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On normality of orthogonal polynomials†

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Abstract

We extend some recent results of Martínez-Finkelshtein and Simon about measures μ on the unit circle for which the corresponding orthonormal polynomials φ_n have the so called normal behavior: $\|\varphi_n'\|/n \to 1$.

Keywords: orthogonal polynomials on the unit circle, doubling weights, normal behavior.

MSC: Primary 42C05; Secondary
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Let μ be a Borel-measure on the unit circle $\mathbf{T}$ (with support that contains infinitely many points) and let $\varphi_n(z) = \kappa_n z^n + \cdots$ be the orthonormal polynomials associated with μ. Thus,

$$\int \varphi_n \overline{\varphi_m} d\mu = 0 \qquad \text{if } n \neq m, \text{ and} \qquad \int |\varphi_n|^2 d\mu = 1.$$

It is a simple fact due to the orthogonality, that here

$$\frac{1}{\kappa_n^2} = \inf\left\{ \int |P_n|^2 d\mu \,\middle|\, P_n(z) = z^n + \cdots \right\}, \tag{1}$$

and if we apply this to $P_n(z) = z\varphi_n'(z)/n\kappa_n$, then we can conclude that

$$\int |\varphi_n'(z)|^2 d\mu \geq n^2. \tag{2}$$

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A. Martínez-Finkelshtein and B. Simon [4] raised the problem: when do we have equality in (2) in asymptotic sense, i.e. when is it true that

$$\lim_{n\to\infty} \frac{\|\varphi_n'\|_{L^2(\mu)}}{n} = 1.$$

When this is the case, they call it normal behavior. The paper [4] contains motivations, different formulations and several criteria for normal/non-normal behavior. The picture is far from complete at this moment, and it is quite intriguing how different properties of the measure influence normal behavior.

It was pointed out in [4] that normal behavior is linked to the Bernstein inequality. In this paper we take this connection further, and with it we get some extensions of some results in [4].

As is usual, we identify the unit circle $\mathbf{T}$ with $\mathbf{R}/(\mathrm{mod} 2\pi)$.

We call a measure μ doubling if there is an L such that for all intervals $I \subset [-\pi, \pi]$ we have

$$\mu(2I) \le L\mu(I),$$

where $2I$ is the interval obtained from I by enlarging it twice from its center. When this is the case and $d\mu(t) = w(t)dt$ is absolutely continuous, then we shall also use the terminology that w is doubling.

In what follows we shall use the decomposition $\mu = \mu_a + \mu_s$, $d\mu_a(t) = w(t)dt$, of μ into its absolutely continuous and singular part, and the letters μ, μ_a, μ_s, w will always be related this way.

One of the general normality criteria of [4] is the following: if w is bounded and it is in the Szegő class, i.e. if

$$\int \log w > -\infty,$$

then for $\mu(t) = w(t)dt$ there is normal behavior (see [4, Theorem 5.1]). Our result is

Theorem 1. *Let w be a doubling weight in the Szegő class such that w is locally bounded outside a set of measure 0, and assume also that μ_s is doubling. Then $d\mu(t) = w(t)dt + d\mu_s(t)$ has normal behavior.*

As an example, let $\{a_n\} \subset [-\pi, \pi]$ be a sequence the cluster points of which are the points of the Cantor-set. Then

$$w(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{|x - a_n|^{1/2}}$$

is in the Szegő class and it is locally bounded outside a set of measure zero (outside the union of $\{a_n\}$ and the set of its cluster points), even though it is unbounded around every point of the Cantor-set.

Furthermore, the weights $|x-a|^{-1/2}$ are uniformly doubling (the doubling constant is independent of a), and it is easy to see that sums and limits of uniformly doubling weights is doubling, so w is doubling. We can also add a nonzero singular doubling measure μ_s. In fact, the existence of singular doubling measures follows from a paper of Beurling and Ahlfors [1] who, in connection with quasiconformal mappings, showed that there is a strictly increasing continuous $\rho:\mathbf{R}\to\mathbf{R}$ for which

$$\frac{1}{M}\le\frac{\rho(x+t)-\rho(x)}{\rho(x)-\rho(x-t)}\le M$$

is true for all x and t, and for which $\rho'=0$ almost everywhere. Clearly, this ρ generates a $d\rho$ which is a singular doubling measure. For completeness, we shall give a direct construction at the end of the paper.

Let us remark that, by a result of Feffermann and Muckenhoupt [3], a doubling weight may vanish on a set of positive measure, so it need not be in the Szegő class. Even then, a doubling measure cannot be too small on intervals, namely there is an s and a $c>0$ such that for all $I\subset[-\pi,\pi]$ we have (see [5, Lemma 2.1])

$$\mu(I)\ge c|I|^s$$

(this property for measurable sets I rather than intervals would be more than sufficient for the Szegő property).

Corollary 2. All generalized Jacobi weigths $d\mu(t)=w(t)dt$ of the form

$$w(t)=h(t)\prod_{1\le k\le N}|t-t_k|^{\alpha_k}$$

where $\alpha_k>-1$ and h is a positive continuous function, have normal behavior.

For Lipschitz continuous h this is [4, Theorem 10.1].

We say that a measure μ on the unit circle $\mathbf{T}$ has the Bernstein property, if there is a constant C_0 such that

$$\int|P_n'|^2d\mu\le C_0n^2\int|P_n|^2d\mu \tag{3}$$

for all polynomials P_n of degree at most $n=1,2,\ldots$. It is easy to see that this is the same that for all trigonometric polynomials S_n of degree at most $n=1,2,\ldots$ we have

$$\int_{-\pi}^{\pi}|S_n'(t)|^2d\mu(t)\le C_0n^2\int_{-\pi}^{\pi}|S_n(t)|^2d\mu(t)$$

(with a possibly different C_0). The L^2-version of a classical theorem of Bernstein says that the Lebesgue-measure has the Bernstein property, and (3) just requires the same for weighted L^2 spaces. It is a remarkable fact that the doubling property alone implies the Bernstein property, see [5] (there absolutely continuous measures were considered, but the theorems and proofs are valid without any change for doubling measures).

The following result shows that a doubling singular part is irrelevant from the point of view of normality provided the absolutely continuous part is also doubling and in the Szegő class.

Theorem 3. Suppose that μ_a is a doubling measure in the Szegő class, and μ_s is also doubling. Then μ is normal if and only if μ_a is normal.

Proof. Since μ is in the Szegő class, Szegő's theorem (see e.g. [7, (12.3.9)] or [6, (1.1.8) and (1.5.22)]) gives that the leading coefficients $\kappa_n(\mu)$ and $\kappa_n(\mu_a)$ have the same positive limit, so if $\eta > 0$ is given, then for large n we have

$$\kappa_n(\mu) \le \kappa_n(\mu_a) \le (1+\eta)\kappa_n(\mu)$$

for all large n no matter how $1 > \eta > 0$ is given. Let Φ_n be the orthonormal polynomial for μ_a. Then (see also (1))

$$\int \left| \frac{\varphi_n/\kappa_n(\mu) - \Phi_n/\kappa_n(\mu_a)}{2} \right|^2 d\mu_a + \int \left| \frac{\varphi_n/\kappa_n(\mu) + \Phi_n/\kappa_n(\mu_a)}{2} \right|^2 d\mu_a$$
$$= \frac{1}{2}\int |\varphi_n/\kappa_n(\mu)|^2 \mu_a + \frac{1}{2}\int |\Phi_n/\kappa_n(\mu_a)|^2 d\mu_a$$
$$\le \frac{1}{2\kappa_n(\mu)^2} + \frac{1}{2\kappa_n(\mu_a)^2} \le \frac{(1+\eta)^2}{\kappa_n(\mu_a)^2}. \tag{4}$$

Since the second term on the left is at least $1/\kappa_n(\mu_a)^2$ (see (1)), it follows that

$$\int \left| \frac{\varphi_n/\kappa_n(\mu) - \Phi_n/\kappa_n(\mu_a)}{2} \right|^2 d\mu_a \le \frac{3\eta}{\kappa_n(\mu_a)^2},$$

i.e.

$$\int |\varphi_n - \Phi_n|^2 d\mu_a \le 24\eta + 2\int |\varphi_n|^2 \left| \frac{\kappa_n(\mu_a)}{\kappa_n(\mu)} - 1 \right|^2 d\mu_a \le 26\eta. \tag{5}$$

Using that μ_a is doubling, therefore it has the Bernstein property, it follows that

$$\int |\varphi_n' - \Phi_n'|^2 d\mu_a \le C_0 n^2 \int |\varphi_n - \Phi_n|^2 d\mu_a \le 26 C_0 n^2 \eta.$$

This gives

$$\|\Phi_n'/n\|_{L^2(\mu_a)} \le \|\varphi_n'/n\|_{L^2(\mu_a)} + \sqrt{26C_0\eta} \le \|\varphi_n'/n\|_{L^2(\mu)} + \sqrt{26C_0\eta},$$

so the normality of μ implies that of μ_a.

In a similar vein,

$$\|\varphi_n'/n\|_{L^2(\mu_a)} \le \|\Phi_n'/n\|_{L^2(\mu_a)} + \sqrt{26C_0\eta}.$$

Since μ_s is also doubling, it has the Bernstein property, therefore

$$\|\varphi_n'/n\|_{L^2(\mu_s)} \le C_0\|\varphi_n\|_{L^2(\mu_s)} \to 0$$

by [6, Theorem 2.2.14,(iv)], so it is less than any given ε if n is large. Hence, for all large n we have

$$\|\varphi_n'/n\|_{L^2(\mu)} \le \|\Phi_n'/n\|_{L^2(\mu_a)} + \sqrt{26C_0\eta} + \varepsilon$$

so the normality of μ_a implies that of μ.

■

To prove Theorem 1 we need

Proposition 4. If μ is in the Szegő class and μ has the Bernstein property (in particular, if μ is doubling), then for sets $E \subseteq \mathbf{T}$ consisting of finitely many arcs

$$\limsup_{n\to\infty} \frac{1}{n^2}\int_E |\varphi_n'|^2 d\mu \le 2C_0|E|.$$

Here $|E|$ is the linear (arc) measure of E.

Proof. For an $\varepsilon > 0$ choose a polynomial S, say of degree m, such that $1 \le |S(z)| \le 2$ on E, $|S(z)| \le \varepsilon$ on $\mathbf{T} \setminus 2E$ ($2E$ is obtained by enlarging each subarc of E twice from its center) and $|S(z)| \le 2$ otherwise. One can get easily such a polynomial from a similar trigonometric polynomial

$$S_{m/2}^*(t) = \sum_{k=-[m/2]}^{[m/2]} c_k e^{ikt}$$

by setting

$$S(z)=z^{[m/2]}\sum_{k=-[m/2]}^{[m/2]} c_k z^k.$$

Then

$$\int_E |\varphi_n'|^2 d\mu \le \int_E |\varphi_n' S|^2 d\mu \le 2\int_E |\varphi_n S'|^2 d\mu + 2\int_E |(\varphi_n S)'|^2 d\mu.$$

The first term on the right is at most a constant times the integral of $|\varphi_n|^2 d\mu$ on E, so it is bounded, and hence the quantity obtained by dividing it by n^2 tends to 0 as n tends to ∞. Using the Bernstein property we obtain for the second term

$$\begin{aligned}\int_E |(\varphi_n S)'|^2 d\mu &\le C_0(n+m)^2 \int |\varphi_n S|^2 d\mu \\ &\le C_0(n+m)^2\varepsilon^2 \int_{\mathbf{T}\setminus 2E} |\varphi_n|^2 d\mu + 4C_0(n+m)^2 \int_{2E} |\varphi_n|^2 d\mu.\end{aligned}$$

To estimate the last factor in the second term of the right-hand side we use that $|\varphi_n(e^{it})|^2 d\mu(t)$ tends weakly to $dt/2\pi$ (see [6, Theorem 2.2.14,(v)]), and so

$$\limsup_{n\to\infty} \int_{2E} |\varphi_n|^2 d\mu = \frac{2|E|}{2\pi}.$$

Plugging this into the preceding estimate, dividing by n^2, letting $n\to\infty$ and then $\varepsilon\to 0$, we obtain what we want.

■

Proof of Theorem 1. In view of Theorem 3 we may assume $\mu=\mu_a$ i.e. that μ is absolutely continuous: $d\mu(x)=w(x)dx$.

We start with a similar argument as in Theorem 3. Let $w_M=\min(w,M)$, and set $d\mu_M(t)=w_M(t)dt$. Then, by the assumption that w is locally finite outside a set of measure 0, this w_M agrees with w outside a set E_M which can be chosen as a finite union of intervals with $|E_M|\to 0$ as $M\to\infty$. From Szegő's theorem (see e.g. [7, (12.3.9)] or [6, (1.1.8) and (1.5.22)]) we get that the corresponding leading coefficients $\kappa_n(\mu)$ and $\kappa_n(\mu_M)$ differ by as small quantity as we wish if M is large and then n is large, i.e. we can have

$$\kappa_n(\mu)\le\kappa_n(\mu_M)\le(1+\eta)\kappa_n(\mu) \tag{6}$$

for all large M and then large n, no matter how $1 > \eta > 0$ is given.

Let Φ_n be the orthonormal polynomial for μ_M. Now repeat the argument (4)–(5) with with μ_a replaced by μ_M (that argument was based on $\mu_a \leq \mu$ and for μ_M we also have $\mu_M \leq \mu$) to conclude

$$\int |\varphi_n - \Phi_n|^2 d\mu_M \leq 26\eta. \tag{7}$$

Let $\varepsilon > 0$ and fix an M_0 such that $|E_{M_0}| \leq \varepsilon$. Let J be a subarc of $\mathbf{T} \setminus 2E_{M_0}$, and J' the subarc of $\mathbf{T} \setminus E_{M_0}$ that contains J. By the local Bernstein inequality for doubling weights [2] we have

$$\int_J |\varphi_n' - \Phi_n'|^2 w \leq C_{J,J'} n^2 \int_{J'} |\varphi_n - \Phi_n|^2 w$$

(in [2] it is assumed that J' is of length at most 1, which is enough for us, for we can apply that result to smaller parts of J' if this is not the case). Taking sum for all subarcs of $\mathbf{T} \setminus 2E_{M_0}$ we can see that

$$\begin{aligned} \int_{\mathbf{T}\setminus 2E_{M_0}} |\varphi_n' - \Phi_n'|^2 w &\leq C_{M_0} n^2 \int_{\mathbf{T}\setminus E_{M_0}} |\varphi_n - \Phi_n|^2 w \\ &= C_{M_0} n^2 \int_{\mathbf{T}\setminus E_{M_0}} |\varphi_n - \Phi_n|^2 w_M \leq C_{M_0} 26\eta n^2, \end{aligned}$$

where, in the last but one step we used that for $M > M_0$ we have $w = w_M$ on $\mathbf{T} \setminus E_{M_0}$ (clearly, we may assume the sets E_M decreasing, so $E_M \subset E_{M_0}$), and in the last step we used (7).

On the other hand, by [4, Theorem 5.1] (note that w_M is a bounded Szegő weight which agrees with w on $\mathbf{T} \setminus E_{M_0}$)

$$\frac{1}{n^2} \int_{\mathbf{T}\setminus 2E_{M_0}} |\Phi_n'|^2 w = \frac{1}{n^2} \int_{\mathbf{T}\setminus 2E_{M_0}} |\Phi_n'|^2 w_M \leq 1 + \varepsilon \tag{8}$$

for large n. A combination of these give for large n

$$\frac{1}{n^2} \int_{\mathbf{T}\setminus 2E_{M_0}} |\varphi_n'|^2 w \leq (\sqrt{C_{M_0} 26\eta} + \sqrt{1+\varepsilon})^2.$$

The integral over $2E_{M_0}$ is handled by Proposition 4, namely

$$\frac{1}{n^2} \int_{2E_{M_0}} |\varphi_n'|^2 d\mu \leq 3C_0 2|E_{M_0}| \leq 6C_0\varepsilon \tag{9}$$

for large n.

All these show that for large n

$$\frac{\|\varphi_n'\|_{L^2(\mu)}}{n} \le \sqrt{C_{M_0} 26\eta} + \sqrt{1+2\varepsilon} + \sqrt{6C_0\varepsilon},$$

and since here $\varepsilon > 0$ is arbitrarily small, and independently of this and M_0, the number $\eta > 0$ can be arbitrarily small, the proof is complete.

For more clarification, this is the order of selection of the parameters: given $\varepsilon > 0$ select M_0 so that $|E_{M_0}| \le \varepsilon$. With this choice of E_{M_0} we get the constant C_{M_0}, and select η so that $C_{M_0} 26\eta < \varepsilon$. Then select the $M > M_0$ and N so large that with this M the inequality (6) is true for $n \ge N$. Finally, there are two more thresholds on n, namely that (8) be true and that (9) be satisfied.

■

We finish the paper by a construction of a singular doubling measure on the unit circle.

We shall construct a 1-periodic singular doubling measure μ on the real line, then its dilation by 2π will be appropriate on the unit circle.

Let h be the a 1-periodic function that is 2 on the interval $[1/3, 2/3)$ and equals $1/2$ on $[0, 1/3) \cup [2/3, 1)$. Then the integral of h over $[0, 1)$ is 1. Let, for $n = 1, 2, \ldots$,

$$g_n(x) = \prod_{k=1}^{n} h(3^k x),$$

and $d\mu_n(x) = g_n(x)dx$. The function g_n is constant on each triadic interval $I_{j,n} = [j/3^n, (j+1)/3^n) \subset [0, 1)$, the constant being $4^{l_{j,n}}/2^n$, where $l_{j,n}$ is the number of those digits $\{\varepsilon_k\}$, $1 \le k \le n$, in the triadic expansion of the center

$$\frac{j+1/2}{3^n} = 0.\varepsilon_1\varepsilon_2\ldots\varepsilon_n\varepsilon_{n+1}\cdots, \qquad \varepsilon_k = 0, 1, 2$$

which equal 1:

$$l_{j,n} = \#\{k \,|\, 1 \le k \le n,\ \varepsilon_k = 1\}. \tag{10}$$

Thus,

$$\mu_n(I_{j,n}) = \frac{4^{l_{j,n}}}{6^n},$$

and from the choice of h (namely from the fact that its integral over $[0,1)$ is 1) we also get

$$\mu_m(I_{j,n}) = \mu_n(I_{j,n}) \qquad \text{for all } m \geq n.$$

Therefore, if μ is a weak*-limit of $\{\mu_m\}_{m=1}^\infty$, then we have

$$\mu(I_{j,n}) = \frac{4^{l_{j,n}}}{6^n}.$$

We fix such a weak* limit μ, and next we show that μ is doubling. If I is a subinterval of $[0,1]$ with $3^{-n} \leq |I| < 3^{n-1}$, $n \geq 2$, then there is an interval $I_{j,n+1} \subset I$, and $2I$ is contained in $I_{k,n-1} \cup I_{k+1,n-1}$ for some k. Let d be the density of μ_{n-1} on $I_{k,n-1}$. Then its density on $I_{k+1,n-1}$ is either $4d$ or $d/4$, so the density of μ_n on any subinterval $I_{s,n}$ of $I_{k,n-1} \cup I_{k+1,n-1}$ lies in between $d/4^2$ and $4^2 d$, and the density of μ_{n+1} on any subinterval $I_{t,n+1}$ of $I_{k,n-1} \cup I_{k+1,n-1}$ lies in between $d/4^3$ and $4^3 d$. In particular, this is true for $I_{j,n+1}$. Thus,

$$\begin{aligned}\mu(2I) \leq \mu(I_{k,n-1} \cup I_{k+1,n-1}) &= \mu_{n-1}(I_{k,n-1} \cup I_{k+1,n-1}) \\ &\leq 4d|I_{k,n-1} \cup I_{k+1,n-1}| = 8d3^{-(n-1)},\end{aligned}$$

while

$$\mu(I) \geq \mu(I_{j,n+1}) = \mu_{n+1}(I_{j,n+1}) \geq \frac{d}{4^3}|I_{j,n+1}| = \frac{d}{4^3} d3^{-(n+1)},$$

so

$$\mu(2I) \leq 8 \cdot 4^3 \cdot 9 \cdot \mu(I).$$

Finally, we prove that μ is singular. Let $\varepsilon > 0$ be given, and for an n consider the set $E_{\varepsilon,n}$ of those points $x \in [0,1)$ for which $g_n(x) > \varepsilon$. If l is an integer with $4^l/2^n > \varepsilon$, then the number of intervals $I_{j,n}$ on which the density is precisely $4^l/2^n$ is (see (10))

$$\binom{n}{l} 2^{n-l},$$

and these have total length

$$\binom{n}{l} \frac{2^{n-l}}{3^n},$$

so

$$|E_{\varepsilon,n}| = \sum_{4^l > \varepsilon 2^n} \binom{n}{l} \frac{2^{n-l}}{3^n} =: \sum_{4^l > \varepsilon 2^n} C_{n,l}.$$

Since, for large n, we have

$$C_{n,l+1} \leq \frac{12}{14} C_{n,l} \qquad \text{for } l \geq 7n/18,$$

and $4^l > \varepsilon 2^n$ implies $l > n/2 + \log \varepsilon > 8n/18$, we get with $q = (12/14)^{1/18}$ that

$$|E_{\varepsilon,n}| \leq C_q \sum_{4^l > \varepsilon 2^n} \left(\frac{12}{14}\right)^{l-7n/18} \leq C'_q q^n. \tag{11}$$

On the complement of $E_{\varepsilon,n}$ (which is a union of intervals $I_{j,n}$) the density of μ_n is $\leq \varepsilon$, so

$$\mu([0,1) \setminus E_{\varepsilon,n}) = \mu_n([0,1) \setminus E_{\varepsilon,n}) \leq \varepsilon.$$

This gives for

$$E_\varepsilon := \limsup_{n\to\infty} E_{\varepsilon,n} = \cap_N \cup_{n\geq N} E_{\varepsilon,n}$$

that $\mu([0,1) \setminus E_\varepsilon) \leq \varepsilon$, and, by (11), E_ε is of measure 0. Thus, if $E = \cup_{m=1}^\infty E_{1/m}$, then E is of measure 0 and $\mu([0,1) \setminus E) = 0$, which shows the singularity of μ.

■

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