, h)$ can be viewed as a random walk on the grid $h^{-1}\mathbb{Z} \times h^{-1}\mathbb{Z}$, starting from $(0, 1)$. It is well known that as the grid size

h^{-1} tends to 0, the discrete random walk on $h^{-1}\mathbb{Z} \times h^{-1}\mathbb{Z}$ tends to the planar Brownian motion (roughly speaking). It is also known that the (abscissa of the) random point where the planar Brownian motion, starting from $(0,1)$, first hits the x-axis follows standard Cauchy distribution with density function $\frac{1}{\pi(1+x^2)}$. Thus we conclude that, for any fixed x,

$$\lim_{h\to\infty} \sum_{k=-\infty}^{\lfloor hx \rfloor} p_k^h = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,\mathrm{d}t;$$

see [S, Chapter 3, p. 156] for a rigorous proof. Since the function $\frac{1}{\pi(1+x^2)}$ is concave on the interval $(0, \frac{1}{\sqrt{3}})$ and convex on $(\frac{1}{\sqrt{3}}, \infty)$, this suggests that, for large h, the probability sequence $\{p_k^h\}_{k=0}^{K}$ is concave and $\{p_k^h\}_{k=K}^{\infty}$ is convex for some constant $K \sim \frac{h}{\sqrt{3}}$. A plausible next step would be to prove concavity for $k \le \alpha h$ with some constant $\alpha > 0$, because this would show that our convexity threshold $h-2$ is optimal up to constant factors. There is also room to sharpen this threshold, i.e. $h-2$ can probably be replaced with βh, for a better constant $\beta < 1$.

It is natural to check whether any of these goals can be achieved by constructing an injective length-preserving function between the sets of Lemma 2, as above. The following theorem shows that the answer is no, and somewhat surprisingly, a "nice" critical length arises. We do not see any combinatorial proof for this fact.

Theorem 4. *Let $h \ge 2$ and k be fixed, and let H_l [and V_l] denote the number of l-length walks in $\mathcal{W}_k^h$ that start with a horizontal (left or right) step [or vertical (up or down) step].*

- *If $l = h^2 - k^2$, then $H_l = V_l$.*
- *If $l \ge h^2 - k^2$, then $H_l \ge V_l$.*
- *If $l \le h^2 - k^2$, then $H_l \le V_l$.*

And if $l \ne h^2 - k^2$, then $H_l = V_l$ can occur only if $H_l = V_l = 0$, i.e. if l is such that $\mathcal{W}_k^h$ does not contain any walk of length l.

Sketch of proof. This can be seen from the closed formula discussed at the end of Section 2. We omit the details, since no ideas are needed, just some elementary but tedious calculation. ∎

We note that a non-constructive proof of Lemma 2 follows from this theorem. Moreover, we obtained that for $0 \le k < h-1$, as $h^2 - k^2 > h + k$ then, both $H_l > V_l$ and $H_l < V_l$ can occur as varying l. (The first inequality always holds for large enough lengths l of the appropriate parity, and in this case the second one holds for $l = h+k$, for example, as $H_l, V_l \ne 0$.) This means that Lemma 2 cannot be strengthened, for $0 \le k < h-1$, no length-preserving injection exists between the sets (in any direction). So one needs more sophisticated estimates of (2) using the weighted sum (3) to handle the convexity of $\{p_k^h\}_{k=0}^{\infty}$ on the interval $[0, k-2]$.

Remark. By an analogous calculation to the proof of Theorem 4, it can be verified that, for $h \ge 2$, among the $(h-k)(2h-1)$-length walks of $\mathcal{W}_k^h$, there are as many walks starting with a right step as there are starting with a down step. (Moreover, there are more walks of the first type for larger lengths, and there are more walks of the second type for smaller lengths in the non-degenerate cases.) We note that the critical length $(h-k)(2h-1)$ is valid for negative k values, too. Thus, by symmetry, the number of walks of $\mathcal{W}_k^h$ starting with a left step can be compared with the number of walks of $\mathcal{W}_k^h$ starting with a down step for any fixed length, the critical length is $(h+k)(2h-1)$ here. And the similar "right versus up" and "left versus up" comparisons can be trivially reduced to the former ones.

We end with the higher-dimensional analogue of our main theorem. Since the 2-dimensional result implies the higher-dimensional one in essentially the same way as in [NT], we only sketch the proof here.

We start with some notations and definitions. The standard basis vectors of the n-dimensional space are denoted by $\boldsymbol{e_1}, \dots, \boldsymbol{e_n}$, where $\boldsymbol{e_i}$ is the vector with ith coordinate 1 and all others zero. For a point $\boldsymbol{k} \in \mathbb{Z}^n$, let $N(\boldsymbol{k})$ denote the set of $2n$ neighbors of $\boldsymbol{k}$ in $\mathbb{Z}^n$, i.e. $N(\boldsymbol{k}) := \{\boldsymbol{k} \pm \boldsymbol{e_i} : i = 1, \dots, n\}$. We say that the discrete function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is *subharmonic* on $U \subset \mathbb{Z}^n$, if for all $\boldsymbol{k} \in U$ such that $N(\boldsymbol{k}) \subset U$,

$$f(\boldsymbol{k}) \le \frac{1}{2n} \sum_{\boldsymbol{j} \in N(\boldsymbol{k})} f(\boldsymbol{j}). \tag{6}$$

Fix an arbitrary dimension $d \ge 2$. The discrete walks on $\mathbb{Z}^d$ are defined analogously to the 2-dimensional case; now there are $2d$ possible steps, the steps $\pm \boldsymbol{e_i}$. For a given $h \in \mathbb{N}$ and $\boldsymbol{k} = (k_1, \dots, k_{d-1}) \in \mathbb{Z}^{d-1}$, let $p_{\boldsymbol{k}}^h$ denote the probability that a symmetric random walk on $\mathbb{Z}^d$, started from $(0, \dots, 0, h)$, first hits the hyperplane $x_d = 0$ at the point $(k_1, \dots, k_{d-1}, 0)$.

In [NT] it has been proved that $p_{\boldsymbol{k}}^1$ is a subharmonic function on $\mathbb{Z}^{d-1} \setminus \{\boldsymbol{0}\}$, of variable $\boldsymbol{k}$. (In fact, slightly more has been showed: inequality (6) holds for all $\boldsymbol{k} \neq \boldsymbol{0}$.) From Lemma 2, a similar result can be obtained for $h \ge 2$ as well, which is a generalization of Theorem 1.

Theorem 5. *For arbitrary fixed $h \ge 2$, the function $\mathbb{Z}^{d-1} \ni \boldsymbol{k} \mapsto p_{\boldsymbol{k}}^h$ is subharmonic on the set $[h-2, \infty)^{d-1} \cap \mathbb{Z}^{d-1}$.*

Sketch of proof. Pick an arbitrary $\boldsymbol{k} = (k_1, \dots, k_{d-1}) \in \mathbb{Z}^{d-1}$ such that $k_i \ge h-1$ for all i. We have to show that

$$p_{\boldsymbol{k}}^h \le \frac{1}{2(d-1)} \sum_{\boldsymbol{j} \in N(\boldsymbol{k})} p_{\boldsymbol{j}}^h.$$

Analogously to the way (2) was derived, we obtain the equivalent inequality

$$(d-1)(p_{\boldsymbol{k}}^{h+1} + p_{\boldsymbol{k}}^{h-1}) \le \sum_{\boldsymbol{j} \in N(\boldsymbol{k})} p_{\boldsymbol{j}}^h,$$

which will follow by summing the inequalities (for $i = 1, \dots, d-1$)

$$p_{\boldsymbol{k}}^{h+1} + p_{\boldsymbol{k}}^{h-1} \le p_{\boldsymbol{k}-\boldsymbol{e_i}}^h + p_{\boldsymbol{k}+\boldsymbol{e_i}}^h. \tag{7}$$

To see (7) for a fixed i, it is enough to give a length-preserving injection from the set of "positive" $(0,\dots,0,h\pm 1)\rightsquigarrow(k_1,\dots,k_{d-1},0)$ walks to the set of "positive" $(0,\dots,0,h)\rightsquigarrow(k_1,\dots,k_{i-1},k_i\pm 1,k_{i+1},\dots,k_{d-1},0)$ walks. (We denote these sets by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.) Such an injection can be easily constructed using Lemma 2. We can think of the d-dimensional steps $\boldsymbol{e_d}$, $-\boldsymbol{e_d}$, $-\boldsymbol{e_i}$, and $\boldsymbol{e_i}$ as up, down, left, and right steps, respectively. (Note that, with a slight abuse of notation, $\boldsymbol{e_i}$ is a $(d-1)$-dimensional vector in (7), while it is a d-dimensional vector here.) These steps, interpreting them as 2-dimensional steps, form a walk of $\mathcal{W}_{k_i}^{h+1}\cup\mathcal{W}_{k_i}^{h-1}$ for the walks in $\mathcal{S}_1$, and they form a walk of $\mathcal{W}_{k_i}^{(-1,h)}\cup\mathcal{W}_{k_i}^{(1,h)}$ for the walks in $\mathcal{S}_2$. It is easy to see that if we convert these four types of steps in the walks of $\mathcal{S}_1$ by applying the injection of Lemma 2 to the walk they form and leaving the other types of steps unchanged, we obtain a length-preserving injection $\mathcal{S}_1\to\mathcal{S}_2$. Recall that $k_i\ge h-1$, that is why we could use Lemma 2. ∎

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