Acta Sci. Math. (*Szeged*)
71 (2005), 427–455

Book Reviews

Communicated by P. Hajnal

Steven R. Finch, **Mathematical constants** (Encyclopedia of Mathematics and Its Applications, Vol. 94), XIX+602 pages, Cambridge University Press, Cambridge, 2003.

This excellent book contains 136 "stories", each devoted to a mathematical constant or a class of constants, from the well known and famous (π) to the highly exotic and obscure ones. Topics covered include the statistics of continued fractions, chaos in nonlinear systems, prime numbers, sum-free sets, isoperimetric problems, approximation theory, self-avoiding walks and the Ising model (from statistical physics), binary and digital search trees (from theoretical computer science), complex analysis, geometric probability, and the traveling salesman problem.

The exact list of the chapters is: 1. Well-known Constants, 2. Constants Associated with Number Theory, 3. Constants Associated with Analytic Inequalities, 4. Constants Associated with the Approximation of Functions, 5. Constants Associated with Enumerating Discrete Structures, 6. Constants Associated with Functional Iteration, 7. Constants Associated with Complex Analysis, 8. Constants Associated with Geometry.

The book is very useful and interesting for all mathematicians looking for information about a lot of specific constants, because it brings together all significant mathematical constants in one place.

J. Németh (Szeged)

Stephen G. Krantz, **A Mathematician's Survival Guide: Graduate School and Early Career Development**, XV+222 pages, American Math. Soc., Providence R.I., 2003.

A graduate student easily becomes discouraged. How should I start a research project? How to find a good advisor and problem? What to do if I can not solve my problem? What to do if I can? What am I supposed to do at the department? What shall I do in the next 3-5 years? And after? Am I gifted enough to be a mathematician? Do my mathematics make any sense at all? Who can help me and how? Such questions arise, and it may be difficult to find the answers. In this frequent situation A Mathematician's Survival Guide is really a need.

"This text was conceived during the years 1998-2001, on the occasion of a course that I taught at the École Normale Supériuere de Lyon. As such every result is accompanied by a detailed proof. During this course I tried to investigate all the principal mathematical aspects of matrices: algebraic, geometric, and analytic. ...

This text was first published in French by Masson (Paris) in 2000, under the title *Les Matrices: théorie et practique*. I have taken the opportunity during the translation process to correct typos and errors, to index a list of symbols, to rewrite some unclear paragraphs, and to add a modest amount of material and exercises. In particular, I added three sections, concerning alternate matrices, the singular value decomposition, and the Moore-Penrose generalized inverse. Therefore, this edition differs from the French one by about 10 percent of the contents."

This book is organized into ten chapters. The chapter titles are: Preface, List of Symbols, 1 Elementary Theory, 2 Square Matrices, 3 Matrices with Real or Complex Entries, 4 Norms, 5 Nonnegative Matrices, 6 Matrices with Entries in a Principal Ideal Domain; Jordan Reduction, 7 Exponential of a Matrix, Polar Decomposition, and Classical Groups, 8 Matrix Factorizations, 9 Iterative Methods for Linear Problems, 10 Approximation of Eigenvalues, References, Index.

As the author writes in the introduction "The first three chapters summarize the basics. The next six chapters can be read of matrix theory and should be known by almost every graduate student in any mathematical field. The other parts can be read more or less independently of each other. However, exercises in a given chapter sometimes refer to the material introduced in another one."

The main motivation of the author is to unify the different approaches, possible presentations for matrix theory and obtain a textbook that is suitable for the wild range of applications and hence for the different audiences deeply interested in matrices. The main goal is certainly achieved. The algorithm driven presentation is useful for those who are mainly interested in practical applications, but suitable for theoreticians too. The carefully chosen exercises give good chance to see deeper connections and see further lines of research.

The complexity theoretical language, the presentation of modern numerical techniques and the classical algebraic basics make this textbook an excellent source for graduate students in each field using matrices. It can be recommended both for use in the classroom and for independent study.

Péter Hajnal (Szeged)

Adi Ben-Israel and Thomas Nall Eden Greville (deceased), **Generalized Inverses** (Theory and Applications. Second Edition (CMS Books in Mathematics)), XV+420 pages, Springer-Verlag, New York – Berlin – Heidelberg – London – Paris – Tokyo – Hong Kong – Barcelona – Budapest, 2003.

The notion of invertibility plays a significant role in the study of linear algebra. Briefly, if A is a complex $m \times n$ matrix of rank r, then A is invertible if and only if $m =$

$n = r$, and in that case, the inverse of A is $A^{-1} = |A|^{-1}A^{ad}$, where $|A|$ is the determinant and A^{ad} is the classical adjoint of A. Extensions of the idea of invertibility may be traced back over several centuries, but the major contribution to today's understanding of the topic was provided by R. Penrose in 1955 [Proc. Cambridge Philos. Soc. **51** (1955), 406–413]. In the article entitled "A Generalized Inverse for Matrices" he showed that there exists one and only one solution to the following list of equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^* = AX$, and (4) $(XA)^* = XA$, where C^* denotes the conjugate transpose of C. The following year, R. Rado noted in the same journal [Proc. Cambridge Philos. Soc. **52** (1956), 600–601] that the solution to these equations was identical to the solution of a list of equations given 36 years earlier by E. H. Moore [Bull. Amer. Math. Soc. **26** (1920), 394–395]. The unique solution to these systems of equations, now called the Moore–Penrose inverse (some writers use the term "pseudoinverse") of A and usually denoted by $A^\dagger$, has since found application in several areas of mathematics and related natural scientific areas.

The book under review which is the second edition of the 30 years ago published one provides a detailed survey of generalized inverses and their main properties, illustrating the theory with applications. The second author, T.E.G. Greville, passed away before the project started. The book itself is the Volume 15 in the series of *Canadian Mathematical Society Books in Mathematics (Ouvrages de mathématiques de la Société mathématique du Canada)* published by Springer-Verlag of New York – Berlin – Heidelberg. It contains the two prefaces by the authors, the introduction, ten chapters, the appendix, and an extensive annotated bibliography.

In the introduction there are some about motivation of the study of generalized inverses (generalized inverse exists for a class of matrices larger than the class of nonsingular matrices; has some of the properties of the usual inverse; and reduces to the usual inverse when A is nonsingular), indication of their diversity, and some historical notes. The newly added preliminary chapter (Chapter 0) provides a collection of the facts, definitions, and notations that are used in successive chapters. The Introduction and the Preliminary chapters can be accessed freely on the internet: http://rutcor.rutgers.edu/˜bisrael/BIG-0.pdf.

For any $m \times n$ complex matrix, let $A\{i, j, \cdots, l\}$ denote the set of all matrices which satisfy equations $(i), (j), \ldots, (l)$ from among the Penrose equations (1), (2), (3), (4). A matrix $X \in A\{i, j, \ldots, l\}$ is called an $\{i, j, \ldots, l\}$-inverse of A. Chapter 1 deals with existence, construction and algebraic properties of $\{1\}$-, $\{1, 2\}$-, $\{1, 2, 3\}$-, $\{1, 2, 4\}$- and $\{1, 2, 3, 4\}$-inverses and includes an explicit formula for the Moore-Penrose $A^\dagger$ based on full rank factorization of A. A method is described for the construction of $\{2\}$-inverses having a prescribed rank between r, the rank of A, and $\min(m, n)$.

In Chapter 2 a consistency condition for the system $AXB = D$ in terms of $\{1\}$-inverses of A and B is described, and development of characterizations of the sets $A\{1, 3\}, A\{1, 4\}, A\{2\}, A\{1, 2\}$ and other subsets of $A\{2\}$ is provided. Special emphasis has been given to the relationship between generalized inverses and idempotent matrices and projectors, and to generalized inverses with prescribed range and null space. Description of constrained linear equation and its solution is given by using restricted

$\{1\}$-inverse, and application of $\{1\}$-inverse for solving interval linear programming and of $\{1, 2\}$-inverse for integral solution of linear equations are described. The Bott-Duffin inverse is developed and applied to electrical networks.

In Chapter 3, minimal properties of generalized inverses are introduced. For the system $Ax = b$ the smallest Euclidean norms are given and used when $X \in A\{1, 3\}$ and $X \in A\{1, 4\}$ in the case of inconsistent linear systems and of existent of multiplicity of solutions, respectively. Some examples of use of Tikhonov regularization are mentioned: constrained least squares problem, ridge estimation, more stable solution if A is ill-conditioned. After that weighted and oblique generalized inverses, essentially strictly convex norms and the associated projectors and generalized inverses are developed. In the end of this chapter an extremal property of the previously investigated Bott-Duffin inverse is treated with application to electrical network.

Chapter 4 deals with spectral generalized inverses. If λ is a nonzero eigenvalue of a singular square matrix A, it is not necessarily true that λ^{-1} is an eigenvalue of the generalized inverse $A^\dagger$. Generalized inverses having some of the spectral properties of the inverse of a nonsingular matrix are studied. To this end, the four Penrose equations are supplemented further by the following equations applicable only to square matrices: $(1^k) A^k X A = A^k$, (5) $AX = XA$, (6) $A^k X = X A^k$, (7) $AX^k = X^k A$, where k is a given positive integer. The authors defined and clearly developed the concept of a spectral inverse. The unique $\{1, 2, 5\}$-inverse called the group inverse had particular attention and its spectral properties are developed in detail (the group inverse exists if and only if the range and null space of A are complementary subspaces). A similar treatment is also given to the Drazin pseudoinverse (equivalently to the $\{1^k, 2, 5\}$-inverse, where k is the index of A), which exists for every square matrix. The spectral properties of the Drazin pseudo-inverse are the same as those of the group inverse with regard to nonzero eigenvalues and the associated eigenvectors, but weaker for 0-vectors.

Chapter 5 entitled "Generalized inverses of partitioned matrices" contains the study of linear equations and matrices in partitioned form, and the associated generalized inverses. It includes partitioning by columns and by rows to solve linear equations and to compute generalized inverses; studying intersections of linear manifolds and using the results to obtain common solutions of pairs of linear equations and to invert matrices partitioned by rows; various results on generalized inverses of bordered and partitioned matrices. In the introduction of this chapter Greville's method for computing $A^\dagger$ was rung in but it is missing.

We can agree that Chapter 6, "Spectral theory for rectangular matrices", is perhaps the most lucid and complete presentation of the topic to be found in any text. It treats singular value decompositions, the Schmidt approximation theorem, partial isometries, the polar decomposition theorem and other decompositions for rectangular matrices, briefly the behavior of the Moore–Penrose inverse of perturbed matrix $A + E$, generalization of the classical spectral theorem for normal matrices to rectangular ones, and the generalized singular value decompositions.

Chapter 7 deals with computational aspects of generalized inverses mentioning some direct methods for the computation of $\{1\}$-, $\{1, 2\}$- and $\{1, 3\}$-inverses and of $\{2\}$-inverses with prescribed range and null space, as well as with iterative methods for computing $A^\dagger$. Greville's method and related results appeared in this chapter. The treatment is very brief and no error analysis is attempted: "these topics have been studied extensively in the numerical analysis literature".

Chapter 8, "Miscellaneous applications", is another newly added chapter and it starts a little bit provocative question: "What can you do with generalized inverses that you could not do without?" Parallel sums with applications in electrical networks; linear statistical model with biased and unbiased estimation; application to Newton methods for solving systems of nonlinear equations, to linear system theory, of group inverse to finite Markov chains, of Drazin inverse for solving singular linear difference equations; and finally the concept of matrix volume applied to surface integrals and to probability distributions may be the answers to the question.

The final chapter is about generalized inverses of linear operator between Hilbert spaces. Generalized inverses of linear integral and differential operators, as well as minimal properties of generalized inverses are studied. Results on integral and series representations of generalized inverses, and iterative methods for their computation, are given without proof.

The Appendix is based on a recently published paper by the first author on Moore of the Moore–Penrose inverse: Moore's lecture to the AMS in 1920 and general reciprocal translated to plain English.

An important feature of this book is the over 600 exercises (many of which are solved in detail) which are primarily intended to impart information rather than to test understanding. These exercises contain several applications beside Chapter 8. Each chapter ends with the section "Suggested further reading". These sections provide excellent additional references on topics treated, as well as reading suggestions on topics which are not considered in the book. The bibliography contains exactly 901 references. The first author compiled a more completed bibliography on generalized inverse containing over 2100 items which can be accessed from the Internet address http://rutcor.rutgers.edu/pub/bisrael/GI.html.

The book is very recommended for reference and self-study or for use as a classroom text. Because only elementary knowledge of linear algebra is assumed it can be used profitably by graduate or advanced undergraduate students of mathematics and computer science, and by PhD students of related scientific area, e.g., chemometrics, chemo/bioinformatics, biometrics, bio/geo-statistics, physics etc.

Róbert Rajkó (Szeged)

Alejandro Adem, R. James Milgram, **Cohomology of finite groups** (Grundlehren der Mathematischen Wissenschaften, Fundamental Principles of Mathematical Sciences 309), VIII+327 pages, Springer-Verlag, Berlin, 1994.

This is the second edition of the book of the same title whose first edition appeared in 1994. This book was a welcome addition to the literature on the cohomology of groups