# Convex polygons and common transversals*

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**Abstract**

It is shown that if two planar convex $n$-gons are oppositely oriented, then the segments joining the corresponding vertices have a common transversal. A different formulation is also given in terms of two cars moving along two convex curves in opposite directions. Some possible analogues in 3-space are also considered, and an example is shown that the full analogue is not true in the space.

## Polygons in the plane

In this paper we discuss a property of planar convex polygons, namely

**Theorem 1** *If $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are convex polygons in the plane with opposite orientation, then there exists a line that intersects each of the line segments $A_1B_1, \ldots, A_nB_n$.*

Thus, the claim is that the segments $A_jB_j$ have a common transversal, see Figure 1 for illustration.

Theorem 1 was proposed by the authors as a problem for the 1999 Miklós Schweitzer Contest for university students — a contest organized every year in Hungary by the János Bolyai Mathematical Society since 1948. About 10 questions are proposed for 10 days, and the students can use any literature they want. Accordingly, the questions are usually more difficult than on other mathematics competitions, see [2] and [6] for the problems and solutions up to 1992.[1]

There is an alternative formulation of Theorem 1 given in

**Theorem 2** *Let $\gamma_1$ and $\gamma_2$ be convex curves in the plane. Suppose that on each curve a car moves around, one of them in the clockwise, the other in the*

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[1]There was only one correct approach for the particular problem we are considering: it was by Péter Frenkel, who basically found the official solution.

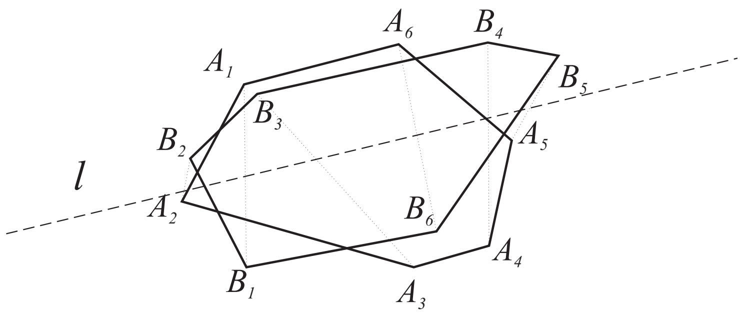


Figure 1: The convex polygons and a common transversal of $A_iB_i$

*counterclockwise direction, returning to the starting point at the same time. Then there exists a line such that the two cars are always on opposite sides of that line.*

The line itself is considered to belong to both of its sides.

The cars may stop in their movement, but they cannot make retreats, see Figure 2.

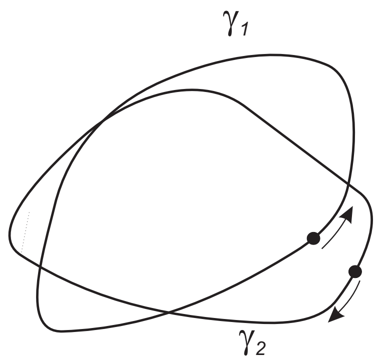


Figure 2: Two convex curves and one-one car on each

Clearly, Theorem 1 follows from Theorem 2. Indeed, given $A_1, \dots, A_n$ and $B_1, \dots, B_n$, let us assume that the two cars start at $A_1$ and $B_1$, and each cover one side of the polygon (for example with uniform speed) in 1 minute. Then, after $i-1$ minutes, the first car is at $A_i$ and the second car is at $B_i$ ($i = 1, \dots, n$). Let $l$ be the line found in Theorem 2. Since, for each $i$, the points $A_i$ and $B_i$ are on the opposite sides of $l$, the line $l$ intersects the segment $A_iB_i$.

A simple approximation argument (take as the vertices of the two polygons the positions of the two cars at times $t = (k/n)T$, $k = 1, \dots, n$, where $T$ is the time needed for the cars to make a full round) shows that conversely, Theorem 1 implies Theorem 2, so these two statements are equivalent.

**Proof of Theorem 2.** For definiteness assume that Car #1, the car on $\gamma_1$, moves in the positive, that is counterclockwise direction. It is also convenient to think of the cars not stopping after making a full round, but continuing their movement periodically forever.

If there is a line that separates the two curves (meaning that the two curves lie on opposite sides of the line including the line itself), then this line clearly satisfies the claim in the theorem. Furthermore, if the intersection of the interiors of two convex curves on the plane is empty, then the two curves can be separated by a line, so in what follows we may assume that there is a common interior point of the two regions enclosed by the curves.

First assume that both curves are strictly convex, i.e. neither of them contains a line segment and that the cars do not make stops.

We shall need to speak of directed lines. A direction $\alpha$ on the plane is given by a unit vector, say by a vector pointing from the origin to a point on the unit circle. We can parametrize such a direction by the angle $\alpha$ that the vector forms with the positive real half-line. So this parameter $\alpha$ lies in $[0, 2\pi)$, but it is convenient to extend the parametrization periodically to the whole real line. A directed line is just a line on a which a direction (parallel with the given line) is given. If we move on the line in the given direction, then we can speak of the left- resp. right-hand side of the line.

We claim that, for every direction $\alpha$, there is a unique directed line $l = l(\alpha)$ with direction $\alpha$ such that Car #1 spends, within one period, exactly as much time on the right side of $l$ as Car #2 spends on the left-side of $l$; note that it is NOT claimed that this happens during the same time interval. (This claim is somewhat similar to the Ham-Sandwich theorem, see [1].) To prove this claim, let us choose a line $l_0$ with direction $\alpha$ so that both curves lie on the right side of $l_0$ and move continuously this line, in one unit of time, using translations, to a position $l_1$, where both curves lie on the left-side of $l_1$. Let $T_1(t)$ be the time that Car #1 spends (in one period) on the right side of $l_t$ (the line at time $t$) and $T_2(t)$ be the time that Car #2 spends on the left-side of $l_t$. The function $f(t) = T_1(t) - T_2(t)$, $f\colon [0, 1] \to R$, is continuous, strictly monotone decreasing, $f(0) = 1$ and $f(1) = -1$ (continuity is due to the fact that we assumed strict convexity of the curves). Therefore, there is a unique point $t_0 \in [0, 1]$ such that $f(t_0) = 0$. The line $l_{t_0}$ is the one we are looking for.

Now we show that the line $l(\alpha)$ intersects both curves in exactly 2 points. Indeed, if this line intersects one of the curves in no or one point, then this curve is completely on one side of $l(\alpha)$, so the other curve must be completely on the other side of $l(\alpha)$. But that would mean that $l(\alpha)$ separates the two curves, which we assumed not to be the case.

Let $B_j(\alpha)$ denote the point where the line $l(\alpha)$ enters the region enclosed by $\gamma_j$, and $K_j(\alpha)$ where it leaves that region for $j = 1, 2$. Clearly, $B_j(\alpha)$ and $K_j(\alpha)$ are $2\pi$-periodic functions from the real line into the complex plane. A simple argument shows that they are continuous. It is heuristically clear that, as $\alpha$ moves from 0 to $2\pi$, the point $B_j(\alpha)$ goes around the curve $\gamma_j$ once in the positive direction. Unfortunately, the points $B_j(\alpha)$ do not necessarily move always in the counterclockwise direction, so this is a subtle point, to which we shall return at the end of the proof.

Let $\phi : \gamma_2 \to \gamma_1$ be the following bijection between the two curves: Car #2 is at the point $P$ on $\gamma_2$ exactly when Car #1 is at $\phi(P)$ on $\gamma_1$ (there is such a bijection since we assumed no stopping of the cars). Since the two cars move in opposite directions, the point $\phi(B_2(\alpha))$ goes around $\gamma_1$ once in the negative (clockwise) direction as $\alpha$ moves from 0 to $2\pi$. It follows that there exists $\alpha_0$ such that $B_1(\alpha_0) = \phi(B_2(\alpha_0))$ (see more explanation at the end of the proof). Note that both $B_1(\alpha_0)$ and $B_2(\alpha_0)$ lie on $l(\alpha_0)$, and the equality $B_1(\alpha_0) = \phi(B_2(\alpha_0))$ means that the two cars are in these two points of $l(\alpha_0)$ precisely at the same moment.

We claim that the line $l(\alpha_0)$ satisfies the condition of the theorem. Indeed, when Car #1 is at the point $B_1(\alpha_0)$, then Car #2 is at the point $B_2(\alpha_0)$. From that position Car #1 moves to the right side of $l(\alpha_0)$ (since Car #1 moves in the counterclockwise direction), while Car #2 moves to the left-side of $l(\alpha_0)$. Since Car #1 spends as much time on the right side of $l(\alpha_0)$ as Car #2 spends on the left-side of $l(\alpha_0)$, we must also have that Car #1 is at the point $K_1(\alpha_0)$ exactly when Car #2 is at the point $K_2(\alpha_0)$. From that position Car #1 moves to the left-side of $l(\alpha_0)$ and Car #2 moves to the right side of $l(\alpha_0)$, and then they hit the line $l(\alpha_0)$ again at the points $B_1(\alpha_0)$ and $B_2(\alpha_0)$. It follows that the two cars are always on opposite sides of the line $l(\alpha_0)$, and we are done.

In the preceding argument we extensively used continuity, which was due to the fact that the curves $\gamma_j$, $j = 1, 2$, are strictly convex. In the general case when this is not necessarily so, we can use approximation and take limit as follows. Still assuming that neither of the cars make a stop, let $O_1$ be a point inside $\gamma_1$ and let $\gamma_1^{(m)}$ be obtained from $\gamma_1$ by shrinking it from $O_1$ by the factor $1 - 1/m$, $m = 2, 3, \ldots$. Select a strictly convex curve $\gamma_{1,m}$ in between $\gamma_1$ and $\gamma_1^{(1)}$. At each moment project Car #1 onto $\gamma_{1,m}$ from $O_1$ — this way we get Car #$1_m$ moving along $\gamma_{1,m}$ in the positive direction. We construct $\gamma_{2,m}$ and Car #$2_m$ in the same way for the second curve. Now $\gamma_{1,m}$ and $\gamma_{2,m}$ are strictly convex, therefore, by the first part of the proof, there is a line $l_m$ that separates Car #$1_m$ and Car #$2_m$ at every moment. We can select a subsequence $\{m_k\}$ of the natural numbers so that the lines $l_{m_k}$ converge to some line $l$ as $m_k \to \infty$, and it is simple to check that then $l$ separates the original two cars at every moment.

The case when the cars make stops can be handled similarly: we approximate such movement by movements without stops, and take limit. We leave the details to the reader.

It remains to prove the heuristic claim that the points $B_j(\alpha)$ traverse (not necessarily monotonically) the curves $\gamma_j$ once in the positive direction as $\alpha$ moves from 0 to $2\pi$, and, due to that, there is an $\alpha_0$ for which $B_1(\alpha_0) = \phi(B_2(\alpha_0))$. Consider $\gamma_1$, and all directed lines in the direction $\alpha$ which hit $\gamma_1$. There are two of them that are supporting lines to $\gamma_1$, one touching $\gamma_1$ on the right side of $l(\alpha)$ at a point $P_{1,\alpha}$ and one touching $\gamma_1$ on the left-side of $l(\alpha)$ at a point $Q_{1,\alpha}$. All other lines enter the interior of $\gamma_1$ somewhere on the arc $I_{1,\alpha} := Q_{1,\alpha}P_{1,\alpha}$ that goes from $Q_{1,\alpha}$ to $P_{1,\alpha}$ in the counterclockwise direction. In particular, $B_1(\alpha)$ is an interior point of that arc, see Figure 3. Note that $P_{1,\alpha}$ and $Q_{1,\alpha}$ move (as $\alpha$ increases) monotonically in the counterclockwise direction, and we express this fact by saying that the arc $I_{1,\alpha}$ moves in the counterclockwise direction.

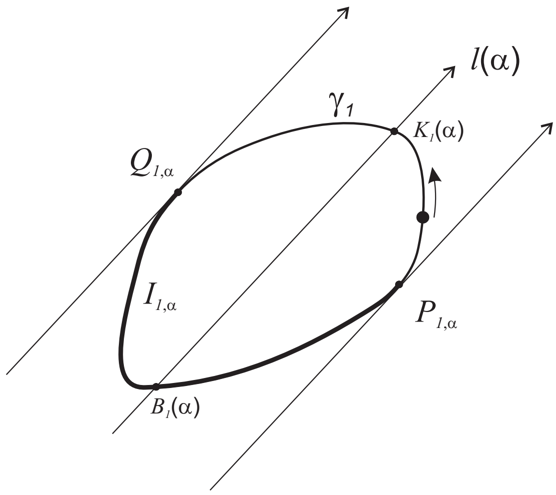


Figure 3: The curve $\gamma_1$, the line $l(\alpha)$, the two supporting lines parallel with it together with the corresponding touching points $P_{1,\alpha}$ and $Q_{1,\alpha}$, and the arc $I_{1,\alpha} = Q_{1,\alpha}P_{1,\alpha}$

Let us denote the analogue of $P_{1,\alpha}, Q_{1,\alpha}, I_{1,\alpha}$ on the curve $\gamma_2$ by $P_{2,\alpha}, Q_{2,\alpha}, I_{2,\alpha}$, and consider their image under the mapping $\phi$. Because the cars move in opposite direction, $\phi(I_{2,\alpha})$ is a proper subarc of $\gamma_1$ with endpoints $\phi(P_{2,\alpha})$ and $\phi(Q_{2,\alpha})$ in the counterclockwise direction, and the arc $\phi(I_{2,\alpha})$ moves in the clockwise direction (again in the sense that its endpoints do so monotonically). Thus, on $\gamma_1$ the proper arc $I_{1,\alpha}$ moves continuously in the counterclockwise direction and the point $B_1(\alpha)$ is always on that arc and moves continuously, while the proper arc $\phi(I_{2,\alpha})$ moves in the clockwise direction and the point $\phi(B_2(\alpha))$ is always on that second arc and moves continuously, see Figure 4. The claim we are dealing with is that then the points $B_1(\alpha)$ and $\phi(B_2(\alpha))$ must meet, which is heuristically clear, since the arcs $I_{1,\alpha}$ and $\phi(I_{2,\alpha})$ "sweep through each other" (actually twice within one full round). A formal proof is as follows.

Let $O$ be a fixed point in the interior of $\gamma_1$, which we may assume to be the origin. If $S(\alpha)$, $\alpha \in \mathbf{R}$, is a continuously and periodically moving point avoiding $O$ (formally $S : \mathbf{R} \to \mathbf{R}^2 \setminus \{O\}$ is a continuous $2\pi$-periodic mapping), then its winding number (relative to $O$) tells us how many times $S(\alpha)$ goes around $O$ within one period (i.e. while $\alpha$ runs through $[0, 2\pi]$), where counterclockwise motions are counted as positive and clockwise motions are counted as negative (see [4, Chapter 3.a] for a precise definition). Now a basic fact is that if $S(\alpha)$ and $\tilde{S}(\alpha)$ are two such points continuously moving along $\gamma_1$ which do not meet,

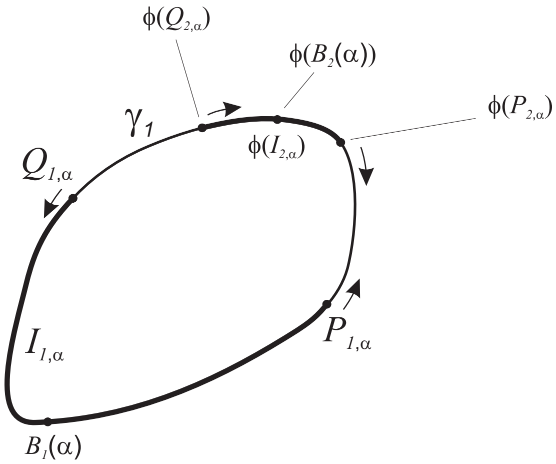


Figure 4: The oppositely moving arcs $I_{1,\alpha}$ and $\phi(I_{2,\alpha})$

then their winding numbers must be the same. Indeed, if $S(\alpha) \neq \tilde{S}(\alpha)$ for any $\alpha$, then the line segments joining $S(\alpha)$ and $-\tilde{S}(\alpha)$ never hit $O$, therefore, by the "Dog-on-a-Leash" theorem [4, Theorem 3.11], $S(\alpha)$ and $-\tilde{S}(\alpha)$ have the same winding numbers, which is the same that $S(\alpha)$ and $\tilde{S}(\alpha)$ have the same winding numbers.

After this, let us return to the points $B_1(\alpha)$ and $\phi(B_2(\alpha))$. Since $B_1(\alpha)$ and $P_{1,\alpha}$ do not meet, and the latter has winding number 1, we conclude that $B_1(\alpha)$ has winding number 1. In a similar fashion, $\phi(B_2(\alpha))$ has winding number $-1$, since it never meets $\phi(P_{2,\alpha})$ and this latter has winding number $-1$. Thus, $B_1(\alpha)$ and $\phi(B_2(\alpha))$ have different winding numbers, so they must meet.

■

## Polytopes in 3-space

In this section, we briefly discuss if Theorem 1 is a purely 2-dimensional result or if it has an analogue in 3-space. As we shall see, there are difficulties in giving the full analogue of Theorem 1 in higher dimensions.

The analogue of a convex polygon in 3-space is a convex polytope $A_1, \ldots, A_n$, and if someone would like to speak of oppositely oriented polytopes, that is not as simple as in the plane. For example, it would be difficult to compare the orientation of a cube with that of a pyramid with a 7-gon base, even though both of them have 8 vertices. However, there is no problem with tetrahedrons: a tetrahedron $ABCD$ is said to be positively (negatively) oriented if the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are right-oriented (left-oriented), i.e. they follow each other as the thumb, forefinger and middle fingers on our right hand (left hand). And for tetrahedrons we have

**Proposition 3** *If $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are oppositely oriented tetrahedrons, then there is a plane that intersects every segment $A_iB_i$, $i = 1, 2, 3, 4$.*

**Proof.** For $t \in [0, 1]$ let a point $P_i(t)$ move continuously on the segment $A_iB_i$ from $A_i$ to $B_i$. Thus, $P_1(0)P_2(0)P_3(0)P_4(0)$ is the tetrahedron $A_1A_2A_3A_4$, while $P_1(1)P_2(1)P_3(1)P_4(1)$ is the tetrahedron $B_1B_2B_3B_4$. The vector triple $\overrightarrow{P_1(t)P_2(t)}$, $\overrightarrow{P_1(t)P_3(t)}$, $\overrightarrow{P_1(t)P_4(t)}$ changes continuously, and at $t = 0$ and at $t = 1$ its orientation is different. So somewhere this orientation must change, and that is possible only if at some $t_0 \in (0, 1)$ these vectors lie in the same plane. Clearly, that plane then intersects each of the segments $A_iB_i$.

For the pedantic readers let us state here the precise meaning of "somewhere this orientation must change": if $t_0$ is the supremum of all $t$ for which the vector triplet $\overrightarrow{P_1(t)P_2(t)}$, $\overrightarrow{P_1(t)P_3(t)}$, $\overrightarrow{P_1(t)P_4(t)}$ is right-oriented, then $t_0 \in (0, 1)$, and for $t = t_0$ all these vectors must line in a plane.

Those familiar with signed volumes (which, for $P_1(t)P_2(t)P_3(t)P_4(t)$ is 1/6-times the mixed product of the vectors $\overrightarrow{P_1(t)P_2(t)}$, $\overrightarrow{P_1(t)P_3(t)}$, $\overrightarrow{P_1(t)P_4(t)}$) will recognize that this proof amounts to the same as with $P_1(t)P_2(t)P_3(t)P_4(t)$ we continuously move from a positive volume to a negative one, so at some point $P_1(t_0)P_2(t_0)P_3(t_0)P_4(t_0)$ must have zero volume, i.e. it must be degenerate: all four points $P_1(t_0)$, $P_2(t_0)$, $P_3(t_0)$, $P_4(t_0)$ lie on the same plane.

■

Next, consider more general polytopes in 3-space, but to avoid the above mentioned problem concerning a cube and a pyramid, let us suppose that $P: A_1 \cdots A_n$ and $Q: B_1 \cdots B_n$ are two polytopes with the property that whenever $A_{i_1} \cdots A_{i_k}$ form a face of $P$ then $B_{i_1} \cdots B_{i_k}$ form a face of $Q$. We are not trying to define in general "opposite orientation" for such polytopes, but any meaningful definition should imply at least that all such corresponding faces $A_{i_1} \cdots A_{i_k}$ and $B_{i_1} \cdots B_{i_k}$ are oppositely oriented (by looking at them, say, from the outside). The simplest case when we can have such a correspondence in between the vertices of $P$ and $Q$ is when $P$ and $Q$ are isometric, and in this case we can state the following proposition, in which "opposite orientation" means that the corresponding faces are oppositely oriented.

**Proposition 4** *If $A_1 \cdots A_n$ and $B_1 \cdots B_n$ are oppositely oriented isometric polytopes, then there is a plane that intersects every segment $A_iB_i$, $i = 1, 2, \ldots, n$.*

Of course, in this formulation we assume that the isometry in between the two polytopes moves $A_i$ into $B_i$.

**Proof.** It is known (see, e.g., [3, Chapter 7]) that in 3-space the isometries are the following:

**a)** screw (a rotation about some axis followed by a translation parallel to that axis),

**b)** glide reflection (reflection onto a plane followed by a translation with a vector that is parallel with the plane),

**c)** rotatory reflection (axial rotation followed by a reflection onto a plane that is perpendicular to the axis of rotation).

The first type preserve orientation, but the latter two types reverse it. Now, if the isometry moving $A_1 \cdots A_n$ into $B_1 \cdots B_n$ is of type **b)** or **c)**, then the plane of reflection in **b)** or **c)** intersects each segment $A_iB_i$.

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Next, we show that a slight distortion of the polytopes in Proposition 4 may result in "almost isometric" polytopes for which the proposition is no longer true. Let $A_1 \cdots A_5A_6'$ and $B_1^* \cdots B_5^*B_6'$ be two isometric regular octahedrons of opposite orientation, see Figure 5. Now move $A_6'$ off the $A_5A_2A_4$ plane towards

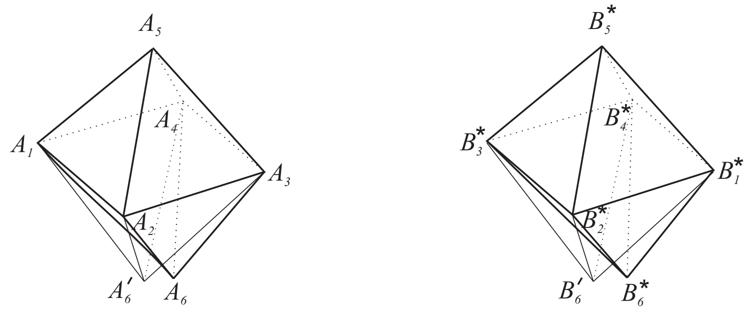


Figure 5: The two regular octahedrons and their modifications

$A_3$ to get the octahedron $A_1 \cdots A_6$, and move $B_6'$ off the $B_5^*B_2^*B_4^*$ plane towards $B_1^*$ to get the octahedron $B_1^* \cdots B_6^*$. Finally move this last octahedron $B_1^* \cdots B_6^*$ so that $B_2^*$ aligns with $A_2$; $B_4^*$ aligns with $A_4$ and $B_5^*$ aligns with $A_5$. Now if $B_1 \cdots B_6$ is the octahedron that we obtain after this last motion, then $A_2 = B_2$, $A_4 = B_4$ and $A_5 = B_5$, so a plane intersecting all segments $A_iB_i$ would have to be the $A_2A_4A_5$ plane. But, by construction, $A_6$ and $B_6$ lie on the same side of that plane, so in this arrangement the segments $A_iB_i$ do not have a common plane transversal.

Finally, we consider a situation that includes both the case of tetrahedrons and the case of isometric polytopes that have been discussed so far. This will also show the strength of algebraic methods in geometry.

An affine mapping is a mapping of $\mathbf{R}^3$ that preserves parallelism. Alternatively, if we use coordinates, then affine transformations can be defined as mappings $T(x_1, x_2, x_3) = (y_1, y_2, y_3)$, where

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1,$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2,$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3$$

with some numbers $a_{ij}$ and $b_i$. Such a $T$ preserves orientation (say of faces) if the determinant $|a_{ij}|_{i,j=1}^3$ is positive, and it reverses orientation if it is negative. If $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two tetrahedra, then there is a unique affine map taking $A_1A_2A_3A_4$ into $B_1B_2B_3B_4$. Isometries are affine maps in which the matrix $(a_{ij})_{i,j=1}^3$ is orthogonal, i.e. $a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3}$ is 0 if $i \neq j$ and is 1 if $i = j$, $1 \leq i, j \leq 3$. So the following statement contains both the tetrahedron and the isometric polytope cases discussed before.

**Proposition 5** *Let $A_1 \cdots A_n$ be a polytope and $B_1 \cdots B_n$ an affine image of it. If these polytopes are oppositely oriented (in the sense that corresponding faces in them are oppositely oriented), then there is a plane that intersects every segment $A_iB_i$, $i = 1, 2, \ldots, m$.*

**Proof.** Let $T$ be the affine mapping in between $A_1 \cdots A_n$ and $B_1 \cdots B_n$ with matrix $\mathcal{A} = (a_{ij})_{i,j=1}^3$, so that if

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

then with some vector $\mathbf{b}$ we have $T\mathbf{x} = \mathcal{A}\mathbf{x} + \mathbf{b}$. Let

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the identity matrix. For $t \in [0, 1]$ set $\mathcal{A}_t = (1-t)I + t\mathcal{A}$ and consider the transformation $T_t\mathbf{x} = \mathcal{A}_t\mathbf{x} + t\mathbf{b}$, so that $T_0$ is the identity and $T_1$ is the mapping $T$. By assumption, the latter reverses orientation, so $\mathcal{A}_1$ has negative determinant, while the determinant of $\mathcal{A}_0$ is 1. Therefore, there must be a $t_0 \in (0, 1)$ such that the determinant of $\mathcal{A}_{t_0}$ is 0, which means that $T_{t_0}$ is singular, i.e., it maps the whole space into a plane $S$. This $S$ intersects every segment $A_iB_i$, the intersection points being $T_{t_0}(A_i)$.

$\blacksquare$

Note that these 3-D results are rather limited, e.g. even though in Proposition 5 convexity is not needed, the proposition itself is a quite restricted analogue of Theorem 1: in it the existence of a common transversal plane is due to the fact that there is a plane that intersects every segment that connects a point with its affine image (under the affine transformation considered). However, the example in Figure 5 shows that, in some sense, this is necessary: a slight distortion of the affine images may lead in 3-space to situations when there are no common transversal planes.

Similar results hold in higher dimensions (in $\mathbf{R}^d$) for simplices and affine polytopes.

The authors thank three unanimous referees whose remarks helped to considerably improve the presentation. In particular, one of the referees suggested

the elegant argument based on comparing the winding numbers of $B_1(\alpha)$ and $P_{1,\alpha}$ ($\phi(B_2(\alpha))$ and $\phi(P_{2,\alpha})$) that was used at the end of the proof of Theorem 1. Two of the referees raised the question of a pure combinatorial proof of Theorem 1 (i.e. not using the continuous reformulation in Theorem 2), and they also pointed out some possibly relevant literature, see [5] and the references there. The authors also thank János Kincses for stimulating discussions.

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