Remarks on a functional equation*

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Dedicated to Professor László Leindler on the occasion of his 80th birthday

Abstract

A functional equation involving pairs of means is considered. It is shown that there are only constant solutions if continuous differentiability is assumed, and there may be non-constant everywhere differentiable solutions. Various other situations are considered, where less smoothness is assumed on the unknown function.

1 Introduction

Throughout this paper let $I \subset \mathbf{R}$ be a non-void open interval. We call the function $M : I \times I \to I$ a *mean* if the condition

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\} \tag{1}$$

holds for all $x, y \in I$. If for all $x, y \in I$, $x \ne y$ the inequalities in (1) are sharp, then M is called a *strict mean*. Two means M and N are called *admissible*, if

$$M(x, y) \ne N(x, y) \text{ if } x \ne y.$$

Examples of admissible pairs:

- $M(x, y) = x$, $N(x, y) = y$, $I \subset \mathbf{R}$,
- $M(x, y) = px + 1(1 - p)y$, $N(x, y) = qx + (1 - q)y$, with $0 \le p < q \le 1$, $I \subset \mathbf{R}$,
- $M(x, y) = \min(x, y)$, $N(x, y) = \max(x, y)$, $I \subset \mathbf{R}$,
- $M(x, y) = (x + y)/2$, $N(x, y) = \sqrt{xy}$, $I \subset \mathbf{R}_+$.

The following problem on a functional equation is investigated (cf. [1], [2], [3]):

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Problem 1 *Let* $M, N : I^2 \to I$ *be admissible means, and let the unknown function* $f : I \to \mathbf{R}$ *satisfy the functional equation*

$$[f(x) - f(y)]\,[f\,(M(x, y)) - f\,(N(x, y))] = 0 \tag{2}$$

for all $x, y \in I$*. Question: What can we say about the function* f*?*

It is obvious, that the constant function $f(x) = c$ for all $x \in I$ ($c \in \mathbf{R}$) is a solution of (2). Hence we ask the following, mathematically more precise questions:

(a) What regularity conditions of f assure that the only solutions of the equation (2) are the constant functions?

(b) For what means M, N are there non-constant solutions f?

Problem 1 is a special case of

Problem 2 *Let* $M_j, N_j : I^2 \to I$*,* $1 \le j \le m$*, be admissible pairs of means, and let the unknown function* $f : I \to \mathbf{R}$ *satisfy the functional equation*

$$\prod_{j=1}^{m} [f\,(M_j(x, y)) - f\,(N_j(x, y))] = 0 \tag{3}$$

for all $x, y \in I$*. Question: What can we say about* f*?*

Clearly, if $m = 2$ and $M_1(x, y) = x$, $N_1(x, y) = y$, then we obtain back our original problem.

2 Differentiable solutions

In this section, we assume the differentiability of f.

Theorem 1 *If the unknown function* f *in Problem 2 is continuously differentiable on* I*, then* f *is constant.*

Note that in this result no more additional property of the means M_j, N_j is required.

Proof. Let $[a, b] \subset I$ ($a < b$) be an arbitrary interval. In view of (3) with $x = a$, $y = b$, for at least one j we must have $f(M_j(a, b)) = f(N_j(a, b))$. Then the closed interval $U := [a', b']$ - determined by $M_j(a, b)$ and $N_j(a, b)$ is a subinterval of $[a, b]$, and by Rolle's theorem, there exists a $\xi \in (a', b') \subset (a, b)$ such that $f'(\xi) = 0$. This means that f' vanishes on a dense subset of I, so from the continuity of f' we have $f'(x) = 0$ for all $x \in I$. Hence f is constant on I.

■

Next, we show that in this theorem continuous differentiability cannot be replaced by pointwise differentiability.

Theorem 2 *There are an everywhere differentiable non-constant f and admissible strict means M, N on $\mathbf{R}$ such that $f(M(x,y)) = f(N(x,y))$ for all x, y.*

Of course, this implies that Problems 1 and 2 have non-constant differentiable solutions for certain means, for if our pair (M, N) is among the means, then one of the factors in (2) or (3) is identically 0.

Proof. The proof is along the note in [2]. Let f be an everywhere differentiable real function which is not monotone on any interval. (Such functions have been constructed by various authors, fist by A. Köpcke [5], [6]. For a relatively simple existence proof using the category theorem see [8].) Since f is not monotone on any interval, for every $x < y$ there are $x < X < Z < Y < y$ such that $f(Z) < f(X), f(Y)$ or $f(Z) > f(X), f(Y)$. As a consequence (look at the $f(Z) + \varepsilon$ resp. $f(Z) - \varepsilon$ level-set of f with some small $\varepsilon > 0$), there are $x < x' < y' < y$ (actually $x' \in (X, Z)$, $y' \in (Z, Y)$) such that $f(x') = f(y')$ (we select one such x', y' for every x, y). Let now $M(x,y) = x'$, $N(x,y) = y'$ if $x < y$, and let $M(x,y) = M(y,x)$, $N(x,y) = N(y,x)$ in the opposite case (and of course, $M(x,x) = N(x,x) \equiv x$). Then M, N are strict means, and $f(M(x,y)) = f(N(x,y))$ by the construction.

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3 Continuous solutions

In this section we assume less on f, namely we only assume its continuity.

Theorem 3 *If M, N are continuous admissible means, then any continuous f that satisfies (2) is constant.*

For a related result see [3] by A. Járai, who proved that if M, N are continuous admissible means, then any (not necessarily continuous) f that satisfies $f(M(x,y)) \equiv f(N(x,y))$ is constant.

Proof. First of all, let us remark that either $M(x,y) < N(x,y)$ for all $x < y$ or $N(x,y) < M(x,y)$ for all $x < y$. Indeed, if, say, $M(x_0,y_0) < N(x_0,y_0)$ for some $x_0 < y_0$, $x_0, y_0 \in I$, then the first case is true, since we can continuously move from (x_0, y_0) to any (x,y), $x < y$, $x, y \in I$, by a moving point (x', y') such that $x' < y'$ is true at any moment, and during this motion we should always have $M(x', y') < N(x', y')$, otherwise the assumption $M(x', y') \neq N(x', y')$ would be violated. Thus, we may assume that $M(x,y) < N(x,y)$ for all $x < y$.

It is enough to prove that f is constant on any subinterval $[a, b]$ of I. Suppose to the contrary that this is not the case. Then the range of f over $[a, b]$ is a non-degenerate interval, and let A be an element of this range which is different from both $f(a)$ and $f(b)$, and which is not a local extremal value of f. (There is such an A since the set of local extremal values of any function is countable, see Problem 9 in Chapter 5 of [2]). Suppose, say, that $f(a) < A$. Then the set

$$\{x \in [a, b] \mid f(x) \geq A\}$$

is a non-empty closed set, let x_0 be its smallest element. Clearly, $f(x_0) = A$, and $a < x_0 < b$ (by the choice of A). Furthermore, $f(x) < A$ for all $a \leq x < x_0$.

Let $\delta > 0$ be such that $x_0 - \delta > a$ and $x_0 + \delta < b$.

We need to distinguish two cases.

Case I, $N(x_0 - \delta, x_0) = x_0$. Then set $x = x_0 - \delta$, $y = x_0$, for which we have $f(x) < A = f(y)$, and since $M(x, y) < N(x, y) = x_0$ also holds, we also have $f(M(x, y)) < A = f(N(x, y))$. Thus, in this case (2) is violated.

Case II. $N(x_0 - \delta, x_0) < x_0$. Note that $f(x) < A$ (and hence $f(x) \leq A$) to the left of x_0, hence this cannot be true in a right-neighborhood of x_0 (otherwise A would be a local maximum value, which is not the case), so there are arbitrarily small $0 < \varepsilon < \delta$ values such that $f(x_0 + \varepsilon) > A$.

We claim that there is an $\eta > 0$ such that for every $0 < \varepsilon < \eta$ there is a $0 < \theta = \theta_\varepsilon < \delta$ for which $N(x_0 - \theta, x_0 + \varepsilon) = x_0$. Indeed, since now $N(x_0 - \delta, x_0) < x_0$, by continuity $N(x_0 - \delta, x_0 + \varepsilon) < x_0$ for all $0 < \varepsilon < \eta$ with some $0 < \eta < \delta$. On the other hand, for all $0 < \varepsilon < \delta$ we have $x_0 \leq M(x_0, x_0 + \varepsilon) < N(x_0, x_0 + \varepsilon)$. Hence, by the intermediate value property of the continuous function $N(x_0 - t, x_0 + \varepsilon)$ over the interval $t \in [0, \delta]$, we must have $N(x_0 - \theta, x_0 + \varepsilon) = x_0$ for some $0 < \theta < \delta$.

To an $0 < \varepsilon < \eta$ with $f(x_0 + \varepsilon) > A$ select a $\theta = \theta_\varepsilon$ as above, and set $x = x_0 - \theta$, $y = x_0 + \varepsilon$. Then we have $f(x) < A < f(y)$, and since $M(x, y) < N(x, y) = x_0$ is also true, we have again $f(M(x, y)) < A = f(N(x, y))$. Thus, (2) is violated again, and this contradiction proves the claim that f must be constant.

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Remark 1 *In this proof the continuity of M and N is needed only in each variable separately.*

4 Non-continuous solutions

Sometimes one can conclude the constancy of f without any smoothness assumption on f. Let us consider, for example, the special case of equation (2) when $M(x, y) := x$ $(x, y \in I)$, that is, the equation

$$[f(x) - f(y)]\,[f(x) - f(N(x, y))] = 0 \tag{4}$$

for all $x, y \in I$ (here $x \neq N(x, y)$ if $x \neq y$).

Proposition 4 *If the mean N in (4) is symmetric (that is, $N(x,y) = N(y,x)$ holds for all $x, y \in I$), then all the solutions $f : I \to \mathbf{R}$ of equation (4) is constant.*

The claim may not be true if N is non-symmetric. As an example, let $N(x,y)$ be a number in between x and y which is rational if x is rational and irrational if x is irrational. Then, clearly, the characteristic function of the set of rationals is a solution of (4).

Proof. Interchanging the variables x and y in equation (4) we get

$$[f(y) - f(x)]\,[f(y) - f(N(y,x))] = 0 \tag{5}$$

for all $x, y \in I$. Because of the symmetry of N, it follows from (4) and (5) that

$$[f(x) - f(y)]\,[f(x) - f(N(x,y)) - f(y) + f(N(y,x))] = [f(x) - f(y)]^2 = 0.$$

Thus f is constant on I.

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Let us go back to equation (2). The simplest non-continuous solution would be one which takes exactly 2 different values. Without loss of generality we may assume that such a solution is the characteristic function of a non-empty set $A \subset I$ ($A \neq I$) (note that if f is a solution, then so is $cf + d$ for any constants c, d). So let

$$f(x) := \chi_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \in \bar{A} := I \setminus A, \end{cases} \tag{6}$$

where $A \neq \emptyset$ and $\bar{A} \neq \emptyset$. The characteristic function (6) is a solution of (2) if and only if the pair $\{A, \bar{A}\}$ has the following property:

(P): *If $x \in A$ and $y \in \bar{A}$ or $x \in \bar{A}$ and $y \in A$, then both $M(x,y)$ and $N(x,y)$ are in A or in $\bar{A}$.*

It is obvious that, if there exists a pair $\{A, \bar{A}\}$ ($A \neq \emptyset$, $\bar{A} \neq \emptyset$, $A \cap \bar{A} = \emptyset$ and $A \cup \bar{A} = I$) with property (P), then the function f defined in (6) is a non-constant solution of (2).

Proposition 5 *If M and N are strict means in the equation (2), then there exists a non-constant solution $f : I \to \mathbf{R}$ of (2).*

By considering $M(x,y) = x$, $N(x,y) = y$ we can see that the strictness of M, N cannot be dropped.

Proof. In this case the singleton $A := \{x_0\}$ ($x_0 \in I$) is a set, for which the pair $\{A, \bar{A}\}$ has property (P). Indeed, if $x \in A$ and $y \in \bar{A}$ (or $x \in \bar{A}$ and $y \in A$), then $x = x_0$ and $y \neq x_0$ (or $x \neq x_0$ and $y = x_0$) and since M and N are strict means, $M(x,y) \neq x_0$ and $N(x,y) \neq x_0$, so $M(x,y) \in \bar{A}$ and $N(x,y) \in \bar{A}$. This proves that $f(x) := \chi_A(x)$ is a non-constant solution of (2).

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The problem to find further pairs $\{A, \bar{A}\}$ with property (P) for given means M and N seems to be difficult. We can find a useful construction in case of the special means M, N from [1].

Proposition 6 *Let $K \subset \mathbf{R}$ be a proper subfield of $\mathbf{R}$ and $A := I \cap K$. Furthermore, let*

$$M(x,y) := px + (1-p)y$$

and

$$N(x,y) := qx + (1-q)y \quad (x, y \in I),$$

where $p, q \in (0,1)$ and $p \neq q$ are fixed. If $p, q \in K \cap (0,1)$, then the pair $\{A, \bar{A}\}$ has property (P).

Proof. Now $\bar{A} = I \setminus A$ is nonempty, since $K \neq \mathbf{R}$. If $x \in A$ and $y \in \bar{A}$ (or $x \in \bar{A}$ and $y \in A$), then $px + (1-p)y$ and $qx + (1-q)y$ are not elements of A, because otherwise y (or x) would also be an element of A. Hence, the pair $\{A, \bar{A}\}$ has property (P) and $f(x) := \chi_A(x)$ $(x \in I)$ is a non-constant solution of the functional equation

$$[f(x) - f(y)]\,[f(px + (1-p)y) - f(qx + (1-q)y)] = 0 \quad (x, y \in I). \tag{7}$$

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Corollary 7 *If $p, q \in K \cap (0,1)$ $(p \neq q)$, then the equation (7) has a solution $f : I \to \mathbf{R}$ with either of the properties below:*

(i) f is non-measurable;

(ii) f equals zero almost everywhere and f is non-zero on a set of continuum cardinality.

Proof. There exists a non-measurable proper subfield K of $\mathbf{R}$ ([1], [7]), hence we get (i). In case of (ii) our result follows from the existence of measurable proper subfields of $\mathbf{R}$ (necessarily with measure zero) which are of cardinality continuum ([1], [7]).

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It is worth mentioning the case

$$M(x,y) := \frac{x+y}{2} \text{ and } N(x,y) := \sqrt{xy}, \tag{8}$$

where $x, y \in I \subset (0, \infty)$. Then (2) takes the form

$$[f(x) - f(y)]\left[f\left(\frac{x+y}{2}\right) - f\left(\sqrt{xy}\right)\right] = 0 \qquad (x, y \in I). \tag{9}$$

Proposition 8 *If $f : I \to \mathbf{R}$ is a continuous solution of (9), then f is constant on I. There exist non-measurable solutions $f : I \to \mathbf{R}$ of (9). There exists a solution $f : I \to \mathbf{R}$ of (9), such that it equals zero almost everywhere and f is non-zero on a set of cardinality continuum.*

Actually, in the second and third parts f can be $\{0, 1\}$-valued.

Proof. The first statement follows from Theorem 3.

To prove the second part let $K \subset \mathbf{R}$ be a proper non-measurable subfield. Then, with the notations $A := I \cap K$ and $\bar{A} := I \setminus A$, the pair $\{A, \bar{A}\}$ has property (P) with the means (8). Indeed, if, for example, $x \in A$ and $y \in \bar{A}$, then both $\frac{x+y}{2}$ and $\sqrt{xy}$ are in $\bar{A}$. Hence, $f(x) := \chi_A(x)$ $(x \in I)$ is non-measurable and it is a solution of (9).

The third statement is valid, because there exists a measurable proper subfield $K \subset \mathbf{R}$ with zero measure, which has cardinality continuum. Then $A := I \cap K$ has the property that $f(x) := \chi_A(x)$ $(x \in I)$ is a solution of (9), it equals zero almost everywhere, and f is non-zero on a set of cardinality continuum.

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