Compression of quasianalytic spectral sets of cyclic contractions*

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Abstract

The class $\mathcal{L}_0(\mathcal{H})$ of cyclic quasianalytic contractions was studied in [K4]. The subclass $\mathcal{L}_1(\mathcal{H})$ consists of those operators T in $\mathcal{L}_0(\mathcal{H})$ whose quasianalytic spectral set $\pi(T)$ covers the unit circle $\mathbb{T}$. The contractions in $\mathcal{L}_1(\mathcal{H})$ have rich invariant subspace lattices. In this paper it is shown that for every operator $T \in \mathcal{L}_0(\mathcal{H})$ there exists an operator $T_1 \in \mathcal{L}_1(\mathcal{H})$ commuting with T. Thus, the hyperinvariant subspace problems for the two classes are equivalent. The operator T_1 is found as an H^∞-function of T. The existence of an appropriate function, compressing $\pi(T)$ to the whole circle, is proved using potential theoretic tools by constructing a suitable regular compact set on $\mathbb{T}$ with absolutely continuous equilibrium measure.

1 Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded, linear operators acting on $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$ let $\{T\}' = \{C \in \mathcal{L}(\mathcal{H}) : CT = TC\}$ denote the commutant of T, and let Hlat $T = \mathrm{Lat}\{T\}'$ stand for the hyperinvariant subspace lattice of T. The Invariant Subspace Problem (ISP) asks whether every operator $T \in \mathcal{L}(\mathcal{H})$ has a non-trivial invariant subspace, that is if Lat $T \neq \{\{0\}, \mathcal{H}\}$. In a similar fashion, the Hyperinvariant Subspace Problem (HSP) is whether every operator $T \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}I$ has a non-trivial hyperinvariant subspace. These problems are arguably the most challenging open questions in operator theory. From the point of view of subspaces one can normalize the operators to have norm at most 1, hence in what follows we shall only consider contractions. In the present work we shall show that for a relatively large class of contractions ($\mathcal{L}_0(\mathcal{H})$, see its definition below) the problem (HSP) is equivalent to (HSP) for a special subclass ($\mathcal{L}_1(\mathcal{H})$), the members of which have rich invariant subspace lattice.

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The reduction will be achieved by establishing that for every $T \in \mathcal{L}_0(\mathcal{H})$ there is a $T_1 \in \mathcal{L}_1(\mathcal{H})$ which commutes with T. This T_1 will be obtained as a function $f(T)$ of T, where f is a special conformal map lying in the disk algebra. The existence of f will be proven via potential theory.

We define some classes of contractions. These concepts were introduced (in the non-cyclic case too) in [K2], where it was shown, among others, that non-quasianalytic contractions (to be defined below) do have proper hyperinvariant subspaces. Thus, in the quest for such subspaces one should concentrate on quasianalytic contractions.

Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction: $\|T\| \leq 1$. We recall that the pair (X, V) is a unitary asymptote of T, if

(i) V is a unitary operator acting on a Hilbert space $\mathcal{K}$,

(ii) $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a contractive mapping intertwining T with V : $\|X\| \leq 1, XT = VX$, and

(iii) for any similar contractive intertwining pair (X', V') there exists a unique contractive transformation $Y \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ such that $YV = V'Y$ and $X' = YX$.

For the existence and uniqueness of unitary asymptotes we refer to [BK] (see also [K1]). We assume that T is of class C_{10}, which means that

- T is asymptotically non-vanishing: $\lim_{n\to\infty} \|T^n x\| > 0$ for every $0 \neq x \in \mathcal{H}$, and
- the adjoint T^* is stable: $\lim_{n\to\infty} \|(T^*)^n x\| = 0$ for every $x \in \mathcal{H}$.

Then the intertwining mapping X is injective, and the unitary operator V is absolutely continuous. Let us also assume that V is cyclic: $\vee_{n=0}^{\infty} V^n y = \mathcal{K}$ for some $y \in \mathcal{K}$. Then, for some measurable subset $\alpha \subset \mathbb{T}$ of the unit circle, V is unitarily equivalent to the multiplication operator M_α on the Hilbert space $L^2(\alpha)$ by the identity function $\chi(\zeta) = \zeta$: $M_\alpha f = \chi f,\ f \in L^2(\alpha)$. So from now on we may assume $\mathcal{K} = L^2(\alpha)$ and $Vf = \chi f,\ f \in L^2(\alpha)$. The set α is uniquely determined up to sets of zero Lebesgue measure, and is called the residual set of T, denoted by $\omega(T)$.

We say that T is quasianalytic on a measurable subset β of $\mathbb{T}$, if $(Xh)(\zeta) \neq 0$ for a.e. $\zeta \in \beta$ whenever $0 \neq h \in \mathcal{H}$. Taking the union of a sequence of quasianalytic sets, whose measures converge to the supremum (of measures of all quasianalytic sets), we obtain that there exists a largest quasianalytic set for T, denoted by $\pi(T)$. The set $\pi(T)$ is determined up to sets of zero Lebesgue measure, and is called the quasianalytic spectral set of T. Clearly, $\pi(T)$ is included in $\omega(T)$. The contraction T is quasianalytic, if $\pi(T) = \omega(T)$.

The paper [K4] introduced distinctive classes of quasianalytic contractions. The class $\mathcal{L}_0(\mathcal{H})$ consists of the operators $T \in \mathcal{L}(\mathcal{H})$ satisfying the conditions:

(i) T is a C_{10}-contraction,

(ii) the unitary operator V is cyclic, and

(iii) T is quasianalytic.

The subclass $\mathcal{L}_1(\mathcal{H})$ consists of those operators $T \in \mathcal{L}_0(\mathcal{H})$, which satisfy also the additional condition:

(iv) $\pi(T) = \mathbb{T}$.

Every operator $T \in \mathcal{L}_1(\mathcal{H})$ has a rich invariant subspace lattice Lat T; see [K3]. Let us consider also the class $\widetilde{\mathcal{L}}(\mathcal{H})$ of those (non-scalar) contractions $T \in \mathcal{L}(\mathcal{H})$, which are non-stable (i.e., $\lim_{n\to\infty} \|T^n x\| > 0$ for some $x \in \mathcal{H}$), and where the unitary asymptote V is cyclic. Clearly $\mathcal{L}_1(\mathcal{H}) \subset \mathcal{L}_0(\mathcal{H}) \subset \widetilde{\mathcal{L}}(\mathcal{H})$.

We emphasize that from the point of view of invariant subspaces these classes are very natural. Namely, we know from [K2] that the (HSP) in the class $\widetilde{\mathcal{L}}(\mathcal{H})$ is equivalent to the (HSP) in the class $\mathcal{L}_0(\mathcal{H})$. Furthermore, if the (HSP) has positive answer in $\widetilde{\mathcal{L}}(\mathcal{H})$, then the (ISP) has an affirmative answer in the large class of contractions T, where T or T^* is non-stable. As was mentioned earlier, the (ISP) in $\mathcal{L}_1(\mathcal{H})$ is answered affirmatively. Actually, a lot of information is at our disposal on the structure of operators in $\mathcal{L}_1(\mathcal{H})$, which may be helpful in the study of the (HSP) in this class; see [K3]. It was proved in [K4] that if $T \in \mathcal{L}_0(\mathcal{H})$ and $\pi(T)$ contains an arc then there exists $T_1 \in \mathcal{L}_1(\mathcal{H})$ such that $\{T\}' = \{T_1\}'$, and so Hlat $T =$ Hlat T_1. In the present paper we show that the whole class $\mathcal{L}_0(\mathcal{H})$ is strongly related to $\mathcal{L}_1(\mathcal{H})$, proving the following theorem.

Theorem 1 *For every operator $T \in \mathcal{L}_0(\mathcal{H})$ there exists $T_1 \in \mathcal{L}_1(\mathcal{H})$ commuting with $T:\ TT_1 = T_1T$.*

Since the commutants $\{T\}'$ and $\{T_1\}'$ are abelian (see e.g. Section 3 in [K4]), the relation $TT_1 = T_1T$ implies $\{T\}' = \{T_1\}'$, and so Hlat $T =$ Hlat T_1. Therefore, we obtain the following corollary.

Corollary 2 *The* (HSP) *in the class $\mathcal{L}_0(\mathcal{H})$ is equivalent to the* (HSP) *in the class $\mathcal{L}_1(\mathcal{H})$.*

These results are related to those in [FPN], [FP], [BFP] and [K3].

We provide an operator T_1 in $\mathcal{L}_1(\mathcal{H}) \cap \{T\}'$ as a function of T, using the Sz.-Nagy–Foias functional calculus; see Chapter III in [NFBK]. We shall apply the spectral mapping theorem established in [K4]. The existence of a function $f \in H^\infty$, satisfying the conditions $f(T) \in \mathcal{L}_0(\mathcal{H})$ and $\pi(f(T)) = f(\pi(T)) = \mathbb{T}$, is based on Theorem 3 below.

Let m denote the linear Lebesgue measure both on the real line and on the unit circle. A domain $G \subset \mathbb{C}$ is called a *circular comb domain* if it is obtained from the open unit disc $\mathbb{D}$ by deleting countably many radial segments of the form $\{r\zeta : \rho < r < 1\}$ with some $0 < \rho < 1$ and $\zeta \in \mathbb{T}$.

Theorem 3 *If Ω is a measurable subset of the unit circle $\mathbb{T}$ of positive (linear) measure, then there are a compact set $\widetilde{\Omega} \subset \Omega$ and a conformal map f from*

$\mathbb{D}$ onto a circular comb domain such that f can be extended to a continuous function on the closed unit disc $\overline{\mathbb{D}}$, $f^{-1}[\mathbb{T}] = \widetilde{\Omega}$, and $m(f[\omega]) = 0$ for every Borel subset ω of $\widetilde{\Omega}$ of zero measure.

Here, and in what follows, $f[A] := \{f(a) : a \in A\}$ is the range of f when restricted to A, and $f^{-1}[B] := \{b : f(b) \in B\}$ is the complete inverse image of the set B under the map f. When $B = \{b\}$ has only one element, then we write $f^{-1}[b]$ instead of $f^{-1}[\{b\}]$.

Theorem 3 will be derived from the subsequent Theorem 4. To formulate it wee need some potential theoretical preliminaries. For all these facts see [R], [GM] or [SaT]. Let K be a compact set on $\mathbb{C}$, and let $\mathcal{P}(K)$ be the system of all probability (Borel) measures supported on K. The potential

$$p_\nu(z) = \int_K \log|z - w| \, d\nu(w)$$

of a measure $\nu \in \mathcal{P}(K)$ is a subharmonic function on $\mathbb{C}$, which is harmonic on $\mathbb{C} \setminus K$. The (logarithmic) capacity of K is defined by $\mathrm{cap}(K) = \exp(M(K))$, where

$$M(K) = \sup\left\{\int_K p_\nu \, d\nu : \nu \in \mathcal{P}(K)\right\}.$$

If $\mathrm{cap}(K) > 0$, then there exists a unique measure $\mu_K \in \mathcal{P}(K)$, called the equilibrium measure of K, which is maximizing the energy integral:

$$\int_K p_{\mu_K} \, d\mu_K = M(K);$$

we write $p_K = p_{\mu_K}$ for short. By Frostman's theorem there is an F_σ-subset F of K with $\mathrm{cap}(F) = 0$ such that $p_K(z) = M(K)$ for all $z \in K \setminus F$, and $p_K(z) > M(K)$ for all $z \in F \cup (\mathbb{C} \setminus K)$. The compact set K is called *regular*, if the potential p_K is continuous on $\mathbb{C}$, or equivalently, if the previous exceptional set F is empty.

Theorem 4 *Let $E \subset \mathbb{R}$ be a compact set of positive Lebesgue measure. Then for every $\varepsilon > 0$, there is a regular compact set $K \subset E$ such that $m(E \setminus K) < \varepsilon$, and μ_K is absolutely continuous with respect to the Lebesgue measure on the real line $\mathbb{R}$.*

Theorems 3 and 4 should be compared to [P, Proposition 9.15]. Here the additional absolute continuity of the extremal measure is the key to our results.

In Section 2 the functional calculus within the class $\mathcal{L}_0(\mathcal{H})$ is discussed, and Theorem 1 is proved relying on Theorem 3. The proofs of Theorems 3 and 4 are given in Section 3.

2 Functional calculus in $\mathcal{L}_0(\mathcal{H})$

In order to get C_{10}-contractions, we consider functions in the Hardy class H^∞ with specific boundary behaviour.

Let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable sets on $\mathbb{T}$. For a complex function f defined on the open unit disc $\mathbb{D}$, let $\Omega(f)$ be the set of those points $\zeta \in \mathbb{T}$, where the radial limit

$$\lim_{r \to 1-0} f(r\zeta) =: f(\zeta)$$

exists and is of modulus 1: $|f(\zeta)| = 1$. It can be easily seen that if f is continuous on $\mathbb{D}$, then $\Omega(f) \in \mathcal{M}$.

For any $f \in H^\infty$ the radial limit exists almost everywhere on $\mathbb{T}$ by Fatou's theorem; see [H]. We recall from [K4] that $f \in H^\infty$ is a *partially inner function*, if

(i) $|f(0)| < 1 = \|f\|_\infty$, and

(ii) $m(\Omega(f)) > 0$.

Note that (i) implies $f[\mathbb{D}] \subset \mathbb{D}$ by the Maximum Principle. Furthermore, Corollary 2 of [K4] states that $m(f^{-1}[\omega]) = 0$ for every $\omega \in \mathcal{M}$ with $m(\omega) = 0$ (recall also that every set of measure 0 is included in a Borel set of measure zero). Hence, for any $\Omega \in \mathcal{M}$, $\Omega \subset \Omega(f)$, the measure $\mu\colon \mathcal{M} \to [0, 2\pi]$, $\mu(\omega) = m(f^{-1}[\omega] \cap \Omega)$ is absolutely continuous with respect to m. The properly essential range of the restriction $f|_\Omega$ is defined by

$$\text{pe-ran}(f|_\Omega) := \{\zeta \in \mathbb{T} : (d\mu/dm)(\zeta) > 0\}.$$

Note that the Radon–Nikodym derivative $d\mu/dm$, and so the Lebesgue measurable set pe-ran$(f|_\Omega)$ too, is determined up to sets of measure zero. The spectral mapping theorems in Section 2 of [K4] are formulated in terms of this kind of range.

The properly essential range is just the range of the function under some regularity conditions. We introduce this regularity property of a partially inner function in a somewhat different (and simpler) manner than in [K4]. We say that a function $g\colon \Omega \to \mathbb{T}$, where $\Omega \subset \mathbb{T}$ is a measurable subset of $\mathbb{T}$, is *weakly absolutely continuous*, if $\omega \subset \Omega$, $m(\omega) = 0$, implies $m(g[\omega]) = 0$. The partially inner function $f \in H^\infty$ is called *regular*, if $f|_{\Omega(f)}$ is a weakly absolutely continuous function. The following lemma shows that this definition is essentially the same as the one given in [K2] and [K4], replacing Borel sets occurring there by Lebesgue measurable sets.

Lemma 5 *Let $f \in H^\infty$ be a partially inner function.*
(a) *Then f is regular if and only if for every measurable set $\Omega \subset \Omega(f)$ the image set $f[\Omega]$ is also measurable.*
(b) *If f is regular and $\Omega \in \mathcal{M}$, $\Omega \subset \Omega(f)$, then* pe-ran$(f|_\Omega) = f(\Omega)$.

Recall that pe-ran$(f|_\Omega)$ is determined only up to measure zero, so the equality pe-ran$(f|_\Omega) = f(\Omega)$ is also understood up to measure zero.

Proof. (a): We sketch the proof of this known equivalence. Suppose that f is regular, and let $\Omega \in \mathcal{M}$, $\Omega \subset \Omega(f)$. Since $f|_\Omega$ is the pointwise limit of a sequence of continuous functions, it follows from Egorov's theorem that $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and $f[\Omega_1]$ are F_σ-sets and $m(\Omega_2) = 0$. Hence, by assumption, $m(f[\Omega_2]) = 0$ and thus $f[\Omega] \in \mathcal{M}$.

Conversely, if f is non-regular, then $m(f[\omega]) = 0$ fails for some $\omega \subset \Omega(f)$ with $m(\omega) = 0$. There is a non-measurable subset Ω' of $f[\omega]$. Thus $\Omega = f^{-1}[\Omega'] \cap \omega \in \mathcal{M}$, while $f[\Omega] = \Omega' \notin \mathcal{M}$.

(b): The sets $\omega_1 = f[\Omega]$ and $\omega_2 =$pe-ran$(f|_\Omega)$ are in $\mathcal{M}$. Let us consider the measure μ occurring in the definition of ω_2, and let $g = d\mu/dm$. Since

$$\int_{\omega_2 \setminus \omega_1} g\, dm = \mu(\omega_2 \setminus \omega_1) = m\left((f|_\Omega)^{-1}[\omega_2 \setminus \omega_1]\right) = m(\emptyset) = 0$$

and $g(\zeta) > 0$ for $\zeta \in \omega_2 \setminus \omega_1$, it follows that $m(\omega_2 \setminus \omega_1) = 0$. On the other hand, we have

$$m\left((f|_\Omega)^{-1}[\omega_1 \setminus \omega_2]\right) = \mu(\omega_1 \setminus \omega_2) = \int_{\omega_1 \setminus \omega_2} g\, dm = 0$$

since $g(\zeta) = 0$ for (almost all) $\zeta \in \omega_1 \setminus \omega_2$; thus $m(\omega_1 \setminus \omega_2) = 0$ by the regularity condition.

□

Applying the functional calculus, for an operator in $\mathcal{L}_0(\mathcal{H})$ we want to get another operator in $\mathcal{L}_0(\mathcal{H})$, which means that the cyclic property should be preserved. Hence, univalent functions will be considered in the sequel. We recall that $f\colon \mathbb{D} \to \mathbb{C}$ is called a univalent function (or a conformal map) if it is analytic and injective. The range $G = f[\mathbb{D}]$ of f is a simply connected domain, different from $\mathbb{C}$. The boundary ∂G of G is a non-empty closed set. It is known that the geometric properties of ∂G are reflected in the analytic properties of f. For example ∂G is a curve (i.e. a continuous image of the unit circle) exactly when f belongs to the disk algebra $\mathcal{A}$, and then $\partial G = f[\mathbb{T}]$ (see Theorem 2.1 in [P]). We recall that the disk algebra $\mathcal{A}$ consists of those analytic complex functions on $\mathbb{D}$, which can be continuously extended to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. We focus our attention to the class

$$\mathcal{A}_1 := \left\{ f \in \mathcal{A} : f|_{\mathbb{D}} \text{ is univalent} \right\}.$$

The following proposition shows that every partially inner function in $\mathcal{A}_1$ has an almost injective unimodular component. The cardinality of a set H is denoted by $|H|$. For distinct points $\zeta_1, \zeta_2 \in \mathbb{T}$, the open arc determined by ζ_1 and ζ_2 is defined by $\widehat{\zeta_1\zeta_2} = \{e^{it} : t_1 < t < t_2\}$, where $t_1 < t_2 < t_1 + 2\pi$ and $\zeta_1 = e^{it_1}, \zeta_2 = e^{it_2}$.

Proposition 6 *Let $f \in \mathcal{A}_1$ be a partially inner function.*

(a) *If $f(\zeta_1) = f(\zeta_2) = w$ holds for distinct points $\zeta_1, \zeta_2 \in \Omega(f)$, then for one of the arcs $I = \widehat{\zeta_1\zeta_2}$ or $I = \widehat{\zeta_2\zeta_1}$ we have $m(I \cap \Omega(f)) = 0$ and $f(\zeta) = w$ for every $\zeta \in I \cap \Omega(f)$.*

(b) *The set* $M = \{w \in \mathbb{T} : |f^{-1}[w]| > 1\}$ *of multiple image points on* $\mathbb{T}$ *is countable.*

(c) *For any Borel subset* Ω *of* $\Omega(f)$ *with* $m(\Omega) > 0$ *we have* $f[\Omega] = \text{pe-ran}(f|_\Omega)$ *if and only if* $f|_\Omega$ *is weakly absolutely continuous.*

Proof. Statement (b) is an easy consequence of statement (a).

We sketch the proof of (a), which is based on ideas taken from the proof of the related Proposition 2.5 in [P]. Let S denote the segment joining ζ_1 with ζ_2. Then $J = f[S]$ is a (closed) Jordan curve in $\mathbb{D} \cup \{w\}$. Let us consider the open sets $G_1 = G \cap \text{int}J$ and $G_2 = G \cap \text{ext}J$, where $G = f[\mathbb{D}]$. It is easy to check that $D_1 = f^{-1}[G_1]$, $D_2 = f^{-1}[G_2]$ are the connected components of $\mathbb{D} \setminus S$, and $G_1 = f[D_1]$, $G_2 = f[D_2]$. We may assume that $\partial D_1 = S \cup \widehat{\zeta_1\zeta_2}$; the other case $\partial D_1 = S \cup \widehat{\zeta_2\zeta_1}$ can be treated similarly. For every $\zeta \in \widehat{\zeta_1\zeta_2} \cap \Omega(f)$, we have $f(\zeta) \in \overline{G_1} \cap \mathbb{T} = \{w\}$. Since $m(f^{-1}[w]) = 0$, the statement follows.

Turning to the proof of (c) notice first that $\Omega(f)$ is a compact set on $\mathbb{T}$. In view of (b) the system

$$\mathcal{S} = \{\omega : \omega \subset \Omega(f),\ \omega, f(\omega) \text{ are Borel measurable}\}$$

is a σ-algebra on $\Omega(f)$ containing compact sets; hence $\mathcal{S}$ consists of the Borel subsets of $\Omega(f)$. Setting $\omega_1 = f[\Omega]$ and $\omega_2 = \text{pe-ran}(f|_\Omega)$ we know that $m(\omega_2 \setminus \omega_1) = 0$ always holds, and $m(\omega_1 \setminus \omega_2) = 0$ whenever $f|_\Omega$ is weakly absolutely continuous; see the proof of Lemma 5. Assuming that $f|_\Omega$ is not weakly absolutely continuous, there exists a Borel set $\omega \subset \Omega$ such that $m(\omega) = 0$ and $m(\omega') > 0$ for $\omega' = f[\omega]$. Applying (b) again, we can see that $\int_{\omega'} g\,dm = \mu(\omega') = m\left((f|_\Omega)^{-1}[\omega']\right) = 0$ holds for $g = d\mu/dm$, and so $m(\omega_2 \cap \omega') = 0$, whence $m(\omega_1 \setminus \omega_2) \geq m(\omega') > 0$ follows.

□

The following theorem describes the functional calculus within the class $\mathcal{L}_0(\mathcal{H})$. It plays crucial role in the proof of Theorem 1.

Theorem 7 *Setting* $T \in \mathcal{L}_0(\mathcal{H})$*, let* $f \in \mathcal{A}_1$ *be a regular partially inner function such that* $m(\pi(T) \cap \Omega(f)) > 0$*. Then* $T_0 = f(T) \in \mathcal{L}_0(\mathcal{H})$ *and we have* $\pi(T_0) = f[\pi(T) \cap \Omega(f)]$*.*

Proof. By Proposition 6 the set $M = \{w \in \mathbb{T} : |f^{-1}[w]| > 1\}$ is countable, hence $m(M) = 0$ yields $m(f^{-1}[M]) = 0$. Deleting $f^{-1}[M]$ from the quasianalytic spectral set (which is determined up to sets of measure zero), we may assume that f is injective on the set $\alpha = \pi(T) \cap \Omega(f) \in \mathcal{M}$. We know also that $\beta = f[\alpha] \in \mathcal{M}$, and $m(\alpha) > 0$, $m(\beta) > 0$. Furthermore, the restriction $\phi = f|_\alpha : \alpha \to \beta$ is a bijection, and for any $\omega \subset \alpha$ we have $\omega \in \mathcal{M}$ if and only if $\phi[\omega] \in \mathcal{M}$, and $m(\omega) = 0$ exactly when $m(\phi[\omega]) = 0$. We use the notation $\tilde{\alpha} = \pi(T) = \omega(T)$. Let $(X, M_{\tilde{\alpha}})$ be a unitary asymptote of T, with a properly chosen contractive intertwining mapping $X : XT = M_{\tilde{\alpha}}X$.

Since T is a completely non-unitary contraction, it follows that $T_0 = f(T)$ is also a completely non-unitary contraction (see Chapter III in [NFBK]). We know that T_0 is quasianalytic and $\pi(T_0) = \beta$ (see Corollary 5 in [K4] and Proposition 6). The condition $m(\pi(T_0)) > 0$ yields $T_0 \in C_{1\cdot}$, and $T \in C_{\cdot 0}$ readily implies $T_0 \in C_{\cdot 0}$. Furthermore, by Theorem 3 in [K4] the pair $(X_0, \phi(M_\alpha))$ is a unitary asymptote of T_0, where $X_0 v = \chi_\alpha X v$ $(v \in \mathcal{H})$ (here χ_α is the characteristic function of the set α). We know that $\phi(M_\alpha)$ is an absolutely continuous unitary operator because T_0 is an absolutely continuous contraction. It remains to show that $\phi(M_\alpha)$ is cyclic.

Let us introduce the measure ν on

$$\mathcal{M}(\beta) = \{\omega \in \mathcal{M} : \omega \subset \beta\}$$

via

$$\nu(\omega) = m(\phi^{-1}[\omega]).$$

The properties of ϕ imply that ν is equivalent to (mutually absolutely continuous with) the Lebesgue measure on β. Let us consider the unitary operator $N_\nu \in \mathcal{L}(L^2(\nu))$, $N_\nu g = \chi g$, which is unitarily equivalent to M_β (see Theorem IX.3.6 in [C]). It is easy to verify that $Z\colon L^2(\nu) \to L^2(\alpha)$, $g \mapsto g \circ \phi$ is a unitary transformation, intertwining N_ν with $\phi(M_\alpha)$: $Z N_\nu = \phi(M_\alpha) Z$. Therefore, $\phi(M_\alpha)$ is unitarily equivalent to M_β, and so it is cyclic.

□

Now we proceed with the proof of Theorem 1 relying on the statement of Theorem 3.

Proof of Theorem 1. Let T be a contraction in the class $\mathcal{L}_0(\mathcal{H})$, and let us consider the quasianalytic spectral set $\Omega = \pi(T)$ of positive measure. By Theorem 3 there exist a compact set $\widetilde{\Omega} \subset \Omega$ and a function $f \in \mathcal{A}_1$ such that $f[\mathbb{D}]$ is a circular comb domain, $f^{-1}[\mathbb{T}] = \widetilde{\Omega}$, and $f|_{\widetilde{\Omega}}$ is weakly absolutely continuous. In other words, f is a regular partially inner function with $\Omega(f) = \widetilde{\Omega}$ and $f[\widetilde{\Omega}] = \mathbb{T}$. Applying Theorem 7 we conclude that $T_1 = f(T) \in \mathcal{L}_0(\mathcal{H})$ and $\pi(T_1) = f[\pi(T) \cap \Omega(f)] = f[\widetilde{\Omega}] = \mathbb{T}$, whence $T_1 \in \mathcal{L}_1(\mathcal{H})$ follows. Being norm-limit of polynomials of T, the operator T_1 commutes with T.

□

3 Absolutely continuous equilibrium measures

First we prove Theorem 3 applying Theorem 4.

Proof of Theorem 3. Let $\Omega \subset \mathbb{T}$ be a set of positive Lebesgue measure, and let $\Omega_1 \subset \Omega$ be a compact subset of positive measure. Applying rotation we may assume that 1 is a density point of Ω_1; let Ω_1' be its reflection onto the real axis. The compact set $\Omega_2 = \Omega_1 \cap \Omega_1'$ is of positive measure and symmetric with respect to $\mathbb{R}$. Let us consider the bijective Joukovskii map $\varphi\colon \mathbb{D} \to \overline{\mathbb{C}} \setminus [-1, 1]$, defined by $\varphi(z) = (z + 1/z)/2$; the continuous extension to $\overline{\mathbb{D}}$ is also denoted by φ. Then $E = \varphi[\Omega_2]$ is a compact subset of $[-1, 1]$ with positive measure, and $\Omega_2 = \varphi^{-1}[\varphi[\Omega_2]]$ because of the symmetry of Ω_2.

By Theorem 4 there is a regular compact subset K of E with an absolutely continuous equilibrium measure μ_K. Let $[a,b]$ be the smallest interval containing K. Consider the analytic function

$$\Phi(z) = \exp\left(-\int_K \log(z-t)\, d\mu_K(t) + \log \operatorname{cap}(K)\right)$$

on the upper half plane $\mathbb{H}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ with that branch of log which is positive on $(0,\infty)$. It is easy to see that for every $x \in \mathbb{R}$ the function ratio $\Phi(z)/|\Phi(z)|$ converges to $\exp\left[-i\pi\mu_K((x,\infty))\right]$ as $z \to x$ from the upper half plane. Since $|\Phi(z)| = \exp(-p_K(z)) \cdot \operatorname{cap}(K)$ and K is regular, it follows that Φ can be continuously extended to the closure of $\mathbb{H}_+$ in $\overline{\mathbb{C}}$; $\Phi(\infty) = 0$. We can see that $\Phi(K)$ coincides with the lower circle $\mathbb{T}_- = \{z \in \mathbb{T} : \Im z \leq 0\}$, $\Phi(\overline{\mathbb{R}} \setminus (a,b)) = [-1,1]$, and each component I of $(a,b) \setminus K$ is mapped by Φ onto a radial segment of the form $\{r\zeta : \rho < r < 1\}$ with some $0 < \rho < 1$ and $\zeta \in \mathbb{T}_-$. It can be shown also that Φ is univalent; see Chapter 2.1 in [A]. Since $\Phi(x) = \exp\left[-i\pi\mu_K((x,\infty))\right]$ for $x \in K$ and μ_K is absolutely continuous, it follows that sets of measure zero on K are mapped by Φ into sets of measure zero.

Let G_+ be the domain $\Phi(\mathbb{H}_+)$, and G_- its reflection onto the real axis. Since $\Phi(z)$ is real for $z \in \mathbb{R} \setminus [a,b]$, using the reflection principle we can extend Φ via the definition $\Phi(z) = \overline{\Phi(\overline{z})}$, $\Im z < 0$ to a conformal map of the domain $\overline{\mathbb{C}} \setminus [a,b]$ onto the circular comb domain $G = G_+ \cup G_- \cup (-1,1)$. Then $f = \Phi \circ \varphi$ is a conformal map from $\mathbb{D}$ onto G, it belongs to the disk algebra, and we have $f[\widetilde{\Omega}] = \mathbb{T}$, $f[\mathbb{T} \setminus \widetilde{\Omega}] \subset \mathbb{D}$ for the compact set $\widetilde{\Omega} = \varphi^{-1}[K] \subset \Omega$. If $\omega \subset \widetilde{\Omega}$ is of zero linear measure, then $f[\omega]$ is also of zero linear measure. Thus $\widetilde{\Omega}$ and f have all the properties set forth in the theorem.

Note also that for compact, symmetric Ω the measure of $\Omega \setminus \widetilde{\Omega}$ can be made as small as we wish.

$\square$

To prove Theorem 4 we need two lemmas.

Lemma 8 *Let $1 \leq \xi_1 < \alpha_1 < \xi_2 < \alpha_2 < \cdots < \xi_l < \alpha_l$. Then for $x, y \in [-1,0]$ we have*

$$\frac{1}{2} \leq \prod_{s=1}^{l} \left(\frac{\xi_s - x}{\alpha_s - x} \Big/ \frac{\xi_s - y}{\alpha_s - y}\right) \leq 2. \tag{1}$$

In a similar manner, if $1 \leq \beta_1 < \xi_1 < \beta_2 < \cdots < \beta_l < \xi_l$, then for $x, y \in [-1,0]$ we have

$$\frac{1}{2} \leq \prod_{s=1}^{l} \left(\frac{\xi_s - x}{\beta_s - x} \Big/ \frac{\xi_s - y}{\beta_s - y}\right) \leq 2. \tag{2}$$

Proof. The inequalities (2) are obtained by taking reciprocal in (1) and switching the role of β_s, ξ_s and ξ_s, α_s. Similarly, in proving (1) we may assume without loss of generality that $y \leq x$. The product in (1) can be written as

$$\prod_{s=1}^{l} \left(\frac{\xi_s - x}{\xi_s - y} \Big/ \frac{\alpha_s - x}{\alpha_s - y}\right) = \left(\frac{\xi_1 - x}{\xi_1 - y} \Big/ \frac{\alpha_l - x}{\alpha_l - y}\right) \prod_{s=1}^{l-1} \left(\frac{\xi_{s+1} - x}{\xi_{s+1} - y} \Big/ \frac{\alpha_s - x}{\alpha_s - y}\right)$$

($l \geq 2$ can be assumed). Since $(t-x)/(t-y)$ is increasing on $(0,\infty)$, it immediately follows from the left hand side that the product in question is at most 1. On the other hand, by the same token the second factor on the right is at least 1, so the product is at least as large as

$$\frac{\xi_1 - x}{\xi_1 - y}\Big/\frac{\alpha_l - x}{\alpha_l - y} \geq \frac{\xi_1 - x}{\xi_1 - y} \geq \frac{1}{2}.$$

□

Let $\beta_1 < \alpha_1 < \cdots < \beta_l < \alpha_l$ be positive integers, and let $\xi_s \in (\beta_s, \alpha_s)$ for every $1 \leq s \leq l$. Taking the geometric mean of the products in (1) and (2) of Lemma 8 it follows that

$$\frac{1}{2} \leq \prod_{s=1}^{l} \left(\frac{|x-\xi_s|}{\sqrt{|x-\alpha_s||x-\beta_s|}} \Big/ \frac{|y-\xi_s|}{\sqrt{|y-\alpha_s||y-\beta_s|}} \right) \leq 2 \tag{3}$$

for every $x, y \in [-1, 0]$. Multiplying everything by (-1), and changing the notation it follows that (3) holds also, when α_s, β_s are negative integers and $x, y \in [0,1]$. Let $\mathbb{Z}$ denote the set of integers. Via scaling (multiplying everything by 2^{-N} ($N \in \mathbb{N}$) and applying translation), we obtain that

(3) *is true if* $\alpha_s, \beta_s \in 2^{-N}\mathbb{Z}$ *for every* $1 \leq s \leq l$ *and* $x, y \in [(j-1)/2^N, j/2^N]$ with some $j \in \mathbb{Z}$ satisfying *the condition* $j/2^N < \beta_1$ *or* $(j-1)/2^N > \alpha_l$. (4)

Given $N \in \mathbb{N}$ let $I_{N,j} = \left[(j-1)2^{-N}, j2^{-N}\right]$ for any $j \in \mathbb{Z}$. Setting a non-empty set $S \subset \{k \in \mathbb{N} : k \leq 2^N\}$ of non-consecutive indeces, let us consider the compact set $F = \cup_{j \in S} I_{N,j}$, which can be written in the form $F = \cup_{s=1}^{n} [a_s, b_s]$ with $a_1 < b_1 < a_2 < b_2 < \cdots < b_n$ ($n \geq 2$). The equilibrium measure μ_F of F is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}$, and its density function is given by the formula

$$\psi(t) = (d\mu_F/dm)(t) = \frac{1}{\pi} \frac{\prod_{s=1}^{n-1} |t-\tau_s|}{\prod_{s=1}^{n} \sqrt{|t-a_s||t-b_s|}}\, dt, \qquad t \in F, \tag{5}$$

where the numbers $\tau_s \in (b_s, a_{s+1})$ $(1 \leq s \leq n-1)$ are the unique solution of the system of equations

$$\int_{b_k}^{a_{k+1}} \frac{\prod_{s=1}^{n-1} (t-\tau_s)}{\prod_{s=1}^{n} \sqrt{|t-a_s||t-b_s|}}\, dt = 0, \qquad 1 \leq k \leq n-1. \tag{6}$$

This is a linear system in the coefficients of the polynomial $\prod_{s=1}^{n-1}(t-\tau_s)$. When $n = 1$ then the product in the numerator (5) is replaced by 1. For all these see Lemma 4.4 in [StT] and Chapter III, (5.8) in [SaT].

Lemma 9 *Let $0 < \eta < 1/2$, $j \in S$, and H a measurable subset of $I_{N,j}$ (N, S, F and $I_{N,j}$ are as before). If*

$$m(H) \geq (1 - 2\eta) m(I_{N,j}), \tag{7}$$

then

$$\mu_F(H) \geq \left(1 - 2^{29}\eta^{1/2}\right) \mu_F(I_{N,j}). \tag{8}$$

Proof. We shall give an estimate of the density function ψ on $I_{N,j}$. Assuming that $I_{N,j} \subseteq [a_r, b_r]$, this estimate depends on the position of $I_{N,j}$ inside $[a_r, b_r]$.

Case I. $a_r, b_r \notin I_{N,j}$, i.e. $I_{N,j}$ lies inside (a_r, b_r). For $x, y \in I_{N,j}$ we can write

$$\frac{\psi(x)}{\psi(y)} = \sqrt{\frac{|y - a_1|}{|x - a_1|} \Big/ \frac{|x - b_n|}{|y - b_n|}} \cdot \frac{\theta_{1,r-1}(x)}{\theta_{1,r-1}(y)} \cdot \frac{\theta_{r,n-1}(x)}{\theta_{r,n-1}(y)}, \tag{9}$$

where

$$\theta_{k,l}(x) = \frac{\prod_{s=k}^{l} |x - \tau_s|}{\prod_{s=k}^{l} \sqrt{|x - a_{s+1}||x - b_s|}}$$

($\theta_{1,0} = \theta_{n,n-1} = 1$ by definition). Since each factor in this decomposition (9) of $\psi(x)/\psi(y)$ lies between $1/2$ and 2 by (4), it follows that

$$\frac{1}{8}\psi(y) \leq \psi(x) \leq 8\psi(y). \tag{10}$$

Case II. Precisely one of a_r, b_r belongs to $I_{N,j}$. Then either $j2^{-N} = b_r$ or $(j-1)2^{-N} = a_r$, say $j2^{-N} = b_r$. We shall consider only the situation when $1 < r < n$, for the other options (i.e. when $r = 1$ or $r = n$) are simpler. In this case

$$\pi\psi(x) = \frac{|x - \tau_r|}{\sqrt{|x - b_r||x - a_{r+1}|}} \cdot \theta_1(x)\theta_2(x), \tag{11}$$

where

$$\theta_1(x) = \frac{1}{\sqrt{|x - a_1|}} \cdot \theta_{1,r-1}(x)$$

and

$$\theta_2(x) = \frac{1}{\sqrt{|x - b_n|}} \cdot \theta_{r+1,n-1}(x).$$

Next we prove that here

$$\tau_r - b_r \geq 2^{-8}2^{-N}. \tag{12}$$

If $\tau_r - b_r \geq 2^{-N}$ then there is nothing to prove, so let us assume that $\tau_r \in [b_r, b_r + 2^{-N}]$. For $t \in [b_r, b_r + 2^{-N}]$ the claim (4) gives the bounds

$$\frac{\theta_i(b_r)}{4} \leq \theta_i(t) \leq 4\theta_i(b_r), \qquad i = 1, 2. \tag{13}$$

For $k = r$ the equation (6) can be written as

$$\int_{b_r}^{a_{r+1}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}}\,\theta_1(t)\theta_2(t)\,dt = 0,$$

so

$$\begin{aligned}\int_{b_r}^{\tau_r} \frac{\tau_r - t}{\sqrt{(t-b_r)(a_{r+1}-t)}}\,\theta_1(t)\theta_2(t)\,dt & \\ &= \int_{\tau_r}^{a_{r+1}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}}\,\theta_1(t)\theta_2(t)\,dt \\ &\geq \int_{\tau_r}^{b_r+2^{-N}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}}\,\theta_1(t)\theta_2(t)\,dt.\end{aligned}$$

In view of (13) this gives after division by $\theta_1(b_r)\theta_2(b_r)$ the inequality

$$\int_{b_r}^{\tau_r} \frac{\tau_r - t}{\sqrt{(t-b_r)(a_{r+1}-t)}} 16\,dt \geq \int_{\tau_r}^{b_r+2^{-N}} \frac{t-\tau_r}{\sqrt{(t-b_r)(a_{r+1}-t)}}\,\frac{1}{16}\,dt.$$

If we make a linear substitution so that $[b_r, b_r + 2^{-N}]$ becomes $[0, 1]$ and make use that for $0 \leq \tau \leq 2^{-8}$ and $l \in \mathbb{N}$ the inequality

$$\int_0^{\tau} \frac{\tau - u}{\sqrt{u(l-u)}} 16\,du < \int_{\tau}^{1} \frac{u-\tau}{\sqrt{u(l-u)}}\,\frac{1}{16}\,du$$

holds, we can conclude (12).

Now (12) immediately gives that for $x, y \in I_{N,j}$

$$\frac{|x-\tau_r|}{|y-\tau_r|} \leq 2^9. \tag{14}$$

Next note that along with (13) the bounds

$$\frac{\theta_i(y)}{4} \leq \theta_i(x) \leq 4\theta_i(y) \qquad (i = 1, 2) \tag{15}$$

are also true for $x, y \in I_{N,j}$ (since $(j-1)2^{-N}$ is not an endpoint of a subinterval of F), so (11), (14) and (15) yield for $x, y \in I_{N,j}$

$$\frac{\psi(x)\sqrt{|x-b_r|}}{\psi(y)\sqrt{|y-b_r|}} \leq 16\frac{|x-\tau_r|}{|y-\tau_r|}\sqrt{\frac{|y-a_{r+1}|}{|x-a_{r+1}|}} \leq 2^{14}.$$

By reversing the role of x and y and then fixing y to be the center of $I_{N,j}$ we can conclude with $c = \sqrt{|b_r - y|}\psi(y)$

$$c2^{-14}\frac{1}{\sqrt{b_r - x}} \leq \psi(x) \leq c2^{14}\frac{1}{\sqrt{b_r - x}}, \qquad x \in I_{N,j}. \tag{16}$$

Case III. $a_r, b_r \in I_{N,j}$. Then $I_{N,j} = [a_r, b_r]$. In this case (15) holds only on the right half $I^+_{N,j}$ of $I_{N,j}$, so we can conclude (16) (with $y = (a_r + b_r)/2$) only there. However, an analogous argument gives that on the left half $I^-_{N,j}$ of $I_{N,j}$ we have

$$c2^{-14}\frac{1}{\sqrt{x-a_r}} \leq \psi(x) \leq c2^{14}\frac{1}{\sqrt{x-a_r}}. \tag{17}$$

Thus, we have the estimates (10), (16) or (17) for ψ on $I_{N,j}$, depending on the position of the interval $I_{N,j}$ in the set F.

Let now H be a measurable subset of $I_{N,j}$ with measure $m(H) \geq (1-2\eta)m(I_{N,j})$ and let $H_0 = I_{N,j} \setminus H$. Assume that the Case III holds for the interval $I_{N,j}$. (In Case II the same argument can be applied, and in Case I the computations based on (10) are actually much simpler, giving a better estimate.) Let I^+ and I^- denote the right half and the left half of the interval $I_{N,j}$, respectively. Then, using (16) on I^+, we can see that

$$\begin{aligned}
\int_{H_0\cap I^+}\psi(x)\,dx &\leq \int_{H_0\cap I^+} c2^{14}\frac{1}{\sqrt{b_r-x}}\,dx \\
&\leq c2^{14}2m(H_0)^{1/2} \leq c2^{15}(2\eta)^{1/2}m(I_{N,j})^{1/2} \\
&\leq c2^{15}\eta^{1/2}2m(I^+)^{1/2} = \eta^{1/2}2^{15}c\int_{I^+}\frac{1}{\sqrt{b_r-x}}\,dx \\
&= \eta^{1/2}2^{29}\int_{I^+}\frac{c2^{-14}}{\sqrt{b_r-x}}\,dx \leq \eta^{1/2}2^{29}\int_{I^+}\psi(x)\,dx.
\end{aligned}$$

Since a similar bound can be given for the integral over $H_0 \cap I^-$ using (17), it follows that $\mu_F(H_0) \leq 2^{29}\eta^{1/2}\mu_F(I_{N,j})$. Then we conclude that $\mu_F(H) \geq (1-2^{29}\eta^{1/2})\mu_F(I_{N,j})$ as was to be proved.

□

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Without loss of generality we may assume that the compact set E of positive Lebesgue measure is contained in $[0,1]$. For an $N \in \mathbb{N}$ and $\delta > 0$ let us consider the finite set

$$S(E,N,\delta) := \{j \in \mathbb{N} : m(E \cap I_{N,j}) \geq (1-\delta)m(I_{N,j})\},$$

and let

$$E(N,\delta) := \bigcup \{I_{N,j} : j \in S(E,N,\delta)\}.$$

By Lebesgue's density theorem almost all $x \in E$ belongs to all $E(N,\delta)$ for sufficiently large N, i.e. to

$$\bigcup_{M=1}^{\infty}\bigcap_{N=M}^{\infty}(E \cap E(N,\delta)).$$

Thus

$$\lim_{M\to\infty} m\left(\bigcap_{N=M}^{\infty}(E \cap E(N,\delta))\right) = m(E),$$

whence

$$\lim_{N\to\infty} m(E\cap E(N,\delta)) = m(E)$$

follows.

Let there be given an $\varepsilon \in (0, m(E)/4)$. Set $\varepsilon_n = \varepsilon/2^n$ for $n \in \mathbb{N}$, and recursively define the positive integers $N_1 < N_2 < \dots$ and the closed sets $E \supset E_1 \supset E_2 \supset \dots$ in the following manner. Let N_1 be so large that

$$m(E \setminus E(N_1, \varepsilon_1)) < \varepsilon_1,$$

and set $E_1 = E \cap E(N_1, \varepsilon_1)$. In general, if N_n, E_n have already been defined, then select a large $N_{n+1} > N_n$ so that

$$m(E_n \setminus E_n(N_{n+1}, \varepsilon_{n+1})) < \varepsilon_{n+1}/2^{N_n},$$

and let $E_{n+1} = E_n \cap E_n(N_{n+1}, \varepsilon_{n+1})$. We obtain the sequences $\{N_n\}_{n=1}^\infty$ and $\{E_n\}_{n=1}^\infty$. The compact subset K of E is defined by $K = \cap_{n=1}^\infty E_n$.

Setting $N_0 = 0$ and $E_0 = E$, we have $m(E_n \setminus E_{n+1}) < \varepsilon_{n+1}/2^{N_n}$ for every $n \geq 0$, hence

$$m(E \setminus K) < \sum_{n=0}^\infty \varepsilon_{n+1}/2^{N_n} = \sum_{n=0}^\infty \varepsilon/2^{n+1+N_n} < \varepsilon,$$

in particular $m(K) > 3m(E)/4 > 0$. Furthermore, given $n \in \mathbb{N}$ for every $j \in S(E_{n-1}, N_n, \varepsilon_n)$ we have $E_{n-1} \cap I_{N_n,j} = E_n \cap I_{N_n,j}$ and so, by the definition of $S(E_{n-1}, N_n, \varepsilon_n)$, we have $m(E_n \cap I_{N_n,j}) \geq (1-\varepsilon_n) m(I_{N_n,j})$. Since for $k \geq 0$

$$m(E_{n+k} \setminus E_{n+k+1}) \leq \varepsilon_{n+k+1}/2^{N_{n+k}} \leq \varepsilon_n/2^{N_n+k+1} = \frac{\varepsilon_n}{2^{k+1}} m(I_{N_n,j}),$$

it follows

$$\begin{aligned} m(K \cap I_{N_n,j}) &\geq m(E_n \cap I_{N_n,j}) - \sum_{k=0}^\infty m(E_{n+k} \setminus E_{n+k+1}) \\ &\geq (1-2\varepsilon_n) m(I_{N_n,j}). \end{aligned} \tag{18}$$

Set $z_0 \in K$, and for any $k \in \mathbb{N}$ let

$$K_k = K \bigcap \left\{ z \in \mathbb{C} : 2^{-k-1} \leq |z - z_0| \leq 2^{-k} \right\}.$$

For every $n \in \mathbb{N}$ there is an index $j_n \in S(E_{n-1}, N_n, \varepsilon_n)$ such that $z_0 \in I_{N_n,j_n}$. Since $\mathrm{cap}(H) \geq m(H)/4$ for any Borel subset of the real line, applying (18) we obtain

$$\mathrm{cap}(K_{N_n+1}) \geq m(K_{N_n+1})/4 \geq \frac{1}{4}\left(\frac{1}{4} - 2\varepsilon_n\right) m(I_{N_n,j_n}) \geq 2^{-N_n-1} \cdot 2^{-4},$$

whence

$$\frac{N_n+1}{\log(1/\mathrm{cap}(K_{N_n+1})} \geq \frac{1}{2}$$

follows (provided $n \geq 3$). Thus

$$\sum_{k=1}^{\infty} \frac{k}{\log(1/\mathrm{cap}(K_k))} = \infty$$

and so Wiener's criterion (see [R, Theorem 5.4.1]) yields that the compact set K is regular.

It remains to show that the measure μ_K is absolutely continuous. Let $V \subset K$ be a set of measure zero, and let $U = K \setminus V$. For $n \in \mathbb{N}$, let us consider the set

$$F_n = E_{n-1}(N_n, \varepsilon_n) = \bigcup \{I_{N_n,j} : j \in S_n\},$$

where $S_n = S(E_{n-1}, N_n, \varepsilon_n)$. We know from (18) that

$$m(U \cap I_{N_n,j}) = m(K \cap I_{N_n,j}) \geq (1 - 2\varepsilon_n) m(I_{N_n,j})$$

holds for every $j \in S_n$. Then Lemma 9 implies

$$\mu_{F_n}(U \cap I_{N_n,j}) \geq (1 - 2^{29}\varepsilon_n^{1/2})\mu_{F_n}(I_{N_n,j}).$$

Summing up for $j \in S_n$ we get

$$\mu_{F_n}(U) \geq 1 - 2^{29}\varepsilon_n^{1/2}.$$

Since $K \subset F_n \subset \mathbb{R}$, the measure μ_K is obtained by adding to the restricition $\mu_{F_n}|K$ the so called balayage of $\mu_{F_n}|(F_n \setminus K)$ onto K (see Theorem IV.1.6(e) in [SaT]). Therefore

$$\mu_K(U) \geq \mu_{F_n}(U) \geq 1 - 2^{29}\varepsilon_n^{1/2},$$

and so

$$\mu_K(V) = 1 - \mu_K(U) \leq 2^{29}\varepsilon_n^{1/2}$$

hold for every $n \in \mathbb{N}$. By letting n tend to infinity we conclude that $\mu_K(V) = 0$.
□

We complete this paper by two comments.

Remarks. 1. The analogue of Theorem 4 is true for sets of positive measure on the unit circle. Actually, the construction that we made on the real line could be done on the unit circle, and then the included compact set can be arbitrarily close in measure. The construction was based on the explicit form (5) of the equilibrium measure for a finite union of intervals. This form has an analogue (see [PS, Lemma4.1]) for a finite union of arcs on the unit circle, but this latter one is more cumbersome to use, and we found it better to work on the real line.

2. Examples for non-regular partially inner functions are induced by compact subsets of $\mathbb{T}$ with equilibrium measures, which are not absolutely continuous. For example, if $\tilde{\Omega} = I \cup \mathcal{C}$ is the disjoint union of an arc I and of the inverse image $\mathcal{C}$ of the Cantor set under the Joukovskii mapping, then $\tilde{\Omega}$ is a regular set of positive measure, but, since $\mathcal{C}$ is of positive capacity, the equilibrium measure $\mu_{\tilde{\Omega}}$ is not identically zero on $\mathcal{C}$, hence it is not absolutely continuous.

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