Bernstein and Markov type inequalities for trigonometric polynomials on general sets

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Abstract

Bernstein and Markov-type inequalities are discussed for the derivatives of trigonometric and algebraic polynomials on general subsets of the real axis and of the unit circle. It has recently been proven by A. Lukashov that the sharp Bernstein factor for trigonometric polynomials is the equilibrium density of the image of the set on the unit circle under the mapping $t \mapsto e^{it}$. In this paper Lukashov’s theorem is extended to entire functions of exponential type using a result of Achieser and Levin. The asymptotically sharp Markov factors for trigonometric polynomials on several intervals is also found via the so called $T$-sets of F. Peherstorfer and R. Steinbauer. This sharp Markov factor is again intimately connected with the equilibrium measure of the aforementioned image set.

1 Introduction

About one hundred years ago in 1912 S. N. Bernstein [7], [8] proved his famous inequality: if $T_n$ is a trigonometric polynomial of degree at most $n$, then

$$\|T'_n\| \leq n\|T_n\|,$$

(1.1)

where $\|\cdot\|$ denotes the supremum norm. Actually, Bernstein had $2n$ instead of $n$, but a very simple argument (which is attributed to Landau in [8, p. 527]) based on his result gives also $n$ ((1.1) was first published by M. Riesz [29] in 1914). This inequality of Bernstein gave rise to converse results in approximation, and it has been applied in thousands of situations. Half a century later, in 1960, V. S. Videnskii [35] proved the analogue of (1.1) on intervals less than a whole period: if $\beta \in (0, \pi)$, then for $\theta \in (-\beta, \beta)$ we have

$$|T_n'(\theta)| \leq n \frac{\cos \theta/2}{\sqrt{\sin^2 \beta/2 - \sin^2 \theta/2}} \|T_n\|_{[-\beta, \beta]},$$

(1.2)
and this is sharp. It turns out that the factor on the right of (1.2) is essentially
the equilibrium density of the arc \( \Gamma_\beta := \{ e^{i\theta} | -\beta \leq \theta \leq \beta \} \), namely if \( \omega_{\Gamma_\beta}(e^{i\theta}) \)
denotes the density of the equilibrium measure of the arc \( \Gamma_\beta \) with respect to arc
measure on the unit circle, then (see [33])

\[
\omega_{\Gamma_\beta}(e^{i\theta}) = \frac{\cos t/2}{2\pi \sqrt{\sin^2 \beta/2 - \sin^2 t/2}}.
\]

(1.3)

The extension of Videnskii’s inequality to general sets was done by A. Lukashov
[19] in 2004, see Theorem A below. He showed that the corresponding Bernstein
factor is the same as \( 2\pi \)-times the equilibrium density of the set on the unit circle
that corresponds to \( E \) under the mapping \( t \to e^{i\theta} \).

In this paper we discuss some consequences of this result and an extension to
etire functions of exponential type. Sharp Bernstein-type inequalities for algebraic polynomials on closed subsets of the unit circle is deduced. We shall also
consider the analogous Markov-type problem that arises around the endpoints
of subintervals of \( E \).

We shall use tools from function theory and from potential theory that can
be found e.g. in the books [4], [13], [18], [28], [30] or [34]. In particular, we shall
need the concept of equilibrium measure \( \mu_E \) of a compact set \( E \) (of positive
logarithmic capacity). If \( E \) lies on the unit circle, then on the one-dimensional
interior (interior relative to the unit circle) the equilibrium measure \( \mu_E \) is
absolutely continuous with respect to the arc measure on the circle, and we shall
denote its density by \( \omega_E \), i.e. \( \omega_E(e^{i\theta})dt = d\mu_E(e^{i\theta}) \). Likewise, if \( E \) lies on the
real line then, in the one dimensional interior of \( E \), \( \omega_E \) denotes the density of the
equilibrium measure \( \mu_E \) with respect to the Lebesgue-measure on \( \mathbb{R} \). In
both cases \( \omega_E \) is an infinitely many times differentiable function on the interior
of \( E \).

The paper is organized as follows. In the next section we discuss the general
Bernstein-Videnskii inequality of Lukashov and prove an extension of it to entire
functions of exponential type using the Levin conformal maps and a theorem
of Aitcher and Levin. Section 3 discusses basic properties of special sets that
arise as inverse images of \([ -1, 1 ] \) under some special trigonometric polynomials.
These sets have nice properties (e.g. every subinterval has rational harmonic
measure) and they can approximate any set on \([ 0, 2\pi ] \) consisting of finitely many
intervals. These special sets will be fundamental in Section 4 in proving the
exact Markov-type inequalities for trigonometric polynomials on sets consisting
of finitely many intervals (on \([ 0, 2\pi ] \)). To illustrate the power of the method, we
give a new and rather elementary proof of Lukashov’s theorem in an Appendix
(Lukashov’s original proof used automorphic forms and Schottky groups, but
since then there has been a different approach by Dubinin and Kalmykov [11],
[12], [15]).

2
2 Bernstein-type inequalities

Let $C_1$ be the unit circle. For a $2\pi$-periodic set $E \subset \mathbb{R}$ let

$$\Gamma_E = \{e^{it} \mid t \in E\}$$

be the set that corresponds to $E$ when we identify $(-\pi, \pi]$ with $C_1$. Let us agree that when we integrate over $E$, then we just integrate over $(0, 2\pi] \cap E$.

The following far-reaching extension of Videnskii’s inequality is a special case of a result of A. Lukashov [19]. In it $\text{Int}(E)$ denotes the one-dimensional interior of $E$.

**Theorem A** Let $E \subset \mathbb{R}$ be a $2\pi$-periodic closed set. If $\theta \in \text{Int}(E)$ is an inner point of $E$, then for any trigonometric polynomial $T_n$ of degree at most $n = 1, 2, \ldots$ we have

$$|T_n'(\theta)| \leq n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_{E},$$

where $\omega_{\Gamma_E}$ denotes the density of the equilibrium measure of $\Gamma_E$ with respect to the arc measure on the unit circle.

[19] contains this estimate as a special case for real trigonometric polynomials on finitely many intervals. The extension to general sets (rather than to $E \cap [0, 2\pi]$ consisting of finitely many intervals) is immediate by simple approximation, and the extension to complex trigonometric polynomials follows by a standard trick: if $T_n$ is an arbitrary trigonometric polynomial, then for fixed $\theta$ there is a complex number $\tau$ of modulus 1 such that $\tau T_n'(\theta) = |T_n'(\theta)|$. Apply (2.1) to the real trigonometric polynomial $T_n^* = \Re(\tau T_n)$ rather than to $T_n$ to get

$$|T_n'(\theta)| = \tau T_n'(\theta) = (T_n^*)'(\theta) \leq n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n^*\|_{E} \leq n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_{E}.$$

For different proofs of Theorem A see the papers [11], [12], [15] by V. N. Dubinin and S. I. Kalmykov.

As an example, consider $E = [-\beta, -\alpha] \cup [\alpha, \beta]$ with some $0 \leq \alpha < \beta \leq \pi$. In this case (see [33, (4.5)])

$$\omega_{\Gamma_E}(e^{i\theta}) = \frac{1}{2\pi} \frac{|\sin \theta|}{\sqrt{\cos \theta - \cos \alpha \cos \theta - \cos \beta}},$$

so we get from Theorem A the sharp inequality

$$|T_n'(\theta)| \leq n \frac{|\sin \theta|}{\sqrt{\cos \theta - \cos \alpha \cos \theta - \cos \beta}} \|T_n\|_{[-\beta, -\alpha] \cup [\alpha, \beta]}.$$  

(2.3)

If $\alpha = 0$, then

$$\frac{|\sin \theta|}{\sqrt{\cos \theta - 1 \cos \theta - \cos \beta}} = \frac{\cos \theta/2}{\sqrt{\sin^2 \theta/2 - \sin^2 \theta/2}},$$

and then (2.3) changes into Videnskii’s inequality (1.2).

Generalizing Theorem A we are going to prove in this section
Theorem 2.1 Let $E \subset \mathbb{R}$ be a $2\pi$-periodic closed set. If $\theta \in E$ is an inner point of $E$, then for any exponential function $f$ of type $\sigma$ we have

$$|f'(\theta)| \leq \sigma 2\pi \omega_{\mathcal{E}}(e^{i\theta})\|f\|_{\mathcal{E}}. \quad (2.4)$$

It is not difficult to prove using the results of [19] and the density theorem (Lemma 3.4) below that (2.1) is sharp:

**Theorem B** If $E \subset \mathbb{R}$ is as before, and $\theta \in E$ is an inner point of $E$, then there are trigonometric polynomials $T_n \not\equiv 0$ of degree at most $n = 1, 2, \ldots$ such that

$$|T_n'(\theta)| \geq (1 - o(1))n2\pi \omega_{\mathcal{E}}(e^{i\theta})\|T_n\|_{E}. \quad (2.5)$$

This sharpness also follows from Theorem 2 of [22].

By the simple $x = \cos t$ substitution we obtain the following sharp Bernstein-type inequality for algebraic polynomials on arbitrary compact subset of the real line. In its formulation, for a compact set $K \subset \mathbb{R}$ of positive capacity, let $\omega_K$ be the density of the equilibrium measure of $K$ with respect to linear Lebesgue measure on $\mathbb{R}$. Let $K \subset \mathbb{R}$ be compact and $x$ an inner point of $K$. Then for algebraic polynomials $P_n$ of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(x)| \leq n\pi \omega_K(x)\|P_n\|_K, \quad x \in K, \quad (2.6)$$

and this is sharp in the sense of Theorem B. This estimate was proved in [5] and [31] by different methods, see also [20].

The estimates (2.1)–(2.5) completely answer the problem of pointwise estimates of the derivative of trigonometric polynomials on closed sets (cf. also [22]). Indeed, let $E$ be a closed set as before, and for $\delta > 0$ let $E_\delta$ be the set of points that are of distance $\leq \delta$ from $E$. Unless $E$ is the whole real line, for sufficiently small $\delta$ the sets $E_\delta$ are strictly decreasing as $\delta$ is decreasing, hence the same is true of the sets $\Gamma_{E_\delta}$. This implies that for small $\delta' < \delta$ we have

$$\omega_{\mathcal{E}_\delta'}(e^{i\theta}) > \omega_{\mathcal{E}_\delta}(e^{i\theta}) \quad (2.7)$$

for $e^{i\theta} \in \Gamma_{E_\delta'}$. Indeed, $\mu_{E_\delta'}$ is the balayage of $\mu_{\mathcal{E}_\delta}$ onto $\Gamma_{E_\delta'}$ (see [30, Theorem IV.1.6,(e)]), hence on $\Gamma_{E_\delta'}$ the measure $\mu_{E_\delta'}$ is strictly bigger than $\mu_{\mathcal{E}_\delta}$. Now (2.7) implies that

$$\tilde{\omega}_E(\theta) = \lim_{\delta \to 0} \omega_{\mathcal{E}_\delta}(e^{i\theta}) \quad (2.8)$$

exists at every point of $E$ (it could be infinite). Since each $\omega_{\mathcal{E}_\delta}$ have integral 1 over the unit circle, it follows from Fatou’s lemma that

$$\int_{\mathcal{E}} \tilde{\omega}_E \leq 1. \quad (2.9)$$

$2\pi \tilde{\omega}_E$ is precisely the quantity

$$\sup_{T_n} \frac{|T_n'(\theta)|}{n\|T_n\|_E}$$

as is shown by
Corollary 2.2 Let $E \subset \mathbb{R}$ be a $2\pi$-periodic closed set. If $\theta \in E$, then for any trigonometric polynomial $T_n$ of degree at most $n = 1, 2, \ldots$ we have

$$|T_n'(\theta)| \leq n2\pi \omega_E(\theta) \|T_n\|_E.$$  \hfill (2.10)

Conversely, if $\gamma < \tilde{\omega}_E(\theta)$, then there are trigonometric polynomials $T_n \neq 0$ of arbitrarily large degree $n$ such that

$$|T_n'(\theta)| \geq n2\pi \gamma \|T_n\|_E.$$  \hfill (2.11)

This corollary along with (2.9) makes a theorem of Privaloff from 1916 more precise: Privaloff [27] proved that if $E \subset [0, 2\pi]$ is of positive Lebesgue measure $m(E)$, then for every $\varepsilon > 0$ there is a constant $B(\varepsilon) < \infty$ and a subset $E' \subset E$ of measure $\geq m(E) - \varepsilon$ with the property that for trigonometric polynomials $T_n$ of degree at most $n = 1, 2, \ldots$ we have

$$|T_n'(\theta)| \leq nB(\varepsilon) \|T_n\|_E, \quad \theta \in E'.$$  \hfill (2.12)

It is clear that (2.12) along with (2.9) is a much more precise result, e.g. it gives that in Privaloff’s theorem one can put $B(\varepsilon) = 2\pi/\varepsilon$.

In a similar manner, one can derive from (2.6) and its sharpness the following

Corollary 2.3 Let $K \subset \mathbb{R}$ be a compact subset of $\mathbb{R}$ and define for $x \in K$

$$\tilde{\omega}_K(x) = \lim_{\delta \to 0} \omega_{K_\delta}(x).$$

If $x \in K$, then for any algebraic polynomial $P_n$ of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(x)| \leq n\pi \tilde{\omega}_K(x) \|P_n\|_K.$$  \hfill (2.13)

Conversely, if $\gamma < \tilde{\omega}_K(x)$, then there are algebraic polynomials $P_n \neq 0$ of arbitrarily large degree $n$ such that

$$|P_n'(x)| \geq n\pi \gamma \|P_n\|_K.$$  \hfill (2.14)

In the rest of this section we are going to prove Theorem 2.1 and Corollary 2.2. The proof of Corollary 2.3 follows the reasoning of Corollary 2.2, but using (2.6) and its sharpness instead of Theorem A.

Proof of Corollary 2.2. If $\theta \in E$, then $\theta$ lies in the interior of every $E_\delta$, $\delta > 0$, so Theorem A gives

$$|T_n'(\theta)| \leq n2\pi \omega_{E_\delta}(\theta) \|T_n\|_{E_\delta} \leq n2\pi \tilde{\omega}_E(\theta) \|T_n\|_{E_\delta}.$$ 

Now if we make here $\delta \to 0$, then we obtain (2.10).

Conversely, if $\gamma < \tilde{\omega}_E(\theta)$, then there is a $\delta > 0$ such that

$$\omega_{E_\delta}(e^{i\theta}) > \gamma,$$
and we can apply Theorem B to conclude that there are trigonometric polynomials $T_n$ of arbitrarily large degree $n$ such that

$$|T_n'(\theta)| \geq n2\pi\gamma \|T_n\|_{E_\delta},$$

which is stronger than (2.11).

**Proof of Theorem 2.1.** We deduce the theorem from a result of N. I. Achieser and B. Ya. Levin [1], [2].

First of all, we may assume that $\Gamma_E$ consists of finitely many arcs on the unit circle. Indeed, suppose (2.4) has been verified for such sets, and for an arbitrary $E$ and for $\delta > 0$ let $E_\delta$ be the set of points lying of distance $\leq \delta$ from $E$. Then $E_\delta \cap [0, 2\pi]$ consists of finitely many intervals, and, as $\delta \to 0$, we have $\omega_{\Gamma_{E_\delta}}(e^{i\theta}) \to \omega_{\Gamma_E}(e^{i\theta})$ at any point $\theta$ in the interior of $E$ (see for example [6, Lemma 3.2], apply it with $\rho_n = \delta\infty$—the Dirac delta at the point infinity—, and use that the balayage of $\delta\infty$ onto $E$ is the equilibrium measure $\mu_E$). Now if we apply (2.4) to this set $E_\delta$ and make $\delta \to 0$, we obtain (2.4).

We start with the function

$$\beta(\zeta) = \zeta(\text{cap}(\Gamma_E))^2 \exp \left(-2 \int \log(1 - \overline{t\zeta})d\mu_{\Gamma_E}(t)\right)$$

(2.15)

(recall that $\mu_{\Gamma_E}$ is the equilibrium measure of $\Gamma_E$). By [26, Proposition 9.15] this maps the unit disk $\Delta$ conformally onto a domain $\Delta^*$ that is obtained from the unit disk by finitely many radial cuts of the form $\{re^{i\tau} | r_{\tau} \leq r \leq 1\}$. Actually, the number of cuts equals the number $m$ of arcs complementary to $\Gamma_E$ (which is the same as the number of arcs in $\Gamma_E$). Furthermore, $\beta$ is continuous on the closed unit disk, it maps $\Gamma_E$ onto the unit circle $C_1$, and the complementary arcs to $\Gamma_E$ are mapped into the radial cuts. Therefore,

$$\alpha(\zeta) = \frac{1}{\beta(\overline{1/\zeta})}$$

is a conformal map from the exterior of the unit circle onto a domain which is obtained from the exterior domain $\mathbb{C} \setminus \overline{\Delta}$ by the finitely many radial cuts $\{re^{i\tau} | 1 \leq r \leq 1/r_{\tau}\}$ (see [3, Sec. 4]). Hence, the function

$$\varphi(z) = i \log \alpha(e^{-iz}),$$

(2.16)

where log is any branch of the logarithm, maps the upper half plane $\mathbb{H}_+$ conformally onto a domain $\mathbb{H}_+^*$ which is obtained from $\mathbb{H}_+$ by vertical cuts of the form $\{a + iy | 0 \leq y \leq y_a\}$. On every interval $(A, A + 2\pi)$ there are $m$ such $a$‘s, $\varphi$ satisfies the property $\varphi(z + 2\pi) = \varphi(z) + 2\pi$, $\varphi$ maps $E$ onto the real line and it maps the intervals complementary to $E$ into the cuts. Since $\beta(z) \sim \text{const} \cdot z$ as $z \to 0$, it also follows that

$$\varphi(z) \sim z + \text{const} \quad \text{as } \Im z \to \infty.$$
The domain $H_+^*$ is called Achieser’s comb domain, and the mapping $\varphi$ is called the Levin conformal map (with the proper normalization (2.17)), see [2]–[1], [3]. It is standard that $\varphi$ can be extended to a continuous function on the closure $H_+$, and this extension is actually $C^\infty$ on every open subinterval of $E$. The continuous extension of $\varphi$ is just Carathéodory’s theorem, and the $C^\infty$ property can be seen as follows. Let $I$ be a closed subinterval of the interior of $E$, and $J \subset I$ a closed subinterval of the interior of $I$. Attach a domain $G \subset H_+$ to $I$ in such a way that $I$ lies on the boundary of $G$, and the boundary of $G$ is a $C^\infty$ Jordan curve. Let $\psi$ be a conformal map from the unit disk onto $G$ and let $I'$, $J'$ be the arcs of the unit circle that correspond to $I$, $J$, respectively under the map $\psi$. The function $\Re \varphi(\psi)$ is a positive harmonic function on $\Delta$ which is continuous on $\bar{\Delta}$ and which vanishes on $I'$. Hence, by Poisson’s formula, it is a $C^\infty$ function on any closed subarc of the interior of $I'$, therefore so is its analytic conjugate. As a consequence, $\varphi(\psi)$ is a $C^\infty$ function on $J'$, and since $\psi$ is $C^\infty$, invertible function on $C_1$ with non-zero derivative by the Kellogg-Warschawskii theorem (see [26, Theorems 3.5, 3.6]), the $C^\infty$ property of $\varphi$ on $J$ follows.

Consider now that mapping

$$\Phi(z) = e^{-i\varphi(z)} = \alpha(e^{-iz}).$$

It was proved by Achieser and Levin [2, Theorem 3], [1, Theorem 2, Sec. 6] that if $f$ is an entire function of exponential type $\sigma$ such that $|f(x)| \leq 1$ for $x \in E$, then

$$|f'(x)| \leq \sigma|\Phi'(x)|, \quad x \in \text{Int}(E).$$

(2.18)

Next, we calculate the derivative on the right-hand side of (2.18).

It is clear that for $x \in \text{Int}(E)$ we have

$$|\Phi'(x)| = |\alpha'(e^{-ix})| = |\beta'(e^{ix})|,$$

(2.19)

and here $\beta'(e^{ix})$ can be obtained by taking the limit $\beta'(r e^{ix})$ as $r \nearrow 1$. Let $\zeta = re^{ix}$. From the form (2.15) of $\beta$ we have

$$\beta'(\zeta) = \frac{\beta(\zeta)}{\zeta} + \beta(\zeta) \int_{\Gamma_E} \frac{2t}{1 - \zeta t} d\mu_E(t) = \frac{\beta(\zeta)}{\zeta} \int_{\Gamma_E} \frac{1 + \zeta}{1 - \zeta} d\mu_E(t).$$

(2.20)

In calculating the limit of this as $r \nearrow 1$ we may assume $x = 0(\in \text{Int}(E))$. Then $\zeta = r$, and

$$\int_{\Gamma_E} \frac{1 + \zeta}{1 - \zeta} d\mu_E(t) = \frac{t + \zeta}{t - \zeta} d\mu_E(t) = \int_{\text{Int}(E)} \frac{e^{iu} + r}{e^{iu} - r} \omega_E(e^{iu}) du,$$

where $\omega_E(e^{iu})$ is the density of the equilibrium measure $\mu_E$ with respect to arc measure on $C_1$ (recall that we assumed $\Gamma_E$ to consist of finitely many arcs). It is easy to see that this $\omega_E(e^{iu})$ is a $C^\infty$ function on $\text{Int}(E)$ (indeed, the Green’s function of $C \setminus \Gamma_E$ with pole at infinity is a $C^\infty$ function on the interior of $\Gamma_E$ by the argument made after (2.17), and $\omega_E$ is obtained from the Green’s
function by taking normal derivatives, see [23, II.(4.1)]. Now the last integral is
\[ \int_E \left( \frac{1 - r^2}{1 - 2r \cos u + r^2} - 2i \frac{r \sin u}{1 - 2r \cos u + r^2} \right) \omega_{E}(e^{iu}) du, \]
and here the real part is \( 2\pi \)-times the Poisson integral of \( \omega_{E}(e^{iu}) \) at the point \( r \), so it converges to \( 2\pi \omega_{E}(1) \) as \( r \to 1 \). The imaginary part equals
\[ -2r \bar{z} \frac{\partial}{\partial r} \int_E \log(r - e^{iu}) \omega_{E}(e^{iu}) du = -2r \frac{\partial}{\partial r} \left( \int_E \log(r - e^{iu}) \omega_{E}(e^{iu}) du \right) \]
with any local branch of the logarithm. Therefore, by the Cauchy-Riemann equations, it equals
\[ 2r \frac{\partial}{\partial y} \Re \left( \int_E \log(r + iy - e^{iu}) \omega_{E}(e^{iu}) du \right) \bigg|_{y=0} = 0 \]
\[ = 2 \frac{\partial}{\partial y} \left( \int_E \log |re^{iy} - e^{iu}| \omega_{E}(e^{iu}) du \right) \bigg|_{y=0} = 0 \]
As \( r \to 1 \), this converges to
\[ 2 \frac{\partial}{\partial y} \left( \int_E \log |e^{iy} - e^{iu}| \omega_{E}(e^{iu}) du \right) \bigg|_{y=0} = 0 \]
which is 0, since the logarithmic potential
\[ - \int_E \log |e^{iy} - e^{iu}| \omega_{E}(e^{iu}) du \]
of the equilibrium measure of \( \Gamma_E \) is constant on \( \Gamma_E \) (and \( 1 \in \text{Int}(\Gamma_E) \) because we assumed that \( 0 \in \text{Int}(E) \)).

In view of the fact that \( |\beta(r\zeta)| \) tends to 1 as \( r \to 1 \), \( \zeta \in \Gamma_E \), the considerations from (2.20) give that
\[ |\beta'(e^{ix})| = 2\pi \omega_{E}(e^{ix}), \]
and then (2.18) and (2.19) prove Theorem 2.1.

\section{T-sets}

In what follows, we shall consider special \( 2\pi \)-periodic sets \( E \) for which \( E \cap [0, 2\pi] \) consists of finitely many intervals. The case \( E = \mathbb{R} \) is trivial, and we may concentrate on \( E \neq \mathbb{R} \), in which case we may assume \( 0 \notin E \). Then \( E \cap [0, 2\pi] = \cup_{j=1}^{m} [a_{2j-1}, a_{2j}] \), \( a_j \in (0, 2\pi) \). The special property we are referring to is that there is a real trigonometric polynomial \( U_N \) of some degree \( N \) such that \( U_N(t) \) runs through the interval \([-1, 1]\) \( 2N \)-times as \( t \) runs through \( E \). In other words,
\[ E = \{ t \mid U_N(t) \in [-1, 1] \} \]
for some real trigonometric polynomial $U_N$ of degree $N$ which takes both the 1 and $-1$ values $2N$-times (recall that a trigonometric polynomial of degree $N$ can take a given value at most $2N$-times). Clearly, in this case $|U_N(a_j)| = 1$ for all $j$.

These sets have been extensively investigated by F. Peherstorfer and R. Steinbauer, and after them let us call a set $E$ with property (3.1) for some $U_N$ a $T$-set. $T$-sets also appear as a special case of the sets in [19, Theorem 2] by A. Lukashov. It turns out that

$$
\Gamma_E := \{e^{it} \mid t \in E\}
$$

for $T$-sets $E$ are also precisely the so-called rational compacts from the beautiful papers [16] and [17] by S. Khrushchev. The papers [16], [17], [24] and [25], proved the basic properties of $T$-sets (and their cousins $\Gamma_E$). The main emphasis in those papers were on orthogonal polynomials with periodic recurrence coefficients and on quadratic irrationalities, and the discussion in [16], [17], [24] and [25] were subject to this emphasis. The present section considers some of the properties of $T$-sets that we need in the next section to establish the Markov inequalities for trigonometric polynomials on several intervals. Not much originality is claimed here, rather we have a discussion that fits our needs. However, to have a concise treatment independent of orthogonal polynomials, we give full proofs.

**Lemma 3.1** Let $E$ be such that there is a real trigonometric polynomial $U_N$ of degree $N$ such that $U_N(t)$ runs through the interval $[-1, 1]$ $2N$-times as $t$ runs through $E$. Then

$$
\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi N} \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}}, \quad t \in E.
$$

Cf. [16, (25)].

**Proof.** There is a polynomial $P_{2N}$ of degree $2N$ such that with $\theta = e^{it}$ we have $U_n(t) = \theta^{-N}P_{2N}(\theta)$. Then

$$
\Gamma_E = \left\{ \zeta \mid |\zeta| = 1, \zeta^{-N}P_{2N}(\zeta) \in [-1, 1] \right\}.
$$

Let $E = \bigcup_{k=1}^{2N} E_k$, where $E_k$’s are intervals, and $U_N(t)$ runs through $[-1, 1]$ precisely once as $t$ runs through $E_k$. For a $t \in \text{Int}(E_1)$ let $t_k \in E_k$ be the point where $U_N(t_k) = U_N(t)$, i.e. where $P_{2N}(e^{it_k}) = U_N(t)e^{iNt_k}$. Hence, $e^{it_1}, \ldots, e^{it_{2N}}$ are the zeros of the equation

$$
P_{2N}(u) - U_N(t)u^N = 0.
$$

Clearly, $t_k = t_k(t)$ is a differentiable and monotone function of $t \in \text{Int}(E_1)$. Consider the integral

$$
\int_E \log |z - e^{it}| \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}} dt = \sum_{k=1}^{2N} \int_{E_k} \log |z - e^{it_k}| \frac{|U'_N(t_k)|}{\sqrt{1 - U_N(t_k)^2}} dt_k.
$$
Here the integral over $E_k$ can be calculated with the substitution $t_k = t_k(t)$, $dt_k = t_k'(t)dt$, and it equals

$$\int_{E_k} \log|z - e^{it_k(t)}| \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}} dt,$$

where we used that $U_N(t_k(t)) = U_N(t)$. Hence, the full integral equals

$$\int_{E_k} \left( \sum_{k=1}^{2N} \log|z - e^{it_k(t)}| \right) \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}} dt.$$

Now if $A_{2N}$ is the leading coefficient of $P_{2N}$, we have

$$\log A_{2N} + \sum_{k=1}^{2N} \log|z - e^{it_k(t)}| = \log |P_{2N}(z) - U_N(t)z^N|,$$

which is the same as $\log|U_N(x) - U_N(t)|$ if $z = e^{ix}$, $x \in \mathbb{R}$. Therefore, for $z = e^{ix} \in \Gamma_E$, i.e. for $x \in E$, or alternatively for $U_N(x) \in [-1, 1]$, we have

$$\int_{E} \log|z - e^{it}| \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}} dt$$

$$= \int_{E_k} \log|U_N(x) - U_N(t)| \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}} dt + \text{const}$$

$$= \int_{-1}^{1} \log|U_N(x) - u| \frac{1}{\sqrt{1 - u^2}} du = -\pi \log 2 + \text{const}.$$

What we have proven is that the logarithmic potential of the measure

$$d\nu(t) = \frac{1}{2\pi N} \frac{|U'_N(t)|}{\sqrt{1 - U_N(t)^2}} dt, \quad t \in E,$$  \hspace{1cm} (3.4)

is constant on $\Gamma_E$. The same calculation that we have just made shows that $\nu$ has total mass 1 on $E$, hence $\nu$ is the equilibrium measure of the set $\Gamma_E$ (see e.g. [30, Theorem I.3.3]).

\[ \blacksquare \]

**Lemma 3.2** Let $E, U_N$ as in Lemma 3.1, and for a $t \in E$ with $U_N(t) \in (-1, 1)$ let $t_1, \ldots, t_{2N}$ be those points in $E$ which satisfy $U_N(t_k) = U_N(t)$. Then, if $V_n$ is a trigonometric polynomial of degree at most $n$, there is an algebraic polynomial $S_{n/N}$ of degree at most $n/N$ such that

$$\sum_{k=1}^{2N} V_n(t_k) = S_{n/N}(U_N(t)).$$  \hspace{1cm} (3.5)
Proof. We use the notations from the proof of Lemma 3.1. We can write
\[ V_n(t) = \theta^{-n} Q_{2n}(\theta) = R_n(\theta) + R_n^*(1/\theta), \quad \theta = e^{it}, \]
with some polynomial \( Q_{2n} \) of degree at most 2\( n \) and with some polynomials \( R_n, R_n^* \) of degree at most \( n \). With \( \theta_k = e^{it_k} \) we have
\[ \sum_{k=1}^{2N} V_n(t_j) = \sum_{k=1}^{2N} \theta_k^{-n} Q_{2n}(\theta_k) = \sum_{k=1}^{2N} R_n(\theta_k) + \sum_{k=1}^{2N} R_n^*(1/\theta_k) = \Sigma_1 + \Sigma_2. \]
Since \( \theta_k \) are the zeros of the equation (3.3), and \( \Sigma_1 \) is a symmetric polynomial of these \( \theta_k \), we get, using the fundamental theorem of symmetric polynomials, that \( \Sigma_1 \) can be written as a polynomial of the elementary symmetric polynomials \( \sigma_1, \ldots, \sigma_{2N} \) of \( \theta_1, \ldots, \theta_{2N} \), and in this representation the exponent of \( \sigma_N \) does not exceed \( n/N \). But, by (3.3) and Viète’s formulae, these \( \sigma_j \)’s are constants, except for \( \sigma_N \), which is \((-1)^N (\text{const} - U_N(t))/A_{2N} \), where \( A_{2N} \) is the leading coefficient of \( P_{2N} \). Therefore, \( \Sigma_1 \) is a polynomial of \( U_N \) of degree at most \( [n/N] \).

Now \( 1/\theta_j \) are the solutions of the reciprocal equation
\[ u^{2N}(P_{2N}(1/u) - U_N(t)(1/u)^N) = u^{2N}P_{2N}(1/u) - U_N(t)u^N = 0, \]
so the preceding argument yields that \( \Sigma_2 \) is also a polynomial of \( U_N \) of degree at most \( [n/N] \).

We need a characterization of \( T \)-sets due to Peherstorfer and Steinbauer, cf. [24, Theorem 4.2] (for a more general statement see [19, Theorem 2]).

**Lemma 3.3** The following are equivalent:

(a) There is a real trigonometric polynomial \( U_N(t) \) of degree \( N \) such that \( U_N \)
runs through the interval \([-1,1]\) 2\( N \)-times as \( t \) runs through \( E \).

(b) For all \( j = 1, 2, \ldots, m \) the harmonic measure \( \mu_{\Gamma_{E_j}}([e^{ia_{2j-1}}, e^{ia_{2j}}]) \) is of the form \( p_j/2N \) with some integer \( p_j \).

**Proof of Lemma 3.3.** The necessity is immediate from Lemma 3.1, since each interval \([a_{2j-1}, a_{2j}]\) is the union of some of the intervals \( E_k \) (see the proof of Lemma 3.1), hence the arcs \([e^{ia_{2j-1}}, e^{ia_{2j}}]\) are the unions of some of the \( \Gamma_{E_k} \)'s. As we have seen in the proof of Lemma 3.1, the integral of \( \omega_E(e^{it}) \) on any of the \( E_k \) is 1/2\( N \), i.e. \( \mu_{\Gamma_E}(\Gamma_{E_k}) = 1/2N \).

Conversely, let us suppose that each \([e^{ia_{2j-1}}, e^{ia_{2j}}]\) carries a mass \( p_j/2N \) of the equilibrium measure with some integers \( p_j \):
\[ \int_{a_{2j-1}}^{a_{2j}} \omega(e^{it}) dt = \frac{p_j}{2N}. \]
Recall that we have assumed $1 \not\in \Gamma_E$. Consider in $C \setminus \Gamma_E$ the function

$$H(z) = \frac{1}{z^N} \exp \left( 2N \int_0^{2\pi} \log(e^{it} - z) \omega_{\Gamma_E}(e^{it}) dt - 2N \log \text{cap}(\Gamma_E) - iN\gamma \right),$$

where

$$\gamma = \int_0^{2\pi} t \omega_{\Gamma_E}(e^{it}) dt,$$

and where $\text{cap}(\Gamma_E)$ denotes the logarithmic capacity of $\Gamma_E$. In this definition we used the main branch of the logarithm. As we circle once with $z$ around the arc $[e^{ia_{2j-1}}, e^{ia_{2j}}]$, the argument changes by $\pm pj 2\pi$, so $H$ is a single-valued analytic function in $C \setminus (\Gamma_E \cup \{0\})$. Since

$$\int_0^{2\pi} \log |e^{it} - z| \omega_{\Gamma_E}(e^{it}) dt = \log \text{cap}(\Gamma_E), \quad z \in \Gamma_E,$$

the absolute value of $H$ is 1 on both sides of $\Gamma_E$. The imaginary part of $\log|e^{it} - e^{ix}|$ (taken on the outer part of the unit circle) is

$$= \begin{cases} \frac{t+x}{2} + \frac{\pi}{2} & \text{if } 0 \leq x < t \leq 2\pi \\ \frac{t+x}{2} - \frac{\pi}{2} & \text{if } 0 \leq t < x \leq 2\pi. \end{cases}$$

When we integrate this against $\omega_{\Gamma_E}(e^{it})$ over $[0, 2\pi]$ and take into account that the argument of $z^N = e^{iNx}$ is $Nx$, we obtain that in the exponent defining $H(z)$ both $iNx$ and $iN\gamma$ cancel, and the argument $A(x)$ of $H(e^{ix})$ (on the outer part of the unit circle) is

$$2N \left( \frac{\pi}{2} \mu_{\Gamma_E}([e^{ix}, e^{i2\pi}]) - \frac{\pi}{2} \mu_{\Gamma_E}([e^{i0}, e^{ix}]) \right) = N\pi - 2N\pi \mu_{\Gamma_E}([1, e^{ix}]).$$

Hence, in view of $1 \not\in \Gamma_E$ and of the assumption that $2N \mu_{\Gamma_E}([e^{ia_{2j-1}}, e^{ia_{2j}}]) = pj$ is an integer for all $j$, we obtain that $H(e^{ix})$ is real on the complementary arcs $[e^{ia_{2j}}, e^{ia_{2j}+1}]$, and its argument changes by $pj\pi$ as $x$ runs through the interval $[a_{2j-1}, a_{2j}]$. Therefore, the function

$$G(z) := \frac{1}{2} \left( H(z) + \frac{1}{H(z)} \right)$$

is real-valued on the (outer part of the) unit circle, and for $x \in [a_{2j-1}, a_{2j}]$ the value $G(e^{ix})$ is $\cos A(x)$, so $U_N(x) := G(e^{ix})$ runs through $[-1, 1]$ precisely $pj$-times as $x$ runs through the interval $[a_{2j-1}, a_{2j}]$. Furthermore, since the logarithmic potential

$$\int_0^{2\pi} \log |e^{it} - z| \omega_{\Gamma_E}(e^{it}) dt$$

is bigger than $\log \text{cap}(\Gamma_E)$ outside $\Gamma_E$, it also follows that $|G(e^{ix})| > 1$ when $e^{ix} \not\in \Gamma_E$. Hence, $E$ is precisely the set of those points $x$ for which $U_N(x) \in$
Therefore, all that remains is to prove that $U_N$ is a trigonometric polynomial of degree $N$.

To this end consider $H(1/z)$. It is

$$
\frac{1}{z^{2N}} \exp \left( 2N \int_0^{2\pi} \log(e^{-it} - 1/z) \omega_{\Gamma_E}(e^{it}) dt - 2N \log \text{cap}(\Gamma_E) + iN\gamma \right).
$$

Now write $z^{2N}$ in the form $\exp(2N \log z)$ and use that

$$
\log z + \log(e^{-it} - 1/z) = \log(z - e^{it}) - it = \log(e^{it} - z) - it + \pi \mod(2\pi i),
$$

which implies

$$
\frac{1}{z^{2N}} \exp \left( 2N \int_0^{2\pi} \log(e^{-it} - 1/z) \omega_{\Gamma_E}(e^{it}) dt \right) = \exp \left( 2N \int_0^{2\pi} \log(e^{it} - z) \omega_{\Gamma_E}(e^{it}) dt - i2N\gamma \right).
$$

As a result, it follows that $H(1/z) = H(z)$. Therefore, $G(1/z) = G(z)$, and since $G$ is real on both sides of $\Gamma_E$, it follows from the extension principle that $G$ can be continued analytically through each arc $(e^{i\alpha_{2j-1}}, e^{i\alpha_{2j}})$. Around $e^{i\alpha_{2j-1}}$ both $H$ and $1/H$ are bounded because $\omega_{\Gamma_E}(e^{it}) \leq C/\sqrt{|t - \alpha_{2j-1}|}$ in a neighborhood of $\alpha_{2j-1}$ (note that $\omega_{\Gamma_E}(e^{it}) \leq \omega_J(e^{it}) = J := [e^{i\alpha_{2j-1}}, e^{i\alpha_{2j}}]$, and apply formula (1.3)), so $G$ is analytic at every $e^{i\alpha_{2j-1}}$. In a similar manner, $G$ is analytic at every $e^{i\alpha_{2j}}$. Hence, $G$ is analytic on $\mathbb{C} \setminus \{0\}$. It is clear that $G$ has a pole of order $N$ both at 0 and at $\infty$, therefore, $G(z)$ is a rational function of the form $P_{2N}(z)/z^N$, which shows that, $U_N(x) = G(e^{ix}) = e^{-iN\pi}P_{2N}(e^{ix})$ is indeed a trigonometric polynomial of degree at most $N$.

Next, we need that an arbitrary $E$ for which $\Gamma_E$ consists of finitely many arcs can be approximated arbitrarily well by $T$-sets, c.f. also [17, Theorem 6.3]. Let $E$ be a $2\pi$-periodic set such that $E \cap [0, 2\pi] = \cup_{j=1}^m [a_{2j-1}, a_{2j}]$ with $a_j \in (0, 2\pi)$, and for some small $x = (x_1, \ldots, x_m)$, let

$$
E_x = \bigcup_{j=1}^m [a_{2j-1}, a_{2j} + x_j].
$$

(3.6)

Let us write $x < \epsilon$ if all $x_j < \epsilon$. Then for small $|x_j|$'s the set $E_x$ consists of $m$ intervals.

**Lemma 3.4** For every $\epsilon > 0$ there are $0 < x_+, x_- < \epsilon$ such that $E_{x_+}$ and $E_{-x_-}$ are $T$-sets.
Note that \( E_{-x} \subseteq E \subseteq E_x \), so we can approximate any such set \( E \) by \( T \)-sets both from the inside and from the outside. Actually, it will turn out that, under this approximation, besides fixing the left-endpoints as in (3.6), we can also fix one of the right-endpoints (any prescribed one), hence \( E \) and \( E_x \) can have a common interval \([a_{2j-1}, a_{2j}]\) (any one of these).

**Proof.** Let \( \mu_{T,x} \) denote the equilibrium measure of the set \( \Gamma_E \), and consider the functions

\[
g_j(x) = \mu_{T,x}(e^{ia_{2j-1}}, e^{ia_{2j}+x_j}]), \quad j = 1, \ldots, m,
\]

i.e. \( g_j(x) \) is the amount of mass that the equilibrium measure of \( \Gamma_{E,x} \) has on the \( j \)-th arc of \( \Gamma_{E,x} \). If we replace \( x \) by some smaller \( x' \), then \( \mu_{T,x'} \) is obtained from \( \mu_{T,x} \) by taking its balayage onto \( \Gamma_{E,x'} \) (see [30, Theorem IV.1.6.(e)]), hence the system \( g_j(x) \) has the following properties (see [32, (2)]):

(A) The \( g_j \)'s are continuous functions on some cube \((-a,a)^m\),

(B) \( g_j \) is strictly increasing in \( x_j \) and strictly decreases in every other variable \( x_k, k \neq j \),

(C) \( \sum_{j=1}^m g_j(x) \equiv 1 \).

Thus, \( \{g_j\}_{j=1}^m \) is a monotone system in the sense of [32]. Now it was proved in [32, Lemma 12] and in the paragraph right after its proof that if one of the \( x_j \)'s, say \( x_m \), is fixed, say \( x_m = 0 \), then the mapping

\[
x \to (g_1(x), \ldots, g_{m-1}(x))
\]

is a homeomorphism from \((-a,a)^{m-1}\) onto some open subset of \( \mathbb{R}^{m-1} \). In particular, there are arbitrarily small \( x > 0 \) for which all \( g_j(x), j = 1, \ldots, m-1 \) are rational. But then \( g_m(x) \) is also rational by property (C) above. However, in view of Lemma 3.3 this means that \( E_x \) is a \( T \)-set.

In a similar manner, there is an \( x < 0 \) with \( x_m = 0 \) lying arbitrarily close to 0 for which \( E_x \) is a \( T \)-set.

Next, we need an explicit form for the equilibrium measure of a set consisting of finitely many arcs on the unit circle. As before, we may assume that \( E \neq \mathbb{R} \), and then that \( 0 \notin E \). Let

\[
E \cap [0, 2\pi] = \bigcup_{j=1}^m [a_{2j-1}, a_{2j}], \quad a_j \in (0, 2\pi).
\]

Then

\[
\Gamma_E = \bigcup_{j=1}^m [e^{ia_{2j-1}}, e^{ia_{2j}}]
\]
consists of the $m$ arcs $I_j := [e^{i\alpha_{2j-1}}, e^{i\alpha_{2j}}]$, and $J_j := (e^{i\alpha_{2j}}, e^{i\alpha_{2j+1}})$, $j = 0, 1, \ldots, m - 1$ (with $\alpha_0 = \alpha_{2m}$) are the complementary arcs to $\Gamma_E$. In what follows we use the main branch of $\sqrt{z}$. Then

$$h(z) = \sqrt[2m]{\prod_{j=1}^{2m} (z - e^{ia_j})}$$  \hspace{1cm} (3.9)$$

is a single-valued analytic function on $\mathbb{C} \setminus \Gamma_E$. We shall need the following form of the equilibrium density $\omega_{\Gamma_E}$ due to Peherstorfer and Steinbauer [24, Lemma 4.1]:

**Lemma 3.5** There are points $e^{i\beta_j} \in J_j$, $j = 0, 1, \ldots, m - 1$, on the complementary arcs $J_j$ with which

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi} \frac{\prod_{j=0}^{m-1} |e^{it} - e^{i\beta_j}|}{\sqrt[2m]{\prod_{j=1}^{2m} |e^{it} - e^{ia_j}|}}, \quad t \in E.  \hspace{1cm} (3.10)$$

The $e^{i\beta_j}$ are the unique points on the unit circle for which

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{j=0}^{m} (e^{it} - e^{i\beta_j})}{\sqrt[2m]{\prod_{j=1}^{2m} (e^{it} - e^{ia_j})}} \, dt = 0, \quad j = 0, 1, \ldots, m - 1,  \hspace{1cm} (3.11)$$

holds, where the denominator is considered as the value of the function (3.9).

Again, since the language here is somewhat different from that of [24], and since we also want to prove the unicity of the $\beta_j$’s with property (3.11), we give a direct proof based on Lemma 3.1 and on the density of $T$-sets. Note that (3.11) is a linear system of equations for the coefficients of the polynomial $\prod(z - e^{i\beta_j})$, but it is not trivial that this system has a unique solution. We shall discuss how to find the $\beta_j$’s after the proof.

**Proof of Lemma 3.5, existence of the $\beta_j$’s.** First of all, it is enough to prove the lemma for $T$-sets. Indeed, suppose that $E = \cup_{j=1}^{m} [a_{2j-1}, a_{2j}]$, $a_j \in (0, 2\pi)$ is an arbitrary set and select $T$-sets of the form $E^{(s)} = \cup_{j=1}^{m} [a_{2j-1}^{(s)}, a_{2j}^{(s)}]$, $a_j^{(s)} \in (0, 2\pi)$ such that $a_{2j-1}^{(s)} = a_{2j-1}$ for all $j$ and $a_{2j}^{(s)} \setminus a_{2j}$ as $s \to \infty$. Thus, if the lemma holds for $T$-sets then there are $\beta_j^{(s)}$ in the complementary intervals $(a_{2j}^{(s)}, a_{2j+1}^{(s)}) \pmod{2\pi}$ such that

$$\omega_{\Gamma_{E^{(s)}}}(e^{it}) = \frac{1}{2\pi} \frac{\prod_{j=0}^{m} |e^{it} - e^{i\beta_j^{(s)}}|}{\sqrt[2m]{\prod_{j=1}^{2m} |e^{it} - e^{ia_j^{(s)}}|}}, \quad t \in E^{(s)}.  \hspace{1cm} (3.12)$$

It is standard that the equilibrium measure of $\Gamma_{E^{(s)}}$ converges in the weak* topology to the equilibrium measure of $\Gamma_E$ as $s \to \infty$. Hence, the measures
Let \( \omega_{E}^{(s)}(t) dt \) in (3.12) converge in the weak* topology, and is easy to see that then each sequence \( \{ \beta_{j}^{(s)} \}_{s=1}^{\infty} \) must converge, say \( \beta_{j}^{(s)} \to \beta_{j} \) with some \( \beta_{j} \in [a_{2j}, a_{2j+1}] \). Therefore, (3.11) is true, and the only problem may be that \( \beta_{j} \) does not belong to the open interval \( (a_{2j}, a_{2j+1}) \). But then \( \beta_{j} \) would be one of the endpoints, say \( \beta_{j} = a_{2j} \), which would mean that \( \omega_{E}^{(t)}(e^{it}) \) has a zero at \( a_{2j} \), which is not the case: Let \( \zeta \in \text{Int}(J) \), \( J := [e^{i(a_{2j}-1)}, e^{i(a_{2j})}] \) and let \( \gamma \) be a smooth closed curve in \( \mathbf{C} \setminus \Gamma_{E} \) circling \( J \) once so that \( \gamma \) does not contain any other point of \( \Gamma_{E} \) in its interior.

Let \( g_{\mathbf{C} \setminus \Gamma_{E}}(\zeta, \infty) \) denote the Green’s function of \( \mathbf{C} \setminus \Gamma_{E} \) with pole at infinity. The density \( \omega_{T_{E}}(\zeta) \) is equal to

\[
\frac{1}{2} \left( \frac{\partial g_{\mathbf{C} \setminus \Gamma_{E}}(\zeta, \infty)}{\partial n_{+}} + \frac{\partial g_{\mathbf{C} \setminus \Gamma_{E}}(\zeta, \infty)}{\partial n_{-}} \right)
\]

(see e.g. [23, II.(4.1)] or [30, Theorem VI.2.3]), and this is

\[
\geq c \frac{1}{2} \left( \frac{\partial g_{\mathbf{C} \setminus \Gamma}^{(\ast)}(\zeta, \infty)}{\partial n_{+}} + \frac{\partial g_{\mathbf{C} \setminus \Gamma}^{(\ast)}(\zeta, \infty)}{\partial n_{-}} \right)
\]

with any constant \( c > 0 \) for which

\[
g_{\mathbf{C} \setminus \Gamma_{E}}(z, \infty) \geq c g_{\mathbf{C} \setminus \Gamma_{E}^{(\ast)}}(z, \infty), \quad z \in \gamma,
\]

because, by the maximum principle, for such a \( c \) we have the same inequality for all \( z \) that lies inside \( \gamma \). Thus, a zero in \( \omega_{T_{E}}(e^{it}) \) at \( a_{2j} \) would yield a zero in \( \omega_{J}(e^{it}) \) at \( a_{2j} \), which is not the case by (1.3). This verifies (3.10) pending its validity for \( T \)-sets.

Thus, let \( E \) be a \( T \)-set, \( E \cap [0, 2\pi] = \bigcup_{j=1}^{m} [a_{2j-1}, a_{2j}] \), \( a_{j} \in (0, 2\pi) \), and let \( U_{N} \) be the trigonometric polynomial appearing in the definition of \( T \)-sets. Let \( E_{1}, \ldots, E_{2N} \) be those subintervals of \( E \) over which \( U_{N} \) runs through the interval \([-1, 1]\). Then their union is \( E \), and suppose that \([a_{2j-1}, a_{2j}]\) contains \( p_{j} \) such subintervals. If two such \( E_{k} \) join each other at a point \( \tau \), then \( U_{N}(\tau) = \pm 1 \), and \( U_{N}'(\tau) = 0 \). There are \( p_{j} - 1 \) such \( \tau \) inside \([a_{2j-1}, a_{2j}]\), so there are altogether \( 2N - m \) such \( \tau \)’s, let these be \( \tau_{1}, \ldots, \tau_{2N-m} \). The trigonometric polynomial \( 1 - U_{N}^{2}(t) \) has a double zero at each \( \tau_{k} \), and, besides these, \( 1-1 \) zero at every \( a_{j} \). These are altogether \((4N-2m) + 2m = 4N \) zeros for \( 1 - U_{N}^{2}(t) \), which is of degree \( 2N \), hence these are all its zeros. As a consequence, \( U_{N} \) does not vanish on the complementary intervals \((a_{2j}, a_{2j+1})\) and it takes the same value (either 1 or \(-1\)) at both endpoints \( a_{2j} \) and \( a_{2j+1} \), therefore \( U_{N} \) must have a zero at some \( \beta_{j} \in (a_{2j}, a_{2j+1}) \), \( j = 0, \ldots, m-1 \). These, together with the \( N - m \) zeros of \( U_{N}' \) at the \( \tau_{k} \)’s give \( N \) zeros for \( U_{N}' \), and these are then all its zeros. Therefore, with some complex numbers \( a, b \) we can write

\[
1 - U_{N}^{2}(t) = ae^{-i2Nt} \prod_{j=1}^{m} \left( e^{it} - e^{ia_{j}} \right) \times \prod_{k=1}^{2N-m} \left( e^{it} - e^{i\tau_{k}} \right)^{2}, \quad (3.13)
\]
\[ U'_N(t) = b e^{-i N t} \prod_{j=0}^{m-1} (e^{i t} - e^{i a_j}) \times \prod_{k=1}^{2N-m} (e^{i t} - e^{i \tau_k}). \] (3.14)

A comparison of the coefficients of \( e^{i 2 N t} \) in (3.13) and of \( e^{i N t} \) in (3.14) gives that \( a = b^2/N^2 \). Now substitute these forms into the right-hand side of (3.2), i.e.

\[ \omega_E(e^{it}) = \frac{1}{2\pi N} \left| U'_N(t) \right| \frac{1}{\sqrt{1 - U_N^2(t)^2}}, \quad t \in E. \]

The factors \( |e^{it} - e^{i \tau_k}| \) cancel and so do \( b \) and \( N \), and we get the form (3.10).

Note also that on the contiguous intervals \((a_{2j}, a_{2j+1})\) the expression \( U_N^2 - 1 \) is positive, and so we can write

\[ \int_{a_{2j}}^{a_{2j+1}} \frac{U'_N(t)}{\sqrt{U_N^2(t)^2 - 1}} \, dt = \left( U_N(t) + \sqrt{U_N^2(t)^2 - 1} \right)_{t=a_{2j+1}}^{t=a_{2j}} = 0, \] (3.15)

since \( U_N \) takes the same value (1 or -1) at \( a_{2j} \) and \( a_{2j+1} \). Using the above substitutions based on (3.13)–(3.14), it is easy to see that (3.15) and (3.11) are the same.

---

**Proof of Lemma 3.5, unicity of the \( \beta_j \)'s.** Note first of all, that

\[ \exp(-\sum_{j=1}^{2m} a_j/2) e^{-im t} \prod_{j=1}^{m-1} (e^{i t} - e^{i a_j}) = \text{const} \prod_{j=1}^{m-1} \sin((t - a_j)/2) \]

is a real trigonometric polynomial of degree \( m \) that vanishes precisely at the endpoints of the subintervals of \( E \). In a similar manner,

\[ T(t) := i^m \exp\left(-\sum_{j=0}^{m-1} \beta_j/2\right) e^{-im(t/2)} \prod_{j=0}^{m-1} (e^{it} - e^{i \beta_j}) \]

is real. It is a trigonometric polynomial of degree at most \( m/2 \) if \( m \) is even, and it is a half-integer trigonometric polynomial of degree at most \( m/2 \) if \( m \) is odd. According to what we have just said, it follows that with it the system (3.11) takes the form

\[ \int_{a_{2j}}^{a_{2j+1}} \frac{T(t)}{\sqrt{\prod_{j=1}^{2m} \sin((t - a_j)/2)}} \, dt = 0, \quad j = 0, 1, \ldots, m - 1. \] (3.16)

Now if we have another system \( \tilde{\beta}_j \in (a_{2j}, a_{2j+1}) \) for which (3.11) is true, then we get another real

\[ \tilde{T}(t) := i^m \exp\left(-\sum_{j=0}^{m-1} \tilde{\beta}_j/2\right) e^{-im(t/2)} \prod_{j=0}^{m-1} (e^{it} - e^{i \tilde{\beta}_j}) \]

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for which
\[ \int_{a_{2j}}^{a_{2j+1}} \frac{\tilde{T}(t)}{\sqrt{\prod_{j=1}^{2m} |\sin((t - a_j)/2)|}} dt = 0, \quad j = 0, 1, \ldots, m - 1. \tag{3.17} \]

Then the same is true of any linear combination \( \cos \tau \cdot T(t) + \sin \tau \cdot \tilde{T}(t) \), hence any such linear combination has a zero on every interval \((a_{2j}, a_{2j+1})\), \( j = 0, 1, \ldots, m - 1 \). On the other hand, any such linear combination is a trigonometric (or half-integer trigonometric) polynomial of degree at most \( m/2 \), therefore it can have at most \( m \) zeros unless it is identically zero, which is certainly not the case (use that non-trivial linear combinations of two polynomials with different zero sets cannot be the zero polynomial). Therefore, all \( \cos \tau \cdot T(t) + \sin \tau \cdot \tilde{T}(t) \) have precisely one zero in every interval \((a_{2j}, a_{2j+1})\), \( j = 0, 1, \ldots, m - 1 \). We shall show that this is enough to conclude that \( \{\beta_j\} = \{\tilde{\beta}_j\} \).

Let us show e.g. that the \( \beta_0 = \tilde{\beta}_0 \). Suppose to the contrary that \( \beta_0 \neq \tilde{\beta}_0 \). Let \( x_\tau \) be the unique zero of \( \cos \tau \cdot T(t) + \sin \tau \cdot \tilde{T}(t) \) that lies in \((a_0, a_1)\). Then \( x_\tau = x_\pi = \beta_0 \), while \( x_{\pi/2} = \tilde{\beta}_0 \). As \( \tau \) moves from 0 to \( \pi/2 \) the point \( x_\tau \) moves continuously from \( \beta_0 \) to \( \tilde{\beta}_0 \), and when \( \tau \) moves further on from \( \pi/2 \) to \( \pi \), then \( x_\tau \) moves continuously back from \( \tilde{\beta}_0 \) to \( \beta_0 \). Thus, there are a \( \tau_1 \in (0, \pi/2) \) and a \( \tau_2 \in (\pi/2, \pi) \) for which \( x_{\tau_1} = x_{\tau_2} = (\beta_0 + \tilde{\beta}_0)/2 \). This means that \( (\beta_0 + \tilde{\beta}_0)/2 \) is a common zero of \( \cos \tau_1 \cdot T(t) + \sin \tau_1 \cdot \tilde{T}(t) \) and \( \cos \tau_2 \cdot T(t) + \sin \tau_2 \cdot \tilde{T}(t) \). But then \( (\beta_0 + \tilde{\beta}_0)/2 \) must be also a common zero of \( T(t) \) and of \( \tilde{T}(t) \), meaning that \( \beta_0 = (\beta_0 + \tilde{\beta}_0)/2 = \tilde{\beta}_0 \), because \( \beta_0 \) is the only zero of \( T(t) \), and \( \tilde{\beta}_0 \) is the only zero of \( \tilde{T}(t) \) in \((a_0, a_1)\).

Next, let us briefly discuss how to determine the numerator in (3.10). Perhaps the simplest is to note that, as we have just seen, (3.10) and (3.11) can be written in the alternate real form
\[ \omega_{m/2}(e^{it}) = \frac{1}{2\pi} \frac{|S_{m/2}(t)|}{\sqrt{\prod_{j=1}^{2m} |\sin((t - a_j)/2)|}}, \quad t \in E, \tag{3.18} \]

and
\[ \int_{a_{2j}}^{a_{2j+1}} \frac{S_{m/2}(t)}{\sqrt{\prod_{j=1}^{2m} |\sin((t - a_j)/2)|}} dt = 0, \quad j = 0, 1, \ldots, m - 1, \tag{3.19} \]

with some real trigonometric (if \( m \) is even) or half-integer trigonometric (if \( m \) is odd) polynomial of the form
\[ S_{m/2}(t) = \left( \cos \delta \cos \left(\frac{m}{2}t\right) + \sin \delta \sin \left(\frac{m}{2}t\right) \right) \tag{3.20} \]
\[ + \left( A_{m/2-1} \cos \left(\frac{m}{2} - 1\right)t + B_{m/2-1} \sin \left(\frac{m}{2} - 1\right)t \right) + \cdots \]
of degree at most $m/2$ with some $\delta$ and some real coefficients

$$A_{m/2-1}, B_{m/2-1}, A_{m/2-2}, \ldots.$$ 

If $m$ is even, then $B_0$ is missing ($\sin 0 \equiv 0$). Furthermore, the $\beta_j$'s are the zeros of $S_m/2$.

For simplicity consider the case when $m$ is even. Then (3.19) gives $m$ equations for the $m$ unknowns $\delta$ and $A_{m/2-1}, B_{m/2-1}, A_{m/2-2}, \ldots$. This is not a linear system, but we have seen in the preceding proof that the zeros of $S_m/2$ are uniquely determined, and this easily implies that the system (3.19) has a unique solution modulo a $(-1)$ sign in the $A_k$'s and $B_k$'s and modulo replacing $\delta$ by $\delta + \pi$. Therefore, in principle one can determine the unknown quantities in the following way: consider $\delta$ as a parameter. Then (3.19) is an $m \times (m-1)$ linear system for the coefficients $A_k, B_k, k \leq m/2 - 1$ $(B_0$ is missing), which has rank $m - 1$. We can select an $(m - 1) \times (m - 1)$ subsystem which has non-zero determinant. Let these $m - 1$ equations refer to the integrals over the intervals, $[a_{2j}, a_{2j+1}], j \neq j_0$, i.e. we skip the $j_0$-th equation in (3.19). We solve this $(m - 1) \times (m - 1)$ system for the $A_k$'s and $B_k$'s and get unique expressions for them in terms of $\cos \delta$ and $\sin \delta$. This way we get two unique trigonometric polynomials $T_{1,\delta}(x)$ and $T_{2,\delta}(x)$ of degree at most $m/2 - 1$ such that

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\cos \left( (m/2)t + T_{1,\delta}(t) \right) + \sin \left( \sin((m/2)t + T_{2,\delta}(t) \right)}{\sqrt{\prod_{j=1}^{2m} \sin((t - a_j)/2)}} \, dt = 0, \quad j \neq j_0.$$ 

The same for $j = j_0$ can be achieved by selecting $\delta$ appropriately (this gives a unique value for $\tan \delta$, so $\delta$ is determined only up to modulo $\pi$).

We finish this section with the following observation that will be used in the next section. Let $a_k$ be one of the endpoints of $E$, and $U_N$ a trigonometric polynomial as in Lemma 3.1. Then we have (recall the values $\beta_j$ from Lemma 3.5)

$$|U_N'(a_k)| = 2N^2 \frac{\prod_{j=0}^{m-1} |e^{i\alpha k} - e^{i\beta j}|^2}{\prod_{j=1, j \neq k}^{2m} |e^{i\alpha k} - e^{i\alpha_j}|},$$ (3.21)

Indeed, we have the two forms (3.10) and (3.2) for $\omega_{E^1}$, and in these let $t \to a_k, t \in E$. Since

$$1 - U_N^2(t) = U_N^2(a_k) - U_N^2(t) = (1 + o(1))|t - a_k|2U_N(a_k)|U_N'(a_k)| = (1 + o(1))2|t - a_k||U_N'(a_k)|,$$

we obtain from comparing what $1/\sqrt{|t-a_k|}$ is multiplied by in (3.10) and (3.2) that

$$\frac{1}{2\pi} \frac{\prod_{j=0}^{m-1} |e^{i\alpha k} - e^{i\beta j}|}{\sqrt{\prod_{j=1, j \neq k}^{2m} |e^{i\alpha k} - e^{i\alpha_j}|}} = \frac{1}{2\pi N} \frac{|U_N'(a_k)|}{\sqrt{2|U_N'(a_k)|}},$$

from which (3.21) follows.
4 Markov-type inequalities for trigonometric polynomials on finitely many intervals

Let $E$ be a $2\pi$-periodic subset of $\mathbb{R}$ such that $[0, 2\pi] \cap E$ consists of finitely many intervals. The right-hand side in the estimate (2.1) given in Theorem A blows up if $\theta \in E$ approaches one of the endpoints, and in this case a Markov-type inequality should replace (2.1). For example, in the case of a single interval $E = [-a, a]$ Videnskii (see e.g. [9, Sec. 5.1, E19,c]) proved that if $T_n(t)$ is a trigonometric polynomial of degree $n$, then

$$||T'_n||_{[-a,a]} \leq (1 + o(1))2n^2 \cot \frac{a}{2} ||T_n||_{[-a,a]}, \quad (4.1)$$

and this is sharp in the sense that one cannot write a constant smaller than $2 \cot \frac{a}{2}$ on the right. This Markov-type behavior is typical around endpoints of $E$, but different endpoints play different roles, so we get different local Markov factors.

If $\rho > 0$ is any fixed number and $E^\rho = \cup_{j}[a_{2j-1} + \rho, a_{2j} - \rho]$ is the set of points of $E$ that are lying of distance $\geq \rho$ from the complementary arcs, then, in view of Theorem A, an estimate of the form

$$||T'_n||_{E^\rho} \leq Cn ||T_n||_E$$

holds with some constant $C$. Thus, in our Markov-type estimate we may restrict our attention to small neighborhoods $E \cap [a_k - \rho, a_k + \rho]$ of the endpoints $a_k$. Let us assume $\rho$ so small that $E \cap [a_k - \rho, a_k + \rho]$ contains only the endpoint $a_k$ and no other $a_j$. Let $M_k$ be the smallest constant such that

$$||T'_n||_{E \cap [a_k - \rho, a_k + \rho]} \leq (1 + o(1))n^2 M_k ||T_n||_E \quad (4.2)$$

is true for all trigonometric polynomials $T_n$ of degree at most $n$. This is asymptotically the best constant in the Markov-inequality around the endpoint $a_k$.

**Theorem 4.1** For all $k = 1, 2, \ldots, 2m$ we have

$$M_k = 2 \prod_{j=1}^{2m} |e^{ia_k} - e^{i\beta_j}|^2 \prod_{j=1, j \neq k}^{2m} |e^{ia_k} - e^{i\beta_j}|.$$

(4.3)

As an example, consider again the set $E = [-\beta, -\alpha] \cup [\alpha, \beta]$ from Section 2 with some $0 \leq \alpha < \beta \leq \pi$, so that $a_1 = -\beta$, $a_2 = -\alpha$, $a_3 = \alpha$ and $a_4 = \beta$. In this case, by symmetry, $\beta_0 = 0$ and $\beta_1 = \pi$, so (4.3) takes the form

$$M_2 = M_3 = 2 \frac{\sin \alpha}{\cos \alpha - \cos \beta},$$

while

$$M_1 = M_4 = 2 \frac{\sin \beta}{\cos \alpha - \cos \beta}.$$
Corollary 4.2 Let $E$ be a closed set such that $E \cap [0, 2\pi]$ consist of finitely many intervals, and let $M_E$ be the maximum of the $M_k$’s defined in (4.3) for all endpoints of the subintervals of $E$. Then for arbitrary trigonometric polynomials of degree at most $n = 1, 2, \ldots$ we have

$$
\|T_n'\|_E \leq (1 + o(1))n^2M_E\|T_n\|_E,
$$

and this is sharp, for there are trigonometric polynomials $T_n \neq 0$ of degree at most $n = 1, 2, \ldots$ for which

$$
\|T_n'\|_E \geq (1 - o(1))n^2M_E\|T_n\|_E.
$$

This is the global Markov inequality for trigonometric polynomials on several intervals.

As an immediate corollary of Theorem 4.1 we get the following for algebraic polynomials. Let $\Gamma = \bigcup_{j=1}^{m} [e^{ia_{2j-1}}, e^{ia_{2j}}]$ be a closed set on the unit circle consisting of finitely many arcs, and let $\beta_j$ be the numbers from (3.10) for this $\Gamma$. Let $e^{ia_k}$ be one of the endpoints of $\Gamma$, and let $H$ be a closed neighborhood of $e^{ia_k}$ which does not contain any other endpoint of $\Gamma$.

Corollary 4.3 If $P_n$ is a polynomial of degree at most $n$, then

$$
\|P_n'\|_{H \cap \Gamma} \leq (1 + o(1))n^2 \frac{\prod_{j=0}^{m-1} |e^{ia_k} - e^{i\beta_j}|^2}{2 \prod_{j=1, j \neq k}^{2m} |e^{ia_k} - e^{i\beta_j}|} \|P_n\|_{\Gamma},
$$

and this is sharp, for there is a sequence of polynomials $P_n \neq 0$ of degree at most $n = 1, 2, \ldots$ such that

$$
|P_n'(e^{ia_k})| \geq (1 - o(1))n^2 \frac{\prod_{j=0}^{m-1} |e^{ia_k} - e^{i\beta_j}|^2}{2 \prod_{j=1, j \neq k}^{2m} |e^{ia_k} - e^{i\beta_j}|} \|P_n\|_{\Gamma},
$$

Indeed, since $(n + 1)^2/n^2 = 1 + o(1)$, we may assume that $n$ is even. Then $T_{n/2}(t) = e^{-itn/2}P_n(e^{it})$ is a trigonometric polynomial of degree at most $n/2$. Now if we apply Theorem 4.1 to this $T_{n/2}$, then we obtain (4.6). The proof of the converse (4.7) is similar, for in the proof of Theorem 4.1 we are going to verify that there are trigonometric polynomials $T_n$ for which

$$
|T_n'(a_k)| \geq (1 - o(1))n^2 M_k\|T_n\|_E.
$$

As another corollary we obtain Markov-type inequalities for algebraic polynomials on a system of intervals on $\mathbb{R}$. Let $K = \bigcup_{j=1}^{m} [A_{2j-1}, A_{2j}]$. Then (see e.g. [31, (2.4)]) there are points $\xi_j \in (A_{2j}, A_{2j+1})$ in the contiguous intervals such that the density of the equilibrium measure of $K$ has the form

$$
\omega_K(x) = \frac{1}{\pi} \frac{\prod_{j=1}^{m-1} |x - \xi_j|}{\sqrt{\prod_{j=1}^{2m} |x - A_j|}},
$$

(4.8)
and \(\xi_j\) are the unique points that satisfy the system of equations

\[
\int_{A_{2j}}^{A_{2j+1}} \frac{\prod_{j=1}^{m-1} (u - \xi_j)}{\sqrt{\prod_{j=1}^m (u - A_j)}} du = 0, \quad j = 1, \ldots, m - 1. \tag{4.9}
\]

**Theorem 4.4** Let \(A_k\) be one of the endpoints of \(K\) and let \(\eta > 0\) be so small that \([A_k - \eta, A_k + \eta]\) does not contain any other endpoint \(A_j\). Then

\[
\|P'_n\|_{K \cap [A_k - \eta, A_k + \eta]} \leq (1 + o(1))2n^2 \frac{\prod_{j=0}^m |A_k - \xi_j|^2}{\prod_{j=1, j \neq k}^{2m} |A_k - A_j|} \|P_n\|_{K}\tag{4.10}
\]

holds for all algebraic polynomials \(P_n\) of degree at most \(n\).

Furthermore, this estimate is sharp in the sense that no smaller constant can be written on the right-hand side of (4.10).

This is Theorem 4.1 from [31]. There are two ways to deduce Theorem 4.4 from Theorem 4.1: either assume that \(K \subseteq [-1, 1]\) and use the substitution \(x = \cos t\) to go to trigonometric polynomials, or place a huge circle \(C_R\) over the real line touching it at the origin, project the set \(K\) from Theorem 4.1: either assume that \(\alpha\) belongs to the last interval, i.e. \(k = 2m - 1\) or \(k = 2m\) (the ordering of the intervals is arbitrary).

In the limit we get Theorem 4.4.

**Proof of Theorem 4.1.** Let \(M_k^* = M_k^*(E)\) be the expression on the right of (4.3), so our task is to show \(M_k = M_k^*\). Without loss of generality we may assume that \(a_k\) belongs to the last interval, i.e. \(k = 2m - 1\) or \(k = 2m\) (the ordering of the intervals is arbitrary).

First we prove that there are \(T_n \neq 0\) such that

\[
|T_n(a_k)| \geq (1 - o(1))n^2 M_k^* \|T_n\|_E, \tag{4.11}
\]

which then proves \(M_k \geq M_k^*\). Indeed, by the proof of Lemma 3.4 there is a nonnegative sequence \(\{x_s = (x_s^{(1)}, \ldots, x_s^{(m)})\}_{s=1}^{\infty}\) of non-zero vectors converging to the 0 vector such that \(x_s^{(m)} = 0\) and \(E_{x_s}\) is a \(T\)-set for all \(s\), i.e. there are \(N_s\) and trigonometric polynomials \(U_{N_s}\) of degree \(N_s\) such that Lemma 3.1 holds for \(E_{x_s}\) and \(U_{N_s}\). It is clear that we must have \(N_s \rightarrow \infty\) as \(s \rightarrow \infty\). Thus, in view of the fact that \(a_k\) is an endpoint of \(E_{x_s}\) (note that the last interval of \(E\) did not change when we moved to \(E_{x_s}\)), we can apply (3.21), according to which

\[
U_{N_s}'(a_k) = 2N_s^2 \frac{\prod_{j=0}^m |e^{ia_k} - e^{i\beta_j^{(s)}}|^2}{\prod_{j=1, j \neq k}^{2m} |e^{ia_k} - e^{i(a_j + x_j^{(s)})}|^4}, \tag{4.12}
\]

where \(e^{i\beta_j^{(s)}}\) are the points from Lemma 3.1 for the set \(E_{x_s}\), and where we set \(x_{2j-1}^{(s)} = 0\) for all \(j\). As \(s \rightarrow \infty\) we have \(x_j^{(s)} \rightarrow 0\) for all \(j = 1, 2, \ldots, 2m\), and it is then easy to show that \(\beta_j^{(s)} \rightarrow \beta_j\) for all \(j = 0, 1, \ldots, m - 1\) (this follows from
the fact that \( \mu_{\Gamma_{E_N}} \) converges to \( \mu_{\Gamma} \) in the weak*-topology and that, in view of (3.10), the equilibrium densities \( \omega_{\Gamma_{E_N}} \) are uniformly equicontinuous on every closed subinterval of the interior of \( \Gamma_E \). From all these and from the fact that \( |U_N| \leq 1 \) on \( E \), we obtain (4.11) along the subsequence \( n = N_s, s = 1, 2, \ldots \) of the natural numbers. Now note that if \( T_l = \cos(l \arccos x) \) are the classical Chebyshev polynomials, then \( T_l(U_N) \) is a trigonometric polynomial of degree \( lN_s \) for which

\[
E_{\chi_s} = \{ x [ T_l(U_N(x)) \in [-1, 1] \}
\]

and

\[
(T_l(U_N))'(a_k) = 2(lN_s)^2 \prod_{j=0}^{m} |e^{ia_k} - e^{i\beta_l}|^2 \prod_{j=1, j \neq k}^{2m} |e^{ia_k} - e^{i(a_j + x)(\text{mod} 1)}|,
\]

i.e. whenever \( N_s, U_N \) is suitable for \( E_{\chi_s} \) in the \( T \)-set definition, so is \( lN_s, T_l(U_N) \) for any \( l = 1, 2, \ldots \). Hence, in the preceding reasoning we can replace \( N_s \) by any \( lN_s, l = 1, 2, \ldots \). For \( n \to \infty \) we can select numbers of the form \( lN_s \) with \( s \to \infty \) and \( n/lN_s \to 1 \), hence (4.11) holds.

Thus, to complete the theorem, we need to show that \( M_k \leq M_k^* \). As before, we may assume that \( k = 2m - 1 \) or \( k = 2m \), say, for definiteness, that \( a_k = a_{2m-1} \). Let \( \gamma > M_k^* \), and select a small \( x_0 \geq 0, x_m = 0 \), such that \( E := E_{x_0} \) is a \( T \)-set and \( M_k^* \) is the expression on the right of (4.3) for the set \( E \). This is clearly possible, since \( M_k^* \) changes continuously with the endpoints of the set \( E \). Recall also that \( E \subset E \) because \(-x \leq 0 \).

Let \( U_N \) be a real trigonometric polynomial of some degree \( N \) such that \( U_N(t) \) runs through the interval \([-1, 1]\) \( 2N \)-times as \( t \) runs through \( E \), and let \( \tilde{E}_1, \ldots, \tilde{E}_{2N} \) be those intervals on which \( U_N(t) \) runs through \([-1, 1]\). Then the union of these \( \tilde{E}_i \)'s is \( E \), and \( a_{2m-1} \) is the left-endpoint of one of these, say of \( \tilde{E}_1 \). Since \( a_{2m-1} \) is also a left-endpoint of a subinterval of \( E \), there is no \( E_s \) attached to \( E \). By (3.21) we have \( U_N'(a_{2m-1}) = N^2 M_k^* \), and select a small \( \eta > 0 \) so that

\[
\| U_N' \| [a_{2m-1}, a_{2m-1} + \eta] < N^2 \gamma.
\]

We may assume \( \eta \) so small that \([a_{2m-1}, a_{2m-1} + \eta] \) lies of positive distance from \( \tilde{E}_1 \). Next, we need the following lemmas that we verify after completing the proof.

Let \( \tau_1, \ldots, \tau_{2N+m} \) be those \( x \in E \) for which \( |U_N(x)| = 1 \) (cf. the proof of Lemma 3.5). Without loss of generality we may assume that \( \tau_1 = a_{2m-1} \).

**Lemma 4.5** For any \( l \) there are real trigonometric polynomials \( Q_l \) of degree at most \( l \) such that for large \( l \) we have \( 0 \leq Q_l(x) \leq 1 \) for all \( x \), \( 0 \leq Q_l(x) \leq l^{-4} \prod_{k>1} \sin^2((x - \tau_k)/2) \) for \( x \in \tilde{E}_1 \), and \( 1 - l^{-4} \leq Q_l(x) \leq 1 \) for \( x \in [a_{2m-1}, a_{2m-1} + \eta] \). Furthermore, \( |Q_l(x)| \leq l^{-2} \) on \( \tilde{E}_1 \) and on \([a_{2m-1}, a_{2m-1} + \eta] \).
Lemma 4.6 If $I$ is an interval then there is a constant $C_I$ such that
\[ \|T_n\|_I \leq C_I n^2 \|T_n\|_I \]
for all trigonometric polynomials $T_n$ of degree at most $n = 1, 2, \ldots$.

Now let $n$ be a large number and $T_n$ a trigonometric polynomial of degree at most $n$. We may assume $\|T_n\|_E \leq 1$. With $l = \sqrt{n}$ consider the trigonometric polynomials $Q_\eta$ from Lemma 4.5, and set $V_n(x) = T_n(x)Q_{\sqrt{n}}(x)$. Then this is a trigonometric polynomial of degree at most $n + \sqrt{n}$ and in it $Q_{\sqrt{n}}(x)$ is

- smaller than $n^{-2}$ on $\tilde{E} \setminus \tilde{E}_1$,
- closer than $n^{-2}$ to 1 on the interval $[a_{2m-1}, a_{2m-1} + \eta]$,
- in absolute value its derivative is smaller than $1/n$ on $\tilde{E} \setminus \tilde{E}_1$ and on $[a_{2m-1}, a_{2m-1} + \eta]$.

For a $t \in \tilde{E}_1$ let $t_1, \ldots, t_{2N}$, $t_1 = t$, be those points in $\tilde{E}$ for which $U_N(t_j) = U_N(t)$, and consider the sum of the values $V_n(t_j)$. According to Lemma 3.2 there is an algebraic polynomial $S_{(n+\sqrt{n})/N}$ of degree at most $(n + \sqrt{n})/N$ such that
\[ \sum_{j=1}^{2N} V_n(t_j) = S_{(n+\sqrt{n})/N}(U_N(t)). \tag{4.14} \]

On $\tilde{E}$ the absolute value on the left-hand side is at most $1 + 2N/n^2$, so we have $\|S_{(n+\sqrt{n})/N}||[-1,1] \leq 1 + 2N/n^2$, since the image of $\tilde{E}$ under $U_N$ is the interval $[-1,1]$. Therefore, by the classical Markov inequality on $[-1,1]$, we have that
\[ \left| \left( S_{(n+\sqrt{n})/N}(U_N(x)) \right)'(x) \right| \leq \left( \frac{n + \sqrt{n}}{N} \right)^2 \left( 1 + \frac{2N}{n^2} \right) |U_N'(x)|, \]
which is smaller than
\[ \left( \frac{n + \sqrt{n}}{N} \right)^2 \left( 1 + \frac{2N}{n^2} \right) N^2 \gamma = (n + \sqrt{n})^2 (1 + 2N/n^2) \gamma \]
on $[a_{2m-1}, a_{2m-1} + \eta]$ by (4.13). Hence, we get from (4.14) on the interval $[a_{2m-1}, a_{2m-1} + \eta]$
\[ \left| T_n'(t)Q_{\sqrt{n}}(t) + \sum_{j=2}^{2N} \frac{dT_n(t_j(t))}{dt}Q_{\sqrt{n}}(t_j) + \sum_{j=1}^{2N} T_n(t_j)Q_{\sqrt{n}}'(t_j) \right| \leq (n + \sqrt{n})^2 (1 + 2N/n^2) \gamma. \tag{4.15} \]

Now we use that for $t \in [a_{2m-1}, a_{2m-1} + \eta]$ we have for $j = 2, 3, \ldots, 2N$ the relation $t_j(t) \in \tilde{E} \setminus \tilde{E}_1$. Therefore, on the left $|Q_{\sqrt{n}}'(t_j)| \leq 1/n$ for all $j$. 

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and $|Q\sqrt{\tau}(t_j)| \leq 1/n^2$ for all $j \geq 2$. Furthermore, in view of Lemma 4.6, the estimate
\[ |T_n'(t_j)| \leq C_0 n^2 \] (4.16)
also holds with some $C_0$. However,
\[ \frac{dT_n(t_j(t))}{dt} = T_n'(t_j) \frac{dt_j(t)}{dt} \]
involves also $dt_j(t)/dt$, and this derivative needs special attention. As before, let $\tau_1, \ldots, \tau_{2N+m}$ be those $x \in \tilde{E}$ for which $|U_N(x)| = 1$, and suppose that $\tau_1 = a_{2m-1}$, is the point around which we are considering the Markov inequality. For $t \in [a_{2m-1}, a_{2m-1} + \eta]$ the point $t_j(t)$ belongs to some interval determined by the $\tau_k$’s, say $t_j(t) \in [\tau_s, \tau_{s+1}]$, and if $\eta > 0$ is small enough, then for all $t \in [a_{2m-1}, a_{2m-1} + \eta]$ the point $t_j(t)$ belongs either to the left-half or to the right-half of that interval, say $t_j(t) \in [\tau_s, (\tau_s + \tau_{s+1})/2]$. Now there are two possibilities:

**I** $\tau_s$ is an endpoint of one of the subintervals making $\tilde{E}$. In that case $U_N'(\tau_s) \neq 0$, and, as a consequence (use the mean value theorem), $|t_j(t) - \tau_s| \sim |t - a_{2m-1}|$ (meaning that the ratio of the two sides lies in between two positive constants),
\[ \left| \frac{dt_j(t)}{dt} \right| \leq C \quad \text{on} \ [a_{2m-1}, a_{2m-1} + \eta]. \]

**II** $\tau_s$ belongs to the interior of $\tilde{E}$. Then $U_N'(\tau_s) = 0$, and $\tau_s$ is a double (and not higher order) zero of $1 - U_N^2$. Hence, in this case (use Taylor’s formula),
\[ \frac{dt_j(t)}{dt} \leq \frac{C_1'}{|t - a_{2m-1}|^{1/2}} \leq \frac{C_1}{|t_j(t) - \tau_s|} \quad \text{on} \ [a_{2m-1}, a_{2m-1} + \eta]. \]

These imply (cf. also (4.16))
\[ \left| \frac{dT_n(t_j(t))}{dt} \right| \leq C_0 n^2 \frac{C_1}{|t_j(t) - \tau_s|}, \]
so by Lemma 4.5
\[ \left| \frac{dT_n(t_j(t))}{dt} \right| Q\sqrt{\tau}(t_j(t)) \leq C_0 n^2 \frac{C_1}{|t_j(t) - \tau_s|} \frac{1}{n^2} \prod_{k=2}^{2N+m} \sin^2((t_j - \tau_k)/2) \leq L, \]
where $L$ is the maximum of all
\[ \max_{x \in E} \frac{\sin^2((x - \tau_s)/2)}{|x - \tau_s|} \prod_{k=2, k \neq s}^{2N+m} \sin^2((x - \tau_k)/2), \quad s = 2, \ldots, 2N + m. \]
Finally, since $|Q_\sqrt{n}(t_1) - 1| \leq 1/n^2$ is also true, we can infer from (4.15) that for all $t \in [a_{2m+1}, a_{2m-1} + \eta]$

$$|T_n(t)| \leq (n + \sqrt{n})^2(1 + N/n^2)\gamma + 2C_0NL + 2N\frac{1}{n} = (1 + o(1))n^2\gamma.$$

Since $\gamma > M_{2m-1}$ was arbitrary, $M_{2m-1} \leq M_2^*$ follows, and this is what we needed to prove.

\[\Box\]

**Proof of Lemmas 4.5 and 4.6.** Let $f$ be the function that is equal to $1/\prod_{k \geq 1} \sin^2((x - \tau_k)/2)$ on the interval $[a_{2m-1}, a_{2m-1} + \eta]$ and is zero on $\tilde{E} \setminus \tilde{E}_1$, and extend this $f$ to a 6 times continuously differentiable $2\pi$ periodic function so that $0 \leq f(x) \leq 1/\prod_{k \geq 1} \sin^2((x - \tau_k)/2)$ holds for the extended function. By Jackson’s theorem [10, Corollary 7.2.4] there are trigonometric polynomials $Q_l^*$ of degree at most $l/2$ such that $|f - Q_l^*| \leq Cl^{-6}$ is true with some constant $C$. Then it is easy to see that

$$Q_l(x) = Q_l^*(x) + Cl^{-6} \frac{2N + m}{1 + 2CMl^{-6}} \prod_{k=2}^{\infty} \sin^2((x - \tau_k)/2),$$

where $M$ is the maximum of $\prod_{k \geq 1} \sin^2((x - \tau_k)/2)$ on $\tilde{E}$, satisfies the requirements not considering those for the derivatives. For example, for $x \in [a_{2m-1}, a_{2m-1} + \eta]$ we have

$$\left|Q_l^*(x) - \frac{1}{\prod_{k \geq 1} \sin^2((x - \tau_k)/2)}\right| \leq Cl^{-6},$$

therefore

$$|1 - Q_l(x)| = \frac{1 + CMl^{-6} - Q_l^*(x) \prod_{k \geq 1} \sin^2((x - \tau_k)/2) + Cl^{-6} \prod_{k \geq 1} \sin^2((x - \tau_k)/2)}{1 + CMl^{-6}} \leq C'l^{-6} \leq l^{-4}$$

for all large $l$.

However, on each interval $\tilde{E}_j$, $j = 2, \ldots, 2N$ we have $|Q_l| \leq C_0l^{-6}$ while on $[a_{2m-1}, a_{2m-1} + \eta]$ we have $|1 - Q_l| \leq C_0l^{-6}$ with some $C_0$ by our construction, hence Lemma 4.6 gives on all these intervals $|Q_l' | \leq C_1l^2l^{-6} = C_1l^{-4}$ with some $C_1$ that may depend on $N$, $\tilde{E}$ and $\eta > 0$. Therefore, for large $l$ we have $|Q_l'| < l^{-2}$ on $(\tilde{E} \setminus \tilde{E}_1) \cup [a_{2m-1}, a_{2m-1} + \eta]$.

Lemma 4.6 is due to D. Jackson [14], and it is also an immediate consequence of Videnskii’s inequality (4.1).

\[\Box\]
5 Appendix
An elementary proof for Theorem A

Having $T$-sets at our disposal, we are in the position to give a relatively simple elementary proof of Theorem A. This is justified by the fact that the original proof given by A. Lukashov in [19] is based on the Schottky-Burnside theory of automorphic forms, while the proof given in Section 2 is based on the Achiezer-Levin theory of conformal maps.

First of all, by Theorem 2.1 of [33] it is enough to prove that for fixed $\theta$

$$|T'_n(\theta)| \leq (1 + o(1))n2\pi\omega_T(e^{i\theta})\|T_n\|_E. \quad (5.1)$$

with $o(1)$ tending to 0 uniformly in $T_n$ as $n \to \infty$.

The proof follows the scheme of [31, Theorem 3.1].

Case 1: $E$ is a $T$-set and $T_n$ is a polynomial of $U_N$

Let $U_N$ be a real trigonometric polynomial of some degree $N$ such that $U_N(t)$ runs through the interval $[-1, 1]$ $2N$-times as $t$ runs through $E$, and assume that $T_n(t) = P_m(U_N)$ with some polynomial $P_m$. Then $n = mN$ and $\|P_m\|_{[-1, 1]} = \|T_n\|_E$. Using Bernstein’s inequality

$$|P'_m(x)| \leq \frac{m}{\sqrt{1 - x^2}}\|P_m\|_{[-1, 1]}, \quad x \in [-1, 1],$$

for $P_m$, we obtain

$$|T'_n(\theta)| = |P'_m(U_N(\theta))U'_N(\theta)| \leq \frac{m}{\sqrt{1 - U_N(\theta)^2}}\|P_m\|_{[-1, 1]}|U'_N(\theta)|$$

$$\leq Nm2\pi\omega_T(\theta)\|T_n\|_E,$$

where, in the last step, we used (3.2).

Case 2: $E$ is a $T$-set and $T_n$ is arbitrary

Let $U_N$ be a real trigonometric polynomial of some degree $N$ such that $U_N(t)$ runs through the interval $[-1, 1]$ $2N$-times as $t$ runs through $E$, and let $E_1, \ldots, E_{2N}$ be those intervals on which $U_N(t)$ runs through $[-1, 1]$. Actually, we need this case only when the fixed $\theta$ lies in the interior of one of the $E_j$’s, say $\theta \in \text{Int}(E_{i_0})$.

With

$$Q_{\sqrt{n}}(t) = \left(\frac{1 - \cos(t - \theta)}{2}\right)^{2[\sqrt{n}/2]}$$

we set $V_n(t) = T_n(t)Q_{\sqrt{n}}(t)$. We have again, as in (4.14),

$$\sum_{j=1}^{2N} V_n(t_j) = S_{(n+\sqrt{n})/N}(U_N(t)), \quad (5.2)$$
where, for a \( t \) in \( E_{i_0} \), the numbers \( t_1, \ldots, t_{2N}, t_{i_0} = t \), are those points in \( E \) for which \( U_N(t_j) = U_N(t) \). Since both \( Q_{\sqrt{\pi}} \) and its derivative tend to zero uniformly outside any neighborhood (mod \( 2\pi \)) of \( \theta \) as \( n \to \infty \), furthermore \( Q'_{\sqrt{\pi}}(\theta) = 0 \), it follows that, as \( n \to \infty \),

\[
\|S_{(n+\sqrt{\pi})/N}(U_N)\|_E \leq (1 + o(1))\|T_n\|_E,
\]

\[
\frac{d}{dt}V_n(t_j) \bigg|_{t = \theta} = o(1)\|T_n\|_E, \quad j \neq j_0,
\]

(see also Lemma 4.6) and

\[
V'_n(\theta) = T'_n(\theta) + o(1)\|T_n\|_E,
\]

hence

\[
\frac{d}{dt}S_{(n+\sqrt{\pi})/N}(U_N(t)) \bigg|_{t = \theta} = \frac{d}{dt} \sum_{j=1}^{2N} V_n(t_j) \bigg|_{t = \theta} = V'_n(\theta) + o(1)\|T_n\|_E
\]

\[
= T'_n(\theta) + o(1)\|T_n\|_E,
\]

where the \( o(1) \) depends only on the set \( E \) and on the location of \( \theta \) inside \( E_{i_0} \). Therefore, the just proven Case 1 gives

\[
|T_n'(\theta)| \leq \left| \frac{d}{dt}S_{(n+\sqrt{\pi})/N}(U_N(t)) \bigg|_{t = \theta} \right| + o(1)\|T_n\|_E
\]

\[
\leq (1 + o(1))(\sqrt{n})2\pi\omega_{T_{E}}(\theta)\|S_{(n+\sqrt{\pi})/N}(U_N)\|_E + o(1)\|T_n\|_E
\]

\[
\leq (1 + o(1))n2\pi\omega_{T_{E}}(\theta)\|T_n\|_E.
\]

Completion of the proof

We may assume \( E \) to consist of finitely many intervals, see the remark after Theorem A.

Let \( \theta \) lie in the interior of \( E \), and for an \( \varepsilon > 0 \) choose a \( T \)-set \( \tilde{E} \subset \text{Int}(E) \) such that \( \theta \) lies inside \( \tilde{E} \) and \( \omega_{T_{E}}(\theta) \leq (1 + \varepsilon)\omega_{T_{E}}(\theta) \). By Lemma 3.4 this is possible (see also Lemma 3.5 and its proof). Let \( U_N \) be a real trigonometric polynomial of some degree \( N \) such that \( U_N(t) \) runs through the interval \([-1, 1]\) \( 2N \)-times as \( t \) runs through \( \tilde{E} \), and let \( \tilde{E}_1, \ldots, \tilde{E}_{2N} \) be those intervals on which \( U_N(t) \) runs through \([-1, 1]\). By slightly shifting \( \tilde{E} \) if necessary, we may assume that \( \theta \) lies in the interior of one of the \( \tilde{E}_j \)’s, so \( \theta \) and \( \tilde{E} \) have the properties that were used in Case 2 above. Therefore, by Case 2,

\[
|T_n'(\theta)| \leq (1 + o(1))n2\pi\omega_{T_{E}}(\theta)\|T_n\|_E \leq (1 + o(1))n2\pi(1 + \varepsilon)\omega_{T_{E}}(\theta)\|T_n\|_E,
\]

where we also used that \( \tilde{E} \subset E \). Since \( \varepsilon > 0 \) is arbitrary, we are done with the proof of (5.1).

\[\blacksquare\]

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References


[22] B. Nagy and V. Totik, Riesz-type inequalities on general sets (manuscript)


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