Erratum to "Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback"

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The purpose of this note is to correct a mistake left in our previous paper [2]. The paper concerns the scalar equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1}$$

with $\mu = 1$ and a special strictly increasing, continuously differentiable f. The natural phase space for Eq. (1) is $C = C([-1,0],\mathbb{R})$ equipped with the supremum norm. For any $\varphi \in C$, there is a unique solution $x^{\varphi} : [-1,\infty) \to \mathbb{R}$ of (1). For each $t \geq 0$, the segment $x_t^{\varphi} \in C$ is defined by $x_t^{\varphi}(s) = x^{\varphi}(t+s)$, $-1 \leq s \leq 0$. Let Φ denote the semiflow induced by E.q. (1):

$$\Phi : [-1,\infty) \times C \ni (t,\varphi) \mapsto x_t^{\varphi} \in C.$$

Theorem 1.1 of paper [2] gives a periodic solution $p : \mathbb{R} \to \mathbb{R}$ of E.q. (1) with $p(-1) = 0$ and $\dot{p}(-1) \neq 0$. The proof of Theorem 1.2 in Section 8 then applies a Poincaré return map defined on a neighborhood of p_0 in H, where $H = \{\varphi : \varphi(-1) = 0\}$ is a hyperplane transversal to the periodic orbit $\mathcal{O}_p = \{p_t : t \in \mathbb{R}\}$. As we shall see, this hyperplane was not selected appropriately.

We evoke results from Floquet theory before pointing at the error and showing its correction.

1. Floquet theory

Let $\omega \in (1,2)$ denote the minimal period of p. Consider the period map $Q = \Phi(\omega,\cdot)$ with fixed point p_0. Consider its derivative $M = D_2\Phi(\omega,p_0)$ at p_0. Then $M\varphi = u_{\omega}^{\varphi}$ for all

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$\varphi \in C$, where $u^{\varphi} : [-1, \infty) \to \mathbb{R}$ is the solution of the variational equation

$$\dot{u}(t) = -u(t) + f'(p(t-1))\, u(t-1) \tag{2}$$

with $u_0^{\varphi} = \varphi$. M is called the monodromy operator. M is a compact operator, 0 belongs to its spectrum $\sigma = \sigma(M)$, and eigenvalues of finite multiplicity – the so called Floquet multipliers – form $\sigma(M) \setminus \{0\}$.

As $\dot{p}$ is a nonzero solution of the variational equation, 1 is a Floquet multiplier with eigenfunction $\dot{p}_0$. The paper [2] proves that $\mathcal{O}_p$ is hyperbolic, which means that the generalized eigenspace of M corresponding to the eigenvalue 1 is one-dimensional, furthermore there are no Floquet multipliers on the unit circle besides 1.

If φ is a nonzero element of the phase space $C = C([-1,0], \mathbb{R})$, let $V(\varphi)$ denote the number of sign changes of φ if it is even or ∞, otherwise let $V(\varphi)$ be the number of sign changes plus one. This is the so-called discrete Lyapunov functional of Mallet-Paret and Sell [4].

By Section 4 of [2], $\mathcal{O}_p$ has two real and simple Floquet multipliers λ_1 and λ_2 outside the unit circle with $\lambda_1 > \lambda_2 > 1$. Regarding the associated eigenspaces, we have the following information from [4] and from Appendix VII of [3]. The eigenvector u_1 of M corresponding to λ_1 is strictly positive. The realified generalized eigenspace $C_{<\lambda_1}$ associated with the spectral set $\{z \in \sigma : |z| < \lambda_1\}$ satisfies

$$C_{<\lambda_1} \cap V^{-1}(0) = \emptyset. \tag{3}$$

Let $C_{\leq \rho}$, $\rho > 0$, denote the realified generalized eigenspace of M associated with the spectral set $\{z \in \sigma : |z| \leq \rho\}$. The set

$$\left\{\rho \in (0, \infty) : \sigma(M) \cap \rho S^1_{\mathbb{C}} \neq \emptyset,\, C_{\leq \rho} \cap V^{-1}(\{0,2\}) = \emptyset\right\}$$

is nonempty and has a maximum r_M. Then

$$C_{\leq r_M} \cap V^{-1}(\{0,2\}) = \emptyset, \quad C_{r_M <} \setminus \left\{\hat{0}\right\} \subset V^{-1}(\{0,2\}) \text{ and } \dim C_{r_M <} \leq 3, \tag{4}$$

where $C_{r_M <}$ is the realified generalized eigenspace of M associated with the nonempty spectral set $\{z \in \sigma : |z| > r_M\}$. It follows from the construction of p in [2] that $V(\dot{p}_0) = 2$. Hence $r_M < 1$ in our case, and $V(u_2) = 2$ for the eigenvector u_2 of M corresponding to λ_2.

2. Poincaré return maps

Choose X to be a hyperplane with codimension 1 so that $\dot{p}_0 \notin X$. An application of the implicit function theorem yields a convex bounded open neighborhood N of p_0 in C, $\varepsilon \in (0, \omega)$ and a C^1-map $\gamma : N \to (\omega - \varepsilon, \omega + \varepsilon)$ with $\gamma(p_0) = \omega$ so that for each $(t, \varphi) \in (\omega - \varepsilon, \omega + \varepsilon) \times N$, the segment x_t^{φ} belongs to $p_0 + X$ if and only if $t = \gamma(\varphi)$ (see

[1], Appendix I in [3]). The Poincaré return map P_X is defined by

$$P_X : N \cap (p_0 + X) \ni \varphi \mapsto \Phi(\gamma(\varphi), \varphi) \in p_0 + X.$$

Then P_X is continuously differentiable with fixed point p_0.

Let $\sigma(P_X)$ and $\sigma(M)$ denote the spectra of $DP_X(p_0) : X \to X$ and the monodromy operator M, respectively. We obtain the following result from Theorem XIV.4.5 in [1].

Lemma.
(i) $\sigma(P_X) \setminus \{0,1\} = \sigma(M) \setminus \{0,1\}$, *and for every* $\lambda \in \sigma(M) \setminus \{0,1\}$, *the projection along* $\mathbb{R}\dot{p}_0$ *onto* X *defines an isomorphism from the realified generalized eigenspace of* λ *and* M *onto the realified generalized eigenspace of* λ *and* $DP_X(p_0)$.
(ii) $1 \notin \sigma(P_X)$.

In Section 8 of [2] we selected the hyperplane $H = \{\varphi : \varphi(-1) = 0\}$ and the associated Poincaré map $P = P_H$. It follows from the above proposition that $DP(p_0)$ has exactly two real eigenvalues $\lambda_1 > \lambda_2 > 1$ outside the unit circle. Let v_1 and v_2 denote the eigenvectors of $DP(p_0)$ corresponding to λ_1 and λ_2, respectively. Section 8 of [2] used the statement that $V(v_1) = 0$ and $V(v_2) = 2$. This is not necessarily true. The mistake can be corrected by selecting a different hyperplane.

Let C_s and C_u be the closed subspaces of C chosen so that $C = C_s \oplus \mathbb{R}\dot{p}_0 \oplus C_u$, C_s and C_u are invariant under M, and the spectra $\sigma_s(M)$ and $\sigma_u(M)$ of the induced maps $C_s \ni x \mapsto Mx \in C_s$ and $C_u \ni x \mapsto Mx \in C_u$ are contained in $\{\mu \in \mathbb{C} : |\mu| < 1\}$ and $\{\mu \in \mathbb{C} : |\mu| > 1\}$, respectively. As $\mathcal{O}_p$ has two real and simple Floquet multipliers λ_1 and λ_2 outside the unit circle with eigenvectors u_1 and u_2, we have $C_u = \{c_1 u_1 + c_2 u_2\}$.

Set $Y = C_s \oplus C_u$. Then Y is a hyperplane in C, $\dot{p}_0 \notin Y$ and $C = Y \oplus \mathbb{R}\dot{p}_0$.

The special choice of Y and Lemma imply that λ_i and u_i is an eigenvalue-eigenvector pair of $DP_Y(p_0)$ for both $i \in \{1,2\}$. In addition, C_s and C_u are invariant under $DP_Y(p_0)$, and the spectra $\sigma_s(P_Y)$ and $\sigma_u(P_Y)$ of the induced maps $C_s \ni x \mapsto DP_Y(p_0)x \in C_s$ and $C_u \ni x \mapsto DP_Y(p_0)x \in C_u$ are contained in $\{\mu \in \mathbb{C} : |\mu| < 1\}$ and $\{\mu \in \mathbb{C} : |\mu| > 1\}$, respectively. Summing up, $DP_Y(p_0)$ has exactly two real and simple eigenvalues $\lambda_1 > \lambda_2 > 1$ outside the unit circle, and for the corresponding eigenvectors u_1 and u_2, we have the desired properties $V(u_1) = 0$ and $V(u_2) = 2$.

In accordance, H and $P = P_H$ should be changed to Y and P_Y in Section 8 of [2]. Then the proof of Theorem 1.2. (found in Section 8 of [2]) becomes correct without any further change.

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