The limit distribution of ratios of jumps and sums of jumps of subordinators

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Abstract

Let V_t be a driftless subordinator, and let denote $m_t^{(1)} \geq m_t^{(2)} \geq \dots$ its jump sequence on interval $[0,t]$. Put $V_t^{(k)} = V_t - m_t^{(1)} - \dots - m_t^{(k)}$ for the k-trimmed subordinator. In this note we characterize under what conditions the limiting distribution of the ratios $V_t^{(k)}/m_t^{(k+1)}$ and $m_t^{(k+1)}/m_t^{(k)}$ exist, as $t \downarrow 0$ or $t \to \infty$.

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1 Introduction and results

Let V_t, $t \geq 0$, be a subordinator with Lévy measure Λ and drift 0. Its Laplace transform is given by

$$\mathbf{E}\mathrm{e}^{-\lambda V_t} = \exp\left\{-t\int_0^\infty \left(1-\mathrm{e}^{-\lambda v}\right)\Lambda(\mathrm{d}v)\right\},$$

where the Lévy measure Λ satisfies

$$\int_0^\infty \min\{1,x\}\Lambda(\mathrm{d}x) < \infty. \tag{1}$$

Put $\overline{\Lambda}(x) = \Lambda((x,\infty))$. Then $\overline{\Lambda}(x)$ is nonincreasing and right continuous on $(0,\infty)$. When $t \downarrow 0$ we also assume that $\overline{\Lambda}(0+) = \infty$, which is necessary and sufficient to assure that there is an infinite number of jumps up to time t, for any $t > 0$.

Denote $m_t^{(1)} \geq m_t^{(2)} \geq \ldots$ the ordered jumps of V_s up to time t, and for $k \geq 0$ consider the trimmed subordinator

$$V_t^{(k)} = V_t - \sum_{j=1}^{k} m_t^{(j)}.$$

We investigate the asymptotic distribution of jump sizes as $t \downarrow 0$ and $t \to \infty$. Specifically, we shall determine a necessary and sufficient condition in terms of the Lévy measure Λ for the convergence in distribution of the ratios $V_t^{(k)}/m_t^{(k+1)}$ and $m_t^{(k+1)}/m_t^{(k)}$. Observe in this notation that $V_t^{(0)} = V_t$ is the subordinator and $m_t^{(1)}$ is the largest jump.

An extended random variable W can take the value ∞ with positive probability, in which case W has a defective distribution function F, meaning that $F(\infty) < 1$. We shall call an extended random variable proper, if it is finite a.s. In this case its F is a probability distribution, i.e. $F(\infty) = 1$. Here we are using the language of the definition given on p. 127 of Feller [8].

Theorem 1. *For any choice of $k \geq 0$ the ratio $V_t^{(k)}/m_t^{(k+1)}$ converges in distribution to an extended random variable W_k as $t \downarrow 0$ ($t \to \infty$) if and only if one of the following holds:*

(i) $\overline{\Lambda}$ is regularly varying at 0 (∞) with parameter $-\alpha$, $\alpha \in (0,1)$, in which case W_k is a proper random variable with Laplace transform

$$g_k(\lambda) = \frac{\mathrm{e}^{-\lambda}}{\left[1 + \alpha \int_0^1 \left(1 - \mathrm{e}^{-\lambda y}\right) y^{-\alpha-1} \mathrm{d}y\right]^{k+1}}; \tag{2}$$

(ii) $\overline{\Lambda}$ is slowly varying at 0 (∞), in which case $W_k = 1$ a.s.;

(iii) the condition

$$\frac{x\overline{\Lambda}(x)}{\int_0^x u\Lambda(\mathrm{d}u)} \longrightarrow 0 \quad \text{as } x \downarrow 0 \ (x \to \infty) \tag{3}$$

holds, in which case $V_t^{(k)}/m_t^{(k+1)} \xrightarrow{\mathbf{P}} \infty$, that is $W_k = \infty$ a.s.

Note that Theorem 1 says that the situation $0 < \mathbf{P}\{W_k = \infty\} < 1$ cannot happen.

The corresponding problem for nonnegative i.i.d. random variables was investigated by Darling [6] and Breiman [4], in the $k = 0$ case. In this case Darling proved the sufficiency parts corresponding to (i) and (ii) (Theorem 5.1 and Theorem 3.2 in [6]), in particular the limit W_0 has the same distribution as given by Darling in his Theorem 5.1, while Breiman proved the necessity parts corresponding to (i), (ii) and (iii) (Theorem 3 (p. 357), Theorem 2 and Theorem 4 in [4]). A special case of Theorem 1 in Teugels [12] gives the sufficiency analog of (i) in the case of i.i.d. nonnegative sums for any $k \geq 0$.

The necessary and sufficient condition in the cases (ii) and (iii), stated in the more general setup of Lévy processes without a normal component, is given by Buchmann, Fan and Maller [5], see their Theorem 3.1 and 5.1.

Next we shall investigate the asymptotic distribution of the ratio of two consecutive ordered jumps $m_t^{(k+1)}/m_t^{(k)}$, $k \geq 1$. We shall obtain the analog for subordinators of a special case of a result that Bingham and Teugels [3] established for i.i.d. nonnegative random variables. This will follow from a general result on the asymptotic distribution of ratios of the form defined for $k \geq 1$ by

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)},\ t > 0,$$

where for each $k \geq 1$, $S_k = \omega_1 + \ldots + \omega_k$, with $\omega_1, \omega_2, \ldots$ being i.i.d. mean 1 exponential random variables and ψ is the nonincreasing and right continuous function defined for $s > 0$ by

$$\psi(s) = \sup\{y : \overline{\Pi}(y) > s\},$$

with Π being a positive measure on $(0, \infty)$ such that $\overline{\Pi}(x) = \Pi((x, \infty)) \to 0$, as $x \to \infty$. Note that we do not require Π to be a Lévy measure. Also whenever we consider the asymptotic distribution of $r_k(t)$ as $t \downarrow 0$ we shall assume that $\overline{\Pi}(0+) = \infty$.

We call a function f *rapidly varying at 0* with index $-\infty$, $f \in \mathrm{RV}_0(-\infty)$, if

$$\lim_{x \downarrow 0} \frac{f(\lambda x)}{f(x)} = \begin{cases} 0, & \text{for } \lambda > 1, \\ 1, & \text{for } \lambda = 1, \\ \infty, & \text{for } \lambda < 1. \end{cases}$$

Correspondingly, a function f is *rapidly varying at* ∞ with index $-\infty$, $f \in \mathrm{RV}_\infty(-\infty)$, if the same holds with $x \to \infty$.

Theorem 2. *For any choice of $k \geq 1$ the ratio $r_k(t)$ converges in distribution as $t \downarrow 0$ ($t \to \infty$) to a random variable Y_k if and only if one of the following holds:*

(i) $\overline{\Pi}$ is regularly varying at 0 (∞) with parameter $-\alpha \in (-\infty, 0)$, in which case Y_k has the Beta$(k\alpha, 1)$ distribution, i.e.

$$G_k(x) = \mathbf{P}\{Y_k \leq x\} = x^{k\alpha}, \quad x \in [0,1]; \tag{4}$$

(ii) $\overline{\Pi}$ is slowly varying at 0 (∞), in which case $Y_k = 0$ a.s.

(iii) $\overline{\Pi}$ is rapidly varying at 0 (∞) with index $-\infty$, in which case $Y_k = 1$ a.s.

Theorem 2 has some important applications to the asymptotic distribution of the ratio of two consecutive ordered jumps $m_t^{(k+1)}/m_t^{(k)}$, $k \geq 1$, of a Lévy process. Let X_t, $t \geq 0$, be a Lévy processes whose Lévy measure Λ is concentrated on $(0, \infty)$. Here in addition to $\overline{\Lambda}(x) \to 0$ as $x \to \infty$, we require that

$$\int_0^\infty \min\{1, x^2\} \Lambda(\mathrm{d}x) < \infty. \tag{5}$$

In this setup one has the distributional representation for $k \geq 1$

$$\left(m_t^{(k)}, m_t^{(k+1)}\right) \stackrel{\mathcal{D}}{=} \left(\varphi(S_k/t), \varphi(S_{k+1}/t)\right), \tag{6}$$

with φ defined for $s > 0$ to be

$$\varphi(s) = \sup\{y : \overline{\Lambda}(y) > s\}. \tag{7}$$

It is readily checked that φ is nonincreasing and right continuous. Moreover, whenever Λ is the Lévy measure of a subordinator V_t, condition (1) holds, which is equivalent to

$$\int_\delta^\infty \varphi(s) \mathrm{d}s < \infty, \text{ for any } \delta > 0. \tag{8}$$

The distributional representation in (6) follows from Proposition 1 in Kevei and Mason [7], see the proof of Theorem 1 below. For general spectrally positive Lévy processes it can be deduced using the same methods that Maller and Mason [9] derived the distributional representation for a Lévy process given in their Proposition 5.7.

When applying Theorem 2 to the asymptotic distribution of consecutive ordered jumps at 0 or ∞ of a Lévy processes X_t whose Lévy measure Λ is concentrated on $(0,\infty)$, we have to keep in mind that (5) must always hold and (1) must be satisfied whenever X_t is a subordinator. For instance in the case of a subordinator V_t, whenever $m_t^{(k+1)}/m_t^{(k)}$ converges in distribution to a random variable Y_k as $t \downarrow 0$, Theorem 2 says that $\overline{\Lambda}$ is regularly varying at 0. Further since (1) must hold, the parameter $-\alpha$ is necessarily in $[-1,0]$, while there is no such restriction when considering convergence in distribution as $t \to \infty$. We note that in case of general Lévy processes for $k=1$ the sufficiency part corresponding to part (ii) in Theorem 2 is given in Theorem 3.1 in [5].

In the special case when V_t is an α-stable subordinator, $\alpha \in (0,1)$, and $m^{(1)} > m^{(2)} > \ldots$ is its jump sequence on $[0,1]$, then $(m^{(1)}/V_1, m^{(2)}/V_1, \ldots)$ has the Poisson–Dirichlet law with parameter $(\alpha, 0)$ (PD$(\alpha,0)$), see Bertoin [1] p. 90. The ratio of the $(k+1)^{\text{th}}$ and k^{th} element of a vector, which has the PD$(\alpha,0)$ law, has the Beta$(k\alpha,1)$ distribution (Proposition 2.6 in [1]).

2 Proofs

In the proofs we only consider the case when $t \downarrow 0$, as the $t \to \infty$ case is nearly identical.

2.1 Proof of Theorem 1

First we calculate the Laplace exponent of the ratio using the notation φ defined in (7). We see by the nonincreasing version of the change of variables formula stated in (4.9) Proposition of Revuz and Yor [10], which is given in Lemma 1 in [7],

$$\begin{aligned}\mathbf{E}\mathrm{e}^{-\lambda V_t} &= \exp\left\{-t\int_0^\infty \left(1-\mathrm{e}^{-\lambda v}\right)\Lambda(\mathrm{d}v)\right\}\\ &= \exp\left\{-t\int_0^\infty \left(1-\mathrm{e}^{-\lambda\varphi(x)}\right)\mathrm{d}x\right\}.\end{aligned}$$

The key ingredient of our proofs is a distributional representation of the subordinator V_t given in Kevei and Mason (Proposition 1 in [7]), which follows from a general representation by Rosiński [11]. It states that for $t>0$

$$V_t \stackrel{\mathcal{D}}{=} \sum_{i=1}^\infty \varphi\left(\frac{S_i}{t}\right). \tag{9}$$

From the proof of this result it is clear that $\varphi(S_i/t)$ corresponds to $m_t^{(i)}$, for $i \geq 1$. Therefore

$$\frac{V_t^{(k)}}{m_t^{(k+1)}} \stackrel{\mathcal{D}}{=} \frac{\sum_{i=k+1}^{\infty} \varphi(S_i/t)}{\varphi(S_{k+1}/t)}.$$

Conditioning on $S_{k+1} = s$ and using the independence we can write

$$\begin{aligned}
\sum_{i=k+2}^{\infty} \varphi(S_i/t) &= \sum_{i=k+2}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i - s}{t}\right) \\
&\stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i}{t}\right) \\
&= \sum_{i=1}^{\infty} \varphi_{s/t}\left(S_i/t\right),
\end{aligned}$$

where $\varphi_y(x) = \varphi(y + x)$. Note that the latter sum has the same form as in (9), therefore it is equal in distribution to a subordinator $V^{(s/t)}(t)$ with Laplace transform

$$\begin{aligned}
\mathbf{E}\mathrm{e}^{-\lambda V_t^{(s/t)}} &= \exp\left\{-t \int_0^{\infty} \left(1 - \mathrm{e}^{-\lambda \varphi_{s/t}(x)}\right) \mathrm{d}x\right\} \\
&= \exp\left\{-t \int_{s/t}^{\infty} (1 - \mathrm{e}^{-\lambda \varphi(x)}) \mathrm{d}x\right\}.
\end{aligned} \tag{10}$$

Now we can compute the Laplace transform of the ratio $V_t^{(k)}/m_t^{(k+1)}$. Since S_{k+1} has Gamma$(k+1, 1)$ distribution, the law of total probability and (10) give

$$\begin{aligned}
&\mathbf{E}\exp\left\{-\lambda \frac{V_t^{(k)}}{m_t^{(k+1)}}\right\} = \mathbf{E}\exp\left\{-\lambda \frac{\sum_{i=k+1}^{\infty} \varphi(S_i/t)}{\varphi(S_{k+1}/t)}\right\} \\
&= \int_0^{\infty} \frac{s^k}{k!} \mathrm{e}^{-s} \left[\mathrm{e}^{-\lambda}\, \mathbf{E}\exp\left\{-\frac{\lambda}{\varphi(s/t)} \sum_{i=1}^{\infty} \varphi_{s/t}(S_i/t)\right\}\right] \mathrm{d}s \\
&= \mathrm{e}^{-\lambda} \int_0^{\infty} \frac{s^k}{k!} \mathrm{e}^{-s} \exp\left\{-t \int_{s/t}^{\infty} \left[1 - \mathrm{e}^{-\frac{\lambda}{\varphi(s/t)}\varphi(x)}\right] \mathrm{d}x\right\} \mathrm{d}s \\
&= \frac{t^{k+1}}{k!} \mathrm{e}^{-\lambda} \int_0^{\infty} u^k \exp\left\{-t\left(u + \int_u^{\infty} \left[1 - \mathrm{e}^{-\lambda \frac{\varphi(x)}{\varphi(u)}}\right] \mathrm{d}x\right)\right\} \mathrm{d}u \\
&= \frac{t^{k+1}}{k!} \mathrm{e}^{-\lambda} \int_0^{\infty} u^k \mathrm{e}^{-t\Psi(u,\lambda)} \mathrm{d}u,
\end{aligned} \tag{11}$$

where

$$\Psi(u,\lambda) = u + \int_u^\infty \left[1 - \mathrm{e}^{-\lambda \frac{\varphi(x)}{\varphi(u)}}\right] \mathrm{d}x. \tag{12}$$

Since φ is right continuous on $(0,\infty)$, $\Psi(\cdot,\lambda)$ is also right continuous on $(0,\infty)$. Further a short calculation shows that this function is strictly increasing for any $\lambda > 0$, moreover for $u_1 > u_2$

$$\Psi(u_1,\lambda) - \Psi(u_2,\lambda) \geq \mathrm{e}^{-\lambda}(u_1 - u_2).$$

Clearly $\Psi(\infty,\lambda) = \infty$ and therefore

$$\Psi_k(u,\lambda) := \Psi\left(((k+1)u)^{1/(k+1)}, \lambda\right)$$

has a right continuous increasing inverse function given by

$$Q_\lambda(s) = \inf\left\{v : \Psi_k\left(v,\lambda\right) > s\right\}, \text{ for } s \geq 0,$$

such that $Q_\lambda(0) = 0$ and $\lim_{x\to\infty} Q_\lambda(x) = \infty$. (For the right continuity part see (4.8) Lemma in Revuz and Yor [10].)

Necessity. Assuming that $V_t^{(k)}/m_t^{(k+1)}$ converges in distribution as $t \to 0$ to some extended random variable W_k, we can apply Theorem 2a on p. 210 of Feller [8] to conclude that its Laplace transform also converges, i.e.

$$\int_0^\infty u^k \mathrm{e}^{-t\Psi(u,\lambda)} \mathrm{d}u = \int_0^\infty \mathrm{e}^{-t\Psi_k(v,\lambda)} \mathrm{d}v$$

$$= \int_0^\infty \mathrm{e}^{-ty} \mathrm{d}Q_\lambda\left(y\right) \sim \frac{\mathrm{e}^\lambda g_k(\lambda) k!}{t^{k+1}}, \text{ as } t \to 0,$$

where $g_k(\lambda) = \mathbf{E}\mathrm{e}^{-\lambda W_k}$, and W_k can possibly have a defective distribution, i.e. possibly $\mathbf{P}\left\{W_k = \infty\right\} > 0$. (Here we used the change of variables formula given in (4.9) Proposition in Revuz and Yor [10].) By Karamata's Tauberian theorem (Theorem 1.7.1 in [2])

$$Q_\lambda(y) \sim \frac{y^{k+1}}{k+1} \mathrm{e}^\lambda g_k(\lambda), \quad \text{as } y \to \infty,$$

and thus by Theorem 1.5.12 in [2]

$$\Psi_k\left(v,\lambda\right) \sim \left(\frac{(k+1)v}{\mathrm{e}^\lambda g_k(\lambda)}\right)^{1/(k+1)}, \quad \text{as } v \to \infty,$$

and hence

$$\Psi(u,\lambda) \sim u\left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}}, \quad \text{as } u \to \infty.$$

Substituting back into (12) we obtain for any $\lambda > 0$

$$\lim_{u\to\infty} \frac{1}{u}\int_u^\infty \left(1 - \mathrm{e}^{-\lambda\frac{\varphi(x)}{\varphi(u)}}\right)\mathrm{d}x = \left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1. \tag{13}$$

Note that the limit W_k is ≥ 1, with probability 1, and so $g_k(\lambda) \leq \mathrm{e}^{-\lambda}$. Thus for any λ

$$\left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1 \geq 0.$$

For any $x \geq 0$ we have $1 - \mathrm{e}^{-x} \leq x$. Therefore by (13) we obtain for any $\lambda > 0$

$$\liminf_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x \geq \frac{1}{\lambda}\left(\left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right). \tag{14}$$

On the other hand, by monotonicity $\varphi(x)/\varphi(u) \leq 1$ for $u \leq x$. Therefore for any $0 < \varepsilon < 1$ there exists a $\lambda_\varepsilon > 0$, such that for all $0 < \lambda < \lambda_\varepsilon$

$$1 - \mathrm{e}^{-\lambda\frac{\varphi(x)}{\varphi(u)}} \geq (1-\varepsilon)\frac{\lambda\varphi(x)}{\varphi(u)}, \text{ for } x \geq u.$$

Using again (13) and keeping (8) in mind, this implies that for such λ

$$\limsup_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x \leq \frac{1}{1-\varepsilon}\frac{1}{\lambda}\left(\left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right). \tag{15}$$

In particular, we obtain that, whenever $g_k(\lambda) \not\equiv 0$ (i.e. $\mathbf{P}\{W_k < \infty\} > 0$)

$$0 \leq \liminf_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x \leq \limsup_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x < \infty.$$

Note that in (14) the greatest lower bound is 0 for all $\lambda > 0$ if and only if $g_k(\lambda) = \mathrm{e}^{-\lambda}$, in which case $W_k = 1$. Then the upper bound for the limsup in (15) is 0, thus

$$\lim_{u\to\infty} \frac{1}{u\varphi(u)}\int_u^\infty \varphi(x)\mathrm{d}x = 0,$$

which by Proposition 2.6.10 in [2] applied to the function $f(x) = x\varphi(x)$ implies that $\varphi \in \mathrm{RV}_\infty(-\infty)$, and so, by Theorem 2.4.7 in [2], $\overline{\Lambda}$ is slowly varying at 0. We have proved that $W_k = 1$ if and only if $\overline{\Lambda}$ is slowly varying at 0.

In the following we assume that $\mathbf{P}\{W_k > 1\} > 0$, therefore the liminf in (14) is strictly positive. Let

$$a = \liminf_{\lambda \downarrow 0} \frac{1}{\lambda}\left(\left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right) \leq \limsup_{\lambda \downarrow 0} \frac{1}{\lambda}\left(\left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right) = b.$$

By (15) and (14), $a > 0$ and $b < \infty$. Moreover

$$b \leq \liminf_{u \to \infty} \frac{1}{u\varphi(u)} \int_u^{\infty} \varphi(x)\mathrm{d}x \leq \limsup_{u \to \infty} \frac{1}{u\varphi(u)} \int_u^{\infty} \varphi(x)\mathrm{d}x \leq a,$$

which forces

$$a = b = \lim_{u \to \infty} \frac{1}{u\varphi(u)} \int_u^{\infty} \varphi(x)\mathrm{d}x = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}\left(\left[\mathrm{e}^{\lambda} g_k(\lambda)\right]^{-\frac{1}{k+1}} - 1\right).$$

By Karamata's theorem (Theorem 1.6.1 (ii) in [2]) we obtain that φ is regularly varying at infinity with parameter $-a^{-1} - 1 =: -\alpha^{-1}$, so Λ is regularly varying with parameter $-\alpha$ at zero with $\alpha \in (0,1)$.

Let us consider the case when $W_k = \infty$ a.s., that is $V_t^{(k)}/m_t^{(k+1)} \xrightarrow{\mathbf{P}} \infty$. All the previous computations are valid, with $g_k(\lambda) = \mathbf{E}\mathrm{e}^{-\lambda\infty} \equiv 0$. Thus, from (14) we have

$$\lim_{u \to \infty} \frac{1}{u\varphi(u)} \int_u^{\infty} \varphi(x)\mathrm{d}x = \infty.$$

From this, through the change of variables formula we obtain (3).

Sufficiency and the limit. Consider first the special case when $\varphi(x) = x^{-\frac{1}{\alpha}}$, $\alpha \in (0,1)$. Then a quick calculation gives

$$\frac{1}{u}\int_u^{\infty}\left(1 - \mathrm{e}^{-\lambda\frac{\varphi(x)}{\varphi(u)}}\right)\mathrm{d}x = \alpha\int_0^1\left(1 - \mathrm{e}^{-\lambda y}\right)y^{-\alpha-1}\mathrm{d}y.$$

By formula (13) for the Laplace transform of the limit we obtain (2).

The sufficiency can be proved by standard arguments for regularly varying functions. Using Potter bounds (Theorem 1.5.6 in [2]) one can show that for $\alpha \in (0,1)$

$$\lim_{u \to \infty} \frac{1}{u}\Psi(u,\lambda) = 1 + \alpha\int_0^1\left(1 - \mathrm{e}^{-\lambda y}\right)y^{-\alpha-1}\mathrm{d}y,$$

from which, through formula (11), the convergence readily follows. As already mentioned, cases (ii) and (iii) are treated in [5].

2.2 Proof of Theorem 2

Using that $\psi(y) \le x$ if and only if $\overline{\Pi}(x) \le y$, for the distribution function of the ratio we have for $x \in (0,1)$

$$\begin{aligned}
\mathbf{P}\left\{r_k(t) \le x\right\} &= \mathbf{P}\left\{\frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \le x\right\} \\
&= \int_0^\infty \frac{s^{k-1}}{(k-1)!}\mathrm{e}^{-s}\,\mathbf{P}\left\{\psi\left(\frac{s+S_1}{t}\right) \le x\psi\left(\frac{s}{t}\right)\right\}\mathrm{d}s \\
&= \int_0^\infty \frac{s^{k-1}}{(k-1)!}\mathrm{e}^{-s}\,\mathrm{e}^{-[t\overline{\Pi}(x\psi(s/t))-s]}\mathrm{d}s \\
&= \frac{t^k}{(k-1)!}\int_0^\infty u^{k-1}\,\mathrm{e}^{-t\overline{\Pi}(x\psi(u))}\mathrm{d}u.
\end{aligned} \tag{16}$$

Necessity. Assume that the limit distribution function G_k exists. Write

$$\frac{t^k}{(k-1)!}\int_0^\infty u^{k-1}\mathrm{e}^{-t\overline{\Pi}(x\psi(u))}\mathrm{d}u = \frac{t^k}{(k-1)!}\int_0^\infty \mathrm{e}^{-t\Phi_k(v,x)}\mathrm{d}v, \tag{17}$$

where $\Phi_k(v,x) = \overline{\Pi}\left(x\psi((kv)^{1/k})\right)$. Note that for each $x \in (0,1)$ the function $\Phi_k(\cdot,x)$ is monotone nondecreasing, since $\overline{\Pi}$ and ψ are both monotone nonincreasing. Let

$$\mathcal{G}_k = \left\{x : x \text{ is a continuity point of } G_k \text{ in } (0,1) \text{ such that } G_k(x) > 0\right\}.$$

First assume that $\mathbf{P}\{Y_k < 1\} > 0$. Clearly we can now proceed as in the proof of Theorem 1 to apply Karamata's Tauberian theorem (Theorem 1.7.1 in [2]) to give that for any $x \in \mathcal{G}_k$,

$$\lim_{u\to\infty}\frac{\overline{\Pi}(x\psi(u))}{u} = [G_k(x)]^{-\frac{1}{k}}. \tag{18}$$

In fact, there is a small difference here compared to the proof of Theorem 1. We have to be more cautious, as $\Phi_k(v,x)$ is not necessarily right-continuous as a function of $v > 0$. To use the machinery from the proof of Theorem 1 we need to consider the right-continuous version $\widetilde{\Phi}_k(v,x) := \Phi_k(v+,x)$. Since, in (17) we integrate with respect to the Lebesgue measure and Φ_k and $\widetilde{\Phi}_k$ are equal almost everywhere, substituting Φ_k with $\widetilde{\Phi}_k$ leaves the integral unchanged. Therefore, proceeding as before we obtain that

$$\widetilde{\Phi}_k(v,x) \sim \left(\frac{kv}{G_k(x)}\right)^{1/k}, \quad \text{as } v \to \infty,$$

and since the right-hand function is continuous, we also get that

$$\Phi_k(v,x) \sim \left(\frac{kv}{G_k(x)}\right)^{1/k}, \quad \text{as } v \to \infty,$$

form which now (18) does indeed follow.

We claim that (18) implies the regular variation of $\overline{\Pi}$. When $\overline{\Pi}$ is continuous and strictly decreasing we get by changing variables to $\psi(u) = t$, $u = \overline{\Pi}(t)$, that we have for any $x \in \mathcal{G}_k$

$$\lim_{t\downarrow 0} \frac{\overline{\Pi}(tx)}{\overline{\Pi}(t)} = [G_k(x)]^{-\frac{1}{k}},$$

which by an easy application of Proposition 1.10.5 in [2] implies that $\overline{\Pi}$ is regularly varying.

Note that the jumps of $\overline{\Pi}$ correspond to constant parts of ψ, and vice versa. Put $\mathcal{J} = \{z : \overline{\Pi}(z-) > \overline{\Pi}(z)\}$ for the jump points of $\overline{\Pi}$. For $z \in \mathcal{J}$ and $y \in \big[\overline{\Pi}(z), \overline{\Pi}(z-)\big)$ we have $\psi(y) = z$. Substituting into (18) we have

$$\lim_{z\downarrow 0, z\in\mathcal{J}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = [G_k(x)]^{-\frac{1}{k}}, \quad \text{and} \quad \lim_{z\downarrow 0, z\in\mathcal{J}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z-)} = [G_k(x)]^{-\frac{1}{k}}. \tag{19}$$

To see how the second limit holds in (19) note that for any $0 < \varepsilon < 1$ and $z \in \mathcal{J}$, we have $\psi\left(\varepsilon\overline{\Pi}(z) + (1-\varepsilon)\,\overline{\Pi}(z-)\right) = z$ and thus

$$\lim_{z\downarrow 0, z\in\mathcal{J}} \frac{\overline{\Pi}(xz)}{\varepsilon\overline{\Pi}(z) + (1-\varepsilon)\,\overline{\Pi}(z-)} = [G_k(x)]^{-\frac{1}{k}}.$$

Since $0 < \varepsilon < 1$ can be chosen arbitrarily close to 0 this implies the validity of the second limit in (19). Therefore by choosing any $x \in \mathcal{G}_k$ we get

$$\lim_{z\downarrow 0} \frac{\overline{\Pi}(z-)}{\overline{\Pi}(z)} = 1. \tag{20}$$

Let

$$\mathcal{A} = \{z > 0 : \overline{\Pi}(z-\varepsilon) > \overline{\Pi}(z) \ \text{ for all } z > \varepsilon > 0\}.$$

This set contains exactly those points z for which $\psi(\overline{\Pi}(z)) = z$. With this notation formula (18) can be written as

$$\lim_{z\downarrow 0, z\in\mathcal{A}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = [G_k(x)]^{-\frac{1}{k}}, \text{ for } x \in \mathcal{G}_k. \tag{21}$$

This together with (20) will allow us to apply Proposition 1.10.5 in [2] to conclude that $\overline{\Pi}$ is regularly varying. We shall need the following technical lemma.

Lemma 1. *Whenever (20) holds, there exists a strictly decreasing sequence $z_n \in \mathcal{A}$ such that $z_n \to 0$ and*

$$\lim_{n\to\infty} \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n)} = 1. \tag{22}$$

Proof. Choose $z_1 \in \mathcal{A}$ such that $\overline{\Pi}(z_1) > 0$, and define for each $n \geq 1$

$$z_{n+1} = \sup\left\{ z > 0 : \overline{\Pi}(z) > \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \right\}.$$

Notice that the sequence $\{z_n\}$ is well-defined, since $\overline{\Pi}(0+) = \infty$ and it is decreasing. Further we have

$$\overline{\Pi}(z_{n+1}-) \geq \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \text{ and } \overline{\Pi}(z_{n+1}) \leq \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-),$$

where the second inequality follows by right continuity of $\overline{\Pi}$. Also note that $z_{n+1} < z_n$, since otherwise if $z_{n+1} = z_n$, then

$$\overline{\Pi}(z_{n+1}-) = \overline{\Pi}(z_n-) \geq \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-),$$

which is impossible. Observe that each z_{n+1} is in $\mathcal{A}$ since by the definition of z_{n+1} for all $0 < \varepsilon < z_{n+1}$

$$\overline{\Pi}(z_{n+1} - \varepsilon) > \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \geq \overline{\Pi}(z_{n+1}).$$

Clearly since $\{z_n\}$ is a decreasing and positive sequence, $\lim_{n\to\infty} z_n = z^*$ exists and is ≥ 0. By construction

$$\overline{\Pi}(z_{n+1}-) \geq \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \ \geq \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) \overline{\Pi}(z_1-).$$

The infinite product $\prod_{n=1}^{\infty}(1+1/n) = \infty$ forces $z^* = 0$. Also by construction we have

$$1 \leq \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n-)} = \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n)} \left(\frac{\overline{\Pi}(z_n)}{\overline{\Pi}(z_n-)} \right) \leq 1 + \frac{1}{n}.$$

By (20) we have

$$\lim_{n\to\infty} \frac{\overline{\Pi}(z_n)}{\overline{\Pi}(z_n-)} = 1.$$

Therefore we get (22). □

According to Proposition 1.10.5 in [2] to establish that $\overline{\Pi}$ is regularly varying at zero it suffices to produce λ_1 and λ_2 in $(0,1)$ such that for $i=1,2$

$$\frac{\overline{\Pi}(\lambda_i z_n)}{\overline{\Pi}(z_n)} \to d_i \in (0,\infty)\,, \text{ as } n\to\infty,$$

where $(\log\lambda_1)\,/\,(\log\lambda_2)$ is finite and irrational. This can clearly be done using (21) and $\mathbf{P}\{Y_k<1\}>0$. Necessarily $\overline{\Pi}$ has index of regular variation parameter $-\alpha\in(-\infty,0]$. For $\alpha\in(0,\infty)$ the limiting distribution function has the form (4). In the case $\alpha=0$, $\overline{\Pi}$ is slowly varying at 0 and we get that $G_k(x)=1$ for $x\in(0,1)$, i.e. $Y_k=0$ a.s.

Now consider the case when $\mathbf{P}\{Y_k=1\}=1$, i.e. $G_k(x)=0$ for any $x\in(0,1)$. We once more use Theorem 1.7.1 in [2], with $c=0$ this time, and as an analog of (18) we obtain

$$\lim_{u\to\infty}\frac{\overline{\Pi}(x\psi(u))}{u}=\infty.$$

This readily implies that

$$\lim_{z\downarrow 0, z\in\mathcal{A}}\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)}=\infty.$$

Moreover, the analogs of formula (19) also hold, i.e.

$$\lim_{z\downarrow 0, z\in\mathcal{J}}\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)}=\infty,\quad \text{and}\quad \lim_{z\downarrow 0, z\in\mathcal{J}}\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z-)}=\infty.$$

(Note, however, that this does not imply (20).) Let $z\notin\mathcal{A}$, and define $z'=\inf\{v: v\in\mathcal{A}, v>z\}$. Clearly, $z'\downarrow 0$ as $z\downarrow 0$. If $z'\in\mathcal{A}$ then necessarily it is a jump point, $z'\in\mathcal{J}$, and $\overline{\Pi}(z'-)=\overline{\Pi}(z)$. Then

$$\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)}=\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z'-)}\geq\frac{\overline{\Pi}(xz')}{\overline{\Pi}(z'-)},$$

and the latter tends to ∞ as $z\downarrow 0$. On the other hand, when $z'\notin\mathcal{A}$ it is simple to see that $\overline{\Pi}(z')=\overline{\Pi}(z)$ and $\overline{\Pi}(z'+\varepsilon)<\overline{\Pi}(z')$ for any $\varepsilon>0$. Moreover, we can find $z<z''\in\mathcal{A}$, such that $\overline{\Pi}(z)\leq\overline{\Pi}(z'')+1\leq 2\overline{\Pi}(z'')$ (we tacitly assumed that z is small enough). Thus

$$\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)}\geq\frac{\overline{\Pi}(xz)}{\overline{\Pi}(z'')+1}\geq\frac{\overline{\Pi}(xz'')}{2\overline{\Pi}(z'')},$$

and the lower bound goes to ∞ as $z \downarrow 0$. Summarizing, we have proved that

$$\lim_{z\downarrow 0} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = \infty,$$

for any $x \in (0,1)$, that is, $\overline{\Pi}$ is rapidly varying at 0 with index $-\infty$.

Sufficiency. Assume that $\overline{\Pi}$ is regularly varying at 0 with index $-\alpha \in (-\infty, 0)$. Then its asymptotic inverse function ψ is regularly varying at ∞ with index $-1/\alpha$, therefore simply

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \to \left(\frac{S_k}{S_{k+1}}\right)^{1/\alpha} \quad \text{a.s., as } t \downarrow 0,$$

which has the distribution G_k in (4). Assume now that $\overline{\Pi}$ is slowly varying at 0. Then $\psi \in \mathrm{RV}_\infty(-\infty)$, therefore

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \to 0 \quad \text{a.s., as } t \downarrow 0.$$

Finally, if $\overline{\Pi} \in \mathrm{RV}_0(-\infty)$ then ψ is slowly varying at infinity, so

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \to 1 \quad \text{a.s., as } t \downarrow 0,$$

and the theorem is completely proved.

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