Chebyshev polynomials on a systems of curves *

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Abstract

The paper is devoted to the problem how close one can get with the n-th Chebyshev numbers of a compact set Γ to the theoretical lower bound $\mathrm{cap}(\Gamma)^n$. For a system of $m \geq 2$ analytic curves it is shown that there is always a subsequence for which the Chebyshev numbers are at least as large as $(1+\alpha)\mathrm{cap}(\Gamma)^n$, while for another subsequence they are at most $(1+O(n^{-1/(m-1)})\mathrm{cap}(\Gamma)^n$. It will also be shown that a better estimate than the last one cannot be given. We shall also discuss how well a system of curves can be approximated by lemniscates in Hausdorff metric. The proofs are based on potential theoretical arguments. Simultaneous Diophantine approximation of harmonic measures lies in the background. To achieve the correct rate, a perturbation of the multi-valued complex Green's function is introduced which makes the n-th power of its exponential single-valued and which allows to construct Faber-like polynomials on multiply connected domains.

1 The norm of monic polynomials on systems of analytic curves

Let K be an infinite compact set on the plane. For every n there is a unique monic polynomial $T_n(z) = z^n + c_1 z^{n-1} + \cdots$, called the Chebyshev polynomial of degree n of K, which minimizes the supremum norm on K:

$$\|T_n\|_K = \min \|z^n + \cdots\|_K.$$

Chebyshev polynomials, being extremal from various points of view, appear in a number of problems: the original motivation of Chebyshev came from mechanics, but since then they made their appearance in potential theory, orthogonal polynomials, number theory, numerical analysis, signal processing, differential

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equations, just to name a few (see e.g. [16], [17], [3]–[5] and the references there). The properties of Chebyshev polynomials have been the subject of many papers, see the extensive literature in [18]. The case when K consists of several intervals on $\mathbf{R}$ have been particularly developed via the theory of elliptic functions, hyperelliptic curves and Riemann surfaces (see for example the papers [1], [18], [3]–[5] and [11]–[14]). Here we shall be primarily interested in the behavior of the Chebyshev numbers $\|T_n\|_K$ when K is the union of a finite number of curves.

By a theorem of Fekete and Szegő (see e.g. [16, Corollary 5.5.5]) for any K we have

$$\|T_n\|_K \geq \mathrm{cap}(K)^n \tag{1.1}$$

where $\mathrm{cap}(K)$ denotes logarithmic capacity, and, as $n \to \infty$, $\|T_n\|_K^{1/n} \to \mathrm{cap}(K)$. The reader can find the concept of logarithmic capacity e.g. in [9] or [16]. For example, the capacity of a disk/circle of radius r is r, while the capacity of a segment of length l is $l/4$.

In this work we address the problem how close the Chebyshev number $\|T_n\|_K$ can get to the theoretical lower limit $\mathrm{cap}(K)^n$.

In a landmark paper [25] Harold Widom described the behavior of various extremal polynomials associated with a system of curves on the complex plane or with some measures on such curves. In particular, he described the $(1+o(1))$-behavior of $\|T_n\|_\Gamma$ when Γ is a family of smooth Jordan curves in terms of the norm of some extremal analytic functions related to n. Although the asymptotic is somewhat implicit, it implies

Theorem A (Widom) *If Γ consists of $m \geq 2$ smooth Jordan curves lying exterior to one another, then there is a C such that for every $n = 1, 2, \ldots$ we have $\|T_n\|_\Gamma \leq C\mathrm{cap}(\Gamma)^n$.*

For an alternating proof based on discretization of the equilibrium measure see [24].

In this paper we are interested in better than $C\mathrm{cap}(\Gamma)^n$ bounds in the sense of [22], where the same problem was considered for finitely many intervals. We shall use some ideas of that paper, but the method here is quite different. We would like to illustrate the method, so not to mix in considerable technical difficulties that would arise for less smooth curves, in this paper we choose the curves to be analytic, and just mention that the results hold for less smooth, say C^2 curves, as well.

The case of a single analytic curve will be excluded below, for then things are different and simpler. Indeed, if Γ is an analytic Jordan curve, $\Phi(z) = z + c + c_{-1}z^{-1} + \cdots$ is the conformal map (of the given form) of the exterior of Γ onto the exterior of the disk $\{z \,|\, |z| = \mathrm{cap}(\Gamma)\}$ and Φ_n are the Faber polynomials associated with Γ (i.e. $\Phi_n(z)$ is the polynomial part of $\Phi(z)^n$), then we have $\|\Phi_n - \Phi^n\|_\Gamma \leq Cq^n \|\Phi^n\|_\Gamma$ with some $0 < q < 1$ ([19, Sec. II.3, (12)]), therefore $\|\Phi_n\|_\Gamma \leq (1 + Cq^n)\mathrm{cap}(\Gamma)^n$, which is in sharp contrast with the results below.

In the formulation of the theorems below T_n will denote the n-th Chebyshev polynomial associated with the set Γ in question. However, the reader should keep in mind that any result on T_n gives a result for all monic polynomials; e.g. if we state $\|T_n\|_\Gamma \geq \gamma_n$, then we get automatically the estimate $\|P_n\| \geq \gamma_n$ for all monic polynomials P_n.

First of all we remark that Theorem A cannot be improved to have $(1 + o(1))\mathrm{cap}(\Gamma)^n$ rate.

Theorem 1.1 *If Γ consists of $m \geq 2$ analytic Jordan curves lying exterior to one another, then there is a $\beta > 0$ and a subsequence $\mathcal{M}$ of the natural numbers such that for every $n \in \mathcal{M}$ we have $\|T_n\|_\Gamma \geq (1+\beta)\mathrm{cap}(\Gamma)^n$.*

Thus, a norm like $(1 + o(1))\mathrm{cap}(\Gamma)^n$ is possible only along some subsequence. That this is indeed the case is the content of

Theorem 1.2 *If Γ consists of $m \geq 2$ analytic Jordan curves lying exterior to one another, then there is a C and a subsequence $\mathcal{N}$ of the natural numbers such that $\|T_n\|_\Gamma \leq (1 + Cn^{-1/(m-1)})\mathrm{cap}(\Gamma)^n$ for every $n \in \mathcal{N}$.*

Finally, we show that Theorem 1.2 is sharp regarding the order $(1+n^{-1/(m-1)})$.

Theorem 1.3 *For every $m \geq 2$ there is set Γ consisting of m disjoint circles such that for every $n = 1, 2, \ldots$ we have $\|T_n\|_\Gamma \geq (1 + cn^{-1/(m-1)})\mathrm{cap}(\Gamma)^n$ with some $c > 0$.*

Actually, if we fix the centers of the circles then the radii for which Theorem 1.3 is true form a dense subset in $[0, a]^m$ for some $a > 0$. On the other hand, there is a dense subset of the radii in $[0, a]^m$ for which there are infinitely many n with the property

$$\|T_n\|_\Gamma \leq (1 + Cq^n)\mathrm{cap}(\Gamma)^n \tag{1.2}$$

with some $0 < q < 1$ (this easily follows from the considerations below since there is a dense set of radii for which each $\mu_\Gamma(\Gamma_j)$ is rational, and for such Γ's Theorem 1.4 gives the estimate (1.2) for infinitely many n).

All these theorems will be easy consequences of the following one. Let μ_Γ be the equilibrium measure of Γ (see e.g. [9], [16] or [20]). Think of μ_Γ as the distribution of a unit charge placed on the conductor Γ (i.e. the charge can move freely in Γ) when it is in equilibrium. Let Γ_k, $k = 1, \ldots, m$ be the components of Γ, and consider the harmonic measures $\mu_\Gamma(\Gamma_k)$, $k = 1, \ldots, m$. For a $\theta > 0$ let $\{\theta\}$ denote its distance from the nearest integer, and set

$$\kappa_n = \max_{1\leq k\leq m} \{n\mu_\Gamma(\Gamma_k)\}. \tag{1.3}$$

Then κ_n/n measures how well each of $\mu_\Gamma(\Gamma_k)$ can be (simultaneously) approximated with rational numbers of the form p/n, $p = 0, 1, 2, \ldots$ (the denumerator is fixed to be n). With this κ_n we can state

Theorem 1.4 *Let Γ be a finite family of analytic Jordan curves lying exterior to one another. Then there are constants $c, C > 0$ and $0 < q < 1$ depending only on Γ such that for every $n = 1, 2, \ldots$*

$$(1 + c\kappa_n)\mathrm{cap}(\Gamma)^n \le \|T_n\|_\Gamma \le (1 + C\kappa_n + Cq^n)\mathrm{cap}(\Gamma)^n. \tag{1.4}$$

Note also that $\kappa_n = 0$ could easily happen for infinitely many n without Γ being a lemniscate (level set of a polynomial), hence the sharper upper estimate

$$\|T_n\|_\Gamma \le (1 + C\kappa_n)\mathrm{cap}(\Gamma)^n$$

is not necessarily true (it can be been shown that the equality $\|T_n\|_\Gamma = \mathrm{cap}(\Gamma)^n$ holds for a particular n if and only if Γ is the level curve of a polynomial of degree n). This is the situation for example, if Γ consists of two circles of the same radius, in which case $\kappa_{2m} = 0$ for all m but $\|T_{2m}\|_\Gamma > \mathrm{cap}(\Gamma)^{2m}$.

We shall see in Section 4 that the results are closely related to the problem of approximation of a system of curves by lemniscates in Hausdorff metric.

Finally, we would like to mention that similar results can be proven for smooth (not analytic) systems of curves. However, that situation is technically very challenging for the following reason: for analytic curves we use the reflection principle, with which we can continue the Green's function of the complement inside the components Γ_k of Γ, and then one can speak of level curves of the Green's function that lie inside Γ (Γ itself arises as the level curve of a Green's function associated with a smaller set). This is what we are going to do, and this is what is no longer true when Γ is not analytic. In that case one needs to imitate the inner level curves (Γ cannot arise then as a level curve of the Green's function associated with a smaller set), which is quite technical. For that reason we skip the case of smooth curves in this paper.

The outline of the paper is as follows. In the next section we list some preliminaries and prove a weaker version of Theorem 1.4 namely we verify

$$(1 + c\kappa_n)\mathrm{cap}(\Gamma)^n \le \|T_n\|_\Gamma \le (1 + C\kappa_n + C/n)\mathrm{cap}(\Gamma)^n. \tag{1.5}$$

Since typically $\kappa_n \succeq n^{1/(m-1)}$ (see the proofs of Theorems 1.2 and 1.3), the additional term C/n on the right is usually bounded by the first term $C\kappa_n$. Even though (1.5) is weaker than (1.4) since instead of q^n we have $1/n$ on the right hand side, it is sufficient to verify Theorems 1.1–1.3, which we shall do in section 3. The sharper form (1.4) is more difficult than (1.5), it will be proven in section 5. While the error $O(1/n)$ in (1.5) will be obtained in section 2 via a relatively simple discretization of the equilibrium potential, the error $O(q^n)$ requires a fairly delicate adjustment of the complex Green's function with which the n-th power of its exponential becomes single-valued, and Cauchy's formula can be applied. The simpler discretization approach of section 2, even though it produces weaker result, is of interest, since it can also be used in the case when

Γ consists of smooth (not necessarily analytic) curves, for which it still yields Theorems 1.1–1.3.

Section 4 will be on approximation of Γ by lemniscates in Hausdorff metric. The proofs for lemniscate approximation are based on the arguments used for Theorems 1.1–1.4. Roughly speaking, we shall get that the error in approximation by lemniscates of degree n is about κ_n/n.

2 Proof of the weaker version of Theorem 1.4

This section is dedicated to the proof of (1.5).

In this work we shall extensively use some basic results from logarithmic potential theory, see e.g. [9], [16] or [20] for the concepts appearing below.

For a compact subset Γ (of positive logarithmic capacity) of the complex plane let $\mathrm{cap}(\Gamma)$ denote its logarithmic capacity and μ_Γ its equilibrium measure. Then, by Frostman's theorem [16, Theorem 3.3.4], for the logarithmic potential

$$U^{\mu_\Gamma}(z) = \int \log \frac{1}{|z-t|} d\mu_\Gamma(t)$$

we have

$$U^{\mu_\Gamma}(z) \le \log \frac{1}{\mathrm{cap}(\Gamma)}, \qquad z \in \mathbf{C}, \tag{2.1}$$

and

$$U^{\mu_\Gamma}(z) = \log \frac{1}{\mathrm{cap}(\Gamma)}, \qquad \text{for quasi-every } z \in \Gamma, \tag{2.2}$$

i.e. with the exception of a set of zero capacity. If Γ consists of finitely many Jordan curves or arcs then (2.2) is true everywhere on Γ by Wiener's criterion [16, Theorem 5.4.1]. Let $\Omega = \Omega_\Gamma$ be the unbounded connected component of $\overline{\mathbf{C}} \setminus \Gamma$ and let $g_{\Omega_\Gamma}(z,\infty)$ be the Green's function in Ω with pole at infinity. For simpler notation we set

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(z,\infty) \equiv g_{\Omega_\Gamma}(z,\infty).$$

Then (see e.g. [16, Sec. 4.4] or [20, (I.4.8)])

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(z,\infty) = \log \frac{1}{\mathrm{cap}(\Gamma)} - U^{\mu_\Gamma}(z). \tag{2.3}$$

The set $\mathrm{Pc}(\Gamma) = \overline{\mathbf{C}} \setminus \Omega_\Gamma$ is called the polynomial convex hull of Γ (it is the union of Γ with all the bounded components of $\mathbf{C} \setminus \Gamma$).

We shall also form balayage out of Ω (see [20, Theorems II.4.1, II.4.4]): if ρ is a finite Borel-measure with compact support in Ω then there is a measure $\widehat{\rho}$ supported on $\partial\Omega$, called its balayage, such that it has the same total mass as ρ and

$$U^{\widehat{\rho}}(z) = U^{\rho}(z) + \mathrm{const}$$

on $\partial\Omega$. The constant is connected with the Green's function, namely we have (see [20, Theorem 4.4])

$$U^{\widehat{\rho}}(z) = U^{\rho}(z) + \int_{\Omega} g_{\Omega}(a,\infty)d\rho(a). \tag{2.4}$$

Proof of the upper bound in (1.5). Let $\Gamma_1,\ldots,\Gamma_m$, $m \ge 2$ be the connected components of Γ, each being an analytic Jordan curve. Let φ_k be a conformal map from the unit disk Δ_1 onto the interior of Γ_k. Then φ_k can be extended to a conformal map of some disk Δ_r of radius $r > 1$ with center at the origin onto some simply connected domain containing Γ_k ([10, Proposition 3.2]).

The function $g_{\overline{\mathbf{C}}\setminus\Gamma}(\varphi_k(z),\infty)$ is a positive harmonic function in the annulus $1 < |z| < r$ which has zero values on the unit circle. Hence, by the reflection principle (see e.g. [8, Sect. X.3] and apply a conformal map from the exterior of the unit disk onto the upper half plane), it can be extended to a harmonic function in $1/r < |z| < 1$ such that it has negative values in the annulus $1/r < |z| < 1$. On applying φ_k^{-1} we get a harmonic extension g of $g_{\overline{\mathbf{C}}\setminus\Gamma}(\cdot,\infty)$ to a neighborhood of Γ_k with negative values inside Γ_k. We can do this for all k, so g is defined in a neighborhood of Γ. But then, for some small $\delta > 0$, the level set $\gamma := \{z \,|\, g(z) = -\delta\}$ consists of analytic Jordan curves γ_k, $k = 1,\ldots,m$ one-one lying inside each Γ_k, $k = 1,\ldots,m$. Since the function $g(z) + \delta$ is harmonic outside γ, it is 0 on γ and it behaves at infinity like const + $\log|z|$, it is the Green's function $g_{\mathbf{C}\setminus\gamma}(z,\infty)$ of the unbounded component Ω_γ of $\overline{\mathbf{C}}\setminus\gamma$. Thus, Γ is the δ-level set of $g_{\mathbf{C}\setminus\gamma}(z,\infty)$. Since the limit of $g_{\mathbf{C}\setminus\gamma}(z,\infty) - \log|z|$ at infinity is $\log 1/\mathrm{cap}(\gamma)$ (see (2.3)), it also follows that

$$\mathrm{cap}(\Gamma) = e^{\delta}\mathrm{cap}(\gamma). \tag{2.5}$$

Finally, from formula (2.3) it follows that

$$U^{\mu_\gamma}(z) = \log\frac{1}{\mathrm{cap}(\gamma)} - \delta = \log\frac{1}{\mathrm{cap}(\Gamma)}, \qquad z \in \Gamma. \tag{2.6}$$

Now let $\theta_k = \mu_\Gamma(\Gamma_k)$ be the amount of mass of the equilibrium measure μ_γ on the k-th component γ_k. Let τ_k be a smooth Jordan curve enclosing Γ_k such that all the other Γ_j's lie outside τ_k, let $\mathbf{n}_-$ denote the inner normal to τ_k and let s_{τ_k} be the arc length measure on τ_k. In view of the formula (2.3) connecting the equilibrium measure and the Green's function, we get from Gauss' theorem (see e.g. [20, Theorem II.1.1])

$$-\frac{1}{2\pi}\oint_{\tau_k}\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}}{\partial\mathbf{n}_-}ds_{\tau_k}(z) = \frac{1}{2\pi}\oint_{\tau_k}\frac{\partial U^{\mu_\Gamma}(z)}{\partial\mathbf{n}_-}ds_{\tau_k}(z) = \mu_\Gamma(\Gamma_k).$$

Since the left-hand side does not change if we replace Γ by γ, it follows that if γ_k is the component of γ that lies in Γ_k, then

$$\mu_\Gamma(\Gamma_k) = \mu_\gamma(\gamma_k), \tag{2.7}$$

so θ_k is also the number $\mu_\gamma(\gamma_k)$. In particular,

$$\kappa_n = \max_{1\le k\le m}\{n\mu_\gamma(\gamma_k)\}. \tag{2.8}$$

For a fixed n let n_k, $k=1,\dots,m-1$ be the closest integer to $n\theta_k$, and we define n_m as $n-(n_1+\dots+n_{m-1})$. Then $n_1+\dots+n_m=n$ and $|n\theta_k-n_k|\le\kappa_n$ for $k=1,\dots,m-1$, while

$$\begin{aligned} |n\theta_m - n_m| &= |n(1-\theta_1-\cdots\theta_{m-1})-(n-n_1-\cdots-n_{m-1})| \\ &= \sum_{k=1}^{m-1}|n\theta_k-n_k| \le (m-1)\kappa_n. \end{aligned}$$

Since μ_γ has $C^{1+\alpha}$ (actually C^∞) smooth density with respect to arc measure on γ (c.f. [24, Proposition 2.2]), we can use the discretization technique of [24]. Divide each γ_k into n_k arcs I_j^k, $j=1,\dots,n_k$, each having equal weight θ_k/n_k with respect to μ_γ, i.e. $\mu_\gamma(I_j^k)=\theta_k/n_k$. Then

$$\left|n-\frac{1}{\mu_\gamma(I_j^k)}\right| = \left|n-\frac{n_k}{\theta_k}\right| = \left|n-\frac{n\theta_k+O(\kappa_n)}{\theta_k}\right| = O(\kappa_n). \tag{2.9}$$

Let

$$\xi_j^k = \frac{1}{\mu_\gamma(I_j^k)}\int_{I_j^k} u\, d\mu_\gamma(u) \tag{2.10}$$

be the center of mass of I_j^k with respect to μ_γ, and consider the polynomials

$$P_n(z) = \prod_{j,k}(z-\xi_j^k) \tag{2.11}$$

of degree at most n. We claim that these polynomials give the upper estimate in (1.5).

It was proved in [24, Proposition 2.2] that the density of μ_γ with respect to arc measure s_γ on γ is positive and continuous, hence $\mathrm{diam}(I_j^k)\sim s_\gamma(I_j^k)\sim \mu_\gamma(I_j^k)\sim 1/n$, where $A\sim B$ means that the ratio A/B is bounded away from zero and infinity. It is also clear that for large n we have $\mathrm{dist}(\xi_j^k, I_j^k)\le \mathrm{diam}(I_j^k)$ for all j,k.

Now for $z\in\Gamma$

$$\int \log^+|z-t|d\mu_\gamma(t)\le \log^+\mathrm{diam}(\Gamma),$$

hence (use also (2.6))

$$\int |\log|z-t||d\mu_\gamma(t)\le 2\log^+\mathrm{diam}(\Gamma)-\log\mathrm{cap}(\Gamma). \tag{2.12}$$

In view of (2.6) we can write for $z \in \Gamma$

$$\begin{aligned} n \log \operatorname{cap}(\Gamma) &= \sum_{j,k} \left(n - \frac{1}{\mu_\gamma(I_j^k)} \right) \int_{I_j^k} \log |z-t| d\mu_\gamma(t) \\ &+ \sum_{j,k} \frac{1}{\mu_\gamma(I_j^k)} \int_{I_j^k} \log |z-t| d\mu_\gamma(t) = \Sigma_1 + \Sigma_2. \end{aligned} \tag{2.13}$$

Here, by (2.9) and (2.12),

$$\begin{aligned} \Sigma_1 &\leq \sum_{k=1}^{m} O(\kappa_n) \sum_{j=1}^{n_k} \left| \int_{I_j^j} \log |z-t| \mu_\gamma(t) \right| \\ &\leq \sum_{k=1}^{m} O(\kappa_n) \int |\log |z-t|| \mu_\gamma(t) = O(\kappa_n). \end{aligned} \tag{2.14}$$

Therefore, to prove that $\|P_n\|_\Gamma \leq (1 + O(\kappa_n + 1/n)) \operatorname{cap}(\Gamma)^n$, it is enough to show that on Γ

$$\log |P_n(z)| - \Sigma_2 = \sum_{j,k} \frac{1}{\mu_\gamma(I_j^k)} \int_{I_j^k} \log \left| \frac{z - \xi_j^k}{z-t} \right| d\mu_\gamma(t) = O(n^{-1}). \tag{2.15}$$

Actually we are going to show that even

$$|\log |P_n(z)| - \Sigma_2| \leq \sum_{j,k} \left| \frac{1}{\mu_\gamma(I_j^k)} \int_{I_j^k} \log \left| \frac{z - \xi_j^k}{z-t} \right| d\mu_\gamma(t) \right| = O(n^{-1}). \tag{2.16}$$

Note that there is a $\rho > 0$ such that for $z \in \Gamma$, $t \in \gamma$ we have $|z - t| \geq \rho$, as well as $|z - \xi_j^k| \geq \rho$ for all j, k (and for all n, of course). For the integrand in (2.16) we write for $t \in I_j^k$

$$\begin{aligned} \log \left| \frac{z - \xi_j^k}{z-t} \right| &= -\log \left| 1 + \frac{\xi_j^k - t}{z - \xi_j^k} \right| = -\Re \frac{\xi_j^k - t}{z - \xi_j^k} + O\left(\left| \frac{\xi_j^k - t}{z - \xi_j^k} \right|^2 \right) \\ &= -\Re \frac{\xi_j^k - t}{z - \xi_j^k} + O\left(\left| \frac{1/n}{\rho^2} \right|^2 \right), \end{aligned}$$

since then $|\xi_j^k - t| \leq 2 \operatorname{diam}(I_j^k) \leq C/n$. Therefore,

$$\frac{1}{\mu_\gamma(I_j^k)} \int_{I_j^k} \log \left| \frac{z - \xi_j^k}{z-t} \right| d\mu_\gamma(t) = \frac{1}{\mu_\gamma(I_j^k)} \int_{I_j^k} O\left(n^{-2}\right) d\mu_\gamma(t) = O(n^{-2}) \tag{2.17}$$

because the integral

$$\int_{I_j^k} \Re\frac{\xi_j^k - t}{z-\xi_j^k} d\mu_\gamma(t) = \Re\frac{1}{z-\xi_j^k}\int_{I_j^k}(\xi_j^k - t)d\mu_\gamma(t)$$

vanishes by the choice of ξ_j^k.

If we sum (2.17) up for all j and k then we obtain that the left-hand side in (2.16) is at most $Cn \cdot n^{-2} \le C/n$, an the proof is complete.

For later use let us mention that we have also proved the following: there are $\rho > 0$, C_0 and a sequence $\{P_n\}_{n\in\mathcal{N}}$ of monic polynomials of exact degree n such that if $\operatorname{dist}(z,\Gamma) < \rho$ then

$$\begin{aligned}\left|ng_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty) + n\log\operatorname{cap}(\gamma) - \log|P_n(z)|\right| &= |U^{n\mu_\gamma}(z) + \log|P_n(z)|| \\ \le C_0(\kappa_n + 1/n). \end{aligned} \tag{2.18}$$

■

Proof of the lower bound in (1.4) and (1.5). Assume to the contrary that there is a subsequence $\mathcal{N}$ of the natural number such that for some positive sequence $\varepsilon_n = o(\kappa_n)$, we have monic polynomials T_n, $n \in \mathcal{N}$ such that $\|T_n\|_\Gamma \le e^{\varepsilon_n}\operatorname{cap}(\Gamma)^n$. In what follows, n will be selected from $\mathcal{N}$. Then we get for the counting measure ν_n on the zeros of T_n the inequality (see (2.1))

$$nU^{\mu_\Gamma}(z) - U^{\nu_n}(z) - \varepsilon_n \le 0, \qquad z \in \Gamma. \tag{2.19}$$

If $\widehat{\nu_n}$ is the balayage of ν_n out of $\Omega = \Omega_\Gamma$ (the unbounded component of $\overline{\mathbf{C}} \setminus \Gamma$) onto Γ, then on Γ the change in the potential is (see (2.4))

$$U^{\widehat{\nu_n}}(z) = U^{\nu_n}(z) + \sum_k g_{\overline{\mathbf{C}}\setminus\Gamma}(z_{k,n},\infty),$$

where the summation extends to all zero $z_{k,n}$ of T_n which lie in Ω. Hence, together with (2.19) we also have

$$nU^{\mu_\Gamma}(z) - U^{\widehat{\nu_n}}(z) + \sum_k g_{\overline{\mathbf{C}}\setminus\Gamma}(z_{k,n},\infty) - \varepsilon_n \le 0, \qquad z \in \Gamma. \tag{2.20}$$

By the principle of domination (see e.g. [20, Theorem II3.2]), this inequality extends to all $z \in \mathbf{C}$, and for $z \to \infty$ we obtain

$$\sum_k g_{\overline{\mathbf{C}}\setminus\Gamma}(z_{k,n},\infty) - \varepsilon_n \le 0,$$

which shows that for large n there cannot be a zero of T_n outside any fixed neighborhood U of Γ (since the Green's function $g_{\overline{\mathbf{C}}\setminus\Gamma}(z,\infty)$ has a positive lower bound there).

Let now $\cup_{j=1}^m \tau_j$, $j=1,\dots$ be the level set $\{z \mid g_{\overline{\mathbf{C}}\setminus\Gamma}(z,\infty)=\rho\}$ with some small $\rho>0$ such that τ_j encloses Γ_j and the τ_j's are lying exterior to one another (for small ρ this is the case), and let V_j be a small closed neighborhood of τ_j. We may assume the neighborhood U of Γ to lie so close to Γ that V_j lies outside U. Now by the principle of domination (2.19) extends to all $z\in\mathbf{C}$, and the left-hand side is a non-positive harmonic function outside U (including the point infinity) with value $-\varepsilon_n$ at infinity. Thus, by Harnack's theorem [16, Theorems 1.3.1 and 1.3.3], there is a $C_1>0$ such that for all $z\in\cup_j V_j$ we have

$$-C_1\varepsilon_n \le U^{n\mu_\Gamma}(z)-U^{\nu_n}(z)-\varepsilon_n\le 0,$$

and so

$$|U^{n\mu_\Gamma}(z)-U^{\nu_n}(z)|\le C_1\varepsilon_n. \tag{2.21}$$

Then for the normal derivative with respect to the inner normal $\mathbf{n}_-$ on τ_j we have

$$\left|\frac{\partial\big(U^{n\mu_\Gamma}(z)-U^{\nu_n}(z)\big)}{\partial\mathbf{n}_-}\right|\le C_2\varepsilon_n \tag{2.22}$$

on τ_j, $j=1,\dots,m$ with some C_2. Indeed, if V_j is a d-neighborhood of τ_j, $j=1,\dots,m$, then for $z\in\gamma_j$ the disk $D_d(z)$ of radius d and with center at z lies in $\cup_{j=1}^m V_j$, hence for the harmonic function $U^{n\mu_\Gamma}-U^{\nu_n}$ the estimate (2.21) is true in $D_d(z)$. Now if we apply Poisson's formula in $D_d(z)$, then (2.22) follows with $C_2=4C_1/d$.

By Gauss' theorem (see e.g. [20, Theorem II.1.1])

$$\frac{1}{2\pi}\oint_{\tau_j}\frac{\partial\big(U^{n\mu_\Gamma}(z)-U^{\nu_n}(z)\big)}{\partial\mathbf{n}_-}ds_{\tau_j}(z)=n\mu_\Gamma(H_j)-\nu_n(H_j), \tag{2.23}$$

where H_j is the domain enclosed by τ_j and s_{τ_j} is the arc-length on τ_j. Here $\mu_\Gamma(H_j)=\mu_\Gamma(\Gamma_j)$ and $\nu_n(H_j)$ is the number of zeros of T_n inside τ_j, so it is an integer n_j (that depends of course on n). Hence, (2.22) and (2.23) give

$$|n\theta_j-n_j|=|n\mu_\Gamma(\Gamma_j)-n_j|\le C_2\varepsilon_n \tag{2.24}$$

for all $j=1,\dots,m$. This, however, means by the definition of the numbers κ_n in (1.3) that $\kappa_n\le C_2\varepsilon_n$, $n\in\mathcal{N}$, which is not the case since we assumed $\varepsilon_n=o(\kappa_n)$. This contradiction proves the lower estimate in (1.4).

■

3 Proofs of Theorems 1.1–1.3

Proof of Theorem 1.2. Let, as before, $\Gamma_1, \ldots, \Gamma_m$, $m \geq 2$ be the connected components of Γ and set $\theta_k = \mu_\Gamma(\theta_k)$. By Kronecker's theorem on simultaneous rational approximation (see e.g. [7, Theorems VI, VII in Chapter I]) there is a $C > 0$ and a subsequence $\mathcal{N}$ of the natural numbers such that if $\{n\theta_k\}$ denotes the distance from $n\theta_k$ to the nearest integer, then $\{n\theta_k\} \leq Cn^{-1/(m-1)}$ for all $k = 1, \ldots, m-1$ as $n \to \infty$, $n \in \mathcal{N}$. Since the sum of the θ_k's is 1, $\{n\theta_m\} \leq (m-1)Cn^{-1/(m-1)}$ is then also true. Hence, $\kappa_n = O(n^{-1/(m-1)})$ along $\mathcal{N}$, and the theorem follows from the upper estimate in (1.5).

■

Proof of Theorem 1.3. By Furtwangler's theorem (see e.g. [7, Theorem III of Chapter V]) there are real numbers $\theta_1, \ldots, \theta_{m-1}$ and a constant $c_0 > 0$ such that for any n and any integers p_j we have $\max_j |n\theta_j - p_j| \geq c_0/n^{1/(m-1)}$. Without loss of generality we may assume $\theta_j > 0$ and $\sum_{j=1}^{m-1} \theta_j < 1$ (just add to θ_j a large integer and then divide the result by another sufficiently large integer number). Set $\theta_m = 1 - \sum_{j=1}^{m-1} \theta_j$. Let $o_1, \ldots, o_m$ be distinct points on the real line and consider circles $C_{x_j}(o_j)$ about o_j of radius $0 < x_j < r_j$, where r_j are so small that the circles $C_{r_j}(o_j)$ lie exterior to one another. We claim that there are x_j's such that if $\Gamma = \cup_{j=1}^m C_{x_j}(o_j)$, then $\mu_\Gamma(C_{x_j}(o_j)) = \theta_j$, and this will be our choice of Γ. Let

$$\Gamma(x_1, \ldots, x_m) = \bigcup_{j=1}^m C_{x_j}(o_j),$$

and set

$$g_j(x_1, \ldots, x_m) = \mu_{\Gamma(x_1,\ldots,x_m)}(C_{x_j}(o_j)), \qquad j = 1, \ldots, m.$$

These are positive continuous functions with sum identically 1. If $x_j' = x_j$ except for $j = k$ and $x_k' > x_k$, then $\Gamma(x_1, \ldots, x_m)$ lies inside $\Gamma(x_1', \ldots, x_m')$ (more precisely inside the polynomial convex hull of $\Gamma(x_1', \ldots, x_m')$), and hence $\mu_{\Gamma(x_1,\ldots,x_m)}$ is the balayage of the measure $\mu_{\Gamma(x_1',\ldots,x_m')}$ onto $\Gamma(x_1, \ldots, x_m)$, i.e. out of $\Omega_{\Gamma(x_1,\ldots,x_m)}$ (see [20, Theorem Iv.1.6,(e)]). This shows that for $j \neq k$ we have $g_j(x_1, \ldots, x_m) > g_j(x_1', \ldots, x_m')$, and, as a consequence (since $\sum_{j=1}^m g_k \equiv 1$), $g_k(x_1, \ldots, x_m) < g_k(x_1', \ldots, x_m')$. Hence, the system $\{g_j(x_1, \ldots, x_m)\}_{j=1}^m$ is a so called monotone system in the sense of [21]. In addition, if x_k is fixed and all other x_j tend to 0, then $g_k(x_1, \ldots, x_m)$ tends to 1, i.e.

$$\lim_{u \searrow 0} g_j(u, u, \ldots, u, x_j, u, \ldots, u) = 1 \qquad \text{for all } j = 1, 2, \ldots, m \text{ and } x_j > 0.$$

Now under these conditions [21, Theorem 10] guarantees that for any positive vector $(\lambda_1, \ldots, \lambda_m)$ with $\sum_j \lambda_j = 1$ there is an $(x_1, \ldots, x_m)$ arbitrarily close to $(0, \ldots, 0)$ such that $g_j(x_1, \ldots, x_m) = \lambda_j$ for each $j = 1, \ldots, m$. With the choice $\lambda_j = \theta_j$ we get our Γ as the corresponding $\Gamma(x_1, \ldots, x_m)$.

By the choice of Γ we have a $c_0 > 0$ with the property that for $n = 1, 2, \ldots$ no matter what integers $n_1, \ldots, n_m$ we chose, always

$$\max_{1 \le j \le m} |n_G(\Gamma_j)\theta_j - n_j| \ge c_0 n^{-1/(m-1)}. \tag{3.1}$$

This means that $\kappa_n \ge c_0 n^{-1/(m-1)}$ for all n, and the statement in Theorem 1.3 follows from the lower bound in Theorem 1.4.

■

Proof of Theorem 1.1. By the assumption $m \ge 2$, and so $0 < \mu_\Gamma(\Gamma_1) < 1$. Then there is an infinite sequence of n's for which $1/3 \le \{n\mu_\Gamma(\Gamma_1)\} \le 1/2$ (this is clear for rational $\mu_\Gamma(\Gamma_1)$, and when $\mu_\Gamma(\Gamma_1)$ is irrational, then actually $\{\mu_\Gamma(\Gamma_1)\}$ is dense in $[0, 1/2]$ by Weyl's theorem). Hence, for an infinite sequence of the n's we have $\kappa_n \ge 1/3$, and the claim follows from the lower estimate in Theorem 1.4.

■

4 Approximation by lemniscates

Call a level set of a polynomial a lemniscate σ. If the polynomial in question is of exact degree n then we denote σ by σ_n. In this section we address the problem how closely a set of analytic Jordan curves Γ can be approximated by a lemniscate σ_n. We shall measure the error of approximation in the Hausdorff distance:

$$d(\sigma_n, \Gamma) = \max\Big(\sup_{z\in\Gamma} \mathrm{dist}(z, \sigma_n); \sup_{z\in\sigma_n} \mathrm{dist}(z, \Gamma)\Big). \tag{4.1}$$

We shall usually require that σ_n and Γ have the same number of components. In addition, we may also request that σ_n lies within Γ (i.e. σ_n lies in the polynomial convex hull $\mathrm{Pc}(\Gamma)$) or vice versa.

D. Hilbert showed in 1897 that if Γ is a single analytic curve then it can be arbitrarily well approximated by lemniscates in the above sense. E. P. Dolzenko (see [6, p.21]) raised the question of the rate of approximation, and in response V. V. Andrievskii [2] proved that for any continuum the rate is $O(\log n/n)$. He also verified that if Γ is a curve with bounded secant variation then the rate of approximation by a σ_n is $O(1/n)$, and better than $O(1/n)$ rate is not possible in general.

Here we prove

Theorem 4.1 *Let Γ consist of $m \ge 2$ analytic Jordan curves lying exterior to one another, and let κ_n be defined by* (1.3)*. Then there are constants $c, C > 0$ and $0 < q < 1$ depending only on Γ such that there is a lemniscate σ_n consisting of m components for which*

$$d(\sigma_n, \Gamma) \le C(\kappa_n/n + q^n), \tag{4.2}$$

and if σ_n is a lemniscate consisting of m components, then

$$c\kappa_n/n \leq d(\sigma_n, \Gamma). \tag{4.3}$$

As a consequence in the sense of the first part of this paper we list

Corollary 4.2 *Let Γ consist of $m \geq 2$ analytic Jordan curves lying exterior to one another.*

(a) *There is a C and for every $n = 1, 2, \ldots$ there is a lemniscate σ_n consisting of precisely m components with $d(\sigma_n, \Gamma) \leq Cn^{-1}$.*

(b) *There is a $c > 0$ and an infinite sequence $\mathcal{M}$ of the natural numbers such that for any lemniscate σ_n, $n \in \mathcal{M}$, that consists of m components we have $d(\sigma_n, \Gamma) \geq cn^{-1}$.*

(c) *There is a C and a subsequence $\mathcal{N}$ of the natural numbers such that for every $n \in \mathcal{N}$ there is a lemniscate σ_n consisting of m components with $d(\sigma_n, \Gamma) \leq Cn^{-m/(m-1)}$.*

(d) *There is a set Γ consisting of m disjoint circles and a constant $c > 0$ such that for every $n = 1, 2, \ldots$ and for every lemniscate σ_n consisting of m components we have $d(\sigma_n, \Gamma) \geq cn^{-m/(m-1)}$.*

Let us also mention that the requirement that σ_n consists of precisely m components is not necessary (in (b) or (d), for in (a) and (c) this is an additional property of the approximating lemniscate), e.g. (b) and (d) hold whenever $\sigma_n \subseteq \mathrm{Pc}(\Gamma)$).

The corollary can be obtained the same way as Theorems 1.1–1.3 were obtained from Theorem 1.4; we shall skip the details.

In this section we prove only the weaker version

$$d(\sigma_n, \Gamma) \leq C(\kappa_n/n + 1/n^2), \tag{4.4}$$

which is enough to deduce the corollary. The sharper form (with q^n instead of $1/n^2$ on the right) will follow the same arguments once we verify in the next section (5.20) instead of (2.18) that we use below.

Proof of (4.4). In the proof of Theorem 1.4 (see (2.18) at the end of the proof) we verified the following (see the notations there): there are $\rho > 0$, C_0 and a sequence $\{P_n\}_{n\in\mathcal{N}}$ of monic polynomials of exact degree n such that if $\mathrm{dist}(z, \Gamma) < \rho$ then

$$\begin{aligned} \left| n g_{\overline{\mathbf{C}}\setminus\gamma}(z, \infty) + n \log \mathrm{cap}(\gamma) - \log |P_n(z)| \right| &= |U^{n\mu_\gamma}(z) + \log |P_n(z)|| \\ &\leq C_0(\kappa_n + 1/n). \end{aligned} \tag{4.5}$$

This implies, in view of $\mathrm{cap}(\Gamma) = e^{\delta}\mathrm{cap}(\gamma)$ (see (2.5)), that the lemniscate $\sigma_n := \{z \,|\, |P_n(z)| = n \log \mathrm{cap}(\Gamma)\}$ lies in between the level curves

$$L_{\pm} := \{z \,\big|\, n g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty) = n\delta \pm C_0(\kappa_n + 1/n)\}.$$

By the Lip 1 property of $g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)$ the Hausdorff distance in between $L_{\pm}$ is $\le C_1(\kappa_n + 1/n)$ with some C_1, and since Γ is the level set $\{z \,\big|\, g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty) = \delta\}$ lying in between $L_{\pm}$, the claim in the theorem follows.

■

Proof of (4.3). Suppose to the contrary that there are lemniscates σ_n (consisting of m components) such that for infinitely many n we have $d(\sigma_n, \Gamma) \le \varepsilon_n^*$ with some $\varepsilon_n^* = o(\kappa_n/n)$. In what follows n will be always selected from this sequence of the n's.

Let $\Gamma_\rho = \{z \,\big|\, g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty) = \delta - \rho\}$, where γ and δ have the same meaning as in the proof of the upper estimate of Theorem 1.4, and further let $(\gamma_\rho)_k$, $k = 1, \ldots, m$ be the m components of Γ_δ. These are level curves of the Green's function $g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)$, $\Gamma_0 = \Gamma$, and it is easy to see that for $0 \le \rho_1 < \rho_2 \le \delta/2$ the distance between $(\Gamma_{\rho_1})_k$ and $(\Gamma_{\rho_2})_k$ is $\sim \rho_2 - \rho_1$ for each $k = 1, \ldots, m$. Indeed, this is immediate from the fact that the normal derivative $\partial g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)/\partial \mathbf{n}$ with respect to the outer normal on Γ_δ is uniformly continuous and positive on each $(\Gamma_{\rho_1})_k$. The uniform continuity is immediate, and the strict positivity follows from the fact that this normal derivative is nothing else (apply [16, Theorem 4.3.14], [20, Theorem II.1.5] to the formula (2.3), see also (5.16) later in this paper) than 2π-times the harmonic measure with respect to the point ∞ in the unbounded component of $\overline{\mathbf{C}} \setminus \Gamma_\rho$ (more precisely, the normal derivative is 2π times the density of this harmonic measure with respect to arc length on Γ_ρ). Note also that this harmonic measure is just the equilibrium measure of Γ_ρ (see [16, Theorem 4.3.14]), and the positivity in question is just the statement that the density of the equilibrium measure (with respect to arc length) cannot vanish; see [24, Proposition 2.2] for more details.

Let now $\Gamma_n^* = \Gamma_{C^*\varepsilon_n^*}$ with some C^* to be chosen in a moment. From the previous discussion and from $d(\sigma_n, \Gamma) \le \varepsilon_n^*$ it follows that if C^* is sufficiently large, then Γ_n^* lies in the polynomial hull $\mathrm{Pc}(\sigma_n)$ of σ_n, and hence $\mu_{\Gamma_n^*}$ is the balayage of μ_{σ_n} onto Γ_n^* ([20, Theorem IV.1.6,(e)]). This gives in view of (2.3)–(2.4)

$$\log \frac{1}{\mathrm{cap}(\Gamma_n^*)} = \log \frac{1}{\mathrm{cap}(\sigma_n)} + \int g_{\overline{\mathbf{C}}\setminus\Gamma_n^*}(a,\infty)\, d\mu_{\sigma_n}(a). \tag{4.6}$$

Since outside Γ_n^* we have

$$g_{\overline{\mathbf{C}}\setminus\Gamma_n^*}(z,\infty) \equiv g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty) - (\delta - C^*\varepsilon_n^*),$$

these functions are uniformly Lip 1 on Γ_n^*, and since σ_n lies in an ε_n^*-neighborhood of Γ (so in a $O(\varepsilon_n^*)$-neighborhood of Γ_n^*), we have for the integral in (4.6) the bound $\leq C_0\varepsilon_n^*$ with some $C_0 > 0$ independent of ε_n^*. This gives

$$\mathrm{cap}(\Gamma_n^*) \geq e^{-C_0\varepsilon_n^*}\mathrm{cap}(\sigma_n).$$

Recall now that $\sigma_n = \{z \mid |P_n(z)| = t_n\}$ with some t_n and polynomial P_n of degree n, and we may assume that this P_n to be a monic polynomial. But then by [16, Theorem 5.2.5] $\mathrm{cap}(\sigma_n) = t_n^{1/n}$, and it follows that

$$\mathrm{cap}(\Gamma_n^*) \geq e^{-C_0\varepsilon_n^*}t_n^{1/n},$$

i.e.

$$t_n \leq e^{C_0\varepsilon_n^* n}\mathrm{cap}(\Gamma_n^*)^n.$$

Since $\|P_n\|_{\Gamma_n^*} \leq \|P_n\|_{\sigma_n} = t_n$, we also have

$$\|P_n\|_{\Gamma_n^*} \leq e^{C_0\varepsilon_n^* n}\mathrm{cap}(\Gamma_n^*)^n.$$

Now copy the proof of Theorem 1.4 from (2.19) to (2.24) with Γ_n^*, P_n and $C_0 n\varepsilon_n^*$ instead of Γ_n, T_n and ε_n, respectively, and conclude for all $k = 1, \ldots, m$ the inequality

$$|n\mu_{\Gamma_n^*}((\Gamma_n^*)_k) - n_k| \leq C_1 n\varepsilon_n^*,$$

with some C_1. Here $\mu_{\Gamma_n^*}((\Gamma_n^*)_k)$ is the same as $\mu_\Gamma(\Gamma_k)$ (see (2.7) which can also be applied to Γ_n^* instead of γ), hence

$$|n\mu_\Gamma(\Gamma_k) - n_k| \leq C_1 n\varepsilon_n^*,$$

that is $\kappa_n \leq C_1 n\varepsilon_n^*$ is also true. However, we have assumed $\varepsilon_n^* = o(\kappa_n/n)$, and this contradiction proves the claim.

■

5 A perturbation of the complex Green's function for a system of curves

In this section we prove (1.4) and (4.2) in its full generality (recall that so far we have proved only the weaker estimates (1.5) and (4.4)).

In the beginning of the paper we indicated that Faber polynomials associated with a single Jordan curve can be very useful; and indeed, they have been used in various situations (see e.g. [19]). However, if the set in question contains several components then there is no conformal map from the (unbounded) complement onto the exterior of a circle and then it is not clear what takes the role of Faber polynomials (the so called Faber-Walsh version that was created for this purpose is not suitable for us). In this section we construct a substitute which allows us to prove (1.4) and (4.2). We believe that the construction can substitute Faber polynomials in other situations, as well.

In the case of a single Jordan curve γ let $\tilde{g}$ be an analytic conjugate of the Green's function $g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)$ of the unbounded component Ω_γ of $\overline{\mathbf{C}}\setminus\gamma$ (i.e. $g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)+i\tilde{g}(z)$ is locally analytic). Then

$$\psi(z) := \exp(g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)+i\tilde{g}(z))$$

maps Ω_γ conformally onto the exterior of the unit circle. When γ has several components then $\tilde{g}$ can still be defined, but $\psi(z)$ is not single-valued. However, if each of the harmonic measures $\mu_\gamma(\gamma_k)$ (where μ_γ is the equilibrium measure of γ and γ_k are the components of γ) are of the form p_k/n with the common denominator n, then $\psi^n(z)$ is single-valued (see the discussion later in between (5.16) and (5.17)). What we are going to show in this section is that this situation can be achieved with small perturbation of the components γ_k. How large perturbation we need for a given n depends on how close the numbers $\mu_\gamma(\gamma_k)$ are from an integer, i.e. on the quantity κ_n in (1.3). The single-valuedness of $\psi^n(z)$ is perfectly enough to construct Faber-like polynomials for Ω_γ by taking its polynomial part in its expansion around the point ∞.

Thus, let $\gamma=\cup_{j=1}^m\gamma_k$ be a family of Jordan curves lying exterior to one another. As before, $g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)$ is the Green's function of the unbounded component Ω_γ of the complement, and for $t>0$ set

$$\gamma_t=\{z\,\big|\,g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)=t\}.$$

For small t this consists of m analytic Jordan curves $\gamma_{k,t}$, $k=1,\ldots,m$, and fist we fix a $c_0>0$ such that γ_{4c_0} still has m components (i.e. $\gamma_{k,4c_0}$ all exist and are disjoint). In what follows t,t_j are always taken from the interval $[0,c_0]$. We shall work with unions of components of different level sets γ_t, so for $\mathbf{t}=(t_1,\ldots,t_m)$ set

$$\gamma(\mathbf{t})=\cup_{j=1}^m\gamma_{j,t_j}.$$

Note that this is not a level curve of the Green's function, it is rather built up from different components of different level curves. As before, let μ_γ denote the equilibrium measure of γ and consider the system of functions

$$F_k(t_1,\ldots,t_m)=\mu_{\gamma(t_1,\ldots,t_m)}(\gamma_{k,t_k}),\qquad k=1,\ldots,m. \tag{5.1}$$

These functions describe how much each part of $\gamma(\mathbf{t})$ carries from the total mass of the equilibrium measure $\mu_{\gamma(\mathbf{t})}$. We clearly have $\sum_k F_k(\mathbf{t})\equiv 1$, and also (see (2.7))

$$F_k(t,t,\ldots,t)\equiv F_k(0,\ldots,0) \tag{5.2}$$

for all $0\le t\le c_0$.

We shall have to work with balayage measures. If G is a domain and ν is a measure supported in G, then let $\mathrm{Bal}(\nu,G;\cdot)$ be the balayage measure obtained by sweeping ν out of G (see section 2). So $\mathrm{Bal}(\nu,G;\cdot)$ is a measure on ∂G with the same mass as ν, and the balayage measure of a Borel set $E \subset \partial G$ is denoted by $\mathrm{Bal}(\nu, G; E)$. Of course,

$$\mathrm{Bal}(\nu, G; E) = \int \mathrm{Bal}(\delta_a, G; E) d\nu(a),$$

and it is known (see e.g. [20, (A.3.3)]) that for $a \in G$ the value $\mathrm{Bal}(\delta_a, G; E)$ agrees with the harmonic measure of E with respect to a and G:

$$\mathrm{Bal}(\delta_a, G; E) = \omega(a, E, G)$$

(recall that $\omega(z, E, G)$ is the harmonic function in G which has boundary values 1 on E and 0 on $\partial G \setminus E$).

By a slight abuse of the language we can also say that $\mathrm{Bal}(\nu, G; \cdot)$ is the balayage of ν onto ∂G, and in this sense if ν is supported outside γ (i.e. in the unbounded component Ω_γ of $\overline{\mathbf{C}} \setminus \gamma$) then we shall write its balayage onto γ as $\mathrm{Bal}(\nu, \overline{\mathbf{C}} \setminus \gamma; \cdot)$ (i.e. $\mathrm{Bal}(\nu, \overline{\mathbf{C}} \setminus \gamma; \cdot) \equiv \mathrm{Bal}(\nu, \Omega_\gamma; \cdot)$). If ν is not supported outside γ but rather on the closure $\overline{\Omega_\gamma}$ of the unbounded component, then in forming the balayage measure $\mathrm{Bal}(\nu, \overline{\mathbf{C}} \setminus \gamma; \cdot)$ we sweep out only the part of ν that lies outside γ and leave its part on γ unchanged. A feature that we shall often use is that if the components γ_k' of γ' include the respective components γ_k of γ (i.e. γ lies in the polynomial convex hull $\mathrm{Pc}(\gamma')$ of γ'), then

$$\mathrm{Bal}(\nu, \overline{\mathbf{C}} \setminus \gamma; \cdot) = \mathrm{Bal}(\mathrm{Bal}(\nu, \overline{\mathbf{C}} \setminus \gamma'; \cdot), \overline{\mathbf{C}} \setminus \gamma; \cdot),$$

i.e taking balayage onto γ amounts the same as taking its balayage first onto γ', and then taking the balayage of this balayage measure onto γ. In particular, this implies that if $\gamma_k' = \gamma_k$ for a particular k, then

$$\mathrm{Bal}(\nu, \overline{\mathbf{C}} \setminus \gamma; \gamma_k) \geq \mathrm{Bal}(\nu, \overline{\mathbf{C}} \setminus \gamma'; \gamma_k). \tag{5.3}$$

The relevance of balayage measures to our subject is the formula: if γ' includes γ in the above sense then

$$\mu_\gamma = \mathrm{Bal}(\mu_{\gamma'}, \overline{\mathbf{C}} \setminus \gamma, \cdot). \tag{5.4}$$

In particular, for $t_j \leq t_j'$, $j = 1, \ldots, m$ we have that $\mu_{\gamma(\mathbf{t})}$ is nothing else than the balayage of $\mu_{\gamma(\mathbf{t}')}$ onto $\gamma_{\mathbf{t}}$. As an immediate consequence it follows that each of the functions F_j, $j \neq k$ in (5.1) is strictly monotone decreasing if $t_k \nearrow$, and since the sum of the F_j's is identically 1, we also get that F_k is strictly increasing as $t_k \nearrow$. Thus, this system $\{F_k\}_{k=1}^m$ is a monotone system in the sense of [21].

Next we remark that there clearly exists a constant $c_1 > 0$ such that for all $k = 1, \ldots, m$

$$\mathrm{Bal}(\delta_w, \overline{\mathbf{C}} \setminus \gamma(c_0, \ldots, c_0, \overset{k}{\breve{0}}, c_0, \ldots, c_0), \gamma_k) \geq c_1, \qquad w \in \gamma_{m, 2c_0}. \tag{5.5}$$

In a similar manner,

$$c_2 := \mu_{\gamma(c_0,\ldots,c_0,0)}(\gamma_m) = F_m(c_0,\ldots,c_0,0) \tag{5.6}$$

is a positive number. With these c_0, c_1, c_2 we set

$$M = \frac{5c_0}{c_1c_2}, \tag{5.7}$$

and prove

Proposition 5.1 *Let $0 < a \le c_0/M$. Then the range of*

$$\mathbf{F}(\mathbf{t}) = (F_1(\mathbf{t}),\ldots,F_{m-1}(\mathbf{t}))\,, \qquad \mathbf{t} \in [0, Ma]^m,$$

includes the cube $F(\mathbf{0}) + [-a,a]^{m-1}$.

Actually, we are going to show that the range includes that cube even for $\mathbf{t} \in [0, Ma]^m$ for which $t_m = Ma/2$ is fixed. Note that this valued vector function $\mathbf{F}$ does not contain the last function F_m.

Proof. Let $k < m$ and $0 \le t_j \le c_0$ for all j. First we estimate from below the quantity

$$F_k(t_1,\ldots,t_{m-1},t_m) - F_k(t_1,\ldots,t_{m-1},t_m+\tau)$$

for some $\tau < c_0$. In view of the definition of F_k and of (5.4), this quantity is equal to

$$\begin{aligned}\mathrm{Bal}\Big(\mu_{\gamma(t_1,\ldots,t_{m-1},t_m+\tau)}\big|_{\gamma_{m,t_m+\tau}}, \gamma(t_1,\ldots,t_{m-1},t_m);\gamma_{k,t_k}\Big)\\ \ge \mu_{\gamma(t_1,\ldots,t_{m-1},t_m+\tau)}(\gamma_{m,t_m+\tau}) \times \\ \times \min_{z\in\gamma_{m,t+\tau_m}} \mathrm{Bal}\Big(\delta_z, \gamma(t_1,\ldots,t_{m-1},t_m);\gamma_{k,t_k}\Big).\end{aligned} \tag{5.8}$$

Because of the monotonicity properties of F_m the first factor on the right is

$$F_m(t_1,\ldots,t_{m-1},t_m+\tau) \ge F_m(c_0,\ldots,c_0,0) = c_2$$

(see (5.6)). Let H be the domain enclosed by the curves γ_{m,t_m} and $\gamma_{m,2c_0}$. The expression

$$\mathrm{Bal}\Big(\delta_z, \gamma(t_1,\ldots,t_{m-1},t_m);\gamma_{k,t_k}\Big)$$

in the second factor in (5.8) is obtained by first taking the balayage of δ_z out of H (onto $\gamma_{m,t_m} \cup \gamma_{m,2c_0}$), and then taking the balayage onto $\gamma(t_1,\ldots,t_{m-1},t_m)$ of that part of this measure that sits on $\gamma_{m,2c_0}$ (the part that sits on γ_{m,t_m} is already on the set $\gamma(t_1,\ldots,t_m)$, os it is not moving when we take the second balayage). In view of (5.3)–(5.5), on γ_{k,t_k} this last balayage is at least as large as the total mass ($=\mathrm{Bal}(\delta_z, H; \gamma_{m,2c_0})$) times

$$\begin{aligned} &\min_{w\in\gamma_{m,2c_0}} \mathrm{Bal}(\delta_w, \overline{\mathbf{C}}\setminus\gamma(t_1,\ldots,t_m),\gamma_{k,t_k}) \\ &\geq \min_{w\in\gamma_{m,2c_0}} \mathrm{Bal}(\delta_w, \overline{\mathbf{C}}\setminus\gamma(c_0,\ldots,c_0,\overset{k}{\breve{t}_k},c_0,\ldots,c_0),\gamma_{k,t_k}) \\ &\geq \min_{w\in\gamma_{m,2c_0}} \mathrm{Bal}(\delta_w, \overline{\mathbf{C}}\setminus\gamma(c_0,\ldots,c_0,\overset{k}{\breve{0}},c_0,\ldots,c_0),\gamma_k) \geq c_1. \end{aligned}$$

Therefore,

$$\mathrm{Bal}\Big(\delta_z, \gamma(t_1,\ldots,t_{m-1},t_m);\gamma_{k,t_k}\Big) \geq c_1 \mathrm{Bal}\Big(\delta_z, H; \gamma_{m,2c_0}\Big). \tag{5.9}$$

Now $g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)$ is harmonic in H, it takes the value t_k on γ_{m,t_m}, it takes the value $2c_0$ on $\gamma_{m,2c_0}$ and these two curves make up the boundary of H, hence $(g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)-t_m)/(2c_0-t_m)$ is the harmonic measure in H corresponding to the boundary arc $\gamma_{m,2c_0}$. As a consequence, for $z\in\gamma_{m,t_m+\tau}$ we have

$$\mathrm{Bal}\Big(\delta_z, H; \gamma_{m,2c_0}\Big) = \omega(z,\gamma_{m,2c_0},H) = \frac{g_{\overline{\mathbf{C}}\setminus\gamma}(z,\infty)-t_m}{2c_0-t_m} = \frac{\tau}{2c_0-t_m} \geq \frac{\tau}{c_0}.$$

All in all, these considerations give (see (5.8))

$$F_k(t_1,\ldots,t_{m-1},t_m) - F_k(t_1,\ldots,t_{m-1},t_m+\tau) \geq c_2 c_1 \frac{\tau}{c_0}. \tag{5.10}$$

After these preparations let $\mathbf{A}=(A_1,\ldots,A_{m-1})\in[-a,a]^{m-1}$, and consider

$$\mathbf{G}(t_1,\ldots,t_{m-1}) = \mathbf{F}(t_1,\ldots,t_{m-1},Ma/2) - (\mathbf{F}(\mathbf{0})+\mathbf{A}).$$

The claim in the proposition follows if we show that his vector valued function $\mathbf{G}$ vanishes somewhere in $[0,Ma]^{m-1}$. For each $k=1,\ldots,m-1$ we have by (5.10) and by the choice of M in (5.7)

$$\begin{aligned} G_k(\overbrace{0,\ldots,0}^{m-1}) &= F_k(\overbrace{0,\ldots,0}^{m-1},Ma/2) - F_k(\overbrace{0,\ldots,0,0}^{m}) - A_k \\ &\leq -c_2c_1\frac{Ma/2}{c_0} - A_k < -2a+a = -a. \end{aligned} \tag{5.11}$$

On the other hand, (5.2) and (5.10) give

$$\begin{aligned} G_k(\overbrace{Ma,\ldots,Ma}^{m-1}) &= F_k(\overbrace{Ma,\ldots,Ma}^{m-1},Ma/2) - F_k(\overbrace{Ma,\ldots,Ma,Ma}^{m}) - A_k \\ &\geq c_2c_1\frac{Ma/2}{c_0} - A_k > 2a-a = a. \end{aligned} \tag{5.12}$$

The system $(G_k(t_1, \ldots, t_{m-1}))_{k=1}^{m-1}$ has the same monotonicity properties as the F_k's had, so for all t_j we can deduce from (5.11) for all $k = 1, \ldots, m-1$

$$G_k(t_1, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_{m-1}) \leq G_k(\mathbf{0}) < -a \tag{5.13}$$

and from (5.12)

$$G_k(t_1, \ldots, t_{k-1}, Ma, t_{k+1}, \ldots, t_{m-1}) \geq G_k(Ma, \ldots, Ma) > a. \tag{5.14}$$

By continuity then these inequalities remain through if 0 resp. Ma on the left is replaced by a t_k with $t_k = \varepsilon$ resp. $t_k = Ma - \varepsilon$ with some small ε. But this means that if $\alpha > 0$ is a small number, then close to the boundary of the cube $[0, Ma]^{m-1}$ the range of the mapping $\mathbf{t} \to \mathbf{t} - \alpha\mathbf{G}(\mathbf{t})$ lies in that same cube, and for sufficiently small α the same holds true when $\mathbf{t}$ is away from the boundary. Hence, this mapping maps $[0, Ma]^{m-1}$ into itself, and the Brower fixed point theorem gives then a $\mathbf{t} \in [0, Ma]^{m-1}$ for which $\mathbf{G}(\mathbf{t}) = \mathbf{0}$, and this is what we needed to prove.

■

We are going to apply what we have done above to the system γ that was constructed from Γ in the proof of Theorem 1.4 at the beginning of section 2. Recall that γ was the level curve of a harmonic function, hence it consists of analytic Jordan curves. Recall also that Γ was the δ-level curve of the Green's function $g_{\overline{\mathbf{C}}\setminus\gamma}(z, \infty)$, and we had $\text{cap}(\Gamma) = \text{cap}(\gamma)e^{\delta}$. We choose the c_0 from the previous discussion so small that γ_{4c_0} lies in the polynomial convex hull of Γ.

Consider the κ_n from (1.3). For a particular n let n_k, $k = 1, \ldots, m$, be the closest integer to $n\mu_\gamma(\gamma_1)$ (so $\{n\mu_\gamma(\gamma_k)\} = |n\mu_\gamma(\gamma_k) - n_k|$), and set

$$\mathbf{A} = \Big(n_1 - n\mu_\gamma(\gamma_1), \ldots, n_{m-1} - n\mu_\gamma(\gamma_{m-1})\Big).$$

Then $\mathbf{A}/n$ is from the cube $[-\kappa_n/n, \kappa_n/n]^{m-1}$, hence, by Proposition 5.1, there is a $\mathbf{t} \in [-\kappa_n/n, \kappa_n/n]^m$ such that the equality

$$nF_k(\mathbf{t}) = n(F_k(\mathbf{0}) + A_k/n) = n\mu_\gamma(\gamma_k) + A_k,$$

holds for all $k = 1, \ldots, m-1$. Since the right-hand side is an integer (n_k) by the definition of A_k, we get that $nF_m(\mathbf{t})$ must also be integer (note that the F_k's sum up to 1). Hence, all $nF_k(\mathbf{t})$ are integer, i.e. if we define $L = \gamma(\mathbf{t})$, then L consists of m analytic Jordan curves L_k and each of $n\mu_L(L_k)$, $k = 1, \ldots, m$ is an integer. Note that this L (and L_k) depends on n, but it is so close to γ that we have

$$\text{cap}(L) \leq \text{cap}(\gamma_{M\kappa_n/n}) = \text{cap}(\gamma)e^{M\kappa_n/n} \tag{5.15}$$

(since L lies in the polynomial convex hull of $\gamma_{M\kappa_n/n}$).

It is known (see e.g. [23, Theorem 3.2] or [20, Theorem II.4.11] and [16, Theorem 4.3.14]) that if $\mathbf{n}$ denotes the normal to L in the direction of the outer domain Ω_L, then for $z \in L$

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus L}(z,\infty)}{\partial \mathbf{n}} = 2\pi \frac{d\mu_L(z)}{ds_L}, \tag{5.16}$$

where s_L is the arc element on L. Thus,

$$\int_{L_k} \frac{\partial g_{\overline{\mathbf{C}}\setminus L}(z,\infty)}{\partial \mathbf{n}} ds_L(z) = 2\pi\mu_L(L_k).$$

Let $\tilde{g}$ be an analytic conjugate of $g_{\overline{\mathbf{C}}\setminus L}(z,\infty)$. By the Cauchy-Riemann equations

$$\frac{\partial g_{\overline{\mathbf{C}}\setminus L}(z,\infty)}{\partial \mathbf{n}} = \frac{\partial \tilde{g}(z)}{\partial \mathbf{e}},$$

where $\mathbf{e}$ is the unit tangent vector to L at $z \in L$ in the direction of positive (counterclockwise) orientation of each component of L. Hence,

$$\int_{L_k} \frac{\partial \tilde{g}(z)}{\partial \mathbf{e}} ds_L(z) = 2\pi\mu_L(L_k)$$

is also true. This means that, as we move once around L_k in the counterclockwise direction, the imaginary part of n-times the complex Green's function, i.e. of

$$n(g_{\overline{\mathbf{C}}\setminus L}(z,\infty) + i\tilde{g}(z)),$$

changes by $2\pi n\mu_L(L_k)$, which, by our construction, is an integer multiple of 2π. Hence

$$\Psi_n(z) = \exp\Big(n(g_{\overline{\mathbf{C}}\setminus L}(z,\infty) + i\tilde{g}(z) + \log \operatorname{cap}(L))\Big) \tag{5.17}$$

is a single-valued analytic function in the unbounded component Ω_L of $\overline{\mathbf{C}} \setminus L$. Since at infinity $g_{\overline{\mathbf{C}}\setminus L}(z,\infty)$ behaves like $\log|z| - \log \operatorname{cap}(L)$, we get that in a neighborhood of infinity

$$\Psi_n(z) = z^n + \beta_1 z^{n-1} + \cdots = S_n(z) + R_n(z),$$

where $S_n(z) = z^n + \cdots$ is a monic polynomial of degree n and R_n is an analytic function in the unbounded component Ω_L with a zero at infinity. Cauchy's formula gives for $z \in \Omega_L$

$$R_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{\Psi_n(\xi)}{\xi - z} d\xi$$

(note that $S_n(\xi)/(\xi - z)$ is analytic in ξ inside each L_k, so its integral over L is 0). In the denominator we have for $z \in \Gamma$ and $\xi \in L$

$$|\xi - z| \geq \operatorname{dist}(\Gamma, L) \geq \operatorname{dist}(\Gamma, \gamma_{2c_0}),$$

and in the numerator

$$|\Psi_n(\xi)| = \exp\Big(n g_{\overline{\mathbf{C}}\setminus L}(\xi,\infty) + n\log \mathrm{cap}(L)\Big) = \mathrm{cap}(L)^n.$$

It is easy to see from the construction of L that the length of L (which depends on n) is bounded (in n). Hence, for $z \in \Gamma$ we obtain

$$|R_n(z)| \le C\mathrm{cap}(\gamma)^n \tag{5.18}$$

with a C independent of n.

On the other hand,

$$|\Psi_n(z)| = \exp\Big(n g_{\overline{\mathbf{C}}\setminus L}(z,\infty) + n\log \mathrm{cap}(L)\Big).$$

Here we can apply (5.15) and the fact (note that γ is contained in the polynomial convex hull of L)

$$g_{\overline{\mathbf{C}}\setminus L}(z,\infty) \le g_{\overline{\mathbf{C}}\setminus \gamma}(z,\infty)$$

to conclude for $z \in \Gamma$

$$\begin{aligned} |\Psi_n(z)| &\le \exp\Big(n g_{\overline{\mathbf{C}}\setminus \gamma}(z,\infty) + n\log \mathrm{cap}(\gamma) + M\kappa_n\Big) \\ &= \exp(n\log \mathrm{cap}(\Gamma) + M\kappa_n) = \mathrm{cap}(\Gamma)^n e^{M\kappa_n}. \end{aligned} \tag{5.19}$$

(5.18) and (5.19) together imply for $z \in \Gamma$

$$|S_n(z)| = |\Psi_n(z) - R_n(z)| \le \mathrm{cap}(\Gamma)^n e^{M\kappa_n} + C\mathrm{cap}(\gamma)^n = \mathrm{cap}(\Gamma)^n(1 + O(\kappa_n + q^n))$$

with $q = \mathrm{cap}(\gamma_{c_0})/\mathrm{cap}(\Gamma)$ (which is bigger than $\mathrm{cap}(L)/\mathrm{cap}(\Gamma)$). This is the upper bound in (1.4), and the proof is complete.

■

The same proof gives the following: there are $\rho > 0$, C_0 and a sequence $\{S_n\}_{n\in\mathcal{N}}$ of monic polynomials of exact degree n such that if $\mathrm{dist}(z,\Gamma) < \rho$ then

$$\Big| n g_{\overline{\mathbf{C}}\setminus \gamma}(z,\infty) + n\log \mathrm{cap}(\gamma) - \log|S_n(z)| \Big| \le C_0(\kappa_n + q^n). \tag{5.20}$$

If we use this instead of (4.5) in the proof of Theorem 4.1 given in Section 4, then we get the sharp form (4.2) stated in Theorem 4.1.

■

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