COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
60. SETS, GRAPHS AND NUMBERS, BUDAPEST (HUNGARY), 1991

# Interval Orders and Shift Graphs

Z. FÜREDI, P. HAJNAL, V. RÖDL and W. T. TROTTER*

**ABSTRACT**

A finite partially ordered set (poset) $\mathbf{P} = (X, P)$ is called an interval order if there is a $1-1$ function which assigns to each element $x \in X$ a closed interval $[a_x, b_x]$ of the real line $\mathbb{R}$ so that $x < y$ in $P$ if and only if $b_x < a_y$ in $\mathbb{R}$. For a poset $(X, P)$, height$(X, P)$ is the maximum number of points in a chain, while $\dim(X, P)$ is the dimension of $(X, P)$, the minimum number of linear orders on $X$ whose intersection is the partial order $P$. In 1972, I. Rabinovitch proved that the function $f(n) = \max\{\dim(X, P) :\ \text{height}(X, P) \leq n,\ P \text{ is an interval order}\}$ is defined and satisfies $f(n) \leq \lceil n+1 \rceil$. In this paper, we show that $f(n) = \lg\lg n + (1/2 + \ o(1))(\lg\lg\lg n)$. The proof techniques include establishing links between the dimension of interval orders, the chromatic number of shift graphs, and the classical problem of counting the number of antichains in the subset lattice.

## 1. Introduction

We consider a *partially ordered set* $\mathbf{P}$ (also called a *poset* or *ordered set*) as a pair $(X, P)$ where $X$ is a set (always finite in this paper) and $P$ is a

* Research supported in part by the National Science Foundation under DMS–8902481.

reflexive, antisymmetric and transitive binary relation on $X$. The *dimension* of a poset $\mathbf{P} = (X, P)$, denoted by $\dim(\mathbf{P})$ or $\dim(X, P)$, is the least $t$ so that the partial order $P$ is the intersection of $t$ linear orders on $X$. We refer the reader to the monograph [14] and the survey articles [6], [11], [12] for additional material on dimension theory for posets.

The *height* of a poset $\mathbf{P} = (X, P)$, denoted by $\mathrm{height}(X, P)$, is the largest $n$ so that $(X, P)$ contains a chain of $n$ points. A poset of height 1 is an antichain and has dimension at most 2. A poset of height 2 can have arbitrarily large dimension. Examples are given in [7].

There is an important class of posets for which the dimension is bounded in terms of height. A poset $\mathbf{P} = (X, P)$ is an *interval order* if there is a $1-1$ function assigning to each element $x \in X$ a closed[†] interval $[a_x, b_x]$ of the real line $\mathbb{R}$ so that $x < y$ in $P$ if and only if $b_x < a_y$ in $\mathbb{R}$. The following well-known theorem of P. Fishburn [4] gives a forbidden subposet characterization of interval orders.

**Theorem 1.1.** (P. Fishburn). *A poset $\mathbf{P} = (X, P)$ is an interval order if and only if $X$ does not contain four points $x_1, x_2, y_1, y_2$ with $x_1 < x_2$, $y_1 < y_2$, $x_1 \not< y_2$, and $y_1 \not< x_2$ in $P$.* ■

In 1972, I. Rabinovitch [9] proved that the function $f(n) = \max\{\dim(X, P) : (X, P)$ is an interval order and height $(X, P) \leq n\}$ is well defined and satisfies $f(n) \leq \lceil n + 1 \rceil$. A logarithmic upper bound on $f(n)$ was proved by Bogart, Rabinovitch and Trotter in [1]. The primary goal of this paper will be to prove a surprisingly tight asymptotic formula for $f(n)$. Hereafter, all logarithms are base 2.

**Theorem 1.2.** *The maximum dimension $f(n)$ of an interval order of height $n$ satisfies:*

$$f(n) = \lg\lg n + \left(\frac{1}{2} + o(1)\right) \lg\lg\lg n. \quad ■$$

In the next section of the paper, we establish combinatorial connections between the dimension of an interval order, the chromatic number of shift graphs, and the number of antichains in the lattice of subsets of a finite set. In order to exploit these connections, we will also extend the concept of shift graphs to hypergraphs.

[†] The essential property is that each $x \in X$ be represented by a connected subset of $\mathbb{R}$. In this paper, we follow the most widely used convention and require that the connected set be a nondegenerate closed interval.

## 2. Shift graphs and antichains in subset lattices

In this paper, we use $[n]$ to denote the finite set $\{1, 2, \ldots, n\}$. For a set $X$ and a nonnegative integer $m$, $\binom{X}{m}$ denotes the set of all $m$-element subsets of $X$. Whenever $S = \{x_1, x_2, \ldots, x_t\}$ is a set of integers, we assume that the elements of $S$ have been labelled so that $x_i < x_j$ whenever $i < j$.

For integers $n, k$ with $n \geq k + 1$, Erdős and Hajnal [2] (see also [5]) defined the *shift graph* $\mathbf{G}(n, k)$ as a graph whose vertex set is $\binom{[n]}{k}$ and whose edge set consists of all pairs of the form $\{\{x_1, x_2, \ldots, x_k\}, \{x_2, x_3, \ldots, x_{k+1}\}\}$ where $S = \{x_1, x_2, \ldots, x_{k+1}\} \in \binom{[n]}{k+1}$. The shift graph $\mathbf{G}(n, 1)$ is a complete graph on $n$ vertices, and of course its chromatic number is $n$. It is an easy exercise to show that $\chi(\mathbf{G}(n, 2)) = \lceil \lg n \rceil$. For larger $k$, Erdős and Hajnal [2] gave the following asymptotic formula.

**Theorem 2.1.** (Erdős and Hajnal) *For fixed $k \geq 3$, $\chi(\mathbf{G}(n, k)) = (1 + o(1)) \lg\lg \ldots \lg n$, where the logarithm function is iterated $k - 1$ times.* ■

In this paper, we are concerned primarily with the family $\{\mathbf{G}(n, 3) : n \geq 4\}$. These graphs are also called *double shift* graphs. For double shift graphs, we can provide a more accurate estimate on the chromatic number than is provided by (2.1). For an integer $t \geq 1$, let $\mathbf{2}^t$ denote the poset consisting of all subsets of $[t]$ partially ordered by inclusion. The poset $\mathbf{2}^t$ is a distributive lattice having dimension $t$, height $t + 1$ and width $\binom{t}{\lfloor \frac{t}{2} \rfloor}$.

The following result is part of the folklore of this subject, but to the best of our knowledge, no one has published a proof. We do so now because it directly motivates arguments which follow.

**Theorem 2.2.** *For each integer $n \geq 4$, the chromatic number of the double shift graph $\mathbf{G}(n, 3)$ is the least $t$ for which there are at least $n$ antichains in $\mathbf{2}^t$, the lattice of all subsets of $[t]$.*

**Proof.** Suppose first that $\chi(\mathbf{G}(n, 3)) = t$. We show that $\mathbf{2}^t$ has at least $n$ antichains. Let $\psi : \binom{[n]}{3} \to [t]$ be a good coloring of $\mathbf{G}(n, 3)$. For each $A = \{i_1, i_2\} \in \binom{[n]}{2}$, let $S_A = \{\alpha \in [t] :$ there exists $i_3 \in [n]$ with $i_3 > i_2$ and $\psi(\{i_1, i_2, i_3\}) = \alpha\}$. For each $i_1 \in [n]$, let $\mathcal{C}_{i_1} = \{S_A :$ there exists $i_2 \in [n]$ with $i_2 > i_1$ and $A = \{i_1, i_2\}\}$. Partial order each $\mathcal{C}_{i_1}$ by inclusion, and let $\mathcal{M}_{i_1}$ be the set of maximal elements. Note that $\mathcal{M}_{i_1}$ is an antichain in the lattice $\mathbf{2}^t$.

Next, we show that the $n$ antichains in $\{\mathcal{M}_{i_1} : i_1 \in [n]\}$ are distinct. Suppose to the contrary that $A = \{i_1, i_2\} \in \binom{[n]}{2}$ and $\mathcal{M}_{i_1} = \mathcal{M}_{i_2}$. Then $S_A \in \mathcal{C}_{i_1}$, so there is some $S_B \in \mathcal{M}_{i_1}$ with $S_A \subseteq S_B$. Since $\mathcal{M}_{i_1} = \mathcal{M}_{i_2}$, we know $S_B \in \mathcal{M}_{i_2} \subseteq \mathcal{C}_{i_2}$, so there is an integer $i_3 \in [n]$ with $i_2 < i_3$ and $S_B = S_C$, where $C = \{i_2, i_3\}$.

Now suppose $\psi(\{i_1, i_2, i_3\}) = \alpha$. Then $\alpha \in S_A \subseteq S_B = S_C$. So there exists $i_4 \in [n]$ with $i_4 > i_3$ such that $\psi(\{i_2, i_3, i_4\}) = \alpha$. This contradicts the assumption that $\psi$ is a good coloring of $\mathbf{G}(n, 3)$. We conclude that $\mathbf{2}^t$ has at least $n$ antichains.

Now suppose that $\mathbf{2}^t$ contains at least $n$ antichains. We show that $\chi(\mathbf{G}(n, 3)) \le t$. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ be antichains in $\mathbf{2}^t$ labelled so that if $1 \le i_1 < i_2 \le n$, then there exists a set $S \in \mathcal{M}_{i_2}$ so that $S \not\subseteq T$, for every $T \in \mathcal{M}_{i_1}$ (such a labelling is easily seen to exist).

For each $A = \{i_1, i_2\} \in \binom{[n]}{2}$, choose a set $S_A \in \mathcal{M}_{i_2}$ so that $S_A \not\subseteq T$ for every $T \in \mathcal{M}_{i_1}$. Then let $\{i_1, i_2, i_3\} \in \binom{[n]}{3}$, $A = \{i_1, i_2\}$ and $B = \{i_2, i_3\}$. Observe that $S_B \not\subseteq S_A$ so we may define a function $\psi : \binom{[n]}{3} \to [t]$ by taking $\psi(\{i_1, i_2, i_3\})$ as an element from $S_B - S_A$.

We claim that $\psi$ is a good coloring of $\mathbf{G}(n, 3)$. To see this, let $\{i_1, i_2, i_3, i_4\} \in \binom{[n]}{4}$, $A = \{i_1, i_2\}$, $B = \{i_2, i_3\}$, and $C = \{i_3, i_4\}$. Then $\psi(\{i_1, i_2, i_3\}) \in S_B - S_A$ and $\psi(\{i_2, i_3, i_4\}) \in S_C - S_B$, so $\psi(\{i_1, i_2, i_3\}) \ne \psi(\{i_2, i_3, i_4\})$. ∎

The problem of counting the number of antichains in $\mathbf{2}^t$ is a classical one and is often called Dedekind's problem. Asymptotic solutions are given in [7] and [8]. For our purposes, we need only the fact that these estimates assert that the total number of antichains in $\mathbf{2}^t$ is approximately the number of subsets of the largest antichain.

**Corollary 2.3.** *The chromatic number of the shift graph $\mathbf{G}(n, 3)$ satisfies:*

$$\chi(\mathbf{G}(n, 3)) = \lg \lg n + \left(\frac{1}{2} + o(1)\right) \lg \lg \lg n. \qquad ∎$$

Further applications of shift graphs in the theory of chromatic numbers can be found in [3].

## 3. Interval orders and shift graphs

For an integer $n \geq 4$, let $\mathbf{I}_n = (I_n, P_n)$ denote the poset whose elements are the nondegenerate closed intervals with integer endpoints from $[n]$ with $[i_1, i_2] < [j_1, j_2]$ in $P_n$ if and only if $i_2 < j_1$. Clearly, each $\mathbf{I}_n$ is an interval order, and we call the posets in the family $\{\mathbf{I}_n : n \geq 4\}$ *canonical* interval orders.

**Theorem 3.1.** *Let $n \geq 4$, let $\mathbf{I}_n = (I_n, P_n)$ be the canonical interval order and let $\mathbf{G}(n, 3)$ be the double shift graph. Then $\dim(\mathbf{I}_n) \geq \chi(\mathbf{G}(n, 3))$.*

**Proof.** Suppose that $\dim(\mathbf{I}_n) = t$. Choose linear extensions $L_1, L_2, \ldots, L_t$ of $P_n$ whose intersection is $P_n$. Now define a coloring $\psi : \binom{[n]}{3} \to [t]$ as follows. For each $\{i_1, i_2, i_3\} \in \binom{[n]}{3}$, choose $\alpha \in [t]$ so that $[i_1, i_2] > [i_2, i_3]$ in $L_\alpha$, and set $\psi(\{i_1, i_2, i_3\}) = \alpha$. If $\{i_1, i_2, i_3, i_4\} \in \binom{[n]}{4}$, there is no $\alpha \in [t]$ for which $\psi(\{i_1, i_2, i_3\}) = \alpha = \psi(\{i_2, i_3, i_4\})$, since this would require $[i_1, i_2] > [i_2, i_3] > [i_3, i_4]$ in $L_\alpha$. However $[i_1, i_2] < [i_3, i_4]$ in $P_n$ requires $[i_1, i_2] < [i_3, i_4]$ in $L_\alpha$. We conclude that $\psi$ is a good coloring of the shift graph $\mathbf{G}(n, 3)$, so $\dim(\mathbf{I}_n) \geq \chi(\mathbf{G}(n, 3))$. ∎

## 4. Some dimension theoretic preliminaries

For a poset $\mathbf{P} = (X, P)$, we write $x \| y$ in $P$ and say $x$ is *incomparable* to $y$ in $P$ when $x \not\leq y$ and $y \not\leq x$ in $P$. Then let $\text{inc}(X, P) = \{(x, y) \in X \times X : x \| y \text{ in } P\}$. A subset $\{(x_i, y_i) : 1 \leq i \leq k\} \subseteq \text{inc}(X, P)$ is called an *alternating cycle* (of length $k$) if $x_{i+1} \leq y_i$ for $i = 1, 2, \ldots, k-1$ and $x_1 \leq y_k$. An alternating cycle $\{(x_i, y_i) : 1 \leq i \leq k\}$ is *strict* if $x_j \leq y_i$ if and only if $j = i + 1$ for all $i = 1, 2, \ldots, k-1$, and $x_j \leq y_k$ if and only if $j = 1$. In Figures 1a and 1b below, $\{(x_i, y_i) : 1 \leq i \leq 4\}$ is an alternating cycle of length 4. The cycle is strict in 1b but not in 1a.

The following elementary lemma is due to Trotter and Moore [16].

**Lemma 4.1.** *Let $(X, P)$ be a poset and let $S \subseteq \text{inc}(X, P)$. Then the following statements are equivalent:*

(1) *There is a linear extension $L$ of $P$ with $x > y$ in $L$ for every $(x, y) \in S$.*
(2) *$S$ does not contain any alternating cycles.*
(3) *$S$ does not contain any strict alternating cycles.* ∎

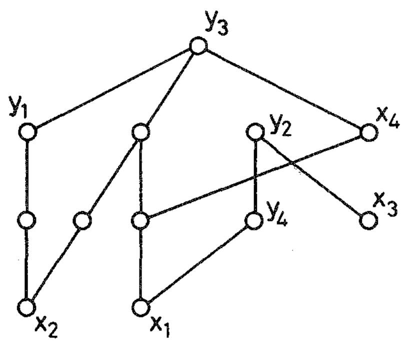


Figure 1a.

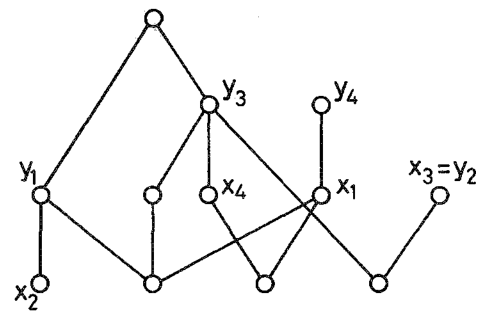


Figure 1b.

Let $\mathbf{P} = (X, P)$ be a poset. An incomparable pair $(x, y)$ is called a *critical* pair if (1) $z < x$ in $P$ implies $z < y$ in $P$ for all $z \in X$ and (2) $y < w$ in $P$ implies $x < w$ in $P$ for all $w \in X$. It is easy to see that if $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is a family of linear extensions of $P$, then $P = L_1 \cap L_2 \cap \ldots \cap L_t$ if and only if for every critical pair $(x, y)$, there is some $\alpha \in [t]$ such that $x > y$ in $L_\alpha$. Let $\mathrm{crit}(X, P) = \mathrm{crit}(\mathbf{P})$ denote the set of all critical pairs in the poset $\mathbf{P} = (X, P)$.

For a poset $\mathbf{P} = (X, P)$, define a hypergraph $\mathcal{H}_\mathbf{P}$ as follows. The vertex set of $\mathcal{H}_\mathrm{p}$ is $\mathrm{crit}(\mathbf{P})$, the set of all critical pairs. The edge set of $\mathcal{H}_\mathbf{P}$ consists of those subsets of $\mathrm{crit}(\mathbf{P})$ which form strict alternating cycles. For emphasis, we state the following result as a lemma, although it follows immediately from the definition of the hypergraph $\mathcal{H}_\mathbf{P}$.

**Lemma 4.2.** *Let $\mathcal{H}_\mathbf{P}$ be the hypergraph of critical pairs associated with a poset $\mathbf{P} = (X, P)$. Then $\chi(\mathcal{H}_\mathbf{P}) = \dim(\mathbf{P})$.* ∎

The following lemma gives us a key property of edges in $\mathcal{H}_\mathbf{P}$ when $\mathbf{P}$ is an interval order.

**Lemma 4.3.** *Let $\mathbf{P} = (X, P)$ be an interval order and let $S = \{(x_i, y_i) : 1 \le i \le k\}$ be a strict alternating cycle of incomparable pairs. If $x_1 < y_k$, then $x_{i+1} = y_i$ for $i = 1, 2, \ldots, k-1$.*

**Proof.** Suppose $x_{i+1} < y_i$ for some $i$ with $1 \le i \le k-1$. Then the four points $x_1, y_k, x_{i+1}, y_i$ violate Fishburn's characterization (1.1) of interval orders. ∎

Now let $\mathbf{P} = (X, P)$ be an interval order as evidenced by the map which assigns to each $x \in X$ a nondegenerate closed interval $[a_x, b_x]$ of $\mathbb{R}$ so that $x < y$ in $P$ if and only if $b_x < a_y$ in $\mathbb{R}$. Set $\text{crit}^*(\mathbf{P}) = \{(x,y) \in \text{crit}(\mathbf{P}) : a_x < a_y \le b_x < b_y \text{ in } \mathbb{R}\}$ and $\dim^*(\mathbf{P})$ = least positive integer $t$ for which there exist $t$ linear extensions $L_1, L_2, \ldots, L_t$ of $P$ so that:

(∗) for every $(x,y) \in \text{crit}^*(\mathbf{P})$, there is some $j \in [t]$ such that $x > y$ in $L_j$.

**Lemma 4.4.** *If* $\mathbf{P} = (X, P)$ *is an interval order, then* $\dim^*(\mathbf{P}) \le \dim(\mathbf{P}) \le 2 + \dim^*(\mathbf{P})$.

**Proof.** The inequality $\dim^*(\mathbf{P}) \le \dim(\mathbf{P})$ is trivial. We now show $\dim(\mathbf{P}) \le 2 + \dim^*(\mathbf{P})$. Let $\dim^*(\mathbf{P}) = t$ and let $L_1, L_2, \ldots, L_t$ be linear extensions of $P$ satisfying property (∗). Then define linear extensions $L_{t+1}$ and $L_{t+2}$ by:

$$(1) \quad x < y \text{ in } L_{t+1} \text{ if } \begin{cases} a_x < a_y, \text{ or} \\ a_x = a_y \text{ and } x > y \text{ in } L_t \end{cases}$$

$$(2) \quad x < y \text{ in } L_{t+2} \text{ if } \begin{cases} b_x < b_y, \text{ or} \\ b_x = b_y \text{ and } x > y \text{ in } L_t \end{cases}$$

It follows easily that $P = L_1 \cap L_2 \cap \cdots \cap L_{t+2}$, so that $\dim(P) \le t+2$. ∎

In view of the preceding lemma, we will be concerned only with estimating $\dim^*(\mathbf{P})$ in the remainder of this paper. For example, Theorem (3.1) actually asserts that $\dim^*(\mathbf{I}_n) \ge \chi(\mathbf{G}(n,3))$.

When $\mathbf{P} = (X, P)$ is an interval order, we let $\mathcal{H}^*_{\mathbf{P}}$ denote the subhypergraph of $\mathcal{H}_{\mathbf{P}}$ induced by $\text{crit}^*(\mathbf{P})$. Observe that $\dim^*(\mathbf{P}) = \chi(\mathcal{H}^*_{\mathbf{P}})$. For the canonical interval order $\mathbf{I}_n$, we abbreviate $\mathcal{H}_{\mathbf{I}_n}$ to $\mathcal{H}_n$ and $\mathcal{H}^*_{\mathbf{I}_n}$ to $\mathcal{H}^*_n$.

**Lemma 4.5.** *Let* $n \ge 4$. *Then a subset* $E \subseteq \text{crit}^*(\mathbf{I}_n)$ *is an edge in* $\mathcal{H}^*_n$ *if and only if there is some integer* $k \ge 2$ *and subsets* $\{a_1, a_2, \ldots, a_{k+1}\}$, $\{b_1, b_2, \ldots, b_{k+1}\} \in \binom{[n]}{k+1}$ *such that:*

1. $a_i < b_i$ *for* $i = 1, 2, \ldots, k+1$,
2. $a_k \le b_1 < a_{k+1} \le b_2$ *and*
3. $E = \{([a_i, b_i], [a_{i+1}, b_{i+1}]) : 1 \le i \le k\}$.

**Proof.** Suppose first that $\{a_1, a_2, \ldots, a_{k+1}\}$ and $\{b_1, b_2, \ldots, b_{k+1}\}$ are subsets from $\binom{[n]}{k+1}$ and $E = \{([a_i, b_i], [a_{i+1}, b_{i+1}]) : 1 \le i \le k\} \subseteq \operatorname{crit}^*(\mathbf{I}_n)$. For $i = 1, 2, \ldots, k$, set $x_i = [a_i, b_i]$ and $y_i = [a_{i+1}, b_{i+1}]$. Then $x_{i+1} = y_i$ in $P_n$ for $i = 1, 2, \ldots, k$. Also, $x_1 = [a_1, b_1] < [a_{k+1}, b_{k+1}] = y_k$, so E is alternating cycle. Clearly, $E$ is a strict alternating cycle.

Conversely, let $E = \{(x_i, y_i) : 1 \le i \le k\}$ be a strict alternating cycle of critical pairs each belonging to $\operatorname{crit}^*(\mathbf{I}_n)$. By Lemma (4.3), we may assume $x_{i+1} = y_i$ for $i = 1, 2, \ldots, k$. For $i = 1, 2, \ldots, k$, set $x_i = [a_i, b_i]$; also, set $y_k = [a_{k+1}, b_{k+1}]$. Since $(x_i, y_i) \in \operatorname{crit}^*(\mathbf{I}_n)$, we know $a_i < a_{i+1} \le b_i < b_{i+1}$ for $i = 1, 2, \ldots, k$, so $\{a_1, a_2, \ldots, a_{k+1}\}, \{b_1, b_2, \ldots, b_{k+1}\} \in \binom{[n]}{k+1}$. Since $x_i, y_i \in I_n$, we know $a_i < b_i$ for $i = 1, 2, \ldots, k+1$. Therefore, $x_1 = y_k$ is impossible. It follows that $x_1 < y_k$ and thus $b_1 < a_{k+1}$. Since $E$ is strict, we conclude that $a_k \le b_1 < a_{k+1} \le b_2$. ∎

## 5. The augmented hypergraph

Let $n \ge 4$. We define the *augmented hypergraph* $\mathcal{H}_n^{**}$ as follows. The vertex set of $\mathcal{H}_n^{**}$ is the same as the vertex set of $\mathcal{H}_n^*$. Also, every edge of $\mathcal{H}_n^*$ is an edge in $\mathcal{H}_n$. However $\mathcal{H}_n^{**}$ contains some additional edges not present in $\mathcal{H}_n^*$.

Let $k \ge 2$ and let $\{a_1, a_2, \ldots, a_{k+1}\}, \{b_1, b_2, \ldots, b_{k+1}\} \in \binom{[n]}{k+1}$. We call this pair of sets *admissible* if:

(1) $a_i < b_i$ for $i = 1, 2, \ldots, k+1$ and

(2) $a_k \le b_1 \le a_{k+1} \le b_2$.

We then take as edges in $\mathcal{H}_n^{**}$ all sets of critical pairs $E = \{([a_i, b_i], [a_{i+1}, b_{i+1}]) : 1 \le i \le k\}$ arising from an admissible pairs of sets. Note that the essential change in the definition of the edges in $\mathcal{H}_n^{**}$ from the characterization in (4.5) is in the weakening of the condition $b_1 < a_{k+1}$ to $b_1 \le a_{k+1}$. Clearly, $\chi(\mathcal{H}_n^*) \le \chi(\mathcal{H}_n^{**})$, so in what follows, we will derive an upper bound on $\chi(\mathcal{H}_n^{**})$.

At this point, the reader may question why we have introduced the augmented hypergraph. The answer will come in Section 6 when we extend the estimate on the dimension of a canonical order to an estimate on the dimension of a general interval order.

The next result is the principal theorem of this paper.

**Theorem 5.1.** *Let $n \geq 4$ and let $\mathcal{H}_n^{**}$ be the augmented hypergraph. If $t$ is an integer with $t \geq 2$, $s = \binom{t}{\lfloor \frac{t}{2} \rfloor}$ and $n \leq 2^s$, then $\chi(\mathcal{H}_n^{**}) \leq t+1$.*

**Proof.** Let $A_1, A_2, \ldots, A_s$ be the subsets of $[t]$ with $|A_j| = \lfloor t/2 \rfloor$ for $j = 1, 2, \ldots, s$. Set $q = 2^s$ and let $f_1, f_2, \ldots, f_q$ be the set of all functions from $[s]$ to $\{0, 1\}$. Without loss of generality, we assume that these functions have been labelled in lexicographic order, i.e., for each $\{i_1, i_2\} \in \binom{[q]}{2}$, if $j$ is the least integer in $[s]$ for which $f_{i_1}(j) \neq f_{i_2}(j)$, then $f_{i_1}(j) = 0$ and $f_{i_2}(j) = 1$.

As a consequence of this labelling, note that the following property is satisfied.

**Lexicographic Property**. If $S \subseteq [q]$ with $|S| \geq 2$ and $j$ is the least integer in $[s]$ for which the elements of $\{f_i(j) : i \in S\}$ are not all identical, then there is a unique element $i_0 \in S$ such that for all $i \in S$, $f_i(j) = 0$ if and only if $i \leq i_0$.

We will now define a good coloring $\psi$ of the augmented hypergraph $\mathcal{H}_n^{**}$ using the colors in $[t+1]$. Let $v$ be a vertex in $\mathcal{H}_n^{**}$. Choose integers $a_1, b_1, a_2, b_2$ from $[n]$ so that $a_1 < a_2 \leq b_1 < b_2$ and $v = ([a_1, b_1], [a_2, b_2])$. Let $S_v = \{a_1, b_1\} \cup \{a_2, b_2\}$ denote the set of endpoints of the two intervals in $v$. Then $S_v$ is either a 3-element or 4-element subset of $[n]$. Let $j$ be the least integer in $[s]$ for which the elements of $\{f_i(j) : i \in S_v\}$ are not all identical. We call $j$ the *distinguishing column* of $v$ and write $j = \mathrm{dc}(v)$. This terminology arises from considering the entries of $\{f_i(j) : i \in [q], j \in [s]\}$ as a $q \times s$ matrix of zeros and ones.

It is an immediate consequence of the lexicographic property that we may classify the vertex $v$ as *balanced*, *zero-dominant* or *one-dominant* by the following scheme:

1. $v$ is *balanced* if $j = \mathrm{dc}(v)$,
$$f_{a_1}(j) = f_{a_2}(j) = 0 \quad \text{and}$$
$$f_{b_1}(j) = f_{b_2}(j) = 1\,;$$

2. $v$ is *zero-dominant* if $j = \mathrm{dc}(v)$,
$$f_{a_1}(j) = f_{a_2}(j) = f_{b_1}(j) = 0 \quad \text{and}$$
$$f_{b_2}(j) = 1 \quad \text{and}$$

3. $v$ is *one-dominant* if $j = \mathrm{dc}(v)$,
$$f_{a_1}(j) = 0 \quad \text{and}$$
$$f_{a_2}(j) = f_{b_1}(j) = f_{b_2}(j) = 1\,.$$

If $v$ is zero-dominant and $j = \mathrm{dc}(v)$ define $T_v = \{i \in S_v : f_i(j) = 0\}$. Then let $j'$ denote the least integer in $[s]$ for which the elements in $\{f_i(j') : i \in T_v\}$ are not all identical. We call $j'$ the *tie-breaking* column of $v$ and write $j' = \mathrm{tb}(v)$. Of course, $\mathrm{dc}(v) < \mathrm{tb}(v)$.

Dually, if $v$ is one-dominant and $j = \mathrm{dc}(v)$, define $T_v = \{i \in S_v : f_i(j) = 1\}$. Then let $j'$ denote the least integer in $[s]$ for which the elements in $\{f_i(j') : i \in T_v\}$ are not all identical. As before, $j'$ is the *tie-breaking* column of $v$ and we write $j' = \mathrm{tb}(v)$. Again, observe that $\mathrm{dc}(v) < \mathrm{tb}(v)$.

If $v$ is balanced, set $\psi(v) = t+1$. If $v$ is zero-dominant, $j = \mathrm{dc}(v)$ and $j' = \mathrm{tb}(v)$, choose $\psi(v)$ as an element from $A_j - A_{j'}$. If $v$ is one-dominant, $j = \mathrm{dc}(v)$ and $j' = \mathrm{tb}(v)$, choose $\psi(v)$ as an element from $A_{j'} - A_j$.

It remains only to show that $\psi$ is a good coloring of $\mathcal{H}_n^{**}$. To the contrary, suppose there exists an element $\alpha \in [t+1]$ and an edge $E$ in $\mathcal{H}_n^{**}$ such that $\psi(v) = \alpha$ for all $v \in E$. Choose an integer $k \geq 2$ and an admissible pair of sets $\{a_1, a_2, \ldots, a_{k+1}\}, \{b_1, b_2, \ldots, b_{k+1}\} \in \binom{[n]}{k+1}$ such that:

(1) $a_i < b_i$ for $i = 1, 2, \ldots, k+1$,

(2) $a_k \leq b_1 \leq a_{k+1} \leq b_2$,

(3) $E = \{v_i : 1 \leq i \leq k\}$ where $v_i = ([a_i, b_i], [a_{i+1}, b_{i+1}])$ for $i = 1, 2, \ldots, k$ and

(4) $\psi(v_i) = \alpha$ for $i = 1, 2, \ldots, k$.

For $i = 1, 2, \ldots, k$, let $j_i = \mathrm{dc}(v_i)$. If $i \in [k]$ and $v_i$ is not balanced, let $j_i' = \mathrm{tb}(v_i)$.

**Case 1.** $\alpha = t+1$.

In this case, each $v_i$ is balanced. Now suppose $i \in [k-1]$. Then $f_{a_{i+1}}(j_i) = 0$ and $f_{b_{i+1}}(j_i) = 1$, which implies that $j_{i+1} \leq j_i$.

If $j_{i+1} < j_i$, then $f_{a_{i+1}}(j_{i+1}) = f_{b_{i+1}}(j_{i+1})$. Since $j_{i+1} = \mathrm{dc}(v_{i+1})$, we also have $f_{a_{i+1}}(j_{i+1}) = 0$. Thus $f_{a_{i+1}}(j_{i+1}) = f_{a_{i+2}}(j_{i+1}) = f_{b_{i+1}}(j_{i+1}) = 0$. This implies that $v_{i+1}$ is zero dominant and that $\psi(v_{i+1}) \neq \alpha$. The contradiction shows $j_{i+1} = j_i$. By induction, $j_i = j_1$ for $i = 1, 2, \ldots, k$.

Set $S = S_{v_1} \cup S_{v_2} \cup \ldots \cup S_{v_k}$. Then $j_1$ is the least integer in $[s]$ for which the elements of $\{f_i(j_1) : i \in S\}$ are not all identical. However, $a_k \leq b_1 \leq a_{k+1} \leq b_2$, $f_{a_k}(j_1) = 0$, $f_{b_1}(j_1) = 1$, $f_{a_{k+1}}(j_1) = 0$ and $f_{b_2}(j_1) = 1$. This is a violation of the lexicographic property.

**Case 2.** $\alpha \in [t]$ and $v_1$ is zero-dominant.

In this case, $f_{a_2}(j_1) = 0$ and $f_{b_2}(j_1) = 1$, so $j_2 \leq j_1$. If $j_2 < j_1$, then $f_{a_1}(j_2) = f_{b_1}(j_2) = f_{a_2}(j_2) = f_{b_2}(j_2) = 0$, so $v_2$ is also zero-dominant.

Furthermore, it is easy to see that $j_1 = j_2' = \mathrm{tb}(v_2)$. Thus $\alpha = \psi(v_1) \in A_{j_1} - A_{j_1'}$ and $\alpha = \psi(v_2) \in A_{j_2} - A_{j_2'} = A_{j_2} - A_{j_1}$. Clearly, this is impossible. Thus $j_2 = j_1$.

Since $f_{a_2}(j_1) = 0$ and $f_{b_2}(j_1) = 1$, we know that $v_2$ is not zero-dominant. Since $\alpha = \psi(v_2) \in [t]$, $v_2$ is not balanced. Now suppose $v_2$ is one-dominant. Then $\psi(v_2) \in A_{j_2'} - A_{j_2} = A_{j_2'} - A_{j_1}$ and $\psi(v_1) \in A_{j_1} - A_{j_1'}$. Again, this is impossible.

**Case 3.** $\alpha \in [t]$ and $v_1$ is one-dominant.

If $j_2 < j_1$, then $f_{a_2}(j_2) = f_{b_2}(j_2) = 0$, so $v_2$ is zero-dominant and $\psi(v_2) \in A_{j_2} - A_{j_2'}$. Since $a_2 \leq a_3 \leq b_2$, $\{a_2, a_3, b_2\} \subseteq T_{v_2}$ and $f_{a_2}(j) = f_{b_2}(j)$ for every $j \leq j_1$, it follows that $j_2' > j_1$. However, $f_{a_2}(j) = f_{b_2}(j)$ for every $j < j_2'$ now implies that $j_1' = j_2'$. Thus $\psi(v_1) \in A_{j_1'} - A_{j_1} = A_{j_2'} - A_{j_1}$, which is impossible. We conclude that $j_2 \geq j_1$.

Now $j_2 = j_1$ is impossible since $f_{a_2}(j_1) = 1$, so it remains only to consider the case where $j_2 > j_1$. Observe that $j_2 \leq j_1'$. If $j_2 < j_1'$, then $v_2$ is zero-dominant and $j_2' = j_1'$. This implies that $\psi(v_1) \in A_{j_1'} - A_{j_1} = A_{j_2'} - A_{j_1}$ while $\psi(v_2) \in A_{j_2} - A_{j_2'}$, which is impossible. Thus $j_2 = j_1'$.

Since $f_{b_2}(j_2) = 1$ and $v_2$ is not balanced, we know that $v_2$ is also one-dominant. Then $\psi(v_1) \in A_{j_1'} - A_{j_1} = A_{j_2} - A_{j_1}$ and $\psi(v_2) \in A_{j_2'} - A_{j_2}$. The contradiction completes the proof of the claim and the theorem. ∎

**Corollary 5.2.** *The dimension of the canonical interval order $\mathbf{I}_n$ satisfies* $\dim(\mathbf{I}_n) = \lg\lg n + \left(\frac{1}{2} + \mathrm{o}(1)\right) \lg\lg\lg n$. ∎

Although it does not take too much additional work to obtain our principal result, the reader should note that Theorem (1.2) does not follow immediately from (5.2). The reason is that there are interval orders of bounded height which are not contained in small canonical interval orders.

## 6. General interval orders

In this section, we give the proof of Theorem (1.2): The maximum dimension $f(n)$ of an interval order of height $n$ satisfies $f(n) = \lg\lg n + \left(\frac{1}{2} + \mathrm{o}(1)\right) \lg\lg\lg n$.

**Proof.** It suffices to show that if $n \geq 2$, then $\chi(\mathbf{G}(2n+1,3)) \leq f(n) \leq 2 + \chi(\mathcal{H}^{**}_{n+1})$. The inequality $f(n) \geq \chi(\mathbf{G}(2n+1,3))$ follows from the fact that the height of the canonical interval order $\mathbf{I}_{2n+1}$ is $n$ and that $\dim(\mathbf{I}_{2n+1}) \geq \chi(\mathbf{G}(2n+1,3))$. It remains only to show that $f(n) \leq 2 + \chi(\mathcal{H}^{**}_{n+1})$.

Without loss of generality, we may assume the points of $X$ are nondegenerate closed intervals of the real line and that no point on the line is the endpoint of more than one interval from $X$. Since height$(X, P) = n$, we may choose a decomposition $X = A_1 \cup A_2 \cup \ldots \cup A_n$ where each $A_i$ is a maximal antichain. It follows that for each $i \in [n]$, the intersection of the intervals belonging to $A_i$ is a nondegenerate closed interval of the real line, so we may choose a nondegenerate closed interval $J_i$ from the interior of this interval. Observe that the intervals $J_1, J_2, \ldots, J_n$ are disjoint. Without loss of generality, we may assume that the antichains in the partition have been labelled so that $J_1 < J_2 < \cdots < J_n$. Also observe that since the antichains in the decomposition are maximal, if an interval from $X$ intersects some $J_i$, then it contains $J_i$ and it belongs to $A_i$.

For each $x \in X$, set

$$a_x = \begin{cases} 1 & \text{if } J_1 \not< x \\ i+1 & \text{if } J_i < x \text{ but } J_{i+1} \not< x \end{cases}$$

and

$$b_x = \begin{cases} n+1 & \text{if } x \not< J_n \\ j & \text{if } x < J_j \text{ but } x \not< J_{j-1}. \end{cases}$$

Then set $F(x) = [a_x, b_x]$. The integers $a_x$ and $b_x$ indicate which gaps between the intervals $J_1, J_2, \ldots, J_n$ contain (respectively) the left and right endpoints of $x$. The function $F$ is a mapping of $(X, P)$ to the canonical interval order $\mathbf{I}_{n+1}$. Although $F$ is not in general an embedding, it does satisfy some key properties.

**Claim 1.** For each $x \in X$, $1 \leq a_x < b_x \leq n+1$.

**Proof.** It is immediate from the definitions that $1 \leq a_x \leq b_x \leq n+1$. The inequality $a_x < b_x$ follows from the observation that each $x \in X$ belongs to some antichain $A_i$ and thus $J_i$ is contained in the interior of $x$. This implies $a_x \leq i$ and $b_x \geq i+1$. ∎

**Claim 2.** If $x < y$ in $P$, then $b_x \leq a_y$.

**Proof.** This claim follows immediately from the observation that the left end point of $y$ is greater than the right end point of $x$. ∎

**Claim 3.** If $F(x) < F(y)$ in $P_{n+1}$, then $x < y$ in $P$.

**Proof.** If $F(x) < F(y)$ in $P_{n+1}$, then $b_x < a_y$. If $b_x = a_y - 1$, then $x < J_{b_x} = J_{a_y-1} < y$. If $b_x < a_y - 1$, then $x < J_{b_x} < J_{a_y-1} < y$. In either case $x < y$ in $P$. ∎

Set $N = \{(x,y) \in \operatorname{crit}(\mathbf{P}) : a_x < a_y \text{ and } b_x < b_y\}$. Observe that if $(x,y) \in N$, then $([a_x,b_x],[a_y,b_y]) \in \operatorname{crit}^*(\mathbf{I}_{n+1})$ and therefore a vertex in $\mathcal{H}^{**}_{n+1}$. Now let $\chi(\mathcal{H}^{**}_{n+1}) = t$ and let $\psi$ be a good coloring of the vertices of $\mathcal{H}^{**}_{n+1}$ using the integers from $[t]$ as colors. For each $\alpha \in [t]$, let $S_\alpha = \{(x,y) \in N : \psi(([a_x,b_x],[a_y,b_y])) = \alpha$.

**Claim 4.** There is no $\alpha \in [t]$ for which $S_\alpha$ contains a strict alternating cycle.

**Proof.** Suppose to the contrary that $\alpha \in [t]$ and that $\{(x_i,y_i) : 1 \le i \le k\}$ is a strict alternating cycle contained in $S_\alpha$. If $x_{i+1} = y_i$ for $i = 1,2,\ldots,k-1$ and $x_1 = y_k$, then $a_{x_1} < a_{y_1} = a_{x_2} < a_{y_2} = a_{x_3} < \cdots < a_{y_{k-1}} = a_{x_k} < a_{y_k} = a_{x_1}$ which is false. We may therefore assume $x_1 < y_k$ and $x_{i+1} = y_i$ for $i = 1,2,\ldots,k-1$.

In this case, we know $a_{x_1} < a_{y_1} = a_{x_2} < a_{y_2} = a_{x_3} < \cdots < a_{y_{k-1}} = a_{x_k} < a_{y_k}$ and $b_{x_1} < b_{y_1} = b_{x_2} < b_{y_2} = b_{x_3} < \cdots < b_{y_{k-1}} = b_{x_k} < b_{y_k}$.

If $b_{x_1} < a_{y_{k-1}} = a_{x_k}$, then $F(x_1) < F(y_{k-1})$ in $P_{m+1}$. Thus $x_1 < y_{k-1}$ in $P$ which is false. We conclude $a_{y_{k-1}} = a_{x_k} \le b_{x_1}$. A similar argument shows $a_{y_k} \le b_{y_1} = b_{x_2}$. Finally, observe that $x_1 < y_k$ requires $b_{x_1} \le a_{y_k}$. However, this implies that $\psi$ is monochromatic on an edge in the augmented hypergraph $\mathcal{H}^{**}_{n+1}$. The contradiction completes the proof of the claim. ∎

Now for each $\alpha \in [t]$, let $L_\alpha$ be a linear extension of $P$ with $x > y$ in $L_\alpha$ for every $(x,y) \in S_\alpha$. Then define two linear orders $L_{t+1}$ and $L_{t+2}$ on $X$ as follows.

$$\text{(1)} \quad x < y \text{ in } L_{t+1} \text{ if } \begin{cases} a_x < a_y, \text{ or} \\ a_x = a_y \text{ and } x > y \text{ in } L_t \end{cases}$$

$$\text{(2)} \quad x < y \text{ in } L_{t+2} \text{ if } \begin{cases} b_x < b_y, \text{ or} \\ b_x = b_y \text{ and } x > y \text{ in } L_t \end{cases}$$

It follows easily that $P = L_1 \cap L_2 \cap \cdots \cap L_{t+2}$ so that $\dim(X, P) \leq t+2$. We conclude that $\dim(X, P) \leq \chi(\mathcal{H}^{**}_{n+1}) + 2$. ■

## 7. Edge patterns and matrix representations

The arguments presented in this paper suggest a broad class of combinatorial problems for graphs and hypergraphs with edges defined by a prescribed set of patterns. In the preceding two sections, we considered a vertex $v$ of the augmented hypergraph $\mathcal{H}^{**}_n$ as a critical pair of the form $([a_x, b_x], [a_y, b_y])$ with $a_x < a_y \leq b_x < b_y$. As noted in the proof of (5.1), the set $S_v = \{a_x, b_x\} \cup \{a_y, b_y\}$ is either a 3-element or 4-element subset of $[n]$. Furthermore, the map $v \to S_v$ is easily seen to be a bijection between the vertex set of $\mathcal{H}^{**}_n$ and $\binom{[n]}{3} \cup \binom{[n]}{4}$. For this reason, we could just as easily consider the vertices in $\mathcal{H}^{**}_n$ to be subsets of $[n]$.

Here is a general setting for coloring problems of a similar nature. Let $\mathcal{P} = \{A_1, A_2, \ldots, A_r\}$ be a family of sets. We call $\mathcal{P}$ an *edge pattern of size* $m$ if $\cup\mathcal{P}$ is an $m$-element set. When $\mathcal{P}$ is an edge pattern of size $m$, $\cup\mathcal{P} = \{x_1, x_2, \ldots, x_m\}$ and $n \geq m$, we define a hypergraph $\mathcal{H}_n(\mathcal{P})$ as follows. The vertices of $\mathcal{H}_n(\mathcal{P})$ are the sets in $\left\{\binom{[n]}{|A|} : |A| \in \mathcal{P}\right\}$. For each subset $B = \{b_1, b_2, \ldots, b_m\} \in \binom{[n]}{m}$, we have an edge $E_B(\mathcal{P})$ in the hypergraph $\mathcal{H}_n(\mathcal{P})$ defined by $B_{A_j} = \{b_i : x_i \in A_j\}$ for each $j \in [r]$, and $E_B(\mathcal{P}) = \{B_{A_j} : j \in [r]\}$.

If $\mathcal{F} = \{\mathcal{P}_\alpha : \alpha \in \mathcal{A}\}$ is a family of patterns, then we define $\mathcal{H}_n(\mathcal{F})$ as the hypergraph whose vertex set (respectively, edge set) is the union of the vertex sets (respectively, edge sets) of the hypergraphs in $\{\mathcal{H}_n(\mathcal{P}_\alpha) : \alpha \in \mathcal{A}\}$.

**Example 7.1.** Let $\mathcal{P} = \{\{1, 2, \ldots, k\}, \{2, 3, \ldots, k+1\}\}$. Then the hypergraph $\mathcal{H}_n(\mathcal{P})$ is the Erdös-Hajnal shift graph $\mathbf{G}(n, k)$ defined in Section 2.

Reflecting on the proof presented in section 5, the essential difficulty is determining the chromatic number of the hypergraph $\mathcal{H}(\mathcal{P})$ where $\mathcal{P}$ is the edge pattern $\{\{1, 2, 3, 5\}, \{2, 4, 5, 6\}\}$. In developing a feel for the chromatic number of hypergraphs specified in edge patterns, we have also found it convenient to represent the pattern in matrix form.

Formally, with an edge pattern $\mathcal{P} = \{A_1, A_2, \ldots, A_r\}$ of size $m$, we associate an $r \times m$ matrix $M$ whose $ij$ entry is 1 if and only if $j \in A_i$. For example, the following 0-1 matrix represents the edge pattern of the double shift graph $\mathbf{G}(n,3)$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

The matrix representation of the edge pattern $\mathcal{P} = \{\{1,2,3,5\}, \{2,4,5,6\}\}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

We have found these matrix representations useful for the research associated with the manuscript. Moreover, we believe that it should be possible to formulate a general theory of the chromatic number of hypergraphs defined in this manner.

## 8. Concluding remarks

There are several interesting open problems involving the dimension of interval orders. Here are three of them.

**Question 8.1.** Is it true that for every $n \geq 4$, there exists some integer $t = t_n$ so that if $(X, P)$ is any interval order with $\dim(X, P) \geq t$, then $(X, P)$ contains a subposet $(Y, Q)$ isomorphic to the canonical interval order $\mathbf{I}_n = (I_n, P_n)$?

A poset $\mathbf{P} = (X, P)$ is *t-irreducible* if $\dim(\mathbf{P}) = t$, but every proper subposet of $\mathbf{P}$ has dimension less than $t$.

**Question 8.2.** For each $t \geq 3$, how many non-isomorphic $t$-irreducible interval orders are there?

**Question 8.3.** Given an interval order $\mathbf{P} = (X, P)$, is it NP-complete to determine the dimension of $\mathbf{P}$?

Other problems and conjectures for posets are given in [13], [14] and [15].

**Acknowledgement.** The authors would like to express their appreciation to the referee for a careful reading of the preliminary version of this paper and for helpful suggestions with the exposition.

## References

[1] K. P. Bogart and W. T. Trotter, A Bound on the Dimension of Interval Orders, *J. Comb. Theory (A)* **21**(1976), 319–328.

[2] P. Erdős and A. Hajnal, On the Chromatic Number of Infinite Graphs, in: *Theory of Graphs* Proceedings of the 1976 Colloquium at Tihany, Academic Press, 1976, 83–98.

[3] P. Erdős, A. Hajnal, Z. Füredi, P. Komjáth, V. Rödl and Á. Seress, Coloring Graphs with Locally Few Colors, *Discrete Math.* **59**(1986), 21–34.

[4] P. C. Fishburn, Intransitive Indifference with Unequal Indifference Intervals, *J. Math. Psych.* **7**(1970), 144–149.

[5] C.C. Harner and R. C. Entringer, $r$-Colorings of Digraphs, *J. Comb. Theory (B)* **13**(1972) 219–225.

[6] D. Kelly and W. T. Trotter, Dimension Theory for Ordered Sets, in: *Proceedings of the Symposium on Ordered Sets,* (eds.: I. Rival et al.), Reidel Publishing, 1982, 171–212.

[7] D.J. Kleitman and G. Markovsky, On Dedekind's Problem: The Number of Isotone Boolean Functions, II, *Trans. Amer. Math. Soc.* **213**(1975), 373–390.

[8] A. Korshunov, Solution of Dedekind's Problem on the Number of Monotonic Boolean Functions, *Dokl. Akad. Nauk* SSSR4 Transl. by Amer. Math. Soc. in: *Soviet Math. Dokl.* **18**(1977), 442–445.

[9] I. Rabinovitch, The Dimension Theory of Semiorders and Interval Orders, Ph.D. Dissertation, Dartmouth College, 1973.

[10] W. T. Trotter, Dimension of the Crown $S_n^k$, *Discrete Math.* **8**(1974), 85–103.

[11] W. T. Trotter, Graphs and Partially Ordered Sets, in: *Selected Topics in Graph Theory II,* (eds.: R. Wilson and L. Beineke), Academic Press, 1983, 237–268.

[12] W. T. Trotter, Partially Ordered Sets, in: *Handbook of Combinatorics,* (eds.: R. Graham, M. Grötschel and L. Lovász), to appear.

[13] W. T. Trotter, Progress and New Directions in Dimension Theory for Finite Partially Ordered Sets, in: *Proceedings of Conference on Extremal Problems for Finite Sets,* Visegrád, Hungary (1991), to appear.

[14] W. T. Trotter, *Combinatorics and Partially Ordered Sets: Dimension Theory,* John Hopkins University Press, to appear.

[15] W. T. Trotter, Problems and Conjectures in the Combinatorial Theory of Ordered Sets, *Annals of Discrete Math.* **41**(1989), 401–416.

[16] W. T. Trotter and J. Moore, The Dimension of Planar Posets, *J. Comb. Theory (B)* **21**(1977), 51–67.

Zoltán Füredi

*Mathematical Institute of the*
*Hungarian Academy of Sciences*
*H-1364 Hungary*
*Budapest, P.O.B. 127*
*and*
*Department of Mathematics*
*University of Illinois*
*Urbana, IL 61801*

Péter Hajnal

*University of Szeged,*
*Bolyai Institute*
*H-6722, Hungary,*
*Szeged*

Vojtech Rödl

*Department of Mathematics and Computer Science*
*Emory University*
*Atlanta, Georgia 30322*
*USA*

William T. Trotter

*Bell Communications Res. 2L-367*
*445 South Street*
*Morristown, NJ 07962-1910*
*and*
*Department of Mathematics*
*Arizona State University*
*Tempe, AZ 85287-1804*
*USA*