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On a Poisson–Lie deformation of the BC_n Sutherland system

L. Fehér [a,b], T.F. Görbe [a,*]

[a] *Department of Theoretical Physics, University of Szeged, Tisza Lajos krt 84-86, H-6720 Szeged, Hungary*
[b] *Department of Theoretical Physics, WIGNER RCP, RMKI, H-1525 Budapest, P.O.B. 49, Hungary*

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Abstract

A deformation of the classical trigonometric BC_n Sutherland system is derived via Hamiltonian reduction of the Heisenberg double of $SU(2n)$. We apply a natural Poisson–Lie analogue of the Kazhdan–Kostant–Sternberg type reduction of the free particle on $SU(2n)$ that leads to the BC_n Sutherland system. We prove that this yields a Liouville integrable Hamiltonian system and construct a globally valid model of the smooth reduced phase space wherein the commuting flows are complete. We point out that the reduced system, which contains 3 independent coupling constants besides the deformation parameter, can be recovered (at least on a dense submanifold) as a singular limit of the standard 5-coupling deformation due to van Diejen. Our findings complement and further develop those obtained recently by Marshall on the hyperbolic case by reduction of the Heisenberg double of $SU(n,n)$.

1. Introduction

Models amenable to exact treatment provide key paradigms for our understanding of natural phenomena and form a fertile field of research crossing the border of physics and mathematics. The study of integrable Hamiltonian systems is a very active subfield with particularly strong

* Corresponding author.
E-mail addresses: lfeher@physx.u-szeged.hu (L. Fehér), tfgorbe@physx.u-szeged.hu (T.F. Görbe).

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ties to group theory and symplectic geometry. For reviews, see e.g. [9,22,30,5,8]. One of the time-honored approaches to such systems consists in viewing them as 'shadows' of natural free systems enjoying high symmetries. This is alternatively known as the projection method or as Hamiltonian reduction [24,25]. The list of the free 'master systems' is monotonically expanding in time. To name a few, it includes free particles on Lie groups together with their Poisson–Lie symmetric deformations and quasi-Hamiltonian analogues. For example, it was shown in the pioneering paper [17] that the integrable many-body system of Sutherland [34], which describes particles on the circle interacting via a pair potential given by the inverse square of the chord-distance, is a reduction of the free particle on the unitary group $\mathrm{U}(n)$. Various deformations of the Sutherland system due to Ruijsenaars and Schneider [31,29] were derived [11,12] from Poisson–Lie symmetric free motion on $\mathrm{U}(n)$, whose phase space is the Heisenberg double [33] of the Poisson–Lie group $\mathrm{U}(n)$, and from the internally fused quasi-Hamiltonian double [2] of $\mathrm{U}(n)$, which arose from Chern–Simons field theory.

The projection method was enriched by an interesting recent contribution of Marshall [20], who obtained an integrable Ruijsenaars–Schneider (RS) type system by reducing the Heisenberg double of $\mathrm{SU}(n,n)$, which directly motivated our present work.[1] Here, we shall deal with a reduction of the Heisenberg double of $\mathrm{SU}(2n)$ and derive a Liouville integrable Hamiltonian system related to Marshall's one in a way similar to the connection between the original trigonometric Sutherland system and its hyperbolic variant. Although this is essentially analytic continuation, it should be noted that the resulting systems are qualitatively different in their dynamical characteristics and global features. In addition, what we hope makes our work worthwhile is that our treatment is different from the one in [20] in several respects and we go considerably further regarding the global characterization of the reduced phase space and the completeness of the relevant Hamiltonian flows.

The main Hamiltonian of the system that we obtain can be displayed as follows

$$
\begin{aligned}
H(\hat{p},\hat{q};x,u,v) = {} & \frac{e^{-2u}+e^{2v}}{2}\sum_{j=1}^{n} e^{-2\hat{p}_j} + \\
& -\sum_{j=1}^{n}\cos(\hat{q}_j)\big[1-(1+e^{2(v-u)})e^{-2\hat{p}_j}+e^{2(v-u)}e^{-4\hat{p}_j}\big]^{\frac{1}{2}} \\
& \times \prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{\sinh^2\left(\frac{x}{2}\right)}{\sinh^2(\hat{p}_j-\hat{p}_k)}\right]^{\frac{1}{2}}.
\end{aligned}
\tag{1.1}
$$

Here u, v and x are real coupling parameters that will be assumed to satisfy

$$u<v, \quad v\neq -u \quad \text{and} \quad x\neq 0. \tag{1.2}$$

The components of $\hat{q}$ parametrize the torus $\mathbb{T}_n$ by $e^{\mathrm{i}\hat{q}}$ and $\hat{p}$ belongs to the domain

$$\mathcal{C}_x := \{\hat{p}\in\mathbb{R}^n \mid 0>\hat{p}_1,\ \hat{p}_k-\hat{p}_{k+1}>|x|/2\ (k=1,\dots,n-1)\}. \tag{1.3}$$

The dynamics is then defined via the symplectic form

$$\hat{\omega}=\sum_{j=1}^{n} d\hat{q}_j\wedge d\hat{p}_j. \tag{1.4}$$

[1] The relation is 'symmetric' as the problem studied by Marshall was originally suggested by one of us.

It will be shown that this system results by restricting a reduced free system on a dense open submanifold of the pertinent reduced phase space. The Hamiltonian flow is complete on the full reduced phase space, but it can leave the submanifold parametrized by $\mathcal{C}_x \times \mathbb{T}_n$. By glancing at the form of the Hamiltonian, one may say that it represents an RS type system coupled to external fields. Since differences of the 'position variables' $\hat{p}_k$ appear, one feels that this Hamiltonian somehow corresponds to an A-type root system.

To better understand the nature of this model, let us now introduce new Darboux variables q_k, p_k following essentially [20] as

$$\exp(\hat{p}_k) = \sin(q_k) \quad \text{and} \quad \hat{q}_k = p_k \tan(q_k). \tag{1.5}$$

In terms of these variables $H(\hat{p}, \hat{q}; x, u, v) = \mathcal{H}_1(q, p; x, u, v)$ with the 'new Hamiltonian'

$$\begin{aligned}\mathcal{H}_1(q, p; x, u, v) = {} & \frac{e^{-2u} + e^{2v}}{2} \sum_{j=1}^{n} \frac{1}{\sin^2(q_j)} \\ & - \sum_{j=1}^{n} \cos(p_j \tan(q_j)) \left[1 - \frac{1 + e^{2(v-u)}}{\sin^2(q_j)} + \frac{4e^{2(v-u)}}{4\sin^2(q_j) - \sin^2(2q_j)}\right]^{\frac{1}{2}} \\ & \times \prod_{\substack{k=1 \\ (k\neq j)}}^{n} \left[1 - \frac{2\sinh^2\left(\frac{x}{2}\right)\sin^2(q_j)\sin^2(q_k)}{\sin^2(q_j - q_k)\sin^2(q_j + q_k)}\right]^{\frac{1}{2}}.\end{aligned} \tag{1.6}$$

Remarkably, only such combinations of the new 'position variables' q_k appear that are naturally associated with the BC_n root system and the Hamiltonian $\mathcal{H}_1$ enjoys symmetry under the corresponding Weyl group. Thus now one may wish to attach the Hamiltonian $\mathcal{H}_1$ to the BC_n root system. Indeed, this interpretation is preferable for the following reason. Introduce the scale parameter (corresponding to the inverse of the velocity of light in the original RS system) $\beta > 0$ and make the substitutions

$$u \to \beta u, \quad v \to \beta v, \quad x \to \beta x, \quad p \to \beta p, \quad \hat{\omega} \to \beta\hat{\omega}. \tag{1.7}$$

Then consider the deformed Hamiltonian

$$\mathcal{H}_\beta(q, p; x, u, v) := \mathcal{H}_1(q, \beta p; \beta x, \beta u, \beta v). \tag{1.8}$$

The point is that one can then verify the following relation:

$$\lim_{\beta\to 0} \frac{\mathcal{H}_\beta(q, p; x, u, v) - n}{\beta^2} = H^{\mathrm{Suth}}_{\mathrm{BC}_n}(q, p; \gamma, \gamma_1, \gamma_2), \tag{1.9}$$

where

$$\begin{aligned}H^{\mathrm{Suth}}_{\mathrm{BC}_n} = {} & \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\leq j<k\leq n} \left[\frac{\gamma}{\sin^2(q_j - q_k)} + \frac{\gamma}{\sin^2(q_j + q_k)}\right] \\ & + \sum_{j=1}^{n} \frac{\gamma_1}{\sin^2(q_j)} + \sum_{j=1}^{n} \frac{\gamma_2}{\sin^2(2q_j)}\end{aligned} \tag{1.10}$$

is the standard trigonometric BC_n Sutherland Hamiltonian with coupling constants

$$\gamma = \frac{x^2}{4}, \quad \gamma_1 = 2uv, \quad \gamma_2 = 2(v-u)^2. \tag{1.11}$$

Note that the domain of the variables $\hat{q}$, $\hat{p}$, and correspondingly that of q, p also depends on β, and in the $\beta \to 0$ limit it is easily seen that we recover the usual BC_n domain

$$\frac{\pi}{2} > q_1 > q_2 > \cdots > q_n > 0, \quad p \in \mathbb{R}^n. \tag{1.12}$$

In conclusion, we see that H in its equivalent form $\mathcal{H}_\beta$ is a 1-parameter deformation of the trigonometric BC_n Sutherland Hamiltonian. We remark in passing that the conditions (1.2) imply that $\gamma_2 > 0$ and $4\gamma_1 + \gamma_2 > 0$, which guarantee that the flows of $H^{\mathrm{Suth}}_{\mathrm{BC}_n}$ are complete on the domain (1.12).

Marshall [20] obtained similar results for an analogous deformation of the hyperbolic BC_n Sutherland Hamiltonian. His deformed Hamiltonian differs from (1.1) above in some important signs and in the relevant domain of the 'position variables' $\hat{p}$. Although in our impression the completeness of the reduced Hamiltonian flows was not treated in a satisfactory way in [20], the completeness proof that we shall present can be adapted to Marshall's case as well.

It is natural to ask how the system studied in the present paper (and its cousin in [20]) is related to van Diejen's [35] 5-coupling trigonometric BC_n system? It was shown already in [35] that the 5-coupling trigonometric system is a deformation of the BC_n Sutherland system, and later [36] several other integrable systems were also derived as its ('Inozemtsev type' [16]) limits. Motivated by this, we can show that the Hamiltonian (1.1) is a singular[2] limit of van Diejen's general Hamiltonian. Incidentally, a Hamiltonian of Schneider [32] can be viewed as a subsequent singular limit of the Hamiltonian (1.1). Schneider's system was mentioned in [20], too, but the relation to van Diejen's system was not described.

The original idea behind the present work and [20] was that a natural Poisson–Lie analogue of the Hamiltonian reduction treatment [13] of the BC_n Sutherland system should lead to a deformation of this system. It was expected that a special case of van Diejen's standard 5-coupling deformation will arise. The expectation has now been confirmed, although it came as a surprise that a singular limit is involved in the connection.

The outline of the paper is as follows. We start in Section 2 by defining the reduction of interest. In Section 3 we observe that several technical results of [11] can be applied for analyzing the reduction at hand, and solve the momentum map constraints by taking advantage of this observation. The heart of the paper is Section 4, where we characterize the reduced system. In Subsection 4.1 we prove that the reduced phase space is smooth, as formulated in Theorem 4.4. Then in Subsection 4.2 we focus on a dense open submanifold on which the Hamiltonian (1.1) lives. The demonstration of the Liouville integrability of the reduced free flows is given in Subsection 4.3. In particular, we prove the integrability of the completion of the system (1.1) carried by the full reduced phase space. Our main result is Theorem 4.9 (proved in Subsection 4.4), which establishes a globally valid model of the reduced phase space. We stress that the global structure of the phase space on which the flow of (1.1) is complete was not considered previously at all, and will be clarified as a result of our group theoretic interpretation. Section 5 contains our conclusions, further comments on the related paper by Marshall [20] and a discussion of open problems. The main text is complemented by four appendices. Appendix A deals with the connection to van Diejen's system; the other 3 appendices contain important details relegated from the main text.

[2] We call the limit singular since it involves sending some shifted position variables to infinity.

2. Definition of the Hamiltonian reduction

We below introduce the 'free' Hamiltonians and define their reduction. We restrict the presentation of this background material to a minimum necessary for understanding our work. The conventions follow [11], which also contains more details. As a general reference, we recommend [7].

2.1. The unreduced free Hamiltonians

We fix a natural number[3] $n \geq 2$ and consider the Lie group $\mathrm{SU}(2n)$ equipped with its standard quadratic Poisson bracket defined by the compact form of the Drinfeld–Jimbo classical r-matrix,

$$r_{\mathrm{DJ}} = \mathrm{i} \sum_{1 \leq \alpha < \beta \leq 2n} E_{\alpha\beta} \wedge E_{\beta\alpha}, \tag{2.1}$$

where $E_{\alpha\beta}$ is the elementary matrix of size $2n$ having a single non-zero entry 1 at the $\alpha\beta$ position. In particular, the Poisson brackets of the matrix elements of $g \in \mathrm{SU}(2n)$ obey Sklyanin's formula

$$\{g \overset{\otimes}{,} g\}_{\mathrm{SU}(2n)} = [g \otimes g, r_{\mathrm{DJ}}]. \tag{2.2}$$

Thus $\mathrm{SU}(2n)$ becomes a Poisson–Lie group, i.e., the multiplication $\mathrm{SU}(2n) \times \mathrm{SU}(2n) \to \mathrm{SU}(2n)$ is a Poisson map. The cotangent bundle $T^* \mathrm{SU}(2n)$ possesses a natural Poisson–Lie analogue, the so-called Heisenberg double [33], which is provided by the real Lie group $\mathrm{SL}(2n, \mathbb{C})$ endowed with a certain symplectic form [1], ω. To describe ω, we use the Iwasawa decomposition and factorize every element $K \in \mathrm{SL}(2n, \mathbb{C})$ in two alternative ways

$$K = g_L b_R^{-1} = b_L g_R^{-1} \tag{2.3}$$

with uniquely determined

$$g_L, g_R \in \mathrm{SU}(2n), \quad b_L, b_R \in \mathrm{SB}(2n). \tag{2.4}$$

Here $\mathrm{SB}(2n)$ stands for the subgroup of $\mathrm{SL}(2n, \mathbb{C})$ consisting of upper triangular matrices with positive diagonal entries. The symplectic form ω reads

$$\omega = \frac{1}{2} \Im \operatorname{tr}(db_L b_L^{-1} \wedge dg_L g_L^{-1}) + \frac{1}{2} \Im \operatorname{tr}(db_R b_R^{-1} \wedge dg_R g_R^{-1}). \tag{2.5}$$

Before specifying free Hamiltonians on the phase space $\mathrm{SL}(2n, \mathbb{C})$, note that any smooth function h on $\mathrm{SB}(2n)$ corresponds to a function $\tilde{h}$ on the space of positive definite Hermitian matrices of determinant 1 by the relation

$$\tilde{h}(bb^\dagger) = h(b), \quad \forall b \in \mathrm{SB}(2n). \tag{2.6}$$

Then introduce the invariant functions

$$C^\infty(\mathrm{SB}(2n))^{\mathrm{SU}(2n)} \equiv \{h \in C^\infty(\mathrm{SB}(2n)) \mid \tilde{h}(bb^\dagger) = \tilde{h}(gbb^\dagger g^{-1}),\ \forall g \in \mathrm{SU}(2n), b \in \mathrm{SB}(2n)\}. \tag{2.7}$$

[3] The $n = 1$ case would need special treatment and is excluded in order to simplify the presentation.

These in turn give rise to the following ring of functions on $\mathrm{SL}(2n,\mathbb{C})$:

$$\mathfrak{H} \equiv \{\mathcal{H} \in C^\infty(\mathrm{SL}(2n,\mathbb{C})) \mid \mathcal{H}(g_L b_R^{-1}) = h(b_R),\ h \in C^\infty(\mathrm{SB}(2n))^{\mathrm{SU}(2n)}\}, \tag{2.8}$$

where we utilized the decomposition (2.3). An important point is that $\mathfrak{H}$ forms an Abelian algebra with respect to the Poisson bracket associated with ω (2.5).

The flows of the 'free' Hamiltonians contained in $\mathfrak{H}$ can be obtained effortlessly. To describe the result, define the derivative $d^R f \in C^\infty(\mathrm{SB}(2n), \mathfrak{su}(2n))$ of any real function $f \in C^\infty(\mathrm{SB}(2n))$ by requiring

$$\left.\frac{d}{ds}\right|_{s=0} f(be^{sX}) = \Im \operatorname{tr}\left(X d^R f(b)\right), \quad \forall b \in \mathrm{SB}(2n),\ \forall X \in \mathrm{Lie}(\mathrm{SB}(2n)). \tag{2.9}$$

The Hamiltonian flow generated by $\mathcal{H} \in \mathfrak{H}$ through the initial value $K(0) = g_L(0) b_R(0)^{-1}$ is in fact given by

$$K(t) = g_L(0) \exp\left[-t d^R h(b_R(0))\right] b_R^{-1}(0), \tag{2.10}$$

where $\mathcal{H}$ and h are related according to (2.8). This means that $g_L(t)$ follows the orbit of a one-parameter subgroup, while $b_R(t)$ remains constant. Actually, $g_R(t)$ also varies along a similar orbit, and $b_L(t)$ is constant.

The constants of motion b_L and b_R generate a Poisson–Lie symmetry, which allows one to define Marsden–Weinstein type [19] reductions.

2.2. *Generalized Marsden–Weinstein reduction*

The free Hamiltonians in $\mathfrak{H}$ are invariant with respect to the action of $\mathrm{SU}(2n) \times \mathrm{SU}(2n)$ on $\mathrm{SL}(2n,\mathbb{C})$ given by left- and right-multiplications. This is a Poisson–Lie symmetry, which means that the corresponding action map

$$\mathrm{SU}(2n) \times \mathrm{SU}(2n) \times \mathrm{SL}(2n,\mathbb{C}) \to \mathrm{SL}(2n,C), \tag{2.11}$$

operating as

$$(\eta_L, \eta_R, K) \mapsto \eta_L K \eta_R^{-1}, \tag{2.12}$$

is a Poisson map. In (2.11) the product Poisson structure is taken using the Sklyanin bracket on $\mathrm{SU}(2n)$ and the Poisson structure on $\mathrm{SL}(2n,\mathbb{C})$ associated with the symplectic form ω (2.5). This Poisson–Lie symmetry admits a momentum map in the sense of Lu [18], given explicitly by

$$\Phi\colon\ \mathrm{SL}(2n,\mathbb{C}) \to \mathrm{SB}(2n) \times \mathrm{SB}(2n), \quad \Phi(K) = (b_L, b_R). \tag{2.13}$$

The key property of the momentum map is represented by the identity

$$\left.\frac{d}{ds}\right|_{s=0} f(e^{sX} K e^{-sY}) = \Im \operatorname{tr}\left(X\{f, b_L\} b_L^{-1} + Y\{f, b_R\} b_R^{-1}\right), \quad \forall X, Y \in \mathfrak{su}(2n), \tag{2.14}$$

where $f \in C^\infty(\mathrm{SL}(2n,\mathbb{C}))$ is an arbitrary real function and the Poisson bracket is the one corresponding to ω (2.5). The map Φ enjoys an equivariance property and one can [18] perform Marsden–Weinstein type reduction in the same way as for usual Hamiltonian actions (for which the symmetry group has vanishing Poisson structure). To put it in a nutshell, any $\mathcal{H} \in \mathfrak{H}$ gives rise to a reduced Hamiltonian system by fixing the value of Φ and subsequently taking quotient with respect to the corresponding isotropy group. The reduced flows can be obtained by the standard restriction–projection algorithm, and under favorable circumstances the reduced phase space is a smooth symplectic manifold.

Now, consider the block-diagonal subgroup

$$G_+ := \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n)) < \mathrm{SU}(2n). \tag{2.15}$$

Since G_+ is also a Poisson submanifold of $\mathrm{SU}(2n)$, the restriction of (2.12) yields a Poisson–Lie action

$$G_+ \times G_+ \times \mathrm{SL}(2n, \mathbb{C}) \to \mathrm{SL}(2n, \mathbb{C}) \tag{2.16}$$

of $G_+ \times G_+$. The momentum map for this action is provided by projecting the original momentum map Φ as follows. Let us write every element $b \in \mathrm{SB}(2n)$ in the block-form

$$b = \begin{bmatrix} b(1) & b(12) \\ \mathbf{0}_n & b(2) \end{bmatrix} \tag{2.17}$$

and define $G_+^* < \mathrm{SB}(2n)$ to be the subgroup for which $b(12) = \mathbf{0}_n$. If $\pi:\ \mathrm{SB}(2n) \to G_+^*$ denotes the projection

$$\pi: \begin{bmatrix} b(1) & b(12) \\ \mathbf{0}_n & b(2) \end{bmatrix} \mapsto \begin{bmatrix} b(1) & \mathbf{0}_n \\ \mathbf{0}_n & b(2) \end{bmatrix}, \tag{2.18}$$

then the momentum map $\Phi_+:\ \mathrm{SL}(2n, \mathbb{C}) \to G_+^* \times G_+^*$ is furnished by

$$\Phi_+(K) = (\pi(b_L), \pi(b_R)). \tag{2.19}$$

Indeed, it is readily checked that the analogue of (2.14) holds with X, Y taken from the block-diagonal subalgebra of $\mathfrak{su}(2n)$ and b_L, b_R replaced by their projections. The equivariance property of this momentum map means that in correspondence to

$$K \mapsto \eta_L K \eta_R^{-1} \quad \text{with} \quad (\eta_L, \eta_R) \in G_+ \times G_+, \tag{2.20}$$

one has

$$\big(\pi(b_L)\pi(b_L)^\dagger, \pi(b_R)\pi(b_R)^\dagger\big) \mapsto \big(\eta_L \pi(b_L)\pi(b_L)^\dagger \eta_L^{-1}, \eta_R \pi(b_R)\pi(b_R)^\dagger \eta_R^{-1}\big). \tag{2.21}$$

We briefly mention here that, as the notation suggests, G_+^* is itself a Poisson–Lie group that can serve as a Poisson dual of G_+. The relevant Poisson structure can be obtained by identifying the block-diagonal subgroup of $\mathrm{SB}(2n)$ with the factor group $\mathrm{SB}(2n)/L$, where L is the block-upper-triangular normal subgroup. This factor group inherits a Poisson structure from $\mathrm{SB}(2n)$, since L is a so-called coisotropic (or 'admissible') subgroup of $\mathrm{SB}(2n)$ equipped with its standard Poisson structure. The projected momentum map Φ_+ is a Poisson map with respect to this Poisson structure on the two factors G_+^* in (2.19). The details are not indispensable for us. The interested reader may find them e.g. in [6].

Inspired by the papers [13,11,20], we wish to study the particular Marsden–Weinstein reduction defined by imposing the following momentum map constraint:

$$\Phi_+(K) = \mu \equiv (\mu_L, \mu_R), \quad \text{where} \quad \mu_L = \begin{bmatrix} e^u \nu(x) & \mathbf{0}_n \\ \mathbf{0}_n & e^{-u}\mathbf{1}_n \end{bmatrix}, \quad \mu_R = \begin{bmatrix} e^v \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & e^{-v}\mathbf{1}_n \end{bmatrix} \tag{2.22}$$

with some real constants u, v and x. Here, $\nu(x) \in \mathrm{SB}(n)$ is the $n \times n$ upper triangular matrix defined by

$$\nu(x)_{jj} = 1, \quad \nu(x)_{jk} = (1 - e^{-x})e^{\frac{(k-j)x}{2}}, \quad j < k, \tag{2.23}$$

whose main property is that $\nu(x)\nu(x)^\dagger$ has the largest possible non-trivial isotropy group under conjugation by the elements of $\mathrm{SU}(n)$.

Our principal task is to characterize the reduced phase space

$$M \equiv \Phi_+^{-1}(\mu)/G_\mu, \tag{2.24}$$

where $\Phi_+^{-1}(\mu) = \{K \in \mathrm{SL}(2n, \mathbb{C}) \mid \Phi_+(K) = \mu\}$ and

$$G_\mu = G_+(\mu_L) \times G_+ \tag{2.25}$$

is the isotropy group of μ inside $G_+ \times G_+$. Concretely, $G_+(\mu_L)$ is the subgroup of G_+ consisting of the special unitary matrices of the form

$$\eta_L = \begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix}, \tag{2.26}$$

where $\eta_L(2)$ is arbitrary and

$$\eta_L(1)\nu(x)\nu(x)^\dagger \eta_L(1)^{-1} = \nu(x)\nu(x)^\dagger. \tag{2.27}$$

In words, $\eta_L(1)$ belongs to the little group of $\nu(x)\nu(x)^\dagger$ in $\mathrm{U}(n)$. We shall see that $\Phi_+^{-1}(\mu)$ and M are smooth manifolds for which the canonical projection

$$\pi_\mu \colon \Phi_+^{-1}(\mu) \to M \tag{2.28}$$

is a smooth submersion. Then M (2.24) inherits a symplectic form ω_M from ω (2.5), which satisfies

$$\iota_\mu^*(\omega) = \pi_\mu^*(\omega_M), \tag{2.29}$$

where $\iota_\mu \colon \Phi_+^{-1}(\mu) \to \mathrm{SL}(2n, \mathbb{C})$ denotes the tautological embedding.

3. Solution of the momentum map constraints

The description of the reduced phase space requires us to solve the momentum map constraints, i.e., we have to find all elements $K \in \Phi_+^{-1}(\mu)$. Of course, it is enough to do this up to the gauge transformations provided by the isotropy group G_μ (2.25). The solution of this problem will rely on the auxiliary equation (3.11) below, which is essentially equivalent to the momentum map constraint, $\Phi_+(K) = \mu$, and coincides with an equation studied previously in great detail in [11]. Thus we start in the next subsection by deriving this equation.

3.1. A crucial equation implied by the constraints

We begin by recalling (e.g. [21]) that any $g \in \mathrm{SU}(2n)$ can be decomposed as

$$g = g_+ \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} h_+, \tag{3.1}$$

where $g_+, h_+ \in G_+$ and $q = \mathrm{diag}(q_1, \ldots, q_n) \in \mathbb{R}^n$ satisfies

$$\frac{\pi}{2} \geq q_1 \geq \cdots \geq q_n \geq 0. \tag{3.2}$$

The vector q is uniquely determined by g, while g_+ and h_+ suffer from controlled ambiguities.

First, apply the above decomposition to g_L in $K = g_L b_R^{-1} \in \Phi_+^{-1}(\mu)$ and use the right-handed momentum constraint $\pi(b_R) = \mu_R$. It is then easily seen that up to gauge transformations every element of $\Phi_+^{-1}(\mu)$ can be represented in the following form:

$$K = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix}. \tag{3.3}$$

Here $\rho \in \mathrm{SU}(n)$ and α is an $n \times n$ complex matrix. By using obvious block-matrix notation, we introduce $\Omega := K_{22}$ and record from (3.3) that

$$\Omega = \mathrm{i}(\sin q)\alpha + e^{v}\cos q. \tag{3.4}$$

For later purpose we introduce also the polar decomposition of the matrix Ω,

$$\Omega = \Lambda T, \tag{3.5}$$

where $T \in \mathrm{U}(n)$ and the Hermitian, positive semi-definite factor Λ is uniquely determined by the relation $\Omega\Omega^\dagger = \Lambda^2$.

Second, by writing $K = b_L g_R^{-1}$ the left-handed momentum constraint $\pi(b_L) = \mu_L$ tells us that b_L has the block-form

$$b_L = \begin{bmatrix} e^{u}\nu(x) & \chi \\ \mathbf{0}_n & e^{-u}\mathbf{1}_n \end{bmatrix} \tag{3.6}$$

with an $n \times n$ matrix χ. Now we inspect the components of the 2×2 block-matrix identity

$$KK^\dagger = b_L b_L^\dagger, \tag{3.7}$$

which results by substituting K from (3.3). We find that the (22) component of this identity is equivalent to

$$\Omega\Omega^\dagger = \Lambda^2 = e^{-2u}\mathbf{1}_n - e^{-2v}(\sin q)^2. \tag{3.8}$$

On account of the condition (1.2), this uniquely determines Λ in terms of q, and shows also that Λ is invertible. A further important consequence is that we must have

$$q_n > 0, \tag{3.9}$$

and therefore $\sin q$ is an invertible diagonal matrix. Indeed, if $q_n = 0$, then from (3.4) and (3.8) we would get $(\Omega\Omega^\dagger)_{nn} = e^{2v} = e^{-2u}$, which is excluded by (1.2).

Next, one can check that in the presence of the relations already established, the (12) and the (21) components of the identity (3.7) are equivalent to the equation

$$\chi = \rho(\mathrm{i}\sin q)^{-1}[e^{-u}\cos q - e^{u+v}\Omega^\dagger]. \tag{3.10}$$

Observe that K uniquely determines q, T and ρ, and conversely K is uniquely defined by the above relations once q, T and ρ are found.

Now one can straightforwardly check by using the above relations that the (11) component of the identity (3.7) translates into the following equation:

$$\rho(\sin q)^{-1}T^\dagger(\sin q)^2 T(\sin q)^{-1}\rho^\dagger = \nu(x)\nu(x)^\dagger. \tag{3.11}$$

This is to be satisfied by q subject to (3.2), (3.9) and $T \in \mathrm{U}(n)$, $\rho \in \mathrm{SU}(n)$. What makes our job relatively easy is that this is the same as equation (5.7) in the paper [11] by Klimčík and one of us. In fact, this equation was analyzed in detail in [11], since it played a crucial role in that work, too. The correspondence with the symbols used in [11] is

$$(\rho, T, \sin q) \Longleftrightarrow (k_L, k_R^\dagger, e^{\hat{p}}). \tag{3.12}$$

This motivates to introduce the variable $\hat{p} \in \mathbb{R}^n$ in our case, by setting

$$\sin q_k = e^{\hat{p}_k}, \quad k = 1, \dots, n. \tag{3.13}$$

Notice from (3.2) and (3.9) that we have

$$0 \geq \hat{p}_1 \geq \cdots \geq \hat{p}_n > -\infty. \tag{3.14}$$

If the components of $\hat{p}$ are all different, then we can directly rely on [11] to establish both the allowed range of $\hat{p}$ and the explicit form of ρ and T. The statement that $\hat{p}_j \neq \hat{p}_k$ holds for $j \neq k$ can be proved by adopting arguments given in [11,12]. This proof requires combining techniques of [11] and [12], whose extraction from [11,12] is rather involved. We present it in Appendix B, otherwise in the next subsection we proceed by simply stating relevant applications of results from [11].

Remark 3.1. In the context of [11] the components of $\hat{p}$ are not restricted to the half-line and both k_L and k_R vary in $\mathrm{U}(n)$. These slight differences do not pose any obstacle to using the results and techniques of [11,12]. We note that essentially the same equation (3.11) surfaced in [20] as well, but the author of that paper refrained from taking advantage of the previous analyses of this equation. In fact, some statements of [20] are not fully correct. This will be specified (and corrected) in Section 5.

3.2. Consequences of equation (3.11)

We start by pointing out the foundation of the whole analysis. For this, we first display the identity

$$v(x)v(x)^\dagger = e^{-x}\mathbf{1}_n + \operatorname{sgn}(x)\hat{v}\hat{v}^\dagger, \tag{3.15}$$

which holds with a certain n-component vector $\hat{v} = \hat{v}(x)$. By introducing

$$w = \rho^\dagger \hat{v} \tag{3.16}$$

and setting $\hat{p} \equiv \operatorname{diag}(\hat{p}_1, \dots, \hat{p}_n)$, we rewrite equation (3.11) as

$$e^{2\hat{p} - x\mathbf{1}_n} + \operatorname{sgn}(x) e^{\hat{p}} w w^\dagger e^{\hat{p}} = T^{-1} e^{2\hat{p}} T. \tag{3.17}$$

The equality of the characteristic polynomials of the matrices on the two sides of (3.17) gives a polynomial equation that contains $\hat{p}$, the absolute values $|w_j|^2$ and a complex indeterminate. Utilizing the requirement that $|w_j|^2 \geq 0$ must hold, one obtains the following result.

Proposition 3.2. *If K given by (3.3) belongs to the constraint surface $\Phi_+^{-1}(\mu)$, then the vector $\hat{p}$ (3.13) is contained in the closed polyhedron*

$$\bar{\mathcal{C}}_x := \{\hat{p} \in \mathbb{R}^n \mid 0 \geq \hat{p}_1,\ \hat{p}_k - \hat{p}_{k+1} \geq |x|/2 \ (k = 1, \dots, n-1)\}. \tag{3.18}$$

Proposition 3.2 can be proved by merging the proofs of Lemma 5.2 of [11] and Theorem 2 of [12]. This is presented in Appendix B.

The above-mentioned polynomial equality permits to find the possible vectors w (3.16) as well. If $\hat{p}$ and w are given, then T is determined by equation (3.17) up to left-multiplication by a diagonal matrix and ρ is determined by (3.16) up to left-multiplication by elements from the little group of $\hat{v}(x)$. Following this line of reasoning and controlling the ambiguities in the same way as in [11], one can find the explicit form of the most general ρ and T at *any* fixed $\hat{p} \in \bar{\mathcal{C}}_x$. In particular, it turns out that the range of the vector $\hat{p}$ equals $\bar{\mathcal{C}}_x$.

Before presenting the result, we need to prepare some notations. First of all, we pick an arbitrary $\hat{p} \in \bar{\mathcal{C}}_x$ and define the $n \times n$ matrix $\theta(x, \hat{p})$ as follows:

$$\theta(x,\hat{p})_{jk} := \frac{\sinh\left(\frac{x}{2}\right)}{\sinh(\hat{p}_k - \hat{p}_j)} \prod_{\substack{m=1\\(m\neq j,k)}}^{n} \left[\frac{\sinh(\hat{p}_j - \hat{p}_m - \frac{x}{2})\sinh(\hat{p}_k - \hat{p}_m + \frac{x}{2})}{\sinh(\hat{p}_j - \hat{p}_m)\sinh(\hat{p}_k - \hat{p}_m)}\right]^{\frac{1}{2}}, \quad j \neq k, \tag{3.19}$$

and

$$\theta(x,\hat{p})_{jj} := \prod_{\substack{m=1\\(m\neq j)}}^{n} \left[\frac{\sinh(\hat{p}_j - \hat{p}_m - \frac{x}{2})\sinh(\hat{p}_j - \hat{p}_m + \frac{x}{2})}{\sinh^2(\hat{p}_j - \hat{p}_m)}\right]^{\frac{1}{2}}. \tag{3.20}$$

All expressions under square root are non-negative and non-negative square roots are taken. Note that $\theta(x, \hat{p})$ is a real orthogonal matrix of determinant 1 for which $\theta(x, \hat{p})^{-1} = \theta(-x, \hat{p})$ holds, too.

Next, define the real vector $r(x, \hat{p}) \in \mathbb{R}^n$ with non-negative components

$$r(x,\hat{p})_j = \sqrt{\frac{1-e^{-x}}{1-e^{-nx}}} \prod_{\substack{k=1\\(k\neq j)}}^{n} \sqrt{\frac{1-e^{2\hat{p}_j - 2\hat{p}_k - x}}{1-e^{2\hat{p}_j - 2\hat{p}_k}}}, \quad j = 1, \dots, n, \tag{3.21}$$

and the real $n \times n$ matrix $\zeta(x, \hat{p})$,

$$\begin{aligned} &\zeta(x,\hat{p})_{aa} = r(x,\hat{p})_a, \quad \zeta(x,\hat{p})_{ij} = \delta_{ij} - \frac{r(x,\hat{p})_i r(x,\hat{p})_j}{1 + r(x,\hat{p})_a},\\ &\zeta(x,\hat{p})_{ia} = -\zeta(x,\hat{p})_{ai} = r(x,\hat{p})_i, \quad i,j \neq a, \end{aligned} \tag{3.22}$$

where $a = n$ if $x > 0$ and $a = 1$ if $x < 0$. Introduce also the vector $v = v(x)$:

$$v(x)_j = \sqrt{\frac{n(e^x - 1)}{1 - e^{-nx}}}\, e^{-\frac{jx}{2}}, \quad j = 1, \dots, n, \tag{3.23}$$

which is related to $\hat{v}$ in (3.15) by

$$\hat{v}(x) = \sqrt{\operatorname{sgn}(x) e^{-x} \frac{e^{nx} - 1}{n}}\, v(x). \tag{3.24}$$

Finally, define the $n \times n$ matrix $\kappa(x)$ as

$$\begin{aligned} &\kappa(x)_{aa} = \frac{v(x)_a}{\sqrt{n}}, \quad \kappa(x)_{ij} = \delta_{ij} - \frac{v(x)_i v(x)_j}{n + \sqrt{n}\, v(x)_a},\\ &\qquad \kappa(x)_{ia} = -\kappa(x)_{ai} = \frac{v(x)_i}{\sqrt{n}}, \quad i,j \neq a, \end{aligned} \tag{3.25}$$

where, again, $a = n$ if $x > 0$ and $a = 1$ if $x < 0$. It can be shown that both $\kappa(x)$ and $\zeta(x, \hat{p})$ are orthogonal matrices of determinant 1 for any $\hat{p} \in \bar{\mathcal{C}}_x$.

Now we can state the main result of this section, whose proof is omitted since it is a direct application of the analysis of the solutions of (3.11) presented in Section 5 of [11].

Proposition 3.3. *Take any $\hat{p} \in \bar{\mathcal{C}}_x$ and any diagonal unitary matrix $e^{\mathrm{i}\hat{q}} \in \mathbb{T}_n$. By using the preceding notations define $K \in \mathrm{SL}(2n, \mathbb{C})$ (3.3) by setting*

$$T = e^{\mathrm{i}\hat{q}}\theta(-x, \hat{p}), \quad \rho = \kappa(x)\zeta(x, \hat{p})^{-1}, \tag{3.26}$$

and also applying the equations (3.4), (3.5), (3.8) and (3.13). Then the element K belongs to the constraint surface $\Phi_+^{-1}(\mu)$, and every orbit of the gauge group G_μ (2.25) in $\Phi_+^{-1}(\mu)$ intersects the set of elements K just constructed.

Remark 3.4. It is worth spelling out the expression of the element K given by Proposition 3.3. Indeed, we have

$$K(\hat{p}, e^{\mathrm{i}\hat{q}}) = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{1}_n - e^{2\hat{p}}} & \mathrm{i}e^{\hat{p}} \\ \mathrm{i}e^{\hat{p}} & \sqrt{\mathbf{1}_n - e^{2\hat{p}}} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix} \tag{3.27}$$

using the above definitions and

$$\alpha = -\mathrm{i}\left[e^{\mathrm{i}\hat{q}}\sqrt{e^{-2u}e^{-2\hat{p}} - e^{-2v}\mathbf{1}_n}\,\theta(-x, \hat{p}) - e^{v}\sqrt{e^{-2\hat{p}} - \mathbf{1}_n}\right]. \tag{3.28}$$

Remark 3.5. Let us call S the set of the elements $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ constructed above, and observe that this set is homeomorphic to

$$\bar{\mathcal{C}}_x \times \mathbb{T}_n = \{(\hat{p}, e^{\mathrm{i}\hat{q}})\} \tag{3.29}$$

by its very definition. This is not a smooth manifold, because of the presence of the boundary of $\bar{\mathcal{C}}_x$. However, this does not indicate any 'trouble' since it is not true (at the boundary of $\bar{\mathcal{C}}_x$) that S intersects every gauge orbit in $\Phi_+^{-1}(\mu)$ in a *single* point. Indeed, it is instructive to verify that if $\hat{p}$ is the special vertex of $\bar{\mathcal{C}}_x$ for which $\hat{p}_k = (1-k)|x|/2$ for $k = 1, \ldots, n$, then all points $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ lie on a single gauge orbit. This, and further inspection, can lead to the idea that the variables $\hat{q}_j$ should be identified with arguments of complex numbers, which lose their meaning at the origin that should correspond to the boundary of $\bar{\mathcal{C}}_x$. Our Theorem 4.9 will show that this idea is correct. It is proper to stress that we arrived at such idea under the supporting influence of previous works [29,11].

4. Characterization of the reduced system

The smoothness of the reduced phase space and the completeness of the reduced free flows follows immediately if we can show that the gauge group G_μ acts in such a way on $\Phi_+^{-1}(\mu)$ that the isotropy group of every point is just the finite center of the symmetry group. In Subsection 4.1, we prove that the factor of G_μ by the center acts freely on $\Phi_+^{-1}(\mu)$. Then in Subsection 4.2 we explain that $\mathcal{C}_x \times \mathbb{T}_n$ provides a model of a dense open subset of the reduced phase space by means of the corresponding subset of $\Phi_+^{-1}(\mu)$ defined by Proposition 3.3. Adopting a key calculation from [20], it turns out that $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}_x \times \mathbb{T}_n$ are Darboux coordinates on this dense open subset. In Subsection 4.3, we demonstrate that the reduction of the Abelian Poisson algebra of free Hamiltonians (2.8) yields an integrable system. Finally, in Subsection 4.4, we present a model of the full reduced phase space, which is our main result.

4.1. Smoothness of the reduced phase space

It is clear that the normal subgroup of the full symmetry group $G_+ \times G_+$ consisting of matrices of the form

$$(\eta, \eta) \quad \text{with} \quad \eta = \mathrm{diag}(z\mathbf{1}_n, z\mathbf{1}_n), \quad z^{2n} = 1 \tag{4.1}$$

acts trivially on the phase space. This subgroup is contained in G_μ (2.25). The corresponding factor group of G_μ is called 'effective gauge group' and is denoted by $\bar{G}_\mu$. We wish to show that $\bar{G}_\mu$ acts freely on the constraint surface $\Phi_+^{-1}(\mu)$.

We need the following elementary lemmas.

Lemma 4.1. *Suppose that*

$$g_+ \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} h_+ = g'_+ \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} h'_+ \tag{4.2}$$

with $g_+, h_+, g'_+, h'_+ \in G_+$ *and* $q = \mathrm{diag}(q_1, \dots, q_n)$ *subject to*

$$\frac{\pi}{2} \geq q_1 > \cdots > q_n > 0. \tag{4.3}$$

Then there exist diagonal matrices $m_1, m_2 \in \mathbb{T}_n$ *having the form*

$$m_1 = \mathrm{diag}(a, \xi), \quad m_2 = \mathrm{diag}(b, \xi), \quad \xi \in \mathbb{T}_{n-1},\ a, b \in \mathbb{T}_1, \quad \det(m_1 m_2) = 1, \tag{4.4}$$

for which

$$(g'_+, h'_+) = (g_+ \mathrm{diag}(m_1, m_2), \mathrm{diag}(m_2^{-1}, m_1^{-1}) h_+). \tag{4.5}$$

If (4.3) *holds with strict inequality* $\frac{\pi}{2} > q_1$*, then* $m_1 = m_2$*, i.e.,* $a = b$*.*

Lemma 4.2. *Pick any* $\hat{p} \in \bar{\mathcal{C}}_x$ *and consider the matrix* $\theta(x, \hat{p})$ *given by* (3.19) *and* (3.20)*. Then the entries* $\theta_{n,1}(x, \hat{p})$ *and* $\theta_{j,j+1}(x, \hat{p})$ *are all non-zero if* $x > 0$ *and the entries* $\theta_{1,n}(x, \hat{p})$ *and* $\theta_{j+1,j}(x, \hat{p})$ *are all non-zero if* $x < 0$*.*

For convenience, we present the proof of Lemma 4.1 in Appendix C. The property recorded in Lemma 4.2 is known [29,11], and is easily checked by inspection.

Proposition 4.3. *The effective gauge group* $\bar{G}_\mu$ *acts freely on* $\Phi_+^{-1}(\mu)$*.*

Proof. Since every gauge orbit intersects the set S specified by Proposition 3.3, it is enough to show that if $(\eta_L, \eta_R) \in G_\mu$ maps $K \in S$ (3.27) to itself, then (η_L, η_R) equals some element (η, η) given in (4.1). For K of the form (3.3), we can spell out $K' \equiv \eta_L K \eta_R^{-1}$ as

$$K' = \begin{bmatrix} \eta_L(1)\rho & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \begin{bmatrix} \eta_R(1)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2)^{-1} \end{bmatrix} \times \begin{bmatrix} e^{-v}\mathbf{1}_n & \eta_R(1)\alpha\eta_R(2)^{-1} \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix}. \tag{4.6}$$

The equality $K' = K$ implies by the uniqueness of the Iwasawa decomposition and Lemma 4.1 that we must have

$$\eta_L(2) = \eta_R(1) = m_2, \quad \eta_R(2) = m_1, \quad \eta_L(1)\rho = \rho m_1, \tag{4.7}$$

with some diagonal unitary matrices having the form (4.4). By using that $\eta_R(1) = m_2$ and $\eta_R(2) = m_1$, the Iwasawa decomposition of $K' = K$ in (3.27) also entails the relation

$$\alpha = m_2 \alpha m_1^{-1}. \tag{4.8}$$

Because of (3.28), the off-diagonal components of the matrix equation (4.8) yield

$$\theta(-x, \hat{p})_{jk} = \left(m_2 \theta(-x, \hat{p}) m_1^{-1}\right)_{jk}, \quad \forall j \neq k. \tag{4.9}$$

This implies by means of Lemma 4.2 and equation (4.4) that $m_1 = m_2 = z\mathbf{1}_n$ is a scalar matrix. But then $\eta_L(1) = m_1$ follows from $\eta_L(1)\rho = \rho m_1$, and the proof is complete. □

Proposition 4.3 and the general results gathered in Appendix D imply the following theorem, which is one of our main results.

Theorem 4.4. *The constraint surface $\Phi_+^{-1}(\mu)$ is an embedded submanifold of* $\mathrm{SL}(2n, \mathbb{C})$ *and the reduced phase space M (2.24) is a smooth manifold for which the natural projection $\pi_\mu \colon \Phi_+^{-1}(\mu) \to M$ is a smooth submersion.*

4.2. Model of a dense open subset of the reduced phase space

Let us denote by $S^o \subset S$ the subset of the elements K given by Proposition 3.3 with $\hat{p}$ in the interior $\mathcal{C}_x$ of the polyhedron $\bar{\mathcal{C}}_x$ (3.18). Explicitly, we have

$$S^o = \{K(\hat{p}, e^{\mathrm{i}\hat{q}}) \mid (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}_x \times \mathbb{T}_n\}, \tag{4.10}$$

where $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ stands for the expression (3.27). Note that S^o is in bijection with $\mathcal{C}_x \times \mathbb{T}_n$. The next lemma says that no two different point of S^o are gauge equivalent.

Lemma 4.5. *The intersection of any gauge orbit with S^o consists of at most one point.*

Proof. Suppose that

$$K' := K(\hat{p}', e^{\mathrm{i}\hat{q}'}) = \eta_L K(\hat{p}, e^{\mathrm{i}\hat{q}}) \eta_R^{-1} \tag{4.11}$$

with some $(\eta_L, \eta_R) \in G_\mu$. By spelling out the gauge transformation as in (4.6), using the shorthand $\sin q = e^{\hat{p}}$, we observe that $\hat{p}' = \hat{p}$ since q in (3.1) does not change under the action of $G_+ \times G_+$. Since now we have $\frac{\pi}{2} > q_1$ (which is equivalent to $0 > \hat{p}_1$), the arguments applied in the proof of Proposition 4.3 permit to translate the equality (4.11) into the relations

$$\eta_L(2) = \eta_R(1) = \eta_R(2) = m, \quad \eta_L(1)\rho = \rho m, \tag{4.12}$$

complemented with the condition

$$\alpha(\hat{p}, e^{\mathrm{i}\hat{q}'}) = m\alpha(\hat{p}, e^{\mathrm{i}\hat{q}}) m^{-1}, \tag{4.13}$$

which is equivalent to

$$e^{\mathrm{i}\hat{q}'}\theta(-x, \hat{p}) = m e^{\mathrm{i}\hat{q}}\theta(-x, \hat{p}) m^{-1}. \tag{4.14}$$

We stress that $m \in \mathbb{T}_n$ and notice from (3.20) that for $\hat{p} \in \mathcal{C}_x$ all the diagonal entries $\theta(-x, \hat{p})_{jj}$ are non-zero. Therefore we conclude from (4.14) that $e^{\mathrm{i}\hat{q}'} = e^{\mathrm{i}q}$. This finishes the proof, but of course we can also confirm that $m = z\mathbf{1}_n$, consistently with Proposition 4.3. □

Now we introduce the map $\mathcal{P}\colon \mathrm{SL}(2n, \mathbb{C}) \to \mathbb{R}^n$ by

$$\mathcal{P}\colon K = g_L b_R^{-1} \mapsto \hat{p}, \tag{4.15}$$

defined by writing g_L in the form (3.1) with $\sin q = e^{\hat{p}}$. The map $\mathcal{P}$ gives rise to a map $\bar{\mathcal{P}}\colon M \to \mathbb{R}^n$ verifying

$$\bar{\mathcal{P}}(\pi_\mu(K)) = \mathcal{P}(K), \quad \forall K \in \Phi_+^{-1}(\mu), \tag{4.16}$$

where π_μ is the canonical projection (2.28). We notice that, since the 'eigenvalue parameters' $\hat{p}_j$ ($j = 1, \dots, n$) are pairwise different for any $K \in \Phi_+^{-1}(\mu)$, $\bar{\mathcal{P}}$ is a smooth map. The continuity of $\bar{\mathcal{P}}$ implies that

$$M^o := \bar{\mathcal{P}}^{-1}(\mathcal{C}_x) = \pi_\mu(S^o) \subset M \tag{4.17}$$

is an open subset. The second equality is a direct consequence of our foregoing results about S and S^o. Note that $\bar{\mathcal{P}}^{-1}(\bar{\mathcal{C}}_x) = \pi_\mu(S) = M$. Since π_μ is continuous (actually smooth) and any point of S is the limit of a sequence in S^o, M^o is *dense* in the reduced phase space M. The dense open subset M^o can be parametrized by $\mathcal{C}_x \times \mathbb{T}_n$ according to

$$(\hat{p}, e^{\mathrm{i}\hat{q}}) \mapsto \pi_\mu(K(\hat{p}, e^{\mathrm{i}\hat{q}})), \tag{4.18}$$

which also allows us to view $S^o \simeq \mathcal{C}_x \times \mathbb{T}_n$ as a model of $M^o \subset M$. In principle, the restriction of the reduced symplectic form to M^o can now be computed by inserting the explicit formula $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ (3.27) into the Alekseev–Malkin form (2.5). In the analogous reduction of the Heisenberg double of $\mathrm{SU}(n, n)$, Marshall [20] found a nice way to circumvent such a tedious calculation. By taking the same route, we have verified that $\hat{p}$ and $\hat{q}$ are Darboux coordinates on M^o.

The outcome of the above considerations is summarized by the next theorem.

Theorem 4.6. *M^o defined by equation (4.17) is a dense open subset of the reduced phase space M. Parameterizing M^o by $\mathcal{C}_x \times \mathbb{T}_n$ according to (4.18), the restriction of reduced symplectic form ω_M (2.29) to M^o is equal to $\hat{\omega} = \sum_{j=1}^n d\hat{q}_j \wedge d\hat{p}_j$ (1.4).*

4.3. Liouville integrability of the reduced free Hamiltonians

The Abelian Poisson algebra $\mathfrak{H}$ (2.8) consists of $(G_+ \times G_+)$-invariant functions[4] generating complete flows, given explicitly by (2.10), on the unreduced phase space. Thus each element of $\mathfrak{H}$ descends to a smooth reduced Hamiltonian on M (2.24), and generates a complete flow via the reduced symplectic form ω_M. This flow is the projection of the corresponding unreduced flow, which preserves the constraint surface $\Phi_+^{-1}(\mu)$. It also follows from the construction that $\mathfrak{H}$ gives rise to an Abelian Poisson algebra, $\mathfrak{H}_M$, on (M, ω_M). Now the question is whether the Hamiltonian vector fields of $\mathfrak{H}_M$ span an n-dimensional subspace of the tangent space at the

[4] More precisely, $\mathfrak{H} = C^\infty(\mathrm{SL}(2n, \mathbb{C}))^{\mathrm{SU}(2n) \times \mathrm{SU}(2n)}$.

points of a dense open submanifold of M. If yes, then $\mathfrak{H}_M$ yields a Liouville integrable system, since $\dim(M) = 2n$.

Before settling the above question, let us focus on the Hamiltonian $\mathcal{H} \in \mathfrak{H}$ defined by

$$\mathcal{H}(K) := \frac{1}{2} \operatorname{tr}\left((K^\dagger K)^{-1}\right) = \frac{1}{2} \operatorname{tr}(b_R^\dagger b_R). \tag{4.19}$$

Using the formula of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ in Remark 3.4, it is readily verified that

$$\mathcal{H}(K(\hat{p}, e^{\mathrm{i}\hat{q}})) = H(\hat{p}, \hat{q}; x, u, v), \quad \forall (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}_x \times \mathbb{T}_n, \tag{4.20}$$

with the Hamiltonian H displayed in equation (1.1). *Consequently, H in (1.1) is identified as the restriction of the reduction of $\mathcal{H}$ (4.19) to the dense open submanifold M^o (4.17) of the reduced phase space, wherein the flow of every element of $\mathfrak{H}_M$ is complete.*

Turning to the demonstration of Liouville integrability, consider the n functions

$$\mathcal{H}_k(K) := \frac{1}{2k} \operatorname{tr}\left((K^\dagger K)^{-1}\right)^k = \frac{1}{2k} \operatorname{tr}(b_R^\dagger b_R)^k, \quad k = 1, \dots, n. \tag{4.21}$$

The restriction of the corresponding elements of $\mathfrak{H}_M$ on $M^o \simeq \mathcal{C}_x \times \mathbb{T}_n$ gives the functions

$$H_k(\hat{p}, \hat{q}) = \frac{1}{2k} \operatorname{tr}\begin{bmatrix} e^{2v}\mathbf{1}_n & -e^v\alpha \\ -e^v\alpha^\dagger & (e^{-2v}\mathbf{1}_n + \alpha^\dagger\alpha) \end{bmatrix}^k, \tag{4.22}$$

where α has the form (3.28). These are real-analytic functions on $\mathcal{C}_x \times \mathbb{T}_n$. It is enough to show that their exterior derivatives are linearly independent on a dense open subset of $\mathcal{C}_x \times \mathbb{T}_n$. This follows if we show that the function

$$f(\hat{p}, \hat{q}) = \det\left[d_{\hat{q}} H_1, d_{\hat{q}} H_2, \dots, d_{\hat{q}} H_n\right] \tag{4.23}$$

is not identically zero on $\mathcal{C}_x \times \mathbb{T}_n$. Indeed, since f is an analytic function and $\mathcal{C}_x \times \mathbb{T}_n$ is connected, if f is not identically zero then its zero set cannot contain any accumulation point. This, in turn, implies that f is non-zero on a dense open subset of $\mathcal{C}_x \times \mathbb{T}_n \simeq M^o$, which is also dense and open in the full reduced phase space M. In other words, the reductions of $\mathcal{H}_k$ $(k = 1, \dots, n)$ possess the property of Liouville integrability. It is rather obvious that the function f is not identically zero, since H_k involves dependence on $\hat{q}$ through $e^{\pm \mathrm{i}k\hat{q}}$ and lower powers of $e^{\pm\mathrm{i}\hat{q}}$. It is not difficult to inspect the function $f(\hat{p}, \hat{q})$ in the 'asymptotic domain' where all differences $|\hat{p}_j - \hat{p}_m|$ $(m \neq j)$ tend to infinity, since in this domain α becomes close to a diagonal matrix. We omit the details of this inspection, whereby we checked that f is indeed not identically zero.

The above arguments prove the Liouville integrability of the reduced free Hamiltonians, i.e., the elements of $\mathfrak{H}_M$. Presumably, there exists a dual set of integrable many-body Hamiltonians that live on the space of action-angle variables of the Hamiltonians in $\mathfrak{H}_M$. The construction of such dual Hamiltonians is an interesting task for the future, which will be further commented upon in Section 5.

4.4. The global structure of the reduced phase space

We here construct a global cross-section of the gauge orbits in the constraint surface $\Phi_+^{-1}(\mu)$. This engenders a symplectic diffeomorphism between the reduced phase space (M, ω_M) and the manifold $(\hat{M}_c, \hat{\omega}_c)$ below. It is worth noting that $(\hat{M}_c, \hat{\omega}_c)$ is symplectomorphic to $\mathbb{R}^{2n}$ carrying the standard Darboux 2-form, and one can easily find an explicit symplectomorphism if desired. Our construction was inspired by the previous papers [29,11], but detailed inspection of the specific example was also required for finding the final result given by Theorem 4.9. After a cursory glance, the reader is advised to go directly to this theorem and follow the definitions backwards as becomes necessary. See also Remark 4.10 for the rationale behind the subsequent definitions.

To begin, consider the product manifold

$$\hat{M}_c := \mathbb{C}^{n-1} \times \mathbb{D}, \tag{4.24}$$

where $\mathbb{D}$ stands for the open unit disk, i.e., $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, and equip it with the symplectic form

$$\hat{\omega}_c = \mathrm{i}\sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j + \frac{\mathrm{i}dz_n \wedge d\bar{z}_n}{1 - z_n\bar{z}_n}. \tag{4.25}$$

The subscript c refers to 'complex variables'. Define the surjective map

$$\hat{\mathcal{Z}}_x : \bar{\mathcal{C}}_x \times \mathbb{T}_n \to \hat{M}_c, \quad (\hat{p}, e^{\mathrm{i}\hat{q}}) \mapsto z(\hat{p}, e^{\mathrm{i}\hat{q}}) \tag{4.26}$$

by the formulae

$$z_j(\hat{p}, e^{\mathrm{i}\hat{q}}) = (\hat{p}_j - \hat{p}_{j+1} - |x|/2)^{\frac{1}{2}} \prod_{k=j+1}^{n} e^{\mathrm{i}\hat{q}_k}, \quad j = 1, \dots, n-1,$$

$$z_n(\hat{p}, e^{\mathrm{i}\hat{q}}) = (1 - e^{\hat{p}_1})^{\frac{1}{2}} \prod_{k=1}^{n} e^{\mathrm{i}\hat{q}_k}. \tag{4.27}$$

Notice that the restriction $\mathcal{Z}_x$ of $\hat{\mathcal{Z}}_x$ to $\mathcal{C}_x \times \mathbb{T}_n$ is a diffeomorphism onto the dense open submanifold

$$\hat{M}_c^o = \{z \in \hat{M}_c \mid \prod_{j=1}^{n} z_j \neq 0\}. \tag{4.28}$$

It verifies

$$\mathcal{Z}_x^*(\hat{\omega}_c) = \hat{\omega} = \sum_{j=1}^{n} d\hat{q}_j \wedge d\hat{p}_j, \tag{4.29}$$

which means that $\mathcal{Z}_x$ is a symplectic embedding of $(\mathcal{C}_x \times \mathbb{T}_n, \hat{\omega})$ into $(\hat{M}_c, \hat{\omega}_c)$. The inverse $\mathcal{Z}_x^{-1} : \hat{M}_c^o \to \mathcal{C}_x \times \mathbb{T}_n$ operates according to

$$\hat{p}_1(z) = \log(1 - |z_n|^2), \quad \hat{p}_j(z) = \log(1 - |z_n|^2) - \sum_{k=1}^{j-1}(|z_k|^2 - |x|/2) \quad (j = 2, \dots, n)$$

$$e^{\mathrm{i}\hat{q}_1}(z) = \frac{z_n\bar{z}_1}{|z_n\bar{z}_1|}, \quad e^{\mathrm{i}\hat{q}_m}(z) = \frac{z_{m-1}\bar{z}_m}{|z_{m-1}\bar{z}_m|} \quad (m = 2, \dots, n-1), \quad e^{\mathrm{i}\hat{q}_n}(z) = \frac{z_{n-1}}{|z_{n-1}|}. \tag{4.30}$$

It is important to remark that the $\hat{p}_k(z)$ $(k = 1, \dots, n)$ given above yield smooth functions on the whole of $\hat{M}_c$, while the angles $\hat{q}_k$ are of course not well-defined on the complementary locus of $\hat{M}_c^o$. Our construction of the global cross-section will rely on the building blocks collected in the following long definition.

Definition 4.7. For any $(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ consider the smooth functions

$$Q_{jk}(x,z) = \left[\frac{\sinh(\sum_{\ell=j}^{k-1} z_\ell \bar{z}_\ell + (k-j)|x|/2 - x/2)}{\sinh(\sum_{\ell=j}^{k-1} z_\ell \bar{z}_\ell + (k-j)|x|/2)}\right]^{\frac{1}{2}}, \quad 1 \le j < k \le n, \tag{4.31}$$

and set $Q_{jk}(x,z) := Q_{kj}(-x,z)$ for $j > k$. Applying these as well as the real analytic function

$$J(y) := \sqrt{\frac{\sinh(y)}{y}}, \quad y \neq 0, \quad J(0) := 1, \tag{4.32}$$

and recalling (3.21), introduce the $n \times n$ matrix $\hat{\zeta}(x,z)$ by the formulae

$$\begin{aligned}
&\hat{\zeta}(x,z)_{aa} = r(x, \hat{p}(z))_a, \quad \hat{\zeta}(x,z)_{aj} = -\overline{\hat{\zeta}(x,z)_{ja}}, \quad j \neq a,\\
&\hat{\zeta}(x,z)_{jn} = \sqrt{\frac{\sinh(\frac{x}{2})}{\sinh(\frac{nx}{2})}}\frac{z_j J(z_j\bar{z}_j)}{\sinh(z_j\bar{z}_j + \frac{x}{2})} \prod_{\substack{\ell=1\\(\ell\neq j,j+1)}}^{n} Q_{j\ell}(x,z), \quad x>0, \quad j\neq n,\\
&\hat{\zeta}(x,z)_{j1} = \sqrt{\frac{\sinh(\frac{x}{2})}{\sinh(\frac{nx}{2})}}\frac{\bar{z}_{j-1} J(z_{j-1}\bar{z}_{j-1})}{\sinh(z_{j-1}\bar{z}_{j-1} - \frac{x}{2})} \prod_{\substack{\ell=1\\(\ell\neq j-1,j)}}^{n} Q_{j\ell}(x,z), \quad x<0, \quad j\neq 1,\\
&\hat{\zeta}(x,z)_{jk} = \delta_{j,k} + \frac{\hat{\zeta}(x,z)_{ja}\hat{\zeta}(x,z)_{ak}}{1+\hat{\zeta}(x,z)_{aa}}, \quad j,k\neq a,
\end{aligned} \tag{4.33}$$

where $a = n$ if $x > 0$ and $a = 1$ if $x < 0$. Then introduce the matrix $\hat{\theta}(x,z)$ for $x > 0$ as

$$\begin{aligned}
&\hat{\theta}(x,z)_{jk} = \frac{\sinh(\frac{nx}{2})\,\mathrm{sgn}(k-j-1)\hat{\zeta}(x,z)_{jn}\hat{\zeta}(-x,z)_{1k}}{\sinh(\sum_{\ell=\min(j,k)}^{\max(j,k)-1} z_\ell\bar{z}_\ell + |k-j-1|\frac{x}{2})}, \quad k \neq j+1,\\
&\hat{\theta}(x,z)_{j,j+1} = \frac{-\sinh(\frac{x}{2})}{\sinh(z_j\bar{z}_j + \frac{x}{2})} \prod_{\substack{\ell=1\\(\ell\neq j,j+1)}}^{n} Q_{j\ell}(x,z)Q_{j+1,\ell}(-x,z),
\end{aligned} \tag{4.34}$$

and for $x < 0$ as

$$\hat{\theta}(x,z) = \hat{\theta}(-x,z)^\dagger. \tag{4.35}$$

Finally, for any $z \in \hat{M}_c$ define the matrix $\hat{\gamma}(x,z) = \mathrm{diag}(\hat{\gamma}_1, \dots, \hat{\gamma}_n)$ with

$$\hat{\gamma}(z)_1 = z_n\sqrt{2 - z_n\bar{z}_n}, \quad \hat{\gamma}(x,z)_j = \left[1 - (1 - z_n\bar{z}_n)e^{-\sum_{\ell=1}^{j-1}(z_\ell\bar{z}_\ell + |x|/2)}\right], \quad j = 2, \dots, n, \tag{4.36}$$

and the matrix

$$\hat{\alpha}(x,u,v,z) = -\mathrm{i}\Big[\sqrt{e^{-2u}e^{-2\hat{p}(z)} - e^{-2v}\mathbf{1}_n}\,\big]\hat{\theta}(-x,z) - e^v e^{-\hat{p}(z)}\hat{\gamma}(x,z)^\dagger\Big], \tag{4.37}$$

using the constants x, u, v subject to (1.2).

Although the variable z_n appears only in $\hat{\gamma}_1$, we can regard all objects defined above as smooth functions on $\hat{M}_c$, and we shall do so below.

The key properties of the matrices $\hat{\zeta}$, $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\gamma}$ are given by the following lemma, which can be verified by straightforward inspection. The role of these identities and their origin will be enlightened by Theorem 4.9.

Lemma 4.8. *Prepare the notations*

$$\tau_{(x)} := \mathrm{diag}(\tau_2, \dots, \tau_n, 1) \quad \textit{if} \quad x > 0 \quad \textit{and} \quad \tau_{(x)} := \mathrm{diag}(1, \tau_1, \dots, \tau_{n-1}) \quad \textit{if} \quad x < 0, \tag{4.38}$$

$$\tilde{\tau}_{(x)} := \mathrm{diag}(1, \tau_2, \dots, \tau_n) \quad \textit{if} \quad x > 0 \quad \textit{and} \quad \tilde{\tau}_{(x)} := \mathrm{diag}(\tau_1, \dots, \tau_{n-1}, 1) \quad \textit{if} \quad x < 0 \tag{4.39}$$

with

$$\tau_j = \prod_{k=j}^{n} e^{\mathrm{i}\hat{q}_k} \quad \textit{if} \quad x > 0 \quad \textit{and} \quad \tau_j = e^{\mathrm{i}\hat{q}_j} \prod_{k=j}^{n} e^{-\mathrm{i}\hat{q}_k} \quad \textit{if} \quad x < 0. \tag{4.40}$$

Then the following identities hold for all $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n$*:*

$$\hat{\zeta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \tau_{(x)} \zeta(x, \hat{p}) \tau_{(x)}^{-1}, \tag{4.41}$$

$$\hat{\theta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \tau_{(x)} \theta(x, \hat{p}) \tilde{\tau}_{(x)}^{-1}, \tag{4.42}$$

$$\hat{\gamma}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = e^{\mathrm{i}\hat{q}} \tau_{(x)} \tilde{\tau}_{(x)}^{-1} \sqrt{\mathbf{1}_n - e^{2\hat{p}}}, \tag{4.43}$$

$$\hat{\alpha}(x, u, v, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = e^{-\mathrm{i}\hat{q}} \tilde{\tau}_{(x)} \alpha(x, u, v, \hat{p}, e^{\mathrm{i}\hat{q}}) \tau_{(x)}^{-1}. \tag{4.44}$$

Here we use Definition 4.7 and the functions on $\bar{\mathcal{C}}_x \times \mathbb{T}_n$ *introduced in Subsection 3.2.*

For the verification of the above identities, we remark that the vector r (3.21) can be expressed as a smooth function of the complex variables as

$$r(x, \hat{p}(z))_j = \sqrt{\frac{\sinh(\frac{x}{2})}{\sinh(\frac{nx}{2})}} \prod_{\substack{k=1 \\ (k \neq j)}}^{n} Q_{jk}(x, z), \qquad j = 1, \dots, n. \tag{4.45}$$

With all necessary preparations now done, we state the main new result of the paper.

Theorem 4.9. *The image of the smooth map* $\hat{K}\colon \hat{M}_c \to \mathrm{SL}(2n, \mathbb{C})$ *given by the formula*

$$\hat{K}(z) = \begin{bmatrix} \kappa(x)\hat{\zeta}(x,z)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \hat{\gamma}(x,z) & \mathrm{i}e^{\hat{p}(z)} \\ \mathrm{i}e^{\hat{p}(z)} & \hat{\gamma}(x,z)^{\dagger} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \hat{\alpha}(x,u,v,z) \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix} \tag{4.46}$$

lies in $\Phi_+^{-1}(\mu)$*, intersects every gauge orbit in precisely one point, and* $\hat{K}$ *is injective. The pull-back of the Alekseev–Malkin 2-form* ω (2.5) *by* $\hat{K}$ *is* $\hat{\omega}_c$ (4.25)*. Consequently,* $\pi_\mu \circ \hat{K}\colon \hat{M}_c \to M$ *is a symplectomorphism, whereby* $(\hat{M}_c, \hat{\omega}_c)$ *provides a model of the reduced phase space* (M, ω_M) *defined in Subsection 2.2.*

Proof. The proof is based upon the identity

$$\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \begin{bmatrix} \kappa(x)\tau_{(x)}\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} \end{bmatrix} K(\hat{p}, e^{\mathrm{i}\hat{q}}) \begin{bmatrix} \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} & \mathbf{0}_n \\ \mathbf{0}_n & \tau_{(x)} \end{bmatrix}^{-1}, \quad \forall(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n, \tag{4.47}$$

which is readily seen to be equivalent to the set of identities displayed in Lemma 4.8. It means that $\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}}))$ is a gauge transform of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ in (3.27). Indeed, the above transformation of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ has the form (2.20) with

$$\eta_L = c\begin{bmatrix} \kappa(x)\tau_{(x)}\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} \end{bmatrix}, \qquad \eta_R = c\begin{bmatrix} \tilde{\tau}_{(x)}e^{-\mathrm{i}\hat{q}} & \mathbf{0}_n \\ \mathbf{0}_n & \tau_{(x)} \end{bmatrix}, \tag{4.48}$$

where c is a harmless scalar inserted to ensure $\det(\eta_L) = \det(\eta_R) = 1$. Using (3.25) and (4.38), one can check that $\kappa(x)\tau_{(x)}\kappa(x)^{-1}\hat{v}(x) = \hat{v}(x)$ for the vector $\hat{v}(x)$ in (3.24), which implies via the relation (3.15) that (η_L, η_R) belongs to the isotropy group G_μ (2.25), the gauge group acting on $\Phi_+^{-1}(\mu)$.

It follows from Proposition 3.3 and the identity (4.47) that the set

$$\hat{S} := \{\hat{K}(z) \mid z \in \hat{M}_c\} \tag{4.49}$$

lies in $\Phi_+^{-1}(\mu)$ and intersects every gauge orbit. Since the dense subset

$$\hat{S}^o := \{\hat{K}(z) \mid z \in \hat{M}_c^o\} \tag{4.50}$$

is gauge equivalent to S^o in (4.10), we obtain the equality

$$\hat{K}^*(\omega) = \hat{\omega} \tag{4.51}$$

by using Theorem 4.6 and equation (4.29). More precisely, we here also utilized that $\hat{K}^*(\omega)$ is (obviously) smooth and $\hat{M}_c^o$ is dense in $\hat{M}_c$.

The only statements that remain to be proved are that the intersection of $\hat{S}$ with any gauge orbit consists of a single point and that $\hat{K}$ is injective. (These are already clear for $\hat{S}^o \subset \hat{S}$ and for $\hat{K}|_{\hat{M}_c^o}$.) Now suppose that

$$\hat{K}(z') = \begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \hat{K}(z) \begin{bmatrix} \eta_R(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2) \end{bmatrix}^{-1} \tag{4.52}$$

for some gauge transformation and $z, z' \in \hat{M}_c$. Let us observe from the definitions that we can write

$$\begin{bmatrix} \hat{\gamma}(x,z) & \mathrm{i}e^{\hat{p}(z)} \\ \mathrm{i}e^{\hat{p}(z)} & \hat{\gamma}(x,z)^\dagger \end{bmatrix} = D(z)\begin{bmatrix} \cos q(z) & \mathrm{i}\sin q(z) \\ \mathrm{i}\sin q(z) & \cos q(z) \end{bmatrix} D(z), \tag{4.53}$$

where $\sin q(z) = e^{\hat{p}(z)}$, with $\frac{\pi}{2} \geq q_1 > \cdots > q_n > 0$, and $D(z)$ is a diagonal unitary matrix of the form $D(z) = \mathrm{diag}(d_1, \mathbf{1}_{n-1}, \bar{d}_1, \mathbf{1}_{n-1})$. Then the uniqueness properties of the Iwasawa decomposition of $\mathrm{SL}(2n, \mathbb{C})$ and the generalized Cartan decomposition (3.1) of $\mathrm{SU}(2n)$ allow to establish the following consequences of (4.52). First,

$$\hat{p}(z) = \hat{p}(z'). \tag{4.54}$$

Second, using Lemma 4.1,

$$\begin{bmatrix} \eta_R(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2) \end{bmatrix} = \begin{bmatrix} m_2 & \mathbf{0}_n \\ \mathbf{0}_n & m_1 \end{bmatrix} \tag{4.55}$$

for some diagonal unitary matrices of the form (4.4). Third, we have

$$\hat{\alpha}(z') = \eta_R(1)\hat{\alpha}(z)\eta_R(2)^{-1} = m_2\hat{\alpha}(z)m_1^{-1}. \tag{4.56}$$

For definiteness, let us focus on the case $x > 0$. Then we see from the definitions that the components $\hat{\alpha}_{k+1,k}$ and $\hat{\alpha}_{1,n}$ depend only on $\hat{p}(z)$ and are non-zero. By using this, we find from (4.56) that $m_1 = m_2 = C\mathbf{1}_n$ with a scalar C, and therefore

$$\hat{\alpha}(z') = \hat{\alpha}(z). \tag{4.57}$$

Inspection of the components $(1,2), \dots, (1,n-1)$ of this matrix equality and (4.54) permit to conclude that $z'_2 = z_2, \dots, z'_{n-1} = z_{n-1}$, respectively. Then, the equality of the $(2,n)$ entries in (4.57) gives $z'_1 = z_1$ which used in the $(1,1)$ position implies $z'_n = z_n$. Thus we see that $z' = z$ and the proof is complete. (Everything written below (4.56) is quite similar for $x < 0$.) □

Remark 4.10. Let us hint at the way the global structure was found. The first point to notice was that all or some of the phases $e^{\mathrm{i}\hat{q}_j}$ cannot encode gauge invariant quantities if $\hat{p}$ belongs to the boundary of $\bar{\mathcal{C}}_x$, as was already mentioned in Remark 3.5. Motivated by [11], then we searched for complex variables by requiring that a suitable gauge transform of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ in (3.27) should be expressible as a smooth function of those variables. Given the similarities to [11], only the definition of z_n was a true open question. After trial and error, the idea came in a flash that the gauge transformation at issue should be constructed from a transformation that appears in Lemma C.1. Then it was not difficult to find the correct result.

Remark 4.11. Let us elaborate on how the trajectories $\hat{p}(t)$ corresponding to the flows of the reduced free Hamiltonians, arising from $\mathcal{H}_k$ (4.21) for $k = 1, \dots, n$, can be obtained. Recall that for $k = 1$ the reduction of $\mathcal{H}_1$ completes the main Hamiltonian H (1.1). Since $\mathcal{H}_k(K) = h_k(b_R)$ with $h_k(b) = \frac{1}{2k}\operatorname{tr}(b^\dagger b)^k$, the free flow generated by $\mathcal{H}_k$ through the initial value $K(0) = g_L(0)b_R^{-1}(0)$ is given by (2.10) with $d^R h_k(b) = \mathrm{i}(b^\dagger b)^k$. Thus the curve $g_L(t)$ (2.10) has the form

$$g_L(t) = g_L(0)\exp(-\mathrm{i}t\mathcal{L}(0)^k) \quad \text{with} \quad \mathcal{L}(0) = b_R(0)^\dagger b_R(0). \tag{4.58}$$

The reduced flow results by the usual projection algorithm. This starts by picking an initial value $z(0) \in \hat{M}_c$ and setting $K(0) = \hat{K}(z(0))$ by applying (4.46), which directly determines $g_L(0)$ and $b_R(0)$ as well. Then the map $\mathcal{P}$ (4.15) gives rise to $\hat{p}(t)$ via the decomposition of $g_L(t) \in \mathrm{SU}(2n)$ as displayed in (3.1), that is

$$\hat{p}(t) = \mathcal{P}(K(t)). \tag{4.59}$$

More explicitly, if $\mathcal{D}(t)$ stands for the (11) block of $g_L(t)$, then the eigenvalues of $\mathcal{D}(t)\mathcal{D}(t)^\dagger$ are

$$\sigma(\mathcal{D}(t)\mathcal{D}(t)^\dagger) = \{\cos^2 q_j(t) \mid j = 1, \dots, n\}, \tag{4.60}$$

from which $\hat{p}_j(t)$ can be obtained using (3.13). In particular, the 'particle positions' evolve according to an 'eigenvalue dynamics' similarly to other many-body systems. This involves the one-parameter group $e^{-\mathrm{i}t\mathcal{L}(0)^k}$, where $\mathcal{L}(0)$ is the initial value of the Lax matrix (cf. (4.22))

$$\mathcal{L}(z) = \begin{bmatrix} e^{2v}\mathbf{1}_n & -e^v\hat{\alpha}(z) \\ -e^v\hat{\alpha}(z)^\dagger & (e^{-2v}\mathbf{1}_n + \hat{\alpha}(z)^\dagger\hat{\alpha}(z)) \end{bmatrix}, \tag{4.61}$$

where we suppressed the dependence of $\hat{\alpha}$ (4.37) on the parameters x, u, v. A more detailed characterization of the dynamics will be provided elsewhere.

5. Discussion and outlook on open problems

In this paper we derived a deformation of the trigonometric BC_n Sutherland system by means of Hamiltonian reduction of a free system on the Heisenberg double of $\mathrm{SU}(2n)$. Our main result is the global characterization of the reduced phase space given by Theorem 4.9. The Liouville integrability of our system holds on this phase space, wherein the reduced free flows are complete. These flows can be obtained by the usual projection method applied to the original free flows described in Section 2.

The local form of our reduced 'main Hamiltonian' (1.1) is similar to the Hamiltonian derived in [20], which deforms the hyperbolic BC_n Sutherland system. However, besides a sign difference corresponding to the difference of the undeformed Hamiltonians, the local domain of our system, $\mathcal{C}_x \times \mathbb{T}_n$ in (1.3), is different from the local domain appearing in [20], which in effect has the form $\mathcal{C}'_x \times \mathbb{T}_n$ with the open polyhedron[5]

$$\mathcal{C}'_x := \{\hat{p} \in \mathbb{R}^n \mid \hat{p}_n > 0,\ \hat{p}_k - \hat{p}_{k+1} > |x|/2\ (k = 1, \dots, n-1)\}. \tag{5.1}$$

We here wish to point out that $\mathcal{C}'_x \times \mathbb{T}_n$ is *not* the full reduced phase space that arises from the reduction considered in [20]. In fact, similarly to our case, the constraint surface contains a submanifold of the form $\bar{\mathcal{C}}'_x \times \mathbb{T}_n$ in the case of [20], where $\bar{\mathcal{C}}'_x$ is the closure of $\mathcal{C}'_x$. Then a global model of the reduced phase space can be constructed by introducing complex variables suitably accommodating the procedure that we utilized in Subsection 4.4. Erroneously, in [20] the full phase space was claimed to be $\mathcal{C}'_x \times \mathbb{T}_n$; the details of the correct description will be presented elsewhere.

Throughout the text we assumed that $n > 1$, but we now note that the reduced system can be specialized to $n = 1$ and the reduction procedure works in this case as well. The assumption was made merely to save words. The formalism actually simplifies for $n = 1$ since the Poisson structure on $G_+ = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) < \mathrm{SU}(2)$ is trivial.

As explained in Appendix A, the Hamiltonian (1.1) is a singular limit of a specialization of the trigonometric van Diejen Hamiltonian [35], which (in addition to the deformation parameter) contains 5 coupling constants. As a result, at least classically, van Diejen's system can be degenerated into the trigonometric BC_n Sutherland system either directly, as described in [35], or in a roundabout way, going through our system. Of course, a similar statement holds in relation to hyperbolic BC_n Sutherland and the system of [20].

Except in the rational limit [26], no Lax matrix is known that would generate van Diejen's commuting Hamiltonians. In the reduction approach a Lax matrix arises automatically, in our case it features in equations (4.22) and (4.61). This might be helpful in searching for a Lax matrix behind van Diejen's 5-coupling Hamiltonian. The search would be easy if one could derive van Diejen's system by Hamiltonian reduction. It is a long standing open problem to find such derivation. Perhaps one should consider some 'classical analogue' of the quantum group interpretation of the Koornwinder (BC_n Macdonald) polynomials found in [23], since those polynomials diagonalize van Diejen's quantized Hamiltonians [37].

Another open problem is to construct action-angle duals of the deformed BC_n Sutherland systems. Duality relations are not only intriguing on their own right, but are also very useful for

[5] The notational correspondence between [20] and the present paper is: $(q, p, \alpha, x, y) \leftrightarrow (\hat{p}, \hat{q}, e^{\frac{x}{2}}, e^{-v}, e^{-u})$.

extracting information about the dynamics [28–30,27]. The duality was used in [4,10] to show that the hyperbolic BC_n Sutherland system is maximally superintegrable, the trigonometric BC_n Sutherland system has precisely n constants of motion, and the relevant dual systems are maximally superintegrable in both cases. These studies, which were heavily influenced by Pusztai's paper [26] (see also [14]), may provide inspiration for a future investigation of the dualities for the deformed BC_n Sutherland systems. We here only remark that deformed dual systems should arise from considering the reduction of alternative sets of commuting free Hamiltonians on the pertinent Heisenberg doubles.

After we finished our work, there appeared a preprint [38] dealing with the quantum mechanics of a lattice version of a 4-parameter Inozemtsev type limit of van Diejen's trigonometric/hyperbolic system. The systems studied in [20] and in our paper correspond to further limits of specializations of this one. The statements about quantum mechanical dualities contained in [38] and its references should be related to classical dualities.

We hope be able to return to some of these questions in the future.

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Appendix A. Links to systems of van Diejen and Schneider

Recall that the trigonometric BC_n van Diejen system [35] has the Hamiltonian

$$H_{\mathrm{vD}}(\lambda,\theta)=\sum_{j=1}^{n}\big(\cosh(\theta_j)\mathsf{V}_j(\lambda)^{1/2}\mathsf{V}_{-j}(\lambda)^{1/2}-[\mathsf{V}_j(\lambda)+\mathsf{V}_{-j}(\lambda)]/2\big), \tag{A.1}$$

with $\mathsf{V}_{\pm j}$ $(j=1,\dots,n)$ defined by

$$\mathsf{V}_{\pm j}(\lambda)=\mathsf{w}(\pm\lambda_j)\prod_{\substack{k=1\\(k\neq j)}}^{n}\mathsf{v}(\pm\lambda_j+\lambda_k)\mathsf{v}(\pm\lambda_j-\lambda_k), \tag{A.2}$$

and v,w denoting the trigonometric potentials

$$\mathsf{v}(z)=\frac{\sin(\mu+z)}{\sin(z)}\quad\text{and}\quad \mathsf{w}(z)=\frac{\sin(\mu_0+z)}{\sin(z)}\frac{\cos(\mu_1+z)}{\cos(z)}\frac{\sin(\mu_0'+z)}{\sin(z)}\frac{\cos(\mu_1'+z)}{\cos(z)}, \tag{A.3}$$

where $\mu,\mu_0,\mu_1,\mu_0',\mu_1'$ are arbitrary parameters. By making the substitutions

$$\begin{array}{c}\lambda_j\to \mathrm{i}(\hat{p}_j+R),\\ \theta_j\to \mathrm{i}\hat{q}_j,\end{array}\quad \forall j\quad\text{and}\quad \mu\to \mathrm{i}g/2,\quad \begin{array}{ll}\mu_0\to \mathrm{i}(g_0+R), & \mu_1\to \mathrm{i}g_1+\pi/2,\\ \mu_0'\to \mathrm{i}(g_0'-R), & \mu_1'\to \mathrm{i}g_1'+\pi/2\end{array} \tag{A.4}$$

the potentials become hyperbolic functions and their $R\to\infty$ limit exists, namely

$$\lim_{R\to\infty} \mathsf{v}(\pm(\lambda_j+\lambda_k)) = e^{\pm g/2},$$
$$\lim_{R\to\infty} \mathsf{v}(\pm(\lambda_j-\lambda_k)) = \frac{\sinh(\pm g/2+\hat{p}_j-\hat{p}_k)}{\sinh(\hat{p}_j-\hat{p}_k)}, \quad \forall j,k \tag{A.5}$$

and

$$\lim_{R\to\infty} \mathsf{w}(\pm\lambda_j) = e^{g_0-g_0'\pm(g_1+g_1')-2\hat{p}_j} - e^{\pm(g_0+g_0'+g_1+g_1')}, \quad \forall j. \tag{A.6}$$

In the 1-particle case we have $V_\pm(\lambda) = \mathsf{w}(\pm\lambda)$, thus H_{vD} takes the following form

$$H_{\mathrm{vD}}(\lambda,\theta) = \cosh(\theta)\mathsf{w}(\lambda)^{1/2}\mathsf{w}(-\lambda)^{1/2} - [\mathsf{w}(\lambda)+\mathsf{w}(-\lambda)]/2. \tag{A.7}$$

By utilizing (A.6) one obtains

$$\lim_{R\to\infty} \mathsf{w}(\lambda)^{1/2}\mathsf{w}(-\lambda)^{1/2} = \big[1-(e^{2g_0}+e^{-2g_0'})e^{-2\hat{p}}+e^{2g_0-2g_0'-4\hat{p}}\big]^{1/2},$$
$$\lim_{R\to\infty} [\mathsf{w}(\lambda)+\mathsf{w}(-\lambda)]/2 = \frac{e^{g_0-g_0'+g_1+g_1'}+e^{g_0-g_0'-g_1-g_1'}}{2}e^{-2\hat{p}} - \cosh(g_0+g_0'+g_1+g_1'). \tag{A.8}$$

Equating the $R\to\infty$ limit of $H_{\mathrm{vD}}(\lambda,\theta)$ (A.7) with the Hamiltonian $H(\hat{p},\hat{q};x,u,v)$ (1.1) yields a system of linear equations involving g_0, g_1, g_0', g_1' as unknowns and u, v as parameters. Actually, four sets of linear equations can be constructed, each with infinitely many solutions depending on one (real) parameter, but these sets are 'equivalent' under the exchanges: $g_0 \leftrightarrow g_0'$ or $g_1 \leftrightarrow g_1'$. Therefore it is sufficient to give only one set of solutions, e.g.

$$g_0 = v-u, \quad g_0' = 0, \quad g_1 = u+v-g_1', \quad g_1' \in \mathbb{R}. \tag{A.9}$$

Setting $g = x$ and $g_1' = 0$ provides the following special choice of couplings in (A.4)

$$\mu = \mathrm{i}x/2, \quad \mu_0 = \mathrm{i}(v-u+R), \quad \mu_0' = -\mathrm{i}R, \quad \mu_1 = \mathrm{i}(u+v)+\pi/2, \quad \mu_1' = \pi/2, \tag{A.10}$$

and one finds the following

$$\lim_{R\to\infty} H_{\mathrm{vD}}\big(\lambda(\hat{p},R),\theta(\hat{q})\big) = -H(\hat{p},\hat{q};x,u,v)+\cosh\big(2u\big). \tag{A.11}$$

In the n-particle case, by using (A.5) and (A.6) it can be shown that with (A.10) one has

$$\lim_{R\to\infty} H_{\mathrm{vD}}\big(\lambda(\hat{p},R),\theta(\hat{q})\big) = -H(\hat{p},\hat{q};x,u,v)+\sum_{j=1}^{n}\cosh\big((j-1)x+2u\big), \tag{A.12}$$

i.e., the Hamiltonian H (1.1) is recovered as a singular limit of H_{vD} (A.1).

Consider now the function $H(\hat{p},\hat{q};x,u,v)$ and introduce the real parameter σ through the substitutions

$$u \to u-\sigma, \quad v \to v-\sigma \tag{A.13}$$

and apply the canonical transformation

$$\hat{p}_j \to -Q_j+\sigma, \quad \hat{q}_j \to -P_j, \quad \forall j. \tag{A.14}$$

Then we have

$$\lim_{\sigma\to\infty} H(\hat{p}(Q,\sigma),\hat{q}(P),x,u(\sigma),v(\sigma)) = H_{\mathrm{Sch}}(Q,P,x,u), \tag{A.15}$$

with Schneider's [32] Hamiltonian

$$H_{\mathrm{Sch}}(Q,P,x,u) = \frac{e^{-2u}}{2}\sum_{j=1}^{n} e^{2Q_j} - \sum_{j=1}^{n}\cos(P_j)\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{\sinh^2\left(\frac{x}{2}\right)}{\sinh^2(Q_j-Q_k)}\right]^{\frac{1}{2}}. \tag{A.16}$$

Remark A.1. (*i*) In (A.4) only two of the four external field couplings $\mu_0, \mu_0', \mu_1, \mu_1'$ are scaled with R. However, scaling all four of these parameters also leads to an integrable Ruijsenaars–Schneider type system with a more general 4-parameter external field. For details, see Section II.B of [36]. (*ii*) The connection to Schneider's Hamiltonian was mentioned in Remark 7.1 of [20] as well, where a singular limit, similar to (A.15) was taken.

Appendix B. Proof of a key result

In this appendix we prove Proposition 3.2 which states that the range of the 'position variable' $\hat{p}$ is contained in the closed thick-walled Weyl chamber $\bar{\mathcal{C}}_x$ (3.18).

Proof of Proposition 3.2. According to (3.17) the matrices $e^{2\hat{p}}$ and $e^{2\hat{p}-x\mathbf{1}_n} + \mathrm{sgn}(x)e^{\hat{p}}ww^\dagger e^{\hat{p}}$ are similar and therefore have the same characteristic polynomial. This gives the identity

$$\prod_{j=1}^{n}(e^{2\hat{p}_j}-\lambda) = \prod_{j=1}^{n}(e^{2\hat{p}_j-x}-\lambda) + \mathrm{sgn}(x)\sum_{j=1}^{n}\left[e^{2\hat{p}_j}|w_j|^2\prod_{\substack{k=1\\(k\neq j)}}^{n}(e^{2\hat{p}_k-x}-\lambda)\right], \tag{B.1}$$

where λ is an arbitrary complex parameter. The constraint on $\hat{p}$ arises from the fact that $|w_m|^2$ $(m=1,\dots,n)$ must be non-negative and not all zero because of the definition (3.16).

Let us assume for a moment that the components of $\hat{p}$ are distinct such that $\hat{p}_1 > \dots > \hat{p}_n$. This enables us to express $|w_m|^2$ for all $m\in\{1,\dots,n\}$ from the above equation by evaluating it at n different values of λ, viz. $\lambda = e^{2\hat{p}_m-x}$, $m=1,\dots,n$. We obtain the following

$$|w_m|^2 = \mathrm{sgn}(x)(1-e^{-x})\prod_{\substack{j=1\\(j\neq m)}}^{n}\frac{e^{2\hat{p}_j+x}-e^{2\hat{p}_m}}{e^{2\hat{p}_j}-e^{2\hat{p}_m}}, \quad m=1,\dots,n. \tag{B.2}$$

For $x>0$ and any $\hat{p}$ with $\hat{p}_1 > \dots > \hat{p}_n$ the formula (B.2) implies that $|w_n|^2>0$ and for $m=1,\dots,n-1$ we have $|w_m|^2\geq 0$ if and only if $\hat{p}_m-\hat{p}_{m+1}\geq x/2$. Similarly, if $x<0$ and $\hat{p}\in\mathbb{R}^n$ with $\hat{p}_1>\dots>\hat{p}_n$, then (B.2) implies $|w_1|^2>0$ and for $m=2,\dots,n$ we have $|w_m|^2\geq 0$ if and only if $\hat{p}_{m-1}-\hat{p}_m\geq -x/2$. In summary, if $\hat{p}_1>\dots>\hat{p}_n$, then $|w_m|^2\geq 0$ $\forall m$ implies that $\hat{p}\in\bar{\mathcal{C}}_x$.

Now, let us prove our assumption, that all components of $\hat{p}$ must be different. Indirectly, suppose that some (or maybe all) of the $\hat{p}_j$'s coincide. This can be captured by a partition of the positive integer

$$n = k_1 + \dots + k_r, \tag{B.3}$$

where $r<n$ (or equivalently, at least one integer $k_1,\dots,k_r$ must be greater than 1) and the indirect assumption can be written as

$$\hat{p}_1 = \cdots = \hat{p}_{k_1}, \quad \hat{p}_{k_1+1} = \cdots = \hat{p}_{k_1+k_2}, \quad \ldots,$$
$$\hat{p}_{k_1+\cdots+k_{r-1}+1} = \cdots = \hat{p}_{k_1+\cdots+k_r} \equiv \hat{p}_n. \tag{B.4}$$

Then (B.1) can be reformulated as

$$\prod_{j=1}^{r} (\Delta_j - \lambda)^{k_j} = \prod_{j=1}^{r} (\Delta_j e^{-x} - \lambda)^{k_j} + \operatorname{sgn}(x) \sum_{m=1}^{r} Z_m \Delta_m (\Delta_m e^{-x} - \lambda)^{k_m - 1} \prod_{\substack{j=1 \\ (j \neq m)}}^{r} (\Delta_j e^{-x} - \lambda)^{k_j}, \tag{B.5}$$

where we introduced r distinct variables

$$\Delta_1 = e^{2\hat{p}_{k_1}}, \quad \Delta_2 = e^{2\hat{p}_{k_1+k_2}}, \quad \ldots, \quad \Delta_r = e^{2\hat{p}_{k_1+\cdots+k_r}} \equiv e^{2\hat{p}_n}, \tag{B.6}$$

and r non-negative real variables

$$Z_1 = |w_1|^2 + \cdots + |w_{k_1}|^2, \quad Z_2 = |w_{k_1+1}|^2 + \cdots + |w_{k_1+k_2}|^2,$$
$$\ldots, \quad Z_r = |w_{k_1+\cdots+k_{r-1}+1}|^2 + \cdots + |w_n|^2. \tag{B.7}$$

Notice that $Z_1 + \cdots + Z_r = |w|^2 = \operatorname{sgn}(x) e^{-x}(e^{nx} - 1) > 0$, therefore at least one of the Z_j's must be positive. Next, we define the rational function of λ

$$Q(\Delta, x, \lambda) = \prod_{j=1}^{r} \frac{(\Delta_j - \lambda)^{k_j}}{(\Delta_j e^{-x} - \lambda)^{k_j - 1}}, \tag{B.8}$$

and use it to rewrite (B.5) as

$$Q(\Delta, x, \lambda) = \prod_{j=1}^{r} (\Delta_j e^{-x} - \lambda) + \operatorname{sgn}(x) \sum_{m=1}^{r} Z_m \Delta_m \prod_{\substack{j=1 \\ (j \neq m)}}^{r} (\Delta_j e^{-x} - \lambda). \tag{B.9}$$

The above equation implies that all poles of Q are apparent, i.e., there must be cancelling factors in its numerator. This observation has a straightforward implication on the Δ's.

(∗) For every index $m \in \{1, \ldots, r\}$ with $k_m > 1$, there exists an index $s \in \{1, \ldots, r\}$ s.t. $\Delta_s = \Delta_m e^{-x}$ and $k_s \geq k_m - 1$.

The quantities $Z_m = Z_m(\Delta, x)$ can be uniquely determined by evaluating (B.9) at r different values of the parameter λ, namely $\lambda_m = \Delta_m e^{-x}$ ($m = 1, \ldots, r$). However, there are 3 disjoint cases which are to be handled separately.

Case 1: $k_m = 1$ and $\nexists s \in \{1, \ldots, r\}$: $\Delta_s = \Delta_m e^{-x}$. Then we find

$$Z_m = \operatorname{sgn}(x)(1 - e^{-x}) e^{(n-1)x} \prod_{\substack{j=1 \\ (j \neq m)}}^{r} \left(\frac{\Delta_j - \Delta_m e^{-x}}{\Delta_j - \Delta_m} \right)^{k_j} > 0. \tag{B.10}$$

Case 2: $k_m > 1$ and $k_s = k_m - 1$. Then we find

$$Z_m = (-1)^{k_m+1} \operatorname{sgn}(x)(1-e^{-x})e^{(n-k_m)x} \prod_{\substack{j=1 \\ (j \neq m,s)}}^{r} \left(\frac{\Delta_j - \Delta_m e^{-x}}{\Delta_j - \Delta_m} \right)^{k_j} > 0. \tag{B.11}$$

Case 3: $k_m = 1$ and $\exists s \in \{1, \dots, r\}$: $\Delta_s = \Delta_m e^{-x}$ or $k_m > 1$ and $k_s > k_m - 1$. Then we get

$$Z_m = 0. \tag{B.12}$$

Since there is at least one Z_m which is positive, the set of indices belonging to Case 1 or Case 2 must be non-empty. Introduce a real positive parameter ε and associate to every degenerate configuration (B.4) a continuous family of configurations, denoted by $\hat{p}(\varepsilon)$, with components $\hat{p}(\varepsilon)_1, \dots, \hat{p}(\varepsilon)_n$ defined by the formulae

$$\begin{aligned} \exp(2\hat{p}(\varepsilon)_a + a\varepsilon) &= \Delta_1, \quad a = 1, \dots, k_1, \\ \exp(2\hat{p}(\varepsilon)_{\sum_{m=1}^{j-1} k_m + a} + a\varepsilon) &= \Delta_j, \quad a = 1, \dots, k_j, \quad j = 2, \dots, r. \end{aligned} \tag{B.13}$$

This way coinciding components of $\hat{p}$ (B.4) are 'pulled apart' to points successively separated by $\varepsilon/2$. It is clear that with sufficiently small separation the configuration $\hat{p}(\varepsilon)$ sits in the chamber $\{\hat{x} \in \mathbb{R}^n \mid 0 > \hat{x}_1 > \cdots > \hat{x}_n\}$. For such non-degenerate configurations $\hat{p}(\varepsilon)$, let us consider the expressions

$$|w_\ell(\hat{p}(\varepsilon), x)|^2 = \operatorname{sgn}(x)(1-e^{-x}) \prod_{\substack{j=1 \\ (j \neq \ell)}}^{n} \frac{e^{2\hat{p}(\varepsilon)_j + x} - e^{2\hat{p}(\varepsilon)_\ell}}{e^{2\hat{p}(\varepsilon)_j} - e^{2\hat{p}(\varepsilon)_\ell}}, \quad \ell = 1, \dots, n, \tag{B.14}$$

which give the unique solution of equation (B.1) at $\hat{p}(\varepsilon)$. The limits $\lim_{\varepsilon \to 0} |w_\ell(\hat{p}(\varepsilon), x)|^2$ exist, and do not vanish for $\ell = k_1 + \cdots + k_m$ if k_m belongs to Case 1 or Case 2. For such $\ell = k_1 + \cdots + k_m$ we must have

$$\lim_{\varepsilon \to 0} |w_{k_1+\cdots+k_m}(\hat{p}(\varepsilon), x)|^2 = Z_m(\Delta, x) > 0, \tag{B.15}$$

where Z_m is given by (B.10) in Case 1 and by (B.11) in Case 2. It can be also seen that

$$|w_\ell(\hat{p}(\varepsilon), x)|^2 \equiv 0 \quad \Longleftrightarrow \quad \begin{cases} \ell \notin \{k_1, k_1 + k_2, \dots, k_1 + \cdots + k_r\} \\ \text{or} \\ \ell = k_1 + \cdots + k_m \text{ with } k_m \text{ from Case 3,} \end{cases} \tag{B.16}$$

i.e., $|w_\ell(\hat{p}(\varepsilon), x)|^2$ vanishes identically except for the components in (B.15). Notice that for a small enough ε some coordinates of $\hat{p}(\varepsilon)$ are separated by less than $|x|/2$. Thus, as it was shown at beginning the proof, we have $|w_\ell(\hat{p}(\varepsilon), x)|^2 < 0$ for some index ℓ, which might depend on ε. Moreover, (B.16) implies that the index in question must have the form $\ell = k_1 + \cdots + k_{m^*}$ for some m^* appearing in (B.15). But since the number of indices is finite, a monotonically decreasing sequence $\{\varepsilon_N\}_{N=1}^{\infty}$ tending to zero can be chosen such that $|w_{k_1+\cdots+k_{m^*}}(\hat{p}(\varepsilon_N), x)|^2 < 0$ for all N. This together with (B.16) gives the contradiction

$$0 \geq \lim_{N \to \infty} |w_{k_1+\cdots+k_{m^*}}(\hat{p}(\varepsilon_N), x)|^2 = Z_{m^*}(\Delta, x) > 0 \tag{B.17}$$

proving that all components of $\hat{p}$ must be distinct. This concludes the proof. □

The above proof is a straightforward adaptation of the proofs of Lemma 5.2 of [11] and Theorem 2 of [12]. We presented it since it could be awkward to extract the arguments from those lengthy papers, and also our notations and the ranges of our variables are different.

Appendix C. Proof of an elementary lemma

We here prove the following equivalent formulation of Lemma 4.1.

Lemma C.1. *Suppose that* $\frac{\pi}{2} \geq q_1 > \cdots > q_n > 0$ *and*

$$\begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \begin{bmatrix} \eta_R(1)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2)^{-1} \end{bmatrix} = \begin{bmatrix} \cos q & \mathrm{i}\sin q \\ \mathrm{i}\sin q & \cos q \end{bmatrix} \tag{C.1}$$

for $\eta_L, \eta_R \in G_+$. *Then*

$$\eta_L(1) = \eta_R(2) = m_1, \quad \eta_L(2) = \eta_R(1) = m_2 \tag{C.2}$$

with some diagonal matrices $m_1, m_2 \in \mathbb{T}_n$ *having the form*

$$m_1 = \mathrm{diag}(a, \xi), \quad m_2 = \mathrm{diag}(b, \xi), \quad \xi \in \mathbb{T}_{n-1},\ a, b \in \mathbb{T}_1, \quad \det(m_1 m_2) = 1. \tag{C.3}$$

If in addition $\frac{\pi}{2} > q_1$, *then* $m_1 = m_2$.

Proof. The block off-diagonal components of the equality (C.1) give

$$\eta_L(1) = (\sin q)\eta_R(2)(\sin q)^{-1}, \quad \eta_L(2) = (\sin q)\eta_R(1)(\sin q)^{-1}. \tag{C.4}$$

Since $\eta_L(1)^{-1} = \eta_L(1)^\dagger$, the first of these relations implies $\eta_R(2) = (\sin q)^2 \eta_R(2)(\sin q)^{-2}$. As the entries of $(\sin q)$ are all different, this entails that $\eta_R(2)$ is diagonal, and consequently we obtain the relations in (C.2) with some diagonal matrices m_1 and m_2. On the other hand, the block-diagonal components of (C.1) require that

$$\cos q = \eta_L(1)(\cos q)\eta_R(1)^{-1}, \quad \cos q = \eta_L(2)(\cos q)\eta_R(2)^{-1}. \tag{C.5}$$

Since $\cos q_k \neq 0$ for $k = 2, \ldots, n$, the formula (C.3) follows. If an addition $\cos q_1 \neq 0$, then we also obtain from (C.5) that $a = b$, i.e., $m_1 = m_2 = m$ with some $m \in \mathbb{T}_n$. □

Appendix D. Auxiliary material on Poisson–Lie symmetry

The statements presented here are direct analogues of well-known results [3,15] about Hamiltonian group actions with zero Poisson bracket on the symmetry group. They are surely familiar to experts, although we could not find them in a reference.

Let us consider a Poisson–Lie group G with dual group G^* and a symplectic manifold P equipped with a left Poisson action of G. Essentially following Lu [18] (cf. Remark D.4), we say that the G-action admits the momentum map $\psi: P \to G^*$ if for any $X \in \mathcal{G}$, the Lie algebra of G, and any $f \in C^\infty(P)$ we have

$$(\mathcal{L}_{X_P} f)(p) = \langle X, \{f, \psi\}(p)\psi(p)^{-1}\rangle, \quad \forall p \in P, \tag{D.1}$$

where X_P is the vector field on P corresponding to X, $\langle ., . \rangle$ stands for the canonical pairing between the Lie algebras of G and G^*, and the notation pretends that G^* is a matrix group. Using the Hamiltonian vector field V_f defined by $\mathcal{L}_{V_f} h = -\{f, h\}$ ($\forall h \in C^\infty(P)$), we can spell out equation (D.1) equivalently as

$$(\mathcal{L}_{X_P} f)(p) = -\big\langle X, \big(D_{\psi(p)} R_{\psi(p)^{-1}}\big)\big((D_p\psi)(V_f(p))\big)\big\rangle, \quad \forall p \in P, \tag{D.2}$$

where $D_p\psi: T_pP \to T_{\psi(p)}G^*$ is the derivative, and $R_{\psi(p)^{-1}}$ denotes the right-translation on G^* by $\psi(p)^{-1}$. Since the vectors of the form $V_f(p)$ span T_pP, we obtain the following characterization of the Lie algebra of the isotropy subgroup $G_p < G$ of $p \in P$.

Lemma D.1. *With the above notations, we have*

$$\mathrm{Lie}(G_p) = \big[\big(D_{\psi(p)} R_{\psi(p)^{-1}}\big)\big(\mathrm{Im}(D_p\psi)\big)\big]^{\perp}. \tag{D.3}$$

This directly leads to the next statement.

Corollary D.2. *An element $\mu \in G^*$ is a regular value of the momentum map ψ if and only if* $\mathrm{Lie}(G_p) = \{0\}$ *for every* $p \in \psi^{-1}(\mu) = \{p \in P \mid \psi(p) = \mu\}$.

Let us further suppose that $\psi\colon P \to G^*$ is G-equivariant, with respect to the appropriate dressing action of G on G^*. Then we have

$$G_p < G_\mu, \quad \forall p \in \psi^{-1}(\mu). \tag{D.4}$$

Here G_p and G_μ refer to the respective actions of G on P and on G^*. Corollary D.2 and equation (D.4) together imply the following useful result.

Corollary D.3. *If G_μ acts locally freely on $\psi^{-1}(\mu)$, then μ is a regular value of the equivariant momentum map ψ. Consequently, $\psi^{-1}(\mu)$ is an embedded submanifold of P.*

We finish by a clarifying remark concerning the momentum map.

Remark D.4. Let B be the Poisson tensor on P, for which $\{f, h\} = B(df, dh) = \mathcal{L}_{V_h} f$. We can write $V_h = B^\sharp(dh)$ with the corresponding bundle map $B^\sharp\colon T^*P \to TP$. Any $X \in \mathcal{G} = T_eG = (T_{e'}G^*)^*$ extends to a unique right-invariant 1-form ϑ_X on G^* ($e \in G$ and $e' \in G^*$ are the unit elements). With this at hand, equation (D.1) can be reformulated as

$$X_P = B^\sharp(\psi^*(\vartheta_X)), \tag{D.5}$$

which is a slight variation of the defining equation of the momentum map found in [18].

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