ON RANDOM DISC-POLYGONS IN SMOOTH CONVEX DISCS

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ABSTRACT. In this paper we generalize some of the classical results of Rényi and Sulanke (cf. [27, 28]) in the context of spindle convexity. A planar convex disc S is spindle convex if it is the intersection of congruent closed circular discs. The intersection of finitely many congruent closed circular discs is called a disc-polygon. We prove asymptotic formulas for the expectation of the number of vertices, missed area and perimeter difference of uniform random disc-polygons contained in a sufficiently smooth spindle convex disc.

1. Introduction and results

In their seminal papers, Rényi and Sulanke [27, 28, 29] investigated the geometric properties of approximations of convex discs by random convex polygons. In particular, they considered the following probability model.

Let K be a convex disc (a compact convex set with nonempty interior) in the Euclidean plane $\mathbf{E}^2$ and let $y_1, y_2, \dots$ be independent random points chosen from K according to the uniform probability distribution. Let K_n denote the convex hull of $Y_n = \{y_1, \dots, y_n\}$. The set K_n is called a uniform random convex polygon in K. We use $\mathbb{E}(\cdot)$ to denote the expectation of a random variable in this probability model.

Rényi and Sulanke [27, 28] proved asymptotic formulas for the expectation of the number of vertices of K_n and the expectation of the missed area of K_n under the assumption that the boundary ∂K of K is twice continuously differentiable. They also proved an asymptotic formula for the expectation of the perimeter difference of K and K_n under stronger differentiability assumptions on ∂K and assuming that the curvature $\kappa(x) > 0$ for all $x \in \partial K$. For later comparison, we state their results below in a slightly modified form.

Let $f_0(K_n)$ denote the number of vertices of K_n, $A(K)$ the area of K and $\Gamma(\cdot)$ Euler's Gamma function. Then (cf. Satz 3 on page 83 in [27])

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}}\,\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \kappa(x)^{1/3} \mathrm{d}x, \tag{1}$$

where integration is with respect to the one-dimensional Hausdorff measure on ∂K. We note that with the help of Efron's identity [12], (1) implies directly the following statement

$$\lim_{n\to\infty} \mathbb{E}(A(K \setminus K_n)) \cdot n^{2/3} = \sqrt[3]{\frac{2A(K)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \kappa(x)^{1/3} \mathrm{d}x. \tag{2}$$

Rényi and Sulanke derived (2) by direct computation, cf. formula (48) in Satz 1 on page 144 in [28].

Assuming that the boundary of K is sufficiently smooth and $\kappa(x) > 0$ for all $x \in \partial K$, Rényi and Sulanke proved the asymptotic formula

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(K) - \mathrm{Per}(K_n)) \cdot n^{2/3} = \frac{1}{12}\Gamma\left(\frac{2}{3}\right)(12A(K))^{2/3} \int_{\partial K} \kappa(x)^{4/3} \mathrm{d}x \tag{3}$$

for the perimeter difference of K and K_n, cf. formula (47) in Satz 1 on page 144 in [28].

For more information about approximations of convex bodies by random polytopes we refer to the recent book by Schneider and Weil [33], and the survey articles by Bárány [2], and by Schneider [32], and by Weil and Wieacker [36].

In this article, we investigate the *R-spindle convex* analogue of the above probability model. Let $R > 0$. R-spindle convex discs are those convex discs that are intersections of (not necessarily finitely many) closed circular discs of radius R. For a precise definition of spindle convexity, see Section 2. The intersection of finitely many closed circular discs of radius R is a closed convex R-disc-polygon. Let X be a compact set which is contained in a closed circular disc of radius R. The intersection of all R-spindle convex discs containing X is called the R-spindle convex hull of X, and it is denoted by $\mathrm{conv}_{s,R}(X)$.

Now we are ready to define our probability model. Let S be an R-spindle convex disc in $\mathbf{E}^2$. Let $x_1, x_2, \ldots$ be independent random points in S chosen according to the uniform probability distribution (the Lebesgue measure in S normalized by the area of S). The R-spindle convex hull $S_n^R = \mathrm{conv}_{s,R}(X_n)$, where $X_n = \{x_1, \ldots, x_n\}$, is called a *uniform random R-disc-polygon* in S. We prove the R-spindle convex analogues of (1), (2) and (3) in this probability model.

The notion of spindle convexity was first introduced by Mayer [22] in 1935 as a generalization of linear convexity in the wider context of Minkowski geometry. In the Euclidean plane $\mathbf{E}^2$, a closed convex set is the intersection of closed half-planes. In the definition of an R-spindle convex set, the radius R closed circular discs play the role of closed half-planes. Thus, formally, the $R = \infty$ case corresponds to linear convexity.

Early investigations of spindle convex sets were done, for example, by Blanc [8], Buter [10], Pasqualini [26], Santaló [30], van der Corput [34], Vincensini [35]. For a short survey of the early history of the subject and references see the paper by Danzer, Grünbaum and Klee [11]. Fejes Tóth proved packing and covering theorems for R-spindle convex discs in [15] and [16]. More recently, Bezdek et al. [6] and Kupitz et al. [20], [21] investigated spindle convex sets and proved numerous results about them, many of which are analogous to those of linearly convex sets. They also considered higher dimensional R-spindle convex sets. Intersections of a finite number of radius R closed balls in $\mathbf{E}^d$ are called *ball-polyhedra* (cf. [6]). Such objects played important roles in the proofs of various results in the last 50 years, for a list see Bezdek et al. [6]. Fodor and Vígh [17] proved asymptotic formulas for best approximations of R-spindle convex discs by R-disc-polygons generalizing some of the corresponding results of Fejes Tóth [14] and McClure and Vitale [25] about best approximations of linearly convex discs by convex polygons. There is a wealth of new information about properties of spindle convex bodies and ball-polyhedra in the recent monographs [4] and [5] by Bezdek.

The notion of spindle convexity is related to diametrical completeness of convex bodies through the so-called spherical intersection property. A convex body K is diametrically complete if for any point $x \notin K$, the diameter of $\mathrm{conv}(K \cup \{x\})$ is strictly larger than that of K. It was proved by Eggleston [13] that in a Banach space exactly those convex bodies are diametrically complete which have the so-called spherical intersection property, that is, they are equal to the intersection of all closed balls whose centre is contained in K and whose radius is equal to the diameter of K. In Euclidean spaces diametrically complete convex bodies are exactly those of constant width, however, in Minkowski spaces this is not the case. Recently, much effort has been expended to investigating the properties of diametrically complete sets in Minkowski spaces where sets that are intersections of congruent closed balls play a fundamental role (see, for example, Moreno and Schneider [24] and the references therein), and to investigating various properties of the ball hull, see, for example, Moreno and Schneider [23] for more information.

Random approximations of R-spindle convex sets by R-disc-polygons naturally appear, for example, in the so-called Diminishing Process of Tóth, see Ambrus et al. [1]. Let $D_0 = B_R$ be the radius R closed circular disc in $\mathbf{E}^2$ centred at the origin. Define the random process (D_n, p_n) for $n \geq 1$ as follows. Let p_{n+1} be a uniform random point in D_n and let $D_{n+1} = D_n \cap (B_R + p_{n+1})$. Then each D_n is a (non-uniform random) R-disc-polygon, and the process converges (in the Hausdorff metric of compact sets) to a set of constant width R with probability 1. This process can be readily generalized for a general convex body $K \subset \mathbf{E}^d$, in place of B_R, that contains the origin. If the body K is symmetric with respect to the origin, then it determines a Minkowski metric and the sets K_n are all (random) spindle convex bodies with respect to K in this Minkowski space.

Finally, we remark that there are various terms used for R-spindle convex sets in the literature. Mayer introduced the word "Überkonvexität" in [22]. Authors of early articles from the 1930s and 1940s used the translations of Mayer's term. Fejes Tóth in [15], [16] called such sets "R-convex". Bezdek et al. [6] and Kupitz et al. [20, 21] used the expression "spindle convex". The notion of spindle convexity arose naturally and was investigated from different points of view, which explains the various names used for these sets and it also indicates their importance.

The main results of this article are described in the following theorems.

Theorem 1. *Let $R > 0$, and let S be an R-spindle convex disc with C^2 smooth boundary and with the property that $\kappa(x) > 1/R$ for all $x \in \partial S$. Then*

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n^R)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(S)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial S} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x, \tag{4}$$

and

$$\lim_{n\to\infty} \mathbb{E}(A(S \setminus S_n^R)) \cdot n^{2/3} = \sqrt[3]{\frac{2A(S)^2}{3}} \Gamma\left(\frac{5}{3}\right) \int_{\partial S} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x. \tag{5}$$

We note that the two statements are connected with an Efron-type relation [12], see (31) in Section 5.

Theorem 2. *Let $R > 0$, and let S be an R-spindle convex disc with C^5 smooth boundary and with the property that $\kappa(x) > 1/R$ for all $x \in \partial S$. Then*

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(S) - \mathrm{Per}(S_n^R)) \cdot n^{2/3} \\ &\qquad = \frac{(12A(S))^{2/3}}{36} \Gamma\left(\frac{2}{3}\right) \int_{\partial S} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \left(3\kappa(x) + \frac{1}{R}\right) \mathrm{d}x. \end{aligned} \tag{6}$$

Theorem 3. *Let $R > 0$, and let $S = B_R$ be a circular disc of radius R. Then*

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n^R)) = \frac{\pi^2}{2}, \tag{7}$$

$$\lim_{n\to\infty} \mathbb{E}(A(B_R \setminus S_n^R)) \cdot n = \frac{R^2 \cdot \pi^3}{2}, \tag{8}$$

and

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(B_R) - \mathrm{Per}(S_n^R)) \cdot n = \frac{R \cdot \pi^3}{2}. \tag{9}$$

It is somewhat surprising that the expectation of the number of the vertices of uniform random spindle convex polygons in circular discs tends to a (very small) constant. Roughly speaking this means that after choosing many random points from a circle, the spindle convex hull will have about 5 vertices. Note that this phenomenon has no analogue in linear convexity.

Furthermore, for a (linearly) convex disc K with C^2 smooth boundary and strictly positive curvature, the asymptotic formulas (1) and (2) of Rényi and Sulanke follow from (4) and (5), respectively. Similarly, for a convex disc with C^5 smooth boundary and strictly positive curvature, the asymptotic formula (3) of Rényi and Sulanke follows from (6). Thus, the results of Theorems 1 and 2 are generalizations of the corresponding results of Rényi and Sulanke.

The rest of the paper is organized as follows. In Section 2, we introduce the necessary notations. In Section 3, we prove how the asymptotic formulas of Rényi and Sulanke follow from our results. In Section 4, we investigate some properties of disc-caps of spindle convex discs that are used in the subsequent arguments. We give the proofs of Theorem 1 and Theorem 2 in Section 5. Finally, in Section 6, we provide an outline of the proof of Theorem 3.

2. Definitions and notation

In this paper we work in the Euclidean plane $\mathbf{E}^2$. We denote points of $\mathbf{E}^2$ by lowercase letters and sets of points by capitals, unless otherwise noted. For a point set $X \subset \mathbf{E}^2$, we write $\mathrm{cl}\, X$ for the closure of X, $\mathrm{int}\, X$ for the interior of X, X^C for the complement set of X, and ∂X for the boundary of X. We use the notation $A(\cdot)$ and $\mathrm{Per}(\cdot)$ for the area and perimeter of compact sets in $\mathbf{E}^2$, respectively, while $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product in $\mathbf{E}^2$. The symbol B_R denotes the closed circular disc of radius R centred at the origin. We use S_R^1 to denote ∂B_R. We tacitly assume that the plane is embedded in $\mathbf{E}^3$ and write $x \times y$ for the cross product of the vectors x and y. For two functions $f(n)$ and $g(n)$, we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. We also use the $O(\cdot)$ and $o(\cdot)$ notations throughout the article.

We say that the boundary of a convex disc K is C^k smooth if it is a k-times continuously differentiable simple closed curve in $\mathbf{E}^2$. We use the notation $\kappa(x)$ for the curvature of ∂K at x. If the boundary of K is C^2 smooth, then at every $x \in \partial K$ there exists a unique outer unit normal vector $u_x \in S^1$ to ∂K.

For a convex disc K, integration on the boundary of K with respect to the one-dimensional Hausdorff measure (the arc-length of ∂K) is denoted by $\int_{\partial K} \cdots \mathrm{d}x$. In the case that the boundary of K is C^2 smooth and $f(u)$ is a measurable function on S^1, $\int_{S^1} f(u)\mathrm{d}u = \int_{\partial K} f(u_x)\kappa(x)\mathrm{d}x$, (cf. formula 2.5.30 in [31]).

Let $x, y \in \mathbf{E}^2$ be such that their distance does not exceed $2R$. We define the *closed R-spindle* $[x, y]_{s,R}$ of x and y as the intersection of all closed circular discs of radius R that contain both x and y. The closed R-spindle of two points whose distance is greater than $2R$ is defined to be the whole plane $\mathbf{E}^2$. The shape of the closed spindle of two points whose distance is less than $2R$ resembles a convex lens or a spindle, which explains its name. A set $S \subseteq \mathbf{E}^2$ is called *R-spindle convex* if from $x, y \in S$ it follows that $[x, y]_{s,R} \subseteq S$. Spindle convex sets are also convex in the usual linear sense. In this paper we restrict our attention to compact spindle convex sets. We call a compact set $S \subset \mathbf{E}^2$ with nonempty interior an *R-spindle convex disc* if it has the R-spindle convex property.

Below, we list those properties of spindle convex discs that are directly relevant to our arguments. For more detailed information about spindle convexity we refer to Bezdek et al. [6].

A compact convex set S is R-spindle convex if and only if it is the intersection of (not necessarily finitely many) congruent closed circular discs of radius R (cf. Corollary 3.4 on page 205 in [6]). If the closed circular disc $B_R + p$ contains an R-spindle convex disc S and there is a point $x \in \partial S$ such that also $x \in \partial B_R + p$, then we say that $B_R + p$ *supports* S *at* x. Let P be a convex R-disc-polygon, and $B_R + p$ a circle supporting P at the set $H = \partial P \cap (\partial B_R + p)$. Then H either consists of only one point, called a *vertex*, or it consists of the points of a closed circular arc, called a *side (or edge)* of P. It is clear that the number of edges of P equals the number of vertices of P (except in the case that P is a circle of radius R); we denote this number by $f_0(P)$.

If S is an R-spindle convex disc with C^2 smooth boundary, then $\kappa(x) \geq 1/R$ for all $x \in \partial S$, and for every unit vector $u \in S^1$, there exists a unique point $x \in \partial S$ such that $u = u_x$; we denote this point by x_u. We also note that if $x \in \partial S$, then $B_R + x - R \cdot u_x$ supports S at x.

3. The limiting case

In this section we show how Theorems 1 and 2 imply the asymptotic formulas (1), (2), and (3) of Rényi and Sulanke.

Let K be a (linearly) convex disc with C^2 smooth boundary and $\kappa(x) > 0$ for all $x \in \partial K$. Let $\kappa_{\min} = \min_{\partial K} \kappa(x) > 0$. It follows from Mayer's results (cf. (Ü4) and (Ü5) on page 521 in [22], or for a more recent and more general reference see also Theorem 2.5.4. in [31]) that K is R-spindle convex for all $R \geq R_0 = 1/\kappa_{\min}$. For $R \geq R_0$ and n sufficiently large, we introduce the following notation

$$\begin{aligned}
\delta_S^R(n) &= \mathbb{E}(A(K \setminus S_n^R)) \cdot n^{\frac{2}{3}}, \\
\delta(n) &= \mathbb{E}(A(K \setminus K_n)) \cdot n^{\frac{2}{3}}, \\
I_S^R &= \sqrt[3]{\frac{2A^2}{3}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{\frac{1}{3}} \mathrm{d}x, \\
I &= \sqrt[3]{\frac{2A^2}{3}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \kappa^{\frac{1}{3}}(x) \mathrm{d}x,
\end{aligned}$$

with $A = A(K)$.

We claim that (5) implies the asymptotic formula (2) of Rényi and Sulanke.

Let $\varepsilon > 0$ be fixed. Then $\lim_{R\to\infty} I_S^R = I$ yields that there exists $R_1(\varepsilon) > R_0$ such that

$$1 - \varepsilon < \frac{I_S^R}{I} < 1 + \varepsilon \tag{10}$$

for all $R > R_1(\varepsilon)$.

Elementary calculations show that there exists $R_2(\varepsilon) \geq R_0$, depending only on K and ε such that for all $R > R_2(\varepsilon)$,

$$\frac{A([p,q]_{s,R})}{A([p,q]_{s,R_0}) - A([p,q]_{s,R})} < \varepsilon, \tag{11}$$

for any points $p, q \in K$.

Let D_m^R denote an R-disc-polygon in K with vertices $p_1, \ldots, p_m$ indexed in the cyclic order, and let P_m denote the (linear) convex hull of $p_1, \ldots, p_m$. Note that this is a polygon with vertices $p_1, \ldots, p_m$. If $R > R_2(\varepsilon)$, then (11) yields

$$\begin{aligned} 1 < \frac{\delta(n)}{\delta_S^R(n)} &= 1 + \frac{\mathbb{E}(A(S_n^R) - A(K_n))}{\mathbb{E}(A(K) - A(S_n^R))} \\ &< 1 + \sup_{\substack{D_m^R \subset K,\\ 2\leq m\leq n}} \frac{A(D_m^R) - A(P_m)}{A(D_m^{R_0}) - A(D_m^R)} < 1 + \varepsilon. \end{aligned} \tag{12}$$

Now assume that $R > \max\{R_1(\varepsilon), R_2(\varepsilon)\}$. It is clear that for any such R, the convergence $\lim_{n\to\infty} \delta_S^R(n)/I_S^R = 1$ yields that there exists $n(R)$ such that

$$1 - \varepsilon < \frac{\delta_S^R(n)}{I_S^R} < 1 + \varepsilon \tag{13}$$

for all $n \geq n(R)$.

Thus, from (10), (12), (13), and from

$$\frac{\delta(n)}{I} = \frac{\delta(n)}{\delta_S^R(n)} \cdot \frac{\delta_S^R(n)}{I_S^R} \cdot \frac{I_S^R}{I},$$

we obtain that

$$1 - 3\varepsilon < \frac{\delta(n)}{I} < 1 + 7\varepsilon$$

for all $R > \max\{R_1(\varepsilon), R_2(\varepsilon)\}$ and $n > n(R)$, which proves that

$$\lim_{n\to\infty} \frac{\delta(n)}{I} = 1.$$

A similar argument shows that (6) implies the asymptotic formula (3) of Rényi and Sulanke. Finally, formula (1) for the number of vertices follows by Efron's equality (31).

4. Caps of spindle convex discs

From now on we restrict our attention to the case when $R = 1$ and we omit R from the notation. We use the simpler terms *spindle convex* and *disc-polygon* in place of 1-*spindle convex* and 1-*disc polygon*, respectively. In particular, $B = B_1$ denotes the unit disc. The R-spindle convex analogues of the following lemmas can be obtained by simple scaling.

Let S be a spindle convex disc with C^2 smooth boundary and assume that $\kappa(x) > 1$ for all $x \in \partial S$. A subset D of S is a *disc-cap of S* if $D = \mathrm{cl}\,(S \cap (B+p)^C)$ for some point $p \in \mathbf{E}^2$. Note that in this case $\partial B + p$ intersects ∂S in at most two points. (This follows, for example, from Theorem 2.5.4. in [31].) Thus, the boundary of a nonempty disc-cap D consists of at most two connected arcs: one arc is a subset of ∂S, and the other arc is a subset of $\partial B + p$. In order to define the vertex and the outer normal of a disc-cap we need the following claim.

Lemma 1. *Let S be a spindle convex disc with C^2 smooth boundary and assume that $\kappa(x) > 1$ for all $x \in \partial S$. Let $D = \mathrm{cl}(S \cap (B+p)^C)$ be a non-empty disc-cap of S (as above). Then there exists a unique point $x_0 \in \partial S \cap \partial D$ such that there exists a $t \geq 0$ with $B + p = B + x_0 - (1+t)u_{x_0}$. We refer to x_0 as the* vertex *of D and to t as the* height *of D.*

Proof. Pick any $x \in \partial S \cap \partial D$, and consider the vectors $\overrightarrow{px}$ and the outer unit normal u_x. We claim that there is a unique x for which $\overrightarrow{px}$ is a positive multiple of u_x. The existence follows from a simple continuity argument since the angles formed by the two vectors have different orientations at the endpoints of $\partial S \cap \partial D$. Uniqueness is proved as follows. Suppose that both $x_1 \neq x_2$ fulfil the requirements. Let φ be the (positive) angle between u_{x_1} and u_{x_2} and denote by I the arc of ∂S between x_1 and x_2 (according to the positive orientation), and by Δs the length of I. By the spindle convexity of S, we obtain that x_1 and x_2 can be joined by a unit circular arc in S. The length of this circular arc is clearly smaller then Δs, on the other hand it is larger than φ, and thus $\Delta s > \varphi$. Using the assumption that the curvature of ∂S is strictly larger than 1, we obtain that

$$\varphi = \int_I \kappa(s)\mathrm{d}s > \int_I \mathrm{d}s = \Delta s > \varphi,$$

a contradiction. □

Let $D(u,t)$ denote the disc-cap with vertex $x_u \in \partial S$ and height t. Note that for each $u \in S^1$, there exists a maximal positive constant $t^*(u)$ such that $(B + x_u - (1+t)u) \cap S \neq \emptyset$ for all $t \in [0, t^*(u)]$. Let $V(u,t) = A(D(u,t))$ and let $\ell(u,t)$ denote the arc-length of $\partial D(u,t) \cap (\partial B + x_u - (1+t)u)$.

Lemma 2. *Let S be a spindle convex disc with C^2 boundary such that $\kappa(x) > 1$ for all $x \in \partial S$. Then for a fixed $x \in \partial S$, the following hold*

$$\lim_{t \to 0^+} \ell(u_x, t) \cdot t^{-1/2} = 2\sqrt{\frac{2}{\kappa(x) - 1}}, \tag{14}$$

and

$$\lim_{t \to 0^+} V(u_x, t) \cdot t^{-3/2} = \frac{4}{3}\sqrt{\frac{2}{\kappa(x) - 1}}. \tag{15}$$

Proof. Assume that $x = (0,0)$ and $u_x = (0,-1)$. Then, in a sufficiently small open neighbourhood of the origin, ∂S is the graph of a C^2 smooth function $f(\sigma)$. Taylor's theorem yields that

$$f(\sigma) = \frac{\kappa(x)}{2}\sigma^2 + o(\sigma^2), \text{ as } \quad \sigma \to 0. \tag{16}$$

In the same open neighbourhood of the origin, the boundary of $B + x - (1+t)u_x$ is the graph of the function $g_t(\sigma) = t + 1 - \sqrt{1 - \sigma^2}$. Simple calculation yields that the positive solution of the equation $g_t(\sigma) = f(\sigma)$ is

$$\sigma_+ = \sqrt{\frac{2}{\kappa(x)-1}} \cdot t^{1/2} + o(t^{1/2}), \text{ as } \quad t \to 0^+.$$

Clearly, $\ell(u_x, t) \sim 2\sigma_+$ as $t \to 0^+$ by the fact that the ratio of the lengths of an arc and the corresponding chord tends to 1 as the length of the arc tends to 0.

Let σ_- denote the negative solution of the equation $g_t(\sigma) = f(\sigma)$. Then

$$\begin{aligned} V(u_x, t) &= \int_{\sigma_-}^{\sigma_+} g_t(\sigma) - f(\sigma) \mathrm{d}\sigma \\ &= 2\int_0^{\sigma_+} \left[t + \frac{\sigma^2}{2} - \frac{\kappa(u_x)}{2}\sigma^2 + o(\sigma^2) \right] \mathrm{d}\sigma \\ &= \frac{4}{3}\sqrt{\frac{2}{\kappa(x)-1}} \cdot t^{3/2} + o(t^{3/2}), \text{ as } \quad t \to 0^+. \end{aligned}$$

This finishes the proof of Lemma 2. □

Let $x_1, x_2 \in S$ be two distinct points. Then there are exactly two disc-caps of S, say $D_-(x_1, x_2) = \operatorname{cl}(S \cap (B + p_-)^C)$ and $D_+(x_1, x_2) = \operatorname{cl}(S \cap (B + p_+)^C)$ with the property that $x_1, x_2 \in \partial B + p_-$ and $x_1, x_2 \in \partial B + p_+$. Let $V_-(x_1, x_2) = A(D_-(x_1, x_2))$ and $V_+(x_1, x_2) = A(D_+(x_1, x_2))$, respectively, and assume that $V_-(x_1, x_2) \le V_+(x_1, x_2)$.

Lemma 3. *Let S be a spindle convex disc with C^2 boundary and $\kappa(x) > 1$ for all $x \in \partial S$. Then there exists a constant $\delta > 0$, depending only on S, such that $V_+(x_1, x_2) > \delta$ for any two distinct points $x_1, x_2 \in S$.*

Proof. We note that $[x_1, x_2]_s$ cannot cover S because of the C^2 smoothness of ∂S and the assumption that $\kappa(x) > 1$ for all $x \in \partial S$. Thus, by compactness, there exists a constant $\delta > 0$, depending only on S, such that $A(S \setminus [x_1, x_2]_s) > 2\delta$ for any two distinct points $x_1, x_2 \in S$. Now, the statement of the lemma readily follows from the fact that $S = D_-(x_1, x_2) \cup D_+(x_1, x_2) \cup [x_1, x_2]_s$. □

Let K be a convex disc with C^2 boundary and with the property that $\kappa(x) > 0$ for all $x \in \partial K$. Let $\kappa_0 > 0$ denote the minimum of the curvature of ∂K. Then there exists an $\varepsilon_0 > 0$, depending only on K, with the property that for any $x \in \partial K$ the (unique) circle of radius $1/\kappa_0$ that is tangent to ∂K at x supports K in a neighbourhood of radius ε_0 of x. Moreover, Mayer proved (see statement (Ü5) on page 521 in [22], or for a more recent and more general reference see also Theorem 2.5.4. in [31]) that in this case the tangent circles of radius $1/\kappa_0$ of ∂K not only locally support K but also contain K and thus they globally support K.

Let S be a spindle convex disc with C^2 smooth boundary and with the property that $\kappa(x) > 1$ for all $x \in \partial K$. Then, by the above, there exists $0 < \hat{\varrho} < 1$, depending only on S, such that S has a supporting circular disc of radius $\hat{\varrho}$ at each $x \in \partial S$. Thus, Lemma 2 yields that there exists a $0 < t_0 \le \hat{\varrho}$ with the property that for any $u \in S^1$

$$\ell(u, t) \le 4\sqrt{\frac{2\hat{\varrho}}{1-\hat{\varrho}}}\, t^{\frac{1}{2}} \quad \text{for } t \in [0, t_0]. \tag{17}$$

A convex disc K has a *rolling ball* if there exists a real number $\varrho > 0$ with the property that any $x \in \partial K$ lies in some closed circular disc of radius ϱ contained in K. Hug proved in [19] that the existence of a rolling ball is equivalent to the exterior unit normal being a Lipschitz function on ∂K. This implies that if the boundary of K is C^2 smooth, then K has a rolling ball. We remark that this last fact was already observed by Blaschke [7].

It follows from the assumption that the boundary of S is C^2 smooth that there exists a rolling ball for S with radius $0 < \varrho < 1$. The existence of the rolling ball and (15) yield that there exists $0 < \hat{t} < \varrho$ such that for any $u \in S^1$

$$V(u,t) \geq \frac{1}{2}\left(\frac{4}{3}\sqrt{\frac{2\varrho}{1-\varrho}}\right) t^{\frac{3}{2}} \quad \text{for } t \in [0,\hat{t}]. \tag{18}$$

Note that although the statements in Lemma 2 are not uniform in u, both (17) and (18) are uniform in u.

5. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. We essentially use the method invented by Rényi and Sulanke [27]. Note that it is enough to prove the theorem for $R = 1$, from that the statement follows by a scaling argument. Thus, from now on we assume that $R = 1$, and omit R from the notation.

Let $A = A(S)$. First, observe that the pair of random points x_1, x_2 determine an edge of S_n if and only if at least one of the disc-caps $D_-(x_1,x_2)$ and $D_+(x_1,x_2)$ does not contain any other points from X_n. Thus

$$\mathbb{E}(f_0(S_n)) = \binom{n}{2} W_n,$$

where

$$W_n = \frac{1}{A^2}\int_S\int_S \left[\left(1 - \frac{V_-(x_1,x_2)}{A}\right)^{n-2} + \left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2}\right] \mathrm{d}x_1 \mathrm{d}x_2. \tag{19}$$

Note that if all points of X_n fall into the closed spindle spanned by x_1 and x_2, then x_1 and x_2 contribute two edges to S_n (since in this case $\operatorname{conv}_s X_n = [x_1,x_2]_S$), and accordingly this event is counted in both terms in the integrand of (19).

Lemma 3 yields that

$$\begin{aligned}
&\lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} \frac{1}{A^2}\int_S\int_S \left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2} \mathrm{d}x_1 \mathrm{d}x_2 \\
&\leq \lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} \frac{1}{A^2}\int_S\int_S e^{-\frac{\delta}{A}(n-2)} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} e^{-\frac{\delta}{A}(n-2)} = 0.
\end{aligned}$$

Thus, the contribution of the second term of (19) is negligible, hence, in what follows, we will consider only the first term. Note that a similar argument yields that in the first term of (19) it is enough to integrate over pairs of random points x_1, x_2 such that $V_-(x_1,x_2) < \delta$. Let $\mathbf{1}(\cdot)$ denote the indicator function of an event.

Then

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n))n^{-\frac{1}{3}} \tag{20}$$
$$= \lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} \frac{1}{A^2} \int_S \int_S \left(1 - \frac{V_-(x_1, x_2)}{A}\right)^{n-2} \mathbf{1}(V_-(x_1, x_2) < \delta) \mathrm{d}x_1 \mathrm{d}x_2.$$

Now, we re-parametrize the pair (x_1, x_2) as follows. Let

$$(x_1, x_2) = \Phi(u, t, u_1, u_2), \tag{21}$$

where $u, u_1, u_2 \in S^1$ and $0 \le t \le t_0(u)$ are chosen such that

$$D(u, t) = D_-(x_1, x_2),$$

and

$$(x_1, x_2) = (x_u - (1 + t)u + u_1, x_u - (1 + t)u + u_2).$$

Note that u_1 and u_2 are the unique outer unit normal vectors of $\partial B + x_u - (1+t)u$ at x_1 and x_2, respectively. This yields that, for fixed u and t, both u_1 and u_2 are in the same arc of length $\ell(u, t)$ in S^1. We denote this unit circular arc by $L(u, t)$.

Note that since $V_-(x_1, x_2) < \delta$, $D_-(x_1, x_2)$ is uniquely determined by Lemma 3. Now, the uniqueness of the vertex and height of a disc-cap guarantees that Φ is well-defined, bijective, and differentiable (see the Appendix) on a suitable domain of (u, t, u_1, u_2). To continue the estimate of W_n we need the Jacobian of the transformation Φ. This calculation can be found in Santaló's paper [30], but for the sake of completeness, we give a sketch in the Appendix.

We obtain that the Jacobian of Φ satisfies

$$|J\Phi| = \left(1 + t - \frac{1}{\kappa(x_u)}\right) |u_1 \times u_2|. \tag{22}$$

We note that $|u_1 \times u_2|$ equals the sine of the length of the unit circular arc between x_1 and x_2 on the boundary of $D(u, t)$. Also note that there exists $t_1 > 0$ with the property that $V(u, t) < \delta$ for all $0 \le t \le t_1$ and for all $u \in S^1$.

Now, (20) and (22) yield that

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n))n^{-\frac{1}{3}} \tag{23}$$
$$= \lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} \frac{1}{A^2} \int_{S^1} \int_0^{t_1} \int_{L(u,t)} \int_{L(u,t)} \left(1 - \frac{V(u, t)}{A}\right)^{n-2}$$
$$\times \left(1 + t - \frac{1}{\kappa(x_u)}\right) |u_1 \times u_2| \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}t \mathrm{d}u.$$

Integration by u_1 and u_2 yields

$$(23) = \lim_{n\to\infty} n^{-\frac{1}{3}} \binom{n}{2} \frac{2}{A^2} \int_{S^1} \int_0^{t_1} \left(1 - \frac{V(u, t)}{A}\right)^{n-2}$$
$$\times \left(1 + t - \frac{1}{\kappa(x_u)}\right) (\ell(u, t) - \sin \ell(u, t)) \mathrm{d}t \mathrm{d}u.$$

Now, we will split the domain of integration with respect to t into two parts. Let $h(n) = (c \ln n/n)^{2/3}$, where c is a positive (absolute) constant to be specified later. From (18) it follows that there exists $n_0 \in \mathbb{N}$ and $\gamma_1 > 0$, depending only on S, such that if $n > n_0$, then $h(n) < t_1$, and $V(u, t) > \gamma_1 \cdot h(n)^{3/2}$ for all $h(n) \le t \le t_1$ and for all $u \in S^1$.

Lemma 4. *Let $h(n)$ be defined as above. Then*

$$\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}\frac{2}{A^2}\int_{S^1}\int_{h(n)}^{t_1}\left(1-\frac{V(u,t)}{A}\right)^{n-2} \times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t\mathrm{d}u = 0.$$

Proof. Note that $t_1 \le 2\pi$, and there exists a universal constant $\gamma_2 > 0$ such that $\ell(u,t)-\sin\ell(u,t) \le \gamma_2$ for all $0 \le t \le t_1$ and $u \in S^1$. Hence, for any fixed $u \in S^1$ and any $n > n_0$, it holds that

$$\begin{aligned}
&\int_{h(n)}^{t_1}\left(1-\frac{V(u,t)}{A}\right)^{n-2}\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t\\
&\le 3\gamma_2\int_{h(n)}^{t_1}\left(1-\frac{\gamma_1 h(n)^{3/2}}{A}\right)^{n-2}\mathrm{d}t\\
&\le 3\gamma_2\int_{0}^{t_1}\left(1-\frac{\gamma_1 c(\ln n/n)}{A}\right)^{n-2}\mathrm{d}t\\
&\le 6\gamma_2 n^{-\frac{c\gamma_1}{A}}.
\end{aligned}$$

Now, let $c > 5A/(3\gamma_1)$. Then

$$\begin{aligned}
&\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}\frac{2}{A^2}\int_{S^1}\int_{h(n)}^{t_1}\left(1-\frac{V(u,t)}{A}\right)^{n-2}\\
&\quad\times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t\mathrm{d}u\\
&\qquad\le \gamma_2\frac{24\pi}{A^2}\lim_{n\to\infty} n^{-\frac{1}{3}}\binom{n}{2}n^{-\frac{c\gamma_1}{A}} = 0.
\end{aligned}$$

□

Now, for $n > n_0$ we define

$$\theta_n(u) = n^{-\frac{1}{3}}\binom{n}{2}\int_0^{h(n)}\left(1-\frac{V(u,t)}{A}\right)^{n-2} \times\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t))\mathrm{d}t \tag{24}$$

and so

$$\lim_{n\to\infty}\mathbb{E}(f_0(S_n))\cdot n^{-\frac{1}{3}} = \lim_{n\to\infty}\frac{2}{A^2}\int_{S^1}\theta_n(u)\,\mathrm{d}u. \tag{25}$$

We recall formula (11) from [9] that states the following. For any $\beta \ge 0$, $\omega > 0$ and $\alpha > 0$ we have that

$$\int_0^{g(n)} t^\beta\left(1-\omega t^\alpha\right)^n\mathrm{d}t \sim \frac{1}{\alpha\omega^{\frac{\beta+1}{\alpha}}}\cdot\Gamma\left(\frac{\beta+1}{\alpha}\right)\cdot n^{-\frac{\beta+1}{\alpha}}, \tag{26}$$

as $n\to\infty$, assuming

$$\left(\frac{(\beta+\alpha+1)\ln n}{\alpha\omega n}\right)^{\frac{1}{\alpha}} < g(n) < \omega^{-\frac{1}{\alpha}},$$

for sufficiently large n.

Formula (17) implies that there exists $\gamma_3 > 0$ such that $\ell(u,t) - \sin\ell(u,t) < \gamma_3 t^{3/2}$ for all $0 < t < t_0$ and $u \in S^1$. We recall that $1 + t - 1/\kappa(x_u) < 3$ for all $u \in S^1$ and $0 \le t \le t_1$. Now (18) and (26) with $\alpha = \beta = 3/2$ and $\omega = (2/(3A))\sqrt{2\rho/(1-\rho)}$ yield that there exists $\gamma_4 > 0$, depending only on S, such that $\theta_n(u) < \gamma_4$ for all $u \in S^1$ and sufficiently large n. Thus, Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}(f_0(S_n)) \cdot n^{-\frac{1}{3}} = \frac{2}{A^2} \int_{S^1} \lim_{n\to\infty} \theta_n(u)\,\mathrm{d}u. \tag{27}$$

Let $u \in S^1$ and $\varepsilon \in (0,1)$. It follows from Lemma 2 that there exists $0 < t_\varepsilon < t_1$ such that

$$(1-\varepsilon)\frac{4}{3}\left(\frac{2}{\kappa(x_u)-1}\right)^{\frac{3}{2}} t^{\frac{3}{2}} \le \ell(u,t) - \sin\ell(u,t) \le (1+\varepsilon)\frac{4}{3}\left(\frac{2}{\kappa(x_u)-1}\right)^{\frac{3}{2}} t^{\frac{3}{2}} \tag{28}$$

and

$$(1-\varepsilon)\frac{4}{3}\sqrt{\frac{2}{\kappa(x_u)-1}}\, t^{\frac{3}{2}} \le V(u,t) \le (1+\varepsilon)\frac{4}{3}\sqrt{\frac{2}{\kappa(x_u)-1}}\, t^{\frac{3}{2}}, \tag{29}$$

for any $t \in (0, t_\varepsilon)$.

Now (28) and (29) yield that

$$\begin{aligned} \lim_{n\to\infty} \theta_n(u) = {} & \frac{4\sqrt{2}}{3}\left(\frac{1}{\kappa(x_u)-1}\right)^{\frac{3}{2}} \\ & \times \left[\frac{\kappa(x_u)-1}{\kappa(x_u)} \lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\, t^{\frac{3}{2}}\right)^{n-2} t^{\frac{3}{2}}\,\mathrm{d}t \right. \\ & \left. \quad + \lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\, t^{\frac{3}{2}}\right)^{n-2} t^{\frac{5}{2}}\,\mathrm{d}t \right]. \end{aligned} \tag{30}$$

Note that (26) with $\alpha = 3/2$, $\beta = 5/2$ implies that the second term of (30) is 0. Now, (26) yields that

$$\begin{aligned} \lim_{n\to\infty} n^{\frac{5}{3}} \int_0^{h(n)} & \left(1 - \frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\, t^{\frac{3}{2}}\right)^{n-2} t^{\frac{3}{2}}\,\mathrm{d}t \\ & = \frac{2}{3}\left(\frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\right)^{-\frac{5}{3}} \Gamma\left(\frac{5}{3}\right). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \theta_n(u) = \frac{8\sqrt{2}}{9}\left(\frac{1}{\kappa(x_u)-1}\right)^{\frac{3}{2}} \frac{\kappa(x_u)-1}{\kappa(x_u)} \left(\frac{4}{3A}\sqrt{\frac{2}{\kappa(x_u)-1}}\right)^{-\frac{5}{3}} \Gamma\left(\frac{5}{3}\right).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E} f_0(S_n)\cdot n^{-\frac{1}{3}} &= \frac{2}{A^2}\int_{S^1}\lim_{n\to\infty}\theta_n(u)\mathrm{d}u\\ &= \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right)\int_{S^1}\frac{1}{\kappa(x_u)}\left(\kappa(x_u)-1\right)^{\frac{1}{3}}\mathrm{d}u\\ &= \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right)\int_{\partial S}\left(\kappa(x)-1\right)^{\frac{1}{3}}\mathrm{d}x.\end{aligned}$$

To compute the expectation of the missed area by S_n, we use the following identity

$$\mathbb{E}(f_0(S_n)) = \frac{n\mathbb{E}(A(S\setminus S_{n-1}))}{A}. \tag{31}$$

(31) is the spindle convex analogue of Efron's identity [12]. The proof of (31) is as follows.

$$\begin{aligned}\mathbb{E}(f_0(S_n)) = \sum_1^n \mathbb{P}(x_i \text{ is a vertex of } S_n) &= n\mathbb{P}(x_1 \text{ is a vertex of } S_n)\\ &= n\mathbb{P}(x_1\notin \mathrm{conv_s}\,(x_2,\dots,x_n)) = \frac{n\mathbb{E}(A(S\setminus S_{n-1}))}{A}\end{aligned}$$

Now, combining (4) and (31) yields (5), thus completing the proof of Theorem 1. □

Now we turn to the proof of Theorem 2. The argument is based on ideas developed by Rényi and Sulanke in [28], and it is similar to the argument of the proof of Theorem 1.

We start with a refinement of Lemma 2 under the hypothesis that the boundary of S is C^5 smooth and that $\kappa(x) > 1$ for all $x\in\partial S$.

Lemma 5. *Let S be a spindle convex disc with C^5 smooth boundary and with the property that $\kappa(x)>1$ for all $x\in\partial S$. Then uniformly in $u\in S^1$*

$$\ell(u,t) = l_1t^{1/2} + l_2t^{3/2} + O(t^{5/2}) \quad \text{as } t\to 0^+, \text{ and} \tag{32}$$

$$V(u,t) = v_1t^{3/2} + v_2t^{5/2} + O(t^{7/2}) \quad \text{as } t\to 0^+, \tag{33}$$

with

$$\begin{aligned}l_1 = l_1(u) &= 2\sqrt{\frac{2}{\kappa(x_u)-1}}\\ l_2 = l_2(u) &= \frac{2^{3/2}\left(15b(x_u)^2 - (\kappa(x_u)-1)(1+6(c(x_u)-1/8)-\kappa(x_u))\right)}{3(\kappa(x_u)-1)^{7/2}}\\ v_1 = v_1(u) &= \frac{4}{3}\sqrt{\frac{2}{\kappa(x_u)-1}}\\ v_2 = v_2(u) &= \frac{2^{5/2}\left(5b(x_u)^2 - 2(c(x_u)-1/8)(\kappa(x_u)-1)\right)}{5(\kappa(x_u)-1)^{7/2}},\end{aligned}$$

where $b(x)$ and $c(x)$ are functions depending only on S and x.

Proof. With the same notation and choice of coordinate system as in the proof of Lemma 2, Taylor's theorem and the C^5 smoothness of the boundary yield that in a sufficiently small neighbourhood of the origin

$$f(\sigma) = \frac{\kappa}{2}\sigma^2 + b\sigma^3 + c\sigma^4 + O(\sigma^5) \quad \text{as } \sigma \to 0,$$

uniformly in $u \in S^1$. We suppress the notation of dependence of the coefficients on u for brevity. Let $g_t(\sigma) = t + 1 - \sqrt{1-\sigma^2}$. From the equation $f(\sigma) = g_t(\sigma)$ we obtain

$$t = \frac{\kappa - 1}{2}\sigma^2 + b\sigma^3 + \left(c - \frac{1}{8}\right)\sigma^4 + O(\sigma^5) \quad \text{as } \sigma \to 0,$$

and routine calculations yield that the positive and negative solutions of the equation $f(\sigma) = g_t(\sigma)$ are

$$\begin{aligned} \sigma_+ &= \sigma_+(t) = d_1 t^{1/2} + d_2 t + d_3 t^{3/2} + O(t^2) \quad \text{as } t \to 0^+, \\ \sigma_- &= \sigma_-(t) = -(d_1 t^{1/2} - d_2 t + d_3 t^{3/2}) + O(t^2) \quad \text{as } t \to 0^+, \end{aligned} \tag{34}$$

where

$$\begin{aligned} d_1 &= \sqrt{\frac{2}{\kappa - 1}}, \\ d_2 &= -\frac{2b}{(\kappa-1)^2}, \\ d_3 &= \frac{\sqrt{2}\left(5b^2 - 2(c - 1/8)(\kappa - 1)\right)}{(\kappa-1)^{7/2}}. \end{aligned}$$

Now, using that $\ell(u,t) = \arcsin \sigma_+ + \arcsin |\sigma_-|$ and that $V(u,t) = \int_{\sigma_-}^{\sigma_+}[g_t(\sigma) - f(\sigma)]\mathrm{d}\sigma$, a short calculation finishes the proof. □

Proof of Theorem 2. Let $L = \mathrm{Per}(S)$ for brevity. Let $x_1, x_2 \in S$, and let $i(x_1, x_2)$ denote the length of the shorter unit circular arc joining x_1 and x_2. We define U_n with

$$\begin{aligned} &\mathbb{E}(\mathrm{Per}(S) - \mathrm{Per}(S_n)) \\ &= L - \binom{n}{2}\mathbb{E}\left[\mathbf{1}(x_1, x_2 \text{ is an edge of } S_n) \cdot i(x_1, x_2)\right] =: L - \binom{n}{2} U_n. \end{aligned}$$

Using the same notation as in the proof of Theorem 1, similar arguments show that

$$U_n = \frac{1}{A^2}\int_S \int_S \left[\left(1 - \frac{V_-(x_1,x_2)}{A}\right)^{n-2} + \left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2}\right] i(x_1,x_2)\, \mathrm{d}x_1 \mathrm{d}x_2,$$

and

$$\lim_{n\to\infty} n^{2/3}\binom{n}{2}\frac{1}{A^2}\int_S\int_S \left(1 - \frac{V_+(x_1,x_2)}{A}\right)^{n-2} i(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = 0,$$

and also that

$$\begin{aligned} \lim_{n\to\infty} n^{2/3}\binom{n}{2}\frac{1}{A^2}\int_S\int_S &\left(1 - \frac{V_-(x_1,x_2)}{A}\right)^{n-2} \\ &\times \mathbf{1}(V_-(x_1,x_2) > \delta) i(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = 0. \end{aligned}$$

Now, the integral transformation Φ in (21) yields that

$$\begin{aligned}&\frac{1}{A^2}\int_S\int_S\left(1-\frac{V_-(x_1,x_2)}{A}\right)^{n-2}\mathbf{1}(V_-(x_1,x_2)\le\delta)i(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2\\&=\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\int_{L(u,t)}\int_{L(u,t)}\left(1-\frac{V(u,t)}{A}\right)^{n-2}\\&\quad\times\left(1+t-\frac{1}{\kappa(x_u)}\right)\cdot|u_1\times u_2|\arccos\langle u_1,u_2\rangle\mathrm{d}u_1\mathrm{d}u_2\,\mathrm{d}t\,\mathrm{d}u,\end{aligned}$$

where $\arccos\langle u_1,u_2\rangle$ is the length of the arc of S^1 spanned by u_1 and u_2. Routine calculations show that

$$\begin{aligned}&\int_{L(u,t)}\int_{L(u,t)}|u_1\times u_2|\arccos\langle u_1,u_2\rangle\mathrm{d}u_1\mathrm{d}u_2\\&=2\left(2-2\cos\ell(u,t)-\ell(u,t)\sin\ell(u,t)\right).\end{aligned}$$

Let $\varepsilon>0$ be arbitrary. According to Lemma 5 we may choose $t_2>0$ such that for all $t\in(0,t_2)$ and for all $u\in S^1$

$$\begin{aligned}\left|\ell(u,t)-(l_1t^{1/2}+l_2t^{3/2})\right|&\le\frac{\varepsilon}{2}t^{3/2},\\ \left|V(u,t)-(v_1t^{3/2}+v_2t^{5/2})\right|&\le\varepsilon t^{5/2}.\end{aligned}\tag{35}$$

For any $\varepsilon'>0$ for sufficiently small x it holds that

$$\left|2\left(2-2\cos x-x\sin x\right)-\left(\frac{x^4}{6}-\frac{x^6}{90}\right)\right|\le\varepsilon' x^6,$$

which, together with (35), implies that there exists $t_3>0$ with the property that for any $t\in(0,t_3)$ and for all $u\in S^1$

$$\left|2\left(2-2\cos\ell(u,t)-\ell(u,t)\sin\ell(u,t)\right)-\frac{1}{6}\left[l_1^4t^2+\left(4l_1^3l_2-\frac{l_1^6}{15}\right)t^3\right]\right|\le\frac{\varepsilon}{6}t^3.\tag{36}$$

The second order Taylor expansion of the function $\log(1-y)$ at $y=0$ yields that there exists $t_4>0$ such that for $0<y\le n\min_{u\in S^1}v_1(u)t_4^{2/3}/A$ and for any $c\in[-a_1,a_1]$, with $a_1=A^{2/3}\max_{u\in S^1}\left|v_2(u)/v_1^{5/3}(u)\right|$, and for all $u\in S^1$

$$e^{-y}e^{-(c+\varepsilon)y^{5/3}n^{-2/3}}\le\left[1-\frac{y}{n}-c\left(\frac{y}{n}\right)^{5/3}\right]^n\le e^{-y}e^{-cy^{5/3}n^{-2/3}}\tag{37}$$

and

$$e^{-(1+\varepsilon)y}\le\left[1-\frac{y}{n}-c\left(\frac{y}{n}\right)^{5/3}\right]^n\le e^{-(1-\varepsilon)y}.\tag{38}$$

Let $\delta=\delta(\varepsilon)$ be small enough such that for all $|y|\le\delta$

$$e^{-y}\le 1-(1-\varepsilon)y,\tag{39}$$

and let n_0 be large enough such that

$$\max_{u\in S^1}\frac{|v_2(u)|A^{2/3}}{v_1^{5/3}(u)}\le n_0^{1/3}\delta.\tag{40}$$

Finally, let $t' := \min\{t_2, t_3, t_4\}$. A similar argument as in the proof of Lemma 4 yields that

$$\lim_{n\to\infty} n^{2/3} \binom{n}{2} \frac{1}{A^2} \int_{S^1} \int_{t'}^{t_1} \left(1 - \frac{V(u,t)}{A}\right)^{n-2} \times 2\left[2 - 2\cos \ell(u,t) - \ell(u,t)\sin \ell(u,t)\right] \left(t + 1 - \frac{1}{\kappa(x_u)}\right) \mathrm{d}t\, \mathrm{d}u = 0.$$

Thus we need to determine the limit

$$\lim_{n\to\infty} n^{2/3} \left[L - \binom{n}{2} \frac{1}{A^2} \int_{S^1} \int_0^{t'} \left(1 - \frac{V(u,t)}{A}\right)^{n} \times 2\left[2 - 2\cos \ell(u,t) - \ell(u,t)\sin \ell(u,t)\right] \left(t + 1 - \frac{1}{\kappa(x_u)}\right) \mathrm{d}t\, \mathrm{d}u \right].$$

By Lemma 5, for sufficiently small t it holds uniformly in $u \in S^1$ that

$$1 \le \left(1 - \frac{V(u,t)}{A}\right)^{-2} \le 1 + \frac{3 \max_{u\in S^1} v_1(u)}{A} t^{3/2}.$$

Therefore changing the exponent from $n-2$ to n in the inner integral above does not affect either the main or the first order term.

By (35) and (36), we have that

$$\begin{aligned}
\hat\theta_n(u) &:= \frac{1}{A^2} \int_0^{t'} \left(1 - \frac{V(u,t)}{A}\right)^{n} 2\left[2 - 2\cos \ell(u,t) - \ell(u,t)\sin \ell(u,t)\right] \left(t + 1 - \frac{1}{\kappa}\right) \mathrm{d}t \\
&\le \frac{1}{6A^2} \int_0^{t'} \left(1 - \frac{v_1}{A} t^{3/2} - \frac{v_2 - \varepsilon}{A} t^{5/2}\right)^n \\
&\quad \times \left[l_1^4 \left(1 - \frac{1}{\kappa}\right) t^2 + \left(l_1^4 + \left(1 - \frac{1}{\kappa}\right)\left(4 l_1^3 l_2 - \frac{l_1^6}{15}\right) + \varepsilon \right) t^3 \right] \mathrm{d}t.
\end{aligned}$$

To shorten the notation put

(41)
$$D_1 = l_1^4\left(1 - \kappa^{-1}\right), \quad D_1 D_2^\varepsilon = l_1^4 + \left(1 - \kappa^{-1}\right)\left(4 l_1^3 l_2 - l_1^6/15\right) + \varepsilon, \quad \text{and } D_2 = D_2^0.$$

Letting $t'' = (t')^{3/2} v_1/A$, the substitution $t^{3/2} v_1 / A = y/n$ yields

$$\begin{aligned}
\hat\theta_n(u) &\le \frac{D_1}{6A^2} \int_0^{nt''} \left[1 - \frac{y}{n} - \frac{v_2 - \varepsilon}{A} \left(\frac{Ay}{nv_1}\right)^{5/3}\right]^n \left(\frac{Ay}{nv_1}\right)^{4/3} \\
&\quad \times \left[1 + D_2^\varepsilon \left(\frac{Ay}{nv_1}\right)^{2/3}\right] \frac{2}{3} y^{-1/3} \left(\frac{A}{nv_1}\right)^{2/3} \mathrm{d}y \\
&= \frac{D_1}{9n^2 v_1^2} \int_0^{nt''} \left[1 - \frac{y}{n} - \frac{(v_2 - \varepsilon)A^{2/3}}{v_1^{5/3}} \left(\frac{y}{n}\right)^{5/3}\right]^n \left[1 + D_2^\varepsilon \left(\frac{Ay}{nv_1}\right)^{2/3}\right] y \mathrm{d}y \\
&=: I_n + J_n,
\end{aligned}$$

where I_n stands for the integral over the interval $[0, n^{1/5}]$, and J_n stands for the integral over the interval $[n^{1/5}, t''n]$. Using (38), for J_n we obtain that

$$J_n \le \frac{D_1}{9n^2 v_1^2} \int_{n^{1/5}}^{nt''} e^{-(1-\varepsilon)y}\, 2nt'' \mathrm{d}y \le \frac{D_1}{9 v_1^2} e^{-(1-\varepsilon)n^{1/5}},$$

which tends to 0 faster than any polynomial of n. For I_n, using (37), (39) and (40) for $n \geq n_0$ we have that

$$\begin{aligned}
I_n &\leq \frac{D_1}{9n^2v_1^2}\int_0^{n^{1/5}} e^{-y}\exp\left\{-\frac{(v_2-\varepsilon)A^{2/3}}{v_1^{5/3}}\frac{y^{5/3}}{n^{2/3}}\right\}\left[1+D_2^\varepsilon\left(\frac{Ay}{nv_1}\right)^{2/3}\right]y\mathrm{d}y\\
&\leq \frac{D_1}{9n^2v_1^2}\int_0^{n^{1/5}} e^{-y}\left(1-(1-\varepsilon)\frac{(v_2-\varepsilon)A^{2/3}}{v_1^{5/3}}\frac{y^{5/3}}{n^{2/3}}\right)\left[1+D_2^\varepsilon\left(\frac{Ay}{nv_1}\right)^{2/3}\right]y\mathrm{d}y\\
&\leq \frac{D_1}{9n^2v_1^2}\int_0^{n^{1/5}} e^{-y}\left[1+n^{-2/3}A^{2/3}\left(\frac{D_2^\varepsilon}{v_1^{2/3}}y^{2/3}-(1-\varepsilon)\frac{v_2-\varepsilon}{v_1^{5/3}}y^{5/3}+\varepsilon\right)\right]y\mathrm{d}y\\
&\leq \frac{D_1}{9n^2v_1^2}\left[1+n^{-2/3}A^{2/3}\left(\frac{D_2^\varepsilon}{v_1^{2/3}}\Gamma(8/3)-(1-\varepsilon)\frac{v_2-\varepsilon}{v_1^{5/3}}\Gamma(11/3)+2\varepsilon\right)\right],
\end{aligned}$$

where in the last inequality we extended the domain of the integration, and used the definition of the $\Gamma(\,\cdot\,)$ function.

We may obtain a lower estimate for $\hat\theta_n(u)$ in a similar way, and as $\varepsilon > 0$ was arbitrary, we have that $\hat\theta_n(u)$ asymptotically equals to the last upper bound with $\varepsilon = 0$. Since $D_1/(18v_1^2) = \kappa^{-1}$ and $\int_{S^1}\kappa^{-1}(x_u)\mathrm{d}u = L$, we have that

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}(L-\mathrm{Per}(S_n))\cdot n^{2/3} = \lim_{n\to\infty} n^{2/3}\left(L-\binom{n}{2}\int_{S^1}\hat\theta_n(u)\mathrm{d}u\right)\\
&=\int_{S^1}\frac{D_1A^{2/3}}{18v_1^2}\left(\frac{D_2}{v_1^{2/3}}\Gamma(8/3)-\frac{v_2}{v_1^{5/3}}\Gamma(11/3)\right)\mathrm{d}u.
\end{aligned}$$

Substituting to the formula above the values of D_1, D_2 from (41) and l_1, l_2, v_1, v_2 from Lemma 5 we obtain that

$$\begin{aligned}
&\frac{D_1A^{2/3}}{18v_1^2}\left(\frac{D_2}{v_1^{2/3}}\Gamma(8/3)-\frac{v_2}{v_1^{5/3}}\Gamma(11/3)\right)\\
&=\frac{A^{2/3}\Gamma(8/3)}{\kappa}\frac{(3/2)^{2/3}\left[60b^2+(\kappa-1)\left(5(\kappa-1)^2+9(\kappa-1)+3-24c\right)\right]}{10(\kappa-1)^{8/3}},
\end{aligned}$$

and thus

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}(L-\mathrm{Per}(S_n))\cdot n^{2/3}\\
&=\frac{(12A)^{2/3}\Gamma(2/3)}{36}\int_{\partial S}\frac{(\kappa-1)\left(24c-5(\kappa-1)^2-9(\kappa-1)-3\right)-60b^2}{(\kappa-1)^{8/3}}\mathrm{d}x. \qquad (42)
\end{aligned}$$

To finish the proof of Theorem 2, we must show that the constant in (42) is the same as in (6). Let $r(s)$ be the arc-length parametrization of ∂S. It is not difficult to verify that

$$\begin{aligned}
b(r(s)) &= \frac{1}{6}\left\langle r'''(s), \frac{r''(s)}{\kappa(r(s))}\right\rangle,\\
c(r(s)) &= \frac{1}{24}\left(\left\langle r^{(4)}(s), \frac{r''(s)}{\kappa(r(s))}\right\rangle - 4\kappa(r(s))\langle r'''(s), r'(s)\rangle\right).
\end{aligned}$$

After substituting these formulas into (42), some tedious but straightforward calculations yield (6). $\square$

6. The case of the unit circular disc

In this section we discuss the case, when $S = B_R$. Note that in the hypotheses of Theorems 1 and 2 it is assumed that $\kappa(x) > 1/R$ for all $x \in \partial S$. This assumption no longer holds in the case that $S = B_R$, and therefore we may not use Lemma 3. However, the arguments of the proofs of Theorems 1 and 2 can be modified slightly to yield a proof of Theorem 3. Below we provide the outline of the proof of Theorem 3 and leave the technical details to the interested reader.

Proof of Theorem 3. As in the previous section, we may and do assume that $R = 1$.

First note that by Efron's identity (31), it is enough to prove (7) and (9). Also note that for any $u \in S^1$ and $0 \leq t \leq 2$ simple calculations yield

$$\ell(u,t) = \ell(t) = 2 \arcsin \sqrt{1 - \frac{t^2}{4}}, \tag{43}$$

and

$$V(u,t) = V(t) = t\sqrt{1 - \frac{t^2}{4}} + 2 \arcsin \frac{t}{2}. \tag{44}$$

Let W_n and U_n be defined as in the proofs of Theorems 1 and 2, respectively, and let Φ and $L(t) = L(u,t)$ be defined as in the proof of Theorem 1. Then

$$\begin{aligned}
W_n &= \frac{1}{\pi^2} \int_{S^1} \int_0^2 \int_{L(t)} \int_{L(t)} \left(1 - \frac{V(t)}{\pi}\right)^{n-2} t|u_1 \times u_2| \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}t \mathrm{d}u, \\
U_n &= \frac{1}{\pi^2} \int_{S^1} \int_0^2 \int_{L(t)} \int_{L(t)} \left(1 - \frac{V(t)}{\pi}\right)^{n-2} t \arccos\langle u_1, u_2\rangle |u_1 \times u_2| \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}t \mathrm{d}u.
\end{aligned}$$

Integration by u_1, u_2 and u yields

$$\begin{aligned}
W_n &= \frac{4}{\pi} \int_0^2 \left(1 - \frac{V(t)}{\pi}\right)^{n-2} t(\ell(t) - \sin \ell(t)) \mathrm{d}t, \\
U_n &= \frac{4}{\pi} \int_0^2 \left(1 - \frac{V(t)}{\pi}\right)^{n-2} t(2 - 2\cos \ell(t) - \ell(t) \sin \ell(t)) \mathrm{d}t.
\end{aligned}$$

Formulas (43), (44) and the substitution $t = 2\sin(\sigma/2)$ yield

$$\begin{aligned}
W_n &= \frac{4}{\pi} \int_0^\pi \sin\sigma\,(\pi - \sigma - \sin\sigma) \left(1 - \frac{\sin\sigma + \sigma}{\pi}\right)^{n-2} \mathrm{d}\sigma\,, \\
U_n &= \frac{4}{\pi} \int_0^\pi \sin\sigma\,(2 + 2\cos\sigma - \sin\sigma(\pi - \sigma)) \left(1 - \frac{\sin\sigma + \sigma}{\pi}\right)^{n-2} \mathrm{d}\sigma.
\end{aligned} \tag{45}$$

Now, by similar arguments as in the proofs of Theorems 1 and 2, we obtain that

$$\begin{aligned}
W_n &\sim \frac{\pi^2}{n^2}, \\
U_n &\sim \frac{4\pi}{(n-2)^2} \left[1 - \frac{1}{n-2}\left(\frac{\pi^2}{4} + 3\right)\right] + O(n^{-3}),
\end{aligned}$$

which yield the statements of Theorem 3. $\square$

7. Appendix

In this section we sketch the calculation of the Jacobian of the transformation Φ defined in (21). We remark that $J\Phi$ was calculated by Santaló in [30].

Let $r\colon [0,2\pi) \to \partial S$ be a parametrization of ∂S such that the outer normal $u_{r(\alpha)} = (\cos\alpha, \sin\alpha)$. We introduce α, ϕ_1 and ϕ_2 such that $u = (\cos\alpha, \sin\alpha)$, $u_1 = (\cos\phi_1, \sin\phi_1)$ and $u_2 = (\cos\phi_2, \sin\phi_2)$. Clearly, $\mathrm{d}u\mathrm{d}u_1\mathrm{d}u_2 = \mathrm{d}\alpha\mathrm{d}\phi_1\mathrm{d}\phi_2$.

To make the calculation more apparent, we add an extra step: let (v,w) be the centre of the unit circle that defines $D_-(x_1,x_2)$ (here $v,w \in \mathbb{R}$). Then $x_1 = (v + \cos\phi_1, w + \sin\phi_1)$ and $x_2 = (v + \cos\phi_2, w + \sin\phi_2)$, and by differentiation we obtain that

$$\mathrm{d}x_1\mathrm{d}x_2 = |(\sin\phi_1\cos\phi_2 - \sin\phi_2\cos\phi_1)|\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}v\mathrm{d}w.$$

Next, observe that $(v,w) = (r_1(\alpha) - (1+t)\cos\alpha, r_2(\alpha) - (1+t)\sin\alpha)$, thus

$$\mathrm{d}v\mathrm{d}w = |(-r_1'(\alpha)\sin\alpha + r_2'(\alpha)\cos\alpha - (1+t))|\mathrm{d}\alpha\mathrm{d}t,$$

and hence

$$\mathrm{d}x_1\mathrm{d}x_2 = |(-r_1'(\alpha)\sin\alpha + r_2'(\alpha)\cos\alpha - (1+t))\sin(\phi_1-\phi_2)|\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\alpha\mathrm{d}t.$$

Using the special choice of $r(\alpha)$ one can see that $-r_1'(\alpha)\sin\alpha + r_2'(\alpha)\cos\alpha = 1/\kappa(r(\alpha))$, and by assumption $\kappa > 1$, thus

$$|(-r_1'(\alpha)\sin\alpha + r_2'(\alpha)\cos\alpha - (1+t))\sin(\phi_1-\phi_2)| = (1+t-1/\kappa(r(\alpha)))\sin(|\phi_1-\phi_2|).$$

We note that $|u_1 \times u_2|$ equals the sine of the length of the unit circular arc between x_1 and x_2 on the boundary of $D(u,t)$, that is, $\sin(|\phi_1-\phi_2|) = |u_1 \times u_2|$, which proves (22).

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References

[1] Ambrus, G., Kevei, P., Vígh, V. (2012). The diminishing segment process. *Statist. Probab. Lett.* **82**, no. 1, 191–195.

[2] Bárány, I. (2008). Random points and lattice points in convex bodies. *Bull. Amer. Math. Soc. (N.S.)* **45,** 339–365.

[3] Bárány, I., Larman, D. (1988). Convex bodies, economic cap coverings, random polytopes. *Mathematika* **35,** 274–291.

[4] Bezdek, K. (2010). *Classical topics in discrete geometry.* CMS Books in Mathematics, Springer, New York.

[5] Bezdek, K. (2013). *Lectures on Sphere Arrangements — the Discrete Geometric Side.* Fields Institute Monographs, Springer, New York.

[6] Bezdek, K., Lángi, Z., Naszódi, M., Papez, P. (2007). Ball-polyhedra. *Discrete Comput. Geom.* **38,** No. 2, 201–230.

[7] Blaschke, W. (1956). *Kreis und Kugel.* Zweite Auflage. Walter de Gruyter AG. Berlin.

[8] Blanc, E. (1943). Les ensembles surconvexes plans. *Ann. Sci. École Norm. Sup. (3)* **60**, 215–246.

[9] Böröczky, K. J., Fodor, F., Reitzner, M., Vígh, V. (2009). Mean width of random polytopes in a reasonably smooth convex body. *J. Multivariate Anal.* **100,** 2287–2295.

[10] Buter, J. (1938). Ueberconvexe Mengen in der Ebene. *Ak. Wetensch. Amsterdam Proc.* **41**.

[11] Danzer, L., Grünbaum, B., Klee, V. (1963). Helly's theorem and its relatives. *Proc. Sympos. Pure Math., Vol. VII*, Amer. Math. Soc., Providence, R.I., pp.101–180.

[12] Efron, B. (1965). The convex hull of a random set of points. *Biometrika* **52,** 331–343.

[13] Eggleston, H. G. (1965). Sets of constant width in finite dimensional Banach spaces. *Israel J. Math.* **3**, 163–172.

[14] Fejes Tóth, L. (1953). *Lagerungen in der Ebene, auf der Kugel und im Raum. (German).* Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXV. Springer-Verlag, Berlin-Göttingen-Heidelberg.

[15] Fejes Tóth, L. (1982). Packing of r-convex discs. *Studia Sci. Math. Hungar.* **17**, 449–452.

[16] Fejes Tóth, L. (1982). Packing and covering with r-convex discs. *Studia Sci. Math. Hungar.* **18**, 69–73.

[17] Fodor, F., Vígh, V. (2012). Disc-polygonal approximations of planar spindle convex sets. *Acta Sci. Math. (Szeged)* **78,** No. 1–2, 331–350.

[18] Gruber, P. M. (1997). Comparisons of best and random approximation of convex bodies by polytopes. (II International Conference in "Stochastic Geometry, Convex Bodies and Empirical Measures" (Agrigento, 1996)) *Rend. Circ. Mat. Palermo (2) Suppl.* 50, 189–216.

[19] Hug, D. (2000). Measures, curvatures and currents in convex geometry. Habilitationsschrift, Univ. Freiburg.

[20] Kupitz, Y. S., Martini, H., Perles, M. A. (2005). Finite sets in $\mathbb{R}^d$ with many diameters — a survey. *Proceedings of the International Conference on Mathematics and Applications (ICMA-MU 2005, Bangkok)*, Mahidol University Press, Bangkok, pp.91–112. (Note: Reprinted in a special volume of the East-West J. Math.: Contributions in Mathematics and Applications (2007), 41–57)

[21] Kupitz, Y. S., Martini, H., Perles, M. A. (2010). Ball polytopes and the Vázsonyi problem. *Acta Math. Hungar.* **126,**, No. 1–2, 99–163.

[22] Mayer, A. E. (1935). Eine Überkonvexität. *Math. Z.* **39,** No. 1, 511–531.

[23] Moreno, J. P., Schneider, R. (2007). Continuity properties of the ball hull mapping. *Nonlinear Anal.* **66**, no. 4, 914–925.

[24] Moreno, J. P., Schneider, R. (2012). Diametrically complete sets in Minkowski spaces. *Israel J. Math.* **191**, no. 2, 701–720.

[25] McClure, D. E., Vitale, R. A. (1975). Polygonal approximation of plane convex bodies. *J. Math. Anal. Appl.* **51**, no. 2, 326–358.

[26] Pasqualini, L. (1939). Superconvexité. *Bull. Cl. Sci. Acad. Belg. (5)* **25**, 18–24.

[27] Rényi, A., Sulanke, R. (1963). Über die konvexe Hülle von n zufällig gewählten Punkten. *Z. Wahrscheinlichkeitsth. verw. Geb.* **2,** 75–84.

[28] Rényi, A., Sulanke, R. (1964). Über die konvexe Hülle von n zufällig gewählten Punkten, II. *Z. Wahrscheinlichkeitsth. verw. Geb.* **3,** 138–147.

[29] Rényi, A., Sulanke, R. (1968). Zufällige konvexe Polygone in einem Ringgebiet. *Z. Wahrscheinlichkeitsth. verw. Geb.* **9,** 146–157.

[30] Santaló, L. A. (1946). Sobre Figuras Planas Hiperconvexas. *Summa Bras. Math.* **1,** 221-239.

[31] Schneider, R. (1993). *Convex bodies: the Brunn-Minkowski theory.* Encyclopedia of Mathematics and its Applications. 44, Cambridge University Press, Cambridge.

[32] Schneider, R. (2008). Recent results on random polytopes. (Survey). *Boll. Un. Mat. Ital., Ser. (9)* **1,** 17–39.

[33] Schneider, R., Weil, W. (2008). *Stochastic and Integral Geometry.* Springer.

[34] van der Corput, J. G. (1938). Ueberkonvexe Mengen in der Ebene. *Ak. Wetensch. Amsterdam Proc.*, **41**.

[35] Vincensini, P. (1936). Sur les figures superconvexes planes. *Bull. Soc. Math. France* **64**, 197–208.

[36] Weil, W., Wieacker, J. A. (1993). Stochastic geometry. In: Handbook of convex geometry, Vol. A, B. North-Holland, Amsterdam, 1391–1438.

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