Riesz-type inequalities on general sets

Béla Nagy[a,*], Vilmos Totik[b,1]

[a]Bolyai Institute, MTA-SZTE Analysis and Stochastics Research Group, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary

[b]Bolyai Institute, MTA-SZTE Analysis and Stochastics Research Group, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary
and
Department of Mathematics, University of South Florida, 4202 E. Fowler Ave, CMC342, Tampa, FL 33620-5700, USA

Abstract

Sharp Riesz–Bernstein-type inequalities are proven for the derivatives of algebraic polynomials on general subsets of unit circle. The sharp Riesz-Bernstein constant involves the equilibrium density of the set in question.

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1. Results

Let C_1 denote the unit circle. The inequality

$$\|P_n'\|_{C_1} \le n\|P_n\|_{C_1} \tag{1}$$

valid for algebraic polynomials of degree at most n was first proved by M. Riesz [12], but since $P_n(e^{it})$ is a trigonometric polynomial of degree at most n, it is also a special case of the classical Bernstein inequality:

$$\|T_n'\| \le n\|T_n\| \tag{2}$$

for trigonometric polynomials T_n of degree at most n. For a comprehensive history of these two inequalities, we refer to the manuscript [8]. The following extension to a subarc was proved in [6]. Let $\Gamma_\beta = \Gamma_{[-\beta,\beta]} = \{e^{it} : t \in [-\beta,\beta]\}$ be the arc on the unit circle that goes from $e^{-i\beta}$ to $e^{i\beta}$ while passing through the point 1. Then, for algebraic polynomials P_n of degree at most $n = 1, 2, \ldots$, we have for any $\zeta = e^{i\theta}$ lying inside Γ_β the estimate

$$|P_n'(\zeta)| \le \frac{n}{2}\left(1 + \frac{\cos\theta/2}{\sqrt{\sin^2\beta/2 - \sin^2\theta/2}}\right)\|P_n\|_{\Gamma_\beta}, \tag{3}$$

*Corresponding author.
Email address: nbela@math.u-szeged.hu (Béla Nagy)

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and this is sharp at every ζ.

The main purpose of this paper is to find and prove the analogue of this on an arbitrary compact subset of the unit circle. In order to formulate the results we need some concepts from potential theory, namely that of the equilibrium measure and of Green's function. See [1], [2], [11], [13] or [16] for all these concepts.

If $\Gamma \subset C_1$ is a closed subset of the unit circle of positive logarithmic capacity, then let ν_Γ be the equilibrium measure of Γ. It is well known that ν_Γ is absolutely continuous with respect to arc measure on any subarc of Γ, and we are going to denote by $\omega_\Gamma(\zeta)$ the density of ν_Γ with respect to arc measure, i.e. on a subarc of Γ we have

$$d\nu_\Gamma(e^{it}) = \omega_\Gamma(e^{it})dt.$$

In what follows "interior" and "inner" are meant with respect to the one-dimensional topology on the unit circle.

Now the analogue of (3) is

Theorem 1. *Let $\Gamma \subseteq C_1$ be a closed subset of the unit circle. If $\zeta \in \Gamma$ is an inner point of Γ (i.e. an inner point of a subarc of Γ), then for algebraic polynomials P_n of degree at most $n = 1, 2, \dots$ we have*

$$|P_n'(\zeta)| \le \frac{n}{2}\big(1 + 2\pi\omega_\Gamma(\zeta)\big)\|P_n\|_\Gamma. \tag{4}$$

This is sharp:

Theorem 2. *If $\zeta \in \Gamma$ is an inner point of the closed set $\Gamma \subseteq C_1$, then there are polynomials $P_n \not\equiv 0$ of degree $n = 1, 2, \dots$ for which*

$$|P_n'(\zeta)| \ge (1 - o(1))\frac{n}{2}\big(1 + 2\pi\omega_\Gamma(\zeta)\big)\|P_n\|_\Gamma. \tag{5}$$

As an example, let

$$\Gamma = \{e^{it} : t \in [-\beta, -\alpha] \cup [\alpha, \beta]\} \tag{6}$$

with some $0 \le \alpha < \beta \le \pi$. Then (see [15, (4.5)])

$$\omega_{\Gamma_E}(e^{i\theta}) = \frac{1}{2\pi}\frac{|\sin\theta|}{\sqrt{|\cos\theta - \cos\alpha||\cos\theta - \cos\beta|}}, \tag{7}$$

so in the special case $\alpha = 0$ Theorem 1 gives back (3).

Exactly as in [6], the proof of the sharpness requires to write Theorem 1 in an equivalent form. We may assume that $\Gamma \neq C_1$, since the $\Gamma = C_1$ case is just the classical Riesz inequality (1) (and then $\omega_{C_1}(t) \equiv 1/2\pi$). Let $g(z) = g_{\overline{\mathbf{C}}\setminus\Gamma}(z, \infty)$ be the Green's function with pole at infinity of the complement $\overline{\mathbf{C}} \setminus \Gamma$ of Γ, and let $g'_\pm(\zeta)$ be the two normal derivatives of g at an inner point ζ of Γ in the direction of the two normals $\mathbf{n}_\pm$ to Γ. With these Theorems 1 and 2 can be written in the alternative form

Theorem 3. *With the assumptions of Theorem 1 we have*

$$|P_n'(\zeta)| \le n\max\bigl(g_+'(\zeta), g_-'(\zeta)\bigr)\|P_n\|_\Gamma. \tag{8}$$

Furthermore, this is sharp: there are polynomials $P_n \not\equiv 0$ of degree $n = 1, 2, \ldots$ for which

$$|P_n'(\zeta)| \ge (1 - o(1))n\max\bigl(g_+'(\zeta), g_-'(\zeta)\bigr)\|P_n\|_\Gamma. \tag{9}$$

It will turn out that the maximum is obtained for the normal derivative in the outward direction.

The estimates (4)–(5) were stated for interior points, but actually they completely answer the problem of pointwise estimates of the derivative of algebraic polynomials on closed subsets of the unit circle at any (not necessarily interior) point. Indeed, let $\Gamma \subset C_1$ be a closed set, and for $\delta > 0$ let Γ_δ be the set of points that are of distance $\le \delta$ from Γ. Unless Γ is the whole circle, for sufficiently small δ the sets Γ_δ are strictly decreasing as δ is decreasing. This implies that for small $\delta' < \delta$ we have

$$\omega_{\Gamma_{\delta'}}(e^{i\theta}) > \omega_{\Gamma_\delta}(e^{i\theta}) \tag{10}$$

for $e^{i\theta} \in \Gamma_{\delta'}$. Indeed, $\nu_{\Gamma_{\delta'}}$ is the balayage of ν_{Γ_δ} onto $\Gamma_{E'_\delta}$ (see [13, Theorem IV.1.6,(e)]), hence on $\Gamma_{\delta'}$ the measure $\nu_{\Gamma_{\delta'}}$ is strictly bigger than ν_{Γ_δ}. Now (10) implies that

$$\tilde\omega_\Gamma(\theta) = \lim_{\delta\to 0}\omega_{\Gamma_\delta}(e^{i\theta}) \tag{11}$$

exists at every point of Γ (it can be infinite). Since each ω_{Γ_δ} have integral 1 over the unit circle, it follows from Fatou's lemma that

$$\int_\Gamma \tilde\omega_\Gamma \le 1 \tag{12}$$

(integration is with respect to arc length). Now the expression $(1 + 2\pi\tilde\omega_\Gamma(\zeta))/2$ is precisely the quantity

$$\sup_{P_n}\frac{|P_n'(\zeta)|}{n\|P_n\|_\Gamma}$$

as is shown by

Corollary 4. *Let $\Gamma \subset C_1$ be a closed set. If $\zeta \in \Gamma$, then for algebraic polynomials P_n of degree at most $n = 1, 2, \ldots$ we have*

$$|P_n'(\zeta)| \le \frac{n}{2}\bigl(1 + 2\pi\tilde\omega_\Gamma(\zeta)\bigr)\|P_n\|_\Gamma. \tag{13}$$

Conversely, if

$$\gamma < \frac{1}{2}\bigl(1 + 2\pi\tilde\omega_\Gamma(\zeta)\bigr),$$

then there are algebraic polynomials $P_n \not\equiv 0$ of arbitrarily large degree n such that

$$|P_n'(\zeta)| \ge n\gamma\|P_n\|_\Gamma. \tag{14}$$

Theorem 1 will be a relatively easy consequence of the following theorem due to A. Lukashov. For a 2π-periodic set $E \subset \mathbf{R}$ let

$$\Gamma_E = \{e^{it} : t \in E\}$$

be the set that corresponds to E when we identify $(-\pi, \pi]$ with C_1. The following far-reaching extension of Bernstein's inequality is a special case of a result of A. Lukashov [4].

Theorem A *Let $E \subset \mathbf{R}$ be a 2π-periodic closed set. If $\theta \in E$ is an inner point of E, then for trigonometric polynomials T_n of degree at most $n = 1, 2, \ldots$ we have*

$$|T_n'(\theta)| \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E. \tag{15}$$

[4] contains this inequality for the special case when real trigonometric polynomials are considered on finitely many intervals. The extension to general sets (rather than to $E \cap [0, 2\pi]$ consisting of finitely many intervals) is immediate by simple approximation (see the sets Γ_δ above), and the extension to complex trigonometric polynomials follows by a standard trick: if T_n is an arbitrary trigonometric polynomial, then for fixed θ there is a complex number τ of modulus 1 such that $\tau T_n'(\theta) = |T_n'(\theta)|$. Apply (15) to the real trigonometric polynomial $T_n^* = \Re(\tau T_n)$ rather than to T_n to get

$$|T_n'(\theta)| = \tau T_n'(\theta) = (T_n^*)'(\theta) \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n^*\|_E \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E.$$

As a corollary of Theorem 2, we get a simple proof of the fact that (15) is sharp. Indeed, suppose to the contrary that for some $\gamma < 1$ we had

$$|T_n'(\theta)| \le \gamma n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E \tag{16}$$

for all trigonometric polynomials T_n of degree at most $n \in \mathcal{N}$, where $\mathcal{N}$ is an infinite subset of the integers. Then the proof of Theorem 1 given below would yield, instead of (4), the inequality

$$|P_n'(\zeta)| \le \frac{n}{2}\bigl(1 + \gamma 2\pi \omega_\Gamma(\zeta)\bigr) \|P_n\|_\Gamma, \qquad n \in \mathcal{N},$$

which is not the case by Theorem 2.

If we compare the Riesz–Bernstein factor

$$\frac{n}{2}\bigl(1 + 2\pi \omega_\Gamma(\zeta)\bigr)$$

in Theorem 1 with the Bernstein factor $n 2\pi \omega_{\Gamma_E}(\zeta)$ from Theorem A, we can see that the former one is smaller than the latter one except in the case when Γ is the whole unit circle (to this note that $2\pi \omega_\Gamma(\zeta) \ge 1$ at every inner point $\zeta \in \Gamma$ because of the aforementioned fact that $\nu_\Gamma \ge \nu_{C_1}$ on Γ). Therefore, the fact that the best constants in (1) and in (2) are 1 both for the trigonometric and for the algebraic polynomials was a mere coincidence, in general the Bernstein constant for trigonometric polynomials is larger than the same for algebraic polynomials.

The preceding fact is also related to a theorem of Szegő, who proved in [14] (see also [5, Theorem 3.1.1, p. 675]) the following beautiful extension of Riesz' inequality 1: if the real part of an algebraic polynomial P_n is at most 1 in absolute value on the whole unit circle, then

$$|P_n'(\zeta)| \le n, \qquad \zeta \in C_1.$$

This seems to be a result pertinent to the whole unit circle, its extension in the sense of Theorem 1 is not valid (at least not with the same factor). As an example, let Γ be the set (6) discussed above with $\beta = \pi - \alpha$, $0 < \alpha < \pi/2$. By (7)

$$\frac{1}{2}\big(1 + 2\pi\omega_\Gamma(i)\big) = \frac{1}{2}\big(1 + \frac{1}{\cos\alpha}\big) \tag{17}$$

Now $P(z) = z/\cos\alpha$ is a polynomial of degree 1 which has real part in between -1 and 1 on Γ and which has derivative $1/\cos\alpha$ at $\zeta = i$. Clearly, this derivative is at most as large as (17) (the factor from Theorem 1) only when $\alpha = 0$, i.e. when Γ is the whole circle.

Proof of Theorem 1. Let $E = \{t : e^{it} \in \Gamma\}$. First of all we need the following fact: Theorem A is also true for half-integer trigonometric polynomials. More precisely, if

$$Q_{n+1/2}(t) = \sum_{j=0}^{n} a_j \cos\left(\left(j + \frac{1}{2}\right)t\right) + b_j \sin\left(\left(j + \frac{1}{2}\right)t\right), \quad a_j, b_j \in \mathbf{C} \tag{18}$$

is a trigonometric polynomial with half-integer frequencies, then the analogue of (15) is true:

$$|Q_n'(\theta)| \le \left(n + \frac{1}{2}\right) 2\pi\omega_{\Gamma_E}(e^{i\theta})\|Q_n\|_E. \tag{19}$$

Indeed, Lukashov's result from [4] implies this precisely as it implied Theorem A, or apply [15, Corollary 2.3] and use the complexifying argument mentioned after Theorem A.

Based on this and on Theorem A, the proof of Theorem 1 is very simple, it coincides with that of [6, Theorem 1]. Indeed, let P_n be an algebraic polynomial of degree at most n, and set

$$Q_{n/2}(t) := e^{-i\frac{n}{2}t} P_n\left(e^{it}\right). \tag{20}$$

When n is even, then this is a trigonometric polynomial of degree at most $n/2$, and for odd n it is a trigonometric polynomial with half-integer frequencies of degree $n/2$. For it we have

$$||Q_{n/2}||_E = ||P_n||_\Gamma,$$

and

$$Q_{n/2}'(\theta) = e^{-i\frac{n}{2}\theta}(-in/2)\, P_n(e^{i\theta}) + e^{-i\frac{n}{2}\theta} P_n'(e^{i\theta})e^{i\theta} i. \tag{21}$$

So

$$|P_n'(e^{i\theta})| \leq |Q_{n/2}'(\theta)| + \frac{n}{2}|P_n(e^{i\theta})|, \qquad \theta \in E,$$

and (4) is an immediate consequence of (15) (in the case when n is even) and (19) (when n is odd), because the second term on the right is $\leq \|P_n\|_\Gamma$. □

Proof of Theorems 2 and 3. Let, at a point $\zeta \in \mathrm{Int}(\Gamma)$, $\mathbf{n}_+ = \zeta$ be the normal to Γ that points to the exterior of the unit circle, and similarly let $\mathbf{n}_- = -\zeta$ be the normal that points to the interior. With the notations of Theorem 3 we show that

$$\frac{1}{2}\left(1 + 2\pi\omega_\Gamma(\zeta)\right) = \max\Big(g_+'(\zeta), g_-'(\zeta)\Big) = g_+'(\zeta), \tag{22}$$

which, in view of Theorem 1, verifies the first part of Theorem 3. Since it is classical (see e.g. [9, II.(4.1)] or [13, Theorem IV.2.3] and [13, (I.4.8)]) that

$$\omega_\Gamma(\zeta) = \frac{1}{2\pi}\Big(g_+'(\zeta) + g_-'(\zeta)\Big), \tag{23}$$

it is sufficient to show that

$$g_+'(\zeta) - g_-'(\zeta) = 1. \tag{24}$$

This formula is known, for example when Γ consists of finitely many arcs it is stated in [3, (46)], and in that case it also easily follows from the explicit form of the Green's functions given in [10], (5.12). From that finite arc case one can easily deduce the validity of (24) for general sets by approximation (see e.g. [6, Lemma 7.1]).

Here is a direct proof. Let ν_Γ be the equilibrium measure of Γ. The complex Green's function

$$f(z) = \int \log(z - e^{it}) d\nu_\Gamma(e^{it}), \qquad z \in \mathbf{C} \setminus \Gamma,$$

is multi-valued, its real part is $g(z) = g_{\overline{\mathbf{C}}\setminus\Gamma}(z, \infty) + \mathrm{const}$, and its derivative

$$f'(z) = \int \frac{1}{z - e^{it}} d\nu_\Gamma(e^{it})$$

is a single-valued analytic function outside Γ. At $\zeta = e^{it_0}$ we have $\mathbf{n}_+ = e^{it_0}$, $\mathbf{n}_- = -e^{it_0}$, and with these for an $\varepsilon > 0$ (with some local branch of f around the point $(1+\varepsilon)e^{it_0}$) write

$$\lim_{h \searrow 0} \frac{f((1+\varepsilon)e^{it_0} + h\mathbf{n}_+) - f((1+\varepsilon)e^{it_0})}{h\mathbf{n}_+} = f'((1+\varepsilon)e^{it_0}).$$

If we multiply through by $\mathbf{n}_+ = e^{it_0}$ and take real parts, then we obtain

$$\begin{aligned}\frac{\partial g((1+\varepsilon)e^{it_0})}{\partial \mathbf{n}_+} &= \Re\left(e^{it_0}\int \frac{1}{(1+\varepsilon)e^{it_0} - e^{it}}d\nu_\Gamma(e^{it})\right)\\ &= \Re\int \frac{1}{(1+\varepsilon) - e^{i(t-t_0)}}d\nu_\Gamma(e^{it})\\ &= \int \frac{(1+\varepsilon) - \cos(t-t_0)}{|(1+\varepsilon) - e^{i(t-t_0)}|^2}d\nu_\Gamma(e^{it})\end{aligned}$$

In a similar manner, since $\mathbf{n}_- = -e^{it_0}$, we have

$$\begin{aligned}\frac{\partial g(e^{it_0}/(1+\varepsilon))}{\partial \mathbf{n}_-} &= -\int \frac{1/(1+\varepsilon) - \cos(t-t_0)}{|1/(1+\varepsilon) - e^{i(t-t_0)}|^2}d\nu_\Gamma(e^{it})\\ &= -\int \frac{(1+\varepsilon) - (1+\varepsilon)^2\cos(t-t_0)}{|(1+\varepsilon) - e^{i(t-t_0)}|^2}d\nu_\Gamma(e^{it}),\end{aligned}$$

where we have used that

$$|(1+\varepsilon) - e^{i(t-t_0)}| = |(1+\varepsilon) - e^{-i(t-t_0)}|.$$

Therefore,

$$\begin{aligned}&\frac{\partial g((1+\varepsilon)e^{it_0})}{\partial \mathbf{n}_+} - \frac{\partial g(e^{it_0}/(1+\varepsilon))}{\partial \mathbf{n}_-}\\ &\qquad= \int \frac{2(1+\varepsilon) - (2(1+\varepsilon)+\varepsilon^2)\cos(t-t_0)}{|(1+\varepsilon) - e^{i(t-t_0)}|^2}d\nu_\Gamma(e^{it}).\end{aligned}$$

Here the integrand is bounded since

$$|(1+\varepsilon) - e^{i(t-t_0)}|^2 \geq \varepsilon^2$$

and

$$|(1+\varepsilon) - e^{i(t-t_0)}|^2 \geq (2\sin(t-t_0)/2)^2 = 2|1-\cos(t-t_0)|.$$

So, in view of Lebesgue's dominated convergence theorem,

$$\begin{aligned}\frac{\partial g(e^{it_0})}{\partial \mathbf{n}_+} - \frac{\partial g(e^{it_0})}{\partial \mathbf{n}_-} &= \lim_{\varepsilon\searrow 0}\left(\frac{\partial g((1+\varepsilon)e^{it_0})}{\partial \mathbf{n}_+} - \frac{\partial g(e^{it_0}/(1+\varepsilon))}{\partial \mathbf{n}_-}\right)\\ &= \int\left(\lim_{\varepsilon\to 0}\frac{2(1+\varepsilon) - (2(1+\varepsilon)+\varepsilon^2)\cos(t-t_0)}{|(1+\varepsilon) - e^{i(t-t_0)}|^2}\right)d\nu_\Gamma(e^{it})\\ &= \int d\nu_\Gamma(e^{it}) = 1,\end{aligned}$$

and (24) has been verified.

Now, to complete the proof of Theorems 2 and 3, it is sufficient to show the existence of a sequence $\{P_n\}$ with the property (9). To do this, we can follow the general procedure outlined in the proof of [6, Theorem 2].

It was proven in [7, Theorem 1.4] that if Γ^* is a finite family of disjoint C^2 Jordan curves, Ω^* is the unbounded component of its complement and $g_{\Omega^*}(z,\infty)$ is the Green's function in Ω^* with pole at infinity, then, for any fixed $\zeta \in \Gamma^*$, there are nonzero polynomials P_n with

$$|P_n'(\zeta)| \geq (1-o(1))n\frac{\partial g_{\Omega^*}(\zeta,\infty)}{\partial \mathbf{n}}\|P_n\|_{\Gamma^*}, \tag{25}$$

where $\mathbf{n}$ denotes the normal to Γ at ζ pointing to the interior of Ω^*. Now consider Γ and an inner point ζ of Γ, and let J be a subarc of Γ that contains ζ in its interior. We enclose Γ into a set G^* with the following properties:

- G^* is a finite family of closed C^2 Jordan domains: there are finitely many disjoint C^2 Jordan curves $S_1,\dots,S_m$ such that if G_j is the bounded connected components of $\overline{\mathbf{C}}\setminus S_j$, then $G^* = \cup_{j=1}^m \overline{G}_j$,
- J is a boundary arc of the boundary ∂G^*,
- the component of G^* that contains ζ lies in the closed unit disk,
- every point of G^* is of distance $\leq \eta$ from a point of Γ, where η is a given positive number.

Then the boundary $\Gamma^* = \partial G^* = \cup_{j=1}^m S_j$ is a C^2 family of disjoint Jordan curves, see Figure 1. Furthermore, $\mathbf{n}_+ = \zeta$ is the normal at ζ to Γ^* pointing to the interior of $\Omega^* := \overline{\mathbf{C}}\setminus G^*$.

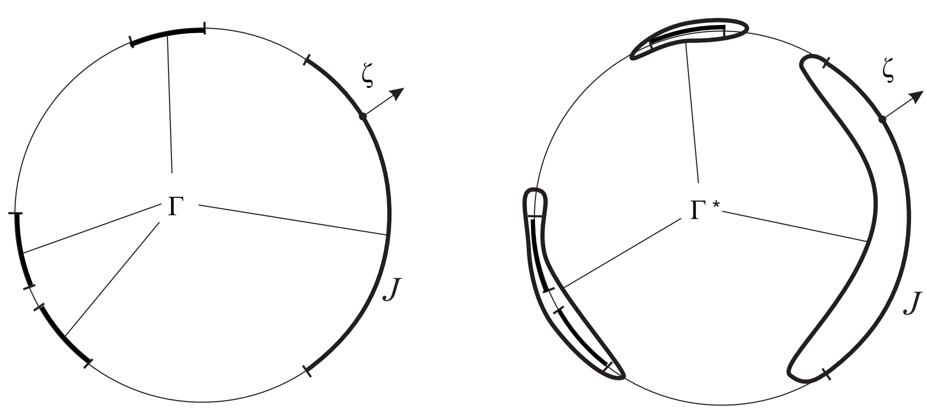

Figure 1: The set G^*

If $\varepsilon > 0$ is given, then for sufficiently small η and for the just given construction we have

$$\frac{\partial g_{\Omega^*}(\zeta,\infty)}{\partial \mathbf{n}} \geq (1-\varepsilon)\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma}(\zeta,\infty)}{\partial \mathbf{n}_+} = (1-\varepsilon) g'_+(\zeta). \tag{26}$$

In fact, since Γ is part of Γ^*, we have $g_{\Omega^*}(\zeta,\infty) \leq g_{\overline{\mathbf{C}}\setminus\Gamma}(\zeta,\infty)$, and at infinity the difference $g_{\overline{\mathbf{C}}\setminus\Gamma}(\zeta,\infty) - g_{\Omega^*}(\zeta,\infty)$ coincides with $\log(\mathrm{cap}(\Gamma^*)/\mathrm{cap}(\Gamma))$ (see [11], Theorem 5.2.1), where $\mathrm{cap}(\cdot)$ denotes logarithmic capacity. As we shrink Γ^* to Γ, the capacity of Γ^* tends to the capacity of Γ, and so the nonnegative harmonic function $g_{\overline{\mathbf{C}}\setminus\Gamma}(\zeta,\infty) - g_{\Omega^*}(\zeta,\infty)$ tends to zero at infinity (this difference is also harmonic there). Now we get from Harnack's theorem ([11], Theorems 1.3.1 and 1.3.3) that this difference tends to 0 uniformly on compact subsets of $\overline{\mathbf{C}}\setminus\Gamma$, and then (26) will be true if Γ^* is sufficiently close to Γ by [6, Lemma 7.1].

Now apply (25) to this Γ^*. For the corresponding polynomials P_n we can write, in view of $||P_n||_\Gamma \leq ||P_n||_{\Gamma^*}$,

$$|P'_n(\zeta)| \geq (1-o(1))n\frac{\partial g_{\Omega^*}(\zeta,\infty)}{\partial \mathbf{n}}\|P_n\|_{\Gamma^*} \geq (1-o(1))n(1-\varepsilon)g'_+(\zeta)||P_n||_\Gamma.$$

Since here $\varepsilon > 0$ is arbitrary, the proof of (9) is complete. □

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