A subharmonicity property of harmonic measures*

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Abstract

Recently it has been established that for compact sets F lying on a circle S, the harmonic measure in the complement of F with respect to any point $a \in S \setminus F$ has convex density on any arc of F. In this note we give an alternative proof of this fact which is based on random walks, and which also yields an analogue in higher dimensions: for compact sets F lying on a sphere S in $\mathbf{R}^n$, the harmonic measure in the complement of F with respect to any point $a \in S \setminus F$ is subharmonic in the interior of F.

1 The result in two dimensions

Let G be a domain $G \subset \mathbf{R}^n$ with compact boundary. In what follows, we denote the n-dimensional harmonic measure for a point $z \in G$ by $\omega(\cdot, z, G)$ (if it exists). So this is a measure on the boundary of G and it is the reproducing measure for harmonic functions in G: if u is harmonic in G (including at infinity if G is unbounded) and continuous on the closure $\overline{G}$, then

$$u(z) = \int_{\partial G} u(t) d\omega(t, z, G).$$

See [4], [7], [9] or [10] for the notion of harmonic measures.

The following convexity result was proved in [2].

Theorem 1 *Let S be a circle on the plane, and $F \subset S$ a closed subset of S. If $a \in S \setminus F$ and $I \subset F$ is an arc, then the density of the harmonic measure $\omega(\cdot, a, \mathbf{R}^2 \setminus F)$ with respect to arc-measure on S is convex on I.*

Actually, the stronger log-convexity was established (i.e. even the logarithm of the density is convex), and that is what we shall also prove below.

The theorem implies that if S is a circle and $F \subset S$ is a closed set with non-empty (one dimensional) interior, then the equilibrium measure of F is convex on any subarc of the interior of F, see [2, Theorem 1.5].

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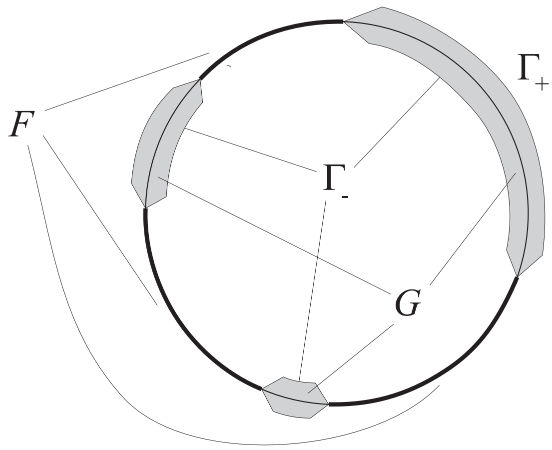

Figure 1: The domain G and its boundary

The circle S in these statements can also be a line and the $F \subset S$ a compact subset of that line (just apply the theorem to circles of radius R with $R \to \infty$). In particular, if $F \subset \mathbf{R}$ is a compact set with non-empty (one dimensional) interior, then the density of the equilibrium measure of F with respect to Lebesgue-measure on $\mathbf{R}$ is log-convex on any interval I that lies in F.

We also mention that Theorem 1 was extended to Riesz potentials in [3].

In [2] the proof of Theorem 1 was given by an iterated balayage technique. In this paper first we reprove Theorem 1 using the connection between harmonic measures and random walks. This proof will allow us in the next section to prove an analogue in higher dimensions.

Proof of Theorem 1. Let S^1 be the unit circle and let D_- resp. D_+ be the inner resp. exterior domains complement to S^1 (i.e D_- is the open unit disk and D_+ is the exterior of the closed unit disk). For a set $E \subset S^1$ let

$$E^* = \{t \in [0, 2\pi) \,|\, e^{it} \in E\}.$$

Let $F \subset S^1$ be a closed set and J a closed subarc in the (one dimensional) interior of F. For some small $\varepsilon > 0$ set

$$G = \{z \,|\, \mathrm{dist}(z, S^1 \setminus F) < \min\{\frac{1}{2}\mathrm{dist}(z, F), \varepsilon\}\}.$$

Then G is a small neighborhood of $S^1 \setminus F$, see Figure 1. Let Γ be the boundary of G and $\Gamma_\pm = \Gamma \cap D_\pm$ the part of that boundary that lies in the unit disk and in its exterior, respectively.

Let $a \in S^1 \setminus F$ be fixed. What we want to show is that the harmonic measure $\omega(\cdot, a, \mathbf{R}^2 \setminus F)$ has log-convex density on J. We can formulate the claim as there is a convex function $\sigma_a(t)$ (convexity in the variable t) on J^* such for any Borel set $E \subset J$ we have

$$\omega(E, a, \mathbf{R}^2 \setminus F) = \int_{E^*} \sigma_a(t)dt.$$

By simple approximation from the outside we may assume that F consists of finitely many arcs on S^1.

Start a 2-dimensional Brownian motion $X(t)$, $t \geq 0$, at a: $X(0) = a$. We are going to use Kakutani's theorem that for $E \subset J$ the harmonic measure $\omega(E, a, \mathbf{R}^2 \setminus F)$ is the probability that X leaves the domain $\mathbf{R}^2 \setminus F$ first at a point of E (see [4, Theorem F6, (F.10)], [6], or [9, Section 3.4]). Let

$$T_X = \min\{t \,|\, X(t) \in F\}.$$

Note that, by the recurrence of the two dimensional Brownian motion, we have $T_X < \infty$ almost surely (because we assumed that F contains and arc), although we are not going to use that. Let

$$T_X^0 = \sup\{t < T_X \,|\, X(t) \in S^1 \setminus F\}$$

and

$$T_X^1 = \min\{t \,|\, t \geq T_X^0,\ X(t) \in \Gamma\}.$$

Again, $T_X^1 < T_X < \infty$ almost surely, and $X(T_X^1) \in \Gamma$. Suppose that, say, $X(T_X^1) \in \Gamma_-$. Then $\{X(t) \,|\, t \geq T_X^1\}$ is a Brownian motion starting at $X(T_X^1) \in D_-$ that leaves the domain D_- in a point of F, so the probability that it leaves D_- at a point of a given Borel set $E \subset J$ is (conditional probability)

$$\frac{\omega(E, X(T_X^1), D_-)}{\omega(F, X(T_X^1), D_-)}.$$

Note that here the denominator is positive since F contains an arc of the unit circle. In a similar fashion, if $X(T_X^1) \in \Gamma_+$, then the probability that the Brownian motion $\{X(t) \,|\, t \geq T_X^1\}$ starting at $X(T_X^1) \in D_+$ leaves the domain D_+ at a point of $E \subset J$ is

$$\frac{\omega(E, X(T_X^1), D_+)}{\omega(F, X(T_X^1), D_+)}.$$

Now

$$\mu_{a,\pm}(A) = \mathbf{P}(X(T_X^1) \in A), \qquad A \subset \Gamma_\pm,\ A \text{ Borel},$$

are two positive Borel measures on $\Gamma_\pm$, respectively, and, according to what we have just explained, we have the formula

$$\omega(E, a, \mathbf{R}^2 \setminus F) = \int_{\Gamma_-} \frac{\omega(E, \zeta, D_-)}{\omega(F, \zeta, D_-)} d\mu_{a,-}(\zeta) + \int_{\Gamma_+} \frac{\omega(E, \zeta, D_+)}{\omega(F, \zeta, D_+)} d\mu_{a,+}(\zeta). \quad (1)$$

But here $\omega(E, \zeta, D_-)$ is given (see [4, Sec. 1.1] or [9, Theorem 3.44]) by the Poisson integral

$$\omega(E, \zeta, D_-) = \int_{E^*} P_\zeta(e^{it})dt$$

with the Poisson kernel

$$P_\zeta(e^{it}) = \frac{1}{2\pi}\frac{1-|\zeta|^2}{|\zeta - e^{it}|^2}, \qquad \zeta \in D_-,$$

and similarly

$$\omega(E, \zeta, D_+) = \int_{E^*} P_\zeta(e^{it})dt$$

with the exterior Poisson kernel

$$P_\zeta(e^{it}) = \frac{1}{2\pi}\frac{|\zeta|^2 - 1}{|\zeta - e^{it}|^2}, \qquad \zeta \in D_+.$$

Thus,

$$\omega(E, a, \mathbf{R}^2 \backslash F) = \int_{E^*}\left(\int_{\Gamma_-}\frac{P_\zeta(e^{it})}{\omega(F,\zeta,D_-)}d\mu_{a,-}(\zeta) + \int_{\Gamma_+}\frac{P_\zeta(e^{it})}{\omega(F,\zeta,D_+)}d\mu_{a,+}(\zeta)\right)dt,$$

i.e. the density σ_a in question is

$$\sigma_a(t) = \int_{\Gamma_-}\frac{P_\zeta(e^{it})}{\omega(F,\zeta,D_-)}d\mu_{a,-}(\zeta) + \int_{\Gamma_+}\frac{P_\zeta(e^{it})}{\omega(F,\zeta,D_+)}d\mu_{a,+}(\zeta). \qquad (2)$$

Now it is easy to see that the sum, and hence the integral of log-convex functions is again log-convex (see [2]), so it is sufficient to show that if ε (appearing in the definition of Γ) is sufficiently small, then for all $\zeta \in \Gamma$ the function $P_\zeta(e^{it})$ is log-convex on J^*. But that is simple: the function

$$\frac{1}{1 - 2\cos v + 1} = \frac{1}{4\sin^2(v/2)}$$

is strictly log-convex on the open interval $(0, 2\pi)$, hence

$$\frac{1}{1 - 2\cos(\theta - t) + 1}$$

is strictly log-convex on any interval not containing θ (mod2π). This and simple compactness implies that if $\theta \in (S^1 \setminus F)^*$, then for r sufficiently close to 1 the functions (in t)

$$\frac{1}{1 - 2r\cos(\theta - t) + r^2}$$

are log-convex for $t \in J^*$. But if $\zeta = re^{i\theta} \in \Gamma$, then $\theta \in (S^1 \setminus F)^*$ and r is to 1 $(1 - \varepsilon \le r \le 1)$, furthermore

$$P_\zeta(e^{it}) = \frac{1}{2\pi}\frac{|1 - r^2|}{1 - 2r\cos(\theta - t) + r^2},$$

so the log-convexity of $P_\zeta(e^{it})$ on J^* for all $\zeta \in \Gamma$ follows.

■

2 Harmonic measures in higher dimensions

In this section, we prove the following higher dimensional analogue of Theorem 1.

Theorem 2 *Let S be an $(n-1)$-dimensional sphere in $\mathbf{R}^n$, $n \geq 3$, and $F \subset S$ a closed set. If $a \in S \setminus F$, then the density of the harmonic measure $\omega(\cdot, a, \mathbf{R}^n \setminus F)$ is subharmonic on the* ($(n-1)$*-dimensional*) *interior of F.*

An explanation is needed for the statement. We may assume S to be S^{n-1}, the unit sphere in $\mathbf{R}^n$. Let $\mathrm{Int}(F)$ be the $(n-1)$-dimensional interior of F. On S we consider the geodesic distant d and geodesic spheres $S_\rho(P) = \{Q \in S \mid d(Q,P) = \rho\}$ (of dimension $n-2$) and geodesic balls $B_\rho(P) = \{Q \in S \mid d(Q,P) = \rho\}$ (of dimension $n-1$). Let also Δ be the Laplace-Beltrami operator on S (see e.g. [1, Section 3.1]), which is the restriction to S^{n-1} of the angular part of the Laplacian on $\mathbf{R}^n$. If dS denotes the surface element on S, then, in the interior of F, the harmonic measure has the form (see the proof below)

$$d\omega(E, a, \mathbf{R}^n \setminus F) = \int_E \sigma_a(\mathbf{x}) dS(\mathbf{x}),$$

where $\sigma_a(\mathbf{x})$ is the density function in the theorem. Now the subharmonicity of σ_a means either of the following:

(i) σ_a has the submean-value property on every geodesic sphere $S_\rho(P)$ lying in $\mathrm{Int}(F)$ together with its interior (i.e. the value $\sigma_a(P)$ is at most as large as the average of σ_a over $S_\rho(P)$),

(ii) σ_a has the submean-value property on every geodesic ball $B_\rho(P)$ lying in $\mathrm{Int}(F)$ (i.e. the value $\sigma_a(P)$ is at most as large as the average of σ_a over $B_\rho(P)$),

(iii) $\Delta\sigma_a(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathrm{Int}(F)$.

Naturally, the averages in question are taken with respect to the corresponding surface or volume elements on $S_\rho(P)$ or $B_\rho(P)$, respectively.

We are going to prove Theorem 2 in the form (iii). From there (ii) and (i) follow from the spherical version of Green's formulae, see e.g. [8, Proposition 2.1, (2.1), (2.2)]. An alternative approach is to apply that Δ-harmonic functions have the mean value property over geodesic balls (see e.g. [5, Corollary X.7.3] as well as the Remark there, or see [11]) and hence also over geodesic spheres, and then showing that (iii) implies that if σ_a agrees with a Δ-harmonic function on a geodesic sphere, then it is below that function inside that sphere (otherwise, if we add to σ_a a small multiple of an appropriate function with strictly positive spherical Laplacian, at a local maximum point (iii) would be violated).

Consider now the equilibrium measure μ_F in the Newtonian case (the kernel is $|z|^{2-n}$) of a non-polar compact set $F \subset \mathbf{R}^n$ in $\mathbf{R}^n$ (see e.g. [9, Sec. 4.3] where it is called harmonic measure from infinity, or [7, Section II.1], where a different normalization is used). Let us record the following

Corollary 3 *If F is a compact subset of a sphere or of $\mathbf{R}^{n-1}$, then the equilibrium measure (with respect to the Newtonian kernel in $\mathbf{R}^n$) μ_F of F has subharmonic density on the ((n − 1)-dimensional) interior of F.*

The density in question is taken with respect to surface measure on the sphere or on $\mathbf{R}^{n-1}$.

Indeed, the equilibrium measure μ_S of the whole sphere S is the uniform distribution on S, and from it the equilibrium measure μ_F of F is obtained by taking the balayage (see [7, Sect. IV.1]) of $\mu_S\Big|_{S\setminus F}$ onto F (followed by a normalization if one uses mass 1 for the equilibrium measure), i.e. $\mu_F = \nu/\nu(F)$, where

$$\nu(\cdot) = \mu_S\Big|_F(\cdot) + \int_{S\setminus F} \omega(\cdot, a, \mathbf{R}^n \setminus F)d\mu_S(a).$$

In view of this formula, Corollary 3 follows from Theorem 2.

Let us also mention that, just as in [2, Corollary 1.6], Theorem 2 implies the subharmonicity of the harmonic measures $\omega(\cdot, a, \mathbf{R}^n\setminus F)$ for a lying in a relatively large n-dimensional domain that contains $S \setminus F$, but the exact description of that domain is not clear and we do not pursue this direction.

Proof of Theorem 2. As before, we may assume that $S = S^{n-1}$ is the unit sphere in $\mathbf{R}^n$, and that the $(n-1)$-dimensional interior of F is not empty (otherwise there is nothing to prove). Also, by approximation from the outside, we may assume (just to avoid irregular sets for the Dirichlet problem) that F is the union of finitely many closed balls.

Again, it is sufficient to show that if the point $(0, \ldots, 0, 1)$ belongs to the interior of F, then the density σ_a of $\omega(\cdot, a, \mathbf{R}^n \setminus F)$ satisfies

$$\Delta\sigma_a(0, \ldots, 0, 1) > 0.$$

If, in a small neighborhood of $(0, \ldots, 0, 1)$, we use the Cartesian coordinates $(x_1, \ldots, x_{n-1})$ as local coordinates for S^{n-1}, then (see e.g. [1, (3.18)] or use [1, (3.4)])

$$\Delta\sigma_a(0, \ldots, 0, 1) = \sum_{i=1}^{n-1} \frac{\partial^2\sigma_a}{\partial x_i^2}(0, \ldots, 0, 1),$$

so we need to prove that the right-hand side is positive.

We follow the argument in the preceding proof. Let X be an n-dimensional Brownian motion in $\mathbf{R}^n$ starting at $a \in S \setminus F$. As before, $\omega(E, a, \mathbf{R}^n \setminus F)$ is the probability that X leaves the domain $\mathbf{R}^n \setminus F$ in a point of E (see [9, Section 3.4]).

Let now $D_\pm$ denote the interior/exterior domains to S^{n-1}, so D_- is the open unit ball in $\mathbf{R}^n$ and D_+ is the (open) exterior of that ball. Define G and its boundary Γ appropriately as in the preceding proof, and introduce the stopping times T_X, T_X^0, T_X^1 as above. It is no longer true that $T_X < \infty$ almost surely, but we may restrict our attention only to Brownian motions hitting F, in which case T_X^0, T_X^1 are finite (under the just set condition $T_X < \infty$). From our point of view this restriction is irrelevant, since that means that instead of the probability that X exits $\mathbf{R}^n \setminus F$ in a point of $E \subset F$ we consider the same probability under the condition $T_X < \infty$ (which has probability $\omega(F, a, \mathbf{R}^n \setminus F)$), so the two densities differ only in the multiplicative constant $\omega(F, a, \mathbf{R}^n \setminus F)$. We have again formula (1) in the form

$$\frac{\omega(E, a, \mathbf{R}^2 \setminus F)}{\omega(F, a, \mathbf{R}^2 \setminus F)} = \int_{\Gamma_-} \frac{\omega(E, \zeta, D_-)}{\omega(F, \zeta, D_-)} d\mu_{a,-}(\zeta) + \int_{\Gamma_+} \frac{\omega(E, \zeta, D_+)}{\omega(F, \zeta, D_+)} d\mu_{a,+}(\zeta),$$

and here the harmonic measures $\omega(E, \zeta, D_\pm)$ are given by integrals on E (against $(n-1)$-dimensional surface element of S^{n-1}) of the n-dimensional Poisson kernels (see [9, Theorem 3.44])

$$P_\zeta(\mathbf{x}) = \tau_n \frac{1 - |\zeta|^2}{|\zeta - \mathbf{x}|^n}, \qquad |\zeta| < 1,$$

and

$$P_\zeta(\mathbf{x}) = \tau_n \frac{|\zeta|^2 - 1}{|\zeta - \mathbf{x}|^n}, \qquad |\zeta| > 1,$$

where τ_n is a normalizing constant, and where $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^n$. Therefore, we get again (2) in the form

$$\frac{\sigma_a(\mathbf{x})}{\omega(F, a, \mathbf{R}^2 \setminus F)} = \int_{\Gamma_-} \frac{P_\zeta(\mathbf{x})}{\omega(F, \zeta, D_-)} d\mu_{a,-}(\zeta) + \int_{\Gamma_+} \frac{P_\zeta(\mathbf{x})}{\omega(F, \zeta, D_+)} d\mu_{a,+}(\zeta)$$

for $\mathbf{x} \in \mathrm{Int}(F)$. With the same argument as in the preceding proof we can see that the claim in the theorem reduces to the fact that for all $\zeta \in \Gamma$ (if the ε in the definition of Γ is sufficiently small) the function $P_\zeta(\mathbf{x})$ has positive spherical Laplacian at $(0, \ldots, 0, 1)$: $\Delta P_\zeta(0, \ldots, 0, 1) > 0$. Again, by compactness, it is enough to prove that if $\zeta = (\zeta_1, \ldots, \zeta_n) \neq (0, \ldots, 0, 1)$ belongs to S^{n-1} and $h(x_1, \ldots, x_{n-1})$ is defined as

$$\frac{1}{\left((\zeta_1 - x_1)^2 + (\zeta_2 - x_2)^2 + \cdots + (\zeta_{n-1} - x_{n-1})^2 + (\zeta_n - \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2})^2\right)^{n/2}},$$

then

$$\sum_{i=1}^{n-1} \frac{\partial^2 h}{\partial x_i^2}(\mathbf{0}) > 0. \tag{3}$$

Simple calculation shows that

$$\frac{\partial^2 h}{\partial x_i^2}(\mathbf{0}) = \frac{n}{2^{\frac{n}{2}+1}} \cdot \frac{-\zeta_n + \zeta_n^2 + (\frac{n}{2} + 1)\zeta_i^2}{(1 - \zeta_n)^{\frac{n}{2}+2}},$$

so in view of

$$\sum_{i=1}^{n-1} \zeta_i^2 = 1 - \zeta_n^2,$$

(3) reduces to showing that

$$\frac{(\frac{n}{2}+1)-(n-1)\zeta_n+(\frac{n}{2}-2)\zeta_n^2}{(1-\zeta_n)^{\frac{n}{2}+2}}$$

is positive for $\zeta_n \in [-1,1]$, $\zeta_n \neq 1$, which can be readily seen.

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It is clear from the just given proof that whatever properties one establishes for the "limit" Poisson kernel $1/|\zeta - \mathbf{x}|^n$ at $\mathbf{x} = (0, \cdots, 1) \neq \zeta$ on S^{n-1}, the same will be true for the density of harmonic measures $\omega(\cdot, a, \mathbf{R}^n \setminus F)$, $a \in S \setminus F$, in the interior of F, provided the property is preserved under summation and taking a limit.

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