Bernstein and Markov type inequalities for trigonometric polynomials on general sets*

Vilmos Totik†

Abstract

Bernstein and Markov-type inequalities are discussed for the derivatives of trigonometric and algebraic polynomials on general subsets of the real axis and of the unit circle. It has recently been proven by A. Lukashov that the sharp Bernstein factor for trigonometric polynomials is the equilibrium density of the image of the set on the unit circle under the mapping $t \to e^{it}$. In this paper Lukashov's theorem is extended to entire functions of exponential type using a result of Achieser and Levin. The asymptotically sharp Markov factors for trigonometric polynomials on several intervals is also found via the so called T-sets of F. Peherstorfer and R. Steinbauer. This sharp Markov factor is again intimately connected with the equilibrium measure of the aforementioned image set.

1 Introduction

About one hundred years ago in 1912 S. N. Bernstein [7], [8] proved his famous inequality: if T_n is a trigonometric polynomial of degree at most n, then

$$\|T_n'\| \le n\|T_n\|, \tag{1.1}$$

where $\|\cdot\|$ denotes the supremum norm. Actually, Bernstein had $2n$ instead of n, but a very simple argument (which is attributed to Landau in [8, p. 527]) based on his result gives also n ((1.1) was first published by M. Riesz [29] in 1914). This inequality of Bernstein gave rise to converse results in approximation, and it has been applied in thousands of situations. Half a century later, in 1960, V. S. Videnskii [35] proved the analogue of (1.1) on intervals less than a whole period: if $\beta \in (0, \pi)$, then for $\theta \in (-\beta, \beta)$ we have

$$|T_n'(\theta)| \le n\frac{\cos\theta/2}{\sqrt{\sin^2\beta/2 - \sin^2\theta/2}}\|T_n\|_{[-\beta,\beta]}, \tag{1.2}$$

*AMS Classification 26D05, 42A05, Keywords: polynomial inequalities, trigonometric polynomials, Bernstein inequality, general sets, equilibrium density, Achiezer-Levin conformal map, T-sets

†Supported by the National Science Foundation grant DMS-1265375

and this is sharp. It turns out that the factor on the right of (1.2) is essentially the equilibrium density of the arc $\Gamma_\beta := \{e^{it} \bigm| -\beta \le t \le \beta\}$, namely if $\omega_{\Gamma_\beta}(e^{it})$ denotes the density of the equilibrium measure of the arc Γ_β with respect to arc measure on the unit circle, then (see [33])

$$\omega_{\Gamma_\beta}(e^{it}) = \frac{1}{2\pi}\frac{\cos t/2}{\sqrt{\sin^2\beta/2 - \sin^2 t/2}}. \tag{1.3}$$

The extension of Videnskii's inequality to general sets was done by A. Lukashov [19] in 2004, see Theorem A below. He showed that the corresponding Bernstein factor is the same as 2π-times the equilibrium density of the set on the unit circle that corresponds to E under the mapping $t \to e^{it}$.

In this paper we discuss some consequences of this result and an extension to entire functions of exponential type. Sharp Bernstein-type inequalities for algebraic polynomials on closed subsets of the unit circle is deduced. We shall also consider the analogous Markov-type problem that arises around the endpoints of subintervals of E.

We shall use tools from function theory and from potential theory that can be found e.g. in the books [4], [13], [18], [28], [30] or [34]. In particular, we shall need the concept of equilibrium measure μ_E of a compact set E (of positive logarithmic capacity). If E lies on the unit circle, then on the one-dimensional interior (interior relative to the unit circle) the equilibrium measure μ_E is absolutely continuous with respect to the arc measure on the circle, and we shall denote its density by ω_E, i.e. $\omega_E(e^{it})dt = d\mu_E(e^{it})$. Likewise, if E lies on the real line then, in the one dimensional interior of E, ω_E denotes the density of the equilibrium measure μ_E with respect to the Lebesgue-measure on $\mathbf{R}$. In both cases ω_E is an infinitely many times differentiable function on the interior of E.

The paper is organized as follows. In the next section we discuss the general Bernstein-Videnskii inequality of Lukashov and prove an extension of it to entire functions of exponential type using the Levin conformal maps and a theorem of Achiezer and Levin. Section 3 discusses basic properties of special sets that arise as inverse images of $[-1, 1]$ under some special trigonometric polynomials. These sets have nice properties (e.g. every subinterval has rational harmonic measure) and they can approximate any set on $[0, 2\pi]$ consisting of finitely many intervals. These special sets will be fundamental in Section 4 in proving the exact Markov-type inequalities for trigonometric polynomials on sets consisting of finitely many intervals (on $[0, 2\pi]$). To illustrate the power of the method, we give a new and rather elementary proof of Lukashov's theorem in an Appendix (Lukashov's original proof used automorphic forms and Schottky groups, but since then there has been a different approach by Dubinin and Kalmykov [11], [12], [15]).

2 Bernstein-type inequalities

Let C_1 be the unit circle. For a 2π-periodic set $E \subset \mathbf{R}$ let

$$\Gamma_E = \{e^{it} \,|\, t \in E\}$$

be the set that corresponds to E when we identify $(-\pi, \pi]$ with C_1. Let us agree that when we integrate over E, then we just integrate over $(0, 2\pi] \cap E$.

The following far-reaching extension of Videnskii's inequality is a special case of a result of A. Lukashov [19]. In it $\mathrm{Int}(E)$ denotes the one-dimensional interior of E.

Theorem A *Let $E \subset \mathbf{R}$ be a 2π-periodic closed set. If $\theta \in \mathrm{Int}(E)$ is an inner point of E, then for any trigonometric polynomial T_n of degree at most $n = 1, 2, \ldots$ we have*

$$|T_n'(\theta)| \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E, \tag{2.1}$$

where ω_{Γ_E} denotes the density of the equilibrium measure of Γ_E with respect to the arc measure on the unit circle.

[19] contains this estimate as a special case for real trigonometric polynomials on finitely many intervals. The extension to general sets (rather than to $E \cap [0, 2\pi]$ consisting of finitely many intervals) is immediate by simple approximation, and the extension to complex trigonometric polynomials follows by a standard trick: if T_n is an arbitrary trigonometric polynomial, then for fixed θ there is a complex number τ of modulus 1 such that $\tau T_n'(\theta) = |T_n'(\theta)|$. Apply (2.1) to the real trigonometric polynomial $T_n^* = \Re(\tau T_n)$ rather than to T_n to get

$$|T_n'(\theta)| = \tau T_n'(\theta) = (T_n^*)'(\theta) \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n^*\|_E \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E.$$

For different proofs of Theorem A see the papers [11], [12]), [15] by V. N. Dubinin and S. I. Kalmykov.

As an example, consider $E = [-\beta, -\alpha] \cup [\alpha, \beta]$ with some $0 \le \alpha < \beta \le \pi$. In this case (see [33, (4.5)])

$$\omega_{\Gamma_E}(e^{i\theta}) = \frac{1}{2\pi} \frac{|\sin\theta|}{\sqrt{|\cos\theta - \cos\alpha||\cos\theta - \cos\beta|}}, \tag{2.2}$$

so we get from Theorem A the sharp inequality

$$|T_n'(\theta)| \le n \frac{|\sin\theta|}{\sqrt{|\cos\theta - \cos\alpha||\cos\theta - \cos\beta|}} \|T_n\|_{[-\beta,-\alpha]\cup[\alpha,\beta]}. \tag{2.3}$$

If $\alpha = 0$, then

$$\frac{|\sin\theta|}{\sqrt{|\cos\theta - 1||\cos\theta - \cos\beta|}} = \frac{\cos\theta/2}{\sqrt{\sin^2\beta/2 - \sin^2\theta/2}},$$

and then (2.3) changes into Videnskii's inequality (1.2).

Generalizing Theorem A we are going to prove in this section

Theorem 2.1 *Let $E \subset \mathbf{R}$ be a 2π-periodic closed set. If $\theta \in E$ is an inner point of E, then for any exponential function f of type σ we have*

$$|f'(\theta)| \leq \sigma 2\pi \omega_{\Gamma_E}(e^{i\theta})\|f\|_E. \tag{2.4}$$

It is not difficult to prove using the results of [19] and the density theorem (Lemma 3.4) below that (2.1) is sharp:

Theorem B *If $E \subset \mathbf{R}$ is as before, and $\theta \in E$ is an inner point of E, then there are trigonometric polynomials $T_n \not\equiv 0$ of degree at most $n = 1, 2, \ldots$ such that*

$$|T_n'(\theta)| \geq (1 - o(1))n 2\pi \omega_{\Gamma_E}(e^{i\theta})\|T_n\|_E. \tag{2.5}$$

This sharpness also follows from Theorem 2 of [22].

By the simple $x = \cos t$ substitution we obtain the following sharp Bernstein-type inequality for algebraic polynomials on arbitrary compact subset of the real line. In its formulation, for a compact set $K \subset \mathbf{R}$ of positive capacity, let ω_K be the density of the equilibrium measure of K with respect to linear Lebesgue measure on $\mathbf{R}$. Let $K \subset \mathbf{R}$ be compact and x an inner point of K. Then for algebraic polynomials P_n of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(x)| \leq n\pi\omega_K(x)\|P_n\|_K, \qquad x \in K, \tag{2.6}$$

and this is sharp in the sense of Theorem B. This estimate was proved in [5] and [31] by different methods, see also [20].

The estimates (2.1)–(2.5) completely answer the problem of pointwise estimates of the derivative of trigonometric polynomials on closed sets (cf. also [22]). Indeed, let E be a closed set as before, and for $\delta > 0$ let E_δ be the set of points that are of distance $\leq \delta$ from E. Unless E is the whole real line, for sufficiently small δ the sets E_δ are strictly decreasing as δ is decreasing, hence the same is true of the sets Γ_{E_δ}. This implies that for small $\delta' < \delta$ we have

$$\omega_{\Gamma_{E_{\delta'}}}(e^{i\theta}) > \omega_{\Gamma_{E_\delta}}(e^{i\theta}) \tag{2.7}$$

for $e^{i\theta} \in \Gamma_{E_{\delta'}}$. Indeed, $\mu_{\Gamma_{E_{\delta'}}}$ is the balayage of $\mu_{\Gamma_{E_\delta}}$ onto $\Gamma_{E_{\delta'}}$ (see [30, Theorem IV.1.6,(e)]), hence on $\Gamma_{E_{\delta'}}$ the measure $\mu_{\Gamma_{E_{\delta'}}}$ is strictly bigger than $\mu_{\Gamma_{E_\delta}}$. Now (2.7) implies that

$$\tilde{\omega}_E(\theta) = \lim_{\delta \to 0} \omega_{\Gamma_{E_\delta}}(e^{i\theta}) \tag{2.8}$$

exists at every point of E (it could be infinite). Since each $\omega_{\Gamma_{E_\delta}}$ have integral 1 over the unit circle, it follows from Fatou's lemma that

$$\int_E \tilde{\omega}_E \leq 1. \tag{2.9}$$

$2\pi\tilde{\omega}_E$ is precisely the quantity

$$\sup_{T_n} \frac{|T_n'(\theta)|}{n\|T_n\|_E}$$

as is shown by

Corollary 2.2 *Let $E \subset \mathbf{R}$ be a 2π-periodic closed set. If $\theta \in E$, then for any trigonometric polynomial T_n of degree at most $n = 1, 2, \ldots$ we have*

$$|T_n'(\theta)| \le n2\pi\tilde{\omega}_E(\theta)\|T_n\|_E. \tag{2.10}$$

Conversely, if $\gamma < \tilde{\omega}_E(\theta)$, then there are trigonometric polynomials $T_n \not\equiv 0$ of arbitrarily large degree n such that

$$|T_n'(\theta)| \ge n2\pi\gamma\|T_n\|_E. \tag{2.11}$$

This corollary along with (2.9) makes a theorem of Privaloff from 1916 more precise: Privaloff [27] proved that if $E \subset [0, 2\pi]$ is of positive Lebesgue measure $m(E)$, then for every $\varepsilon > 0$ there is a constant $B(\varepsilon) < \infty$ and a subset $E' \subset E$ of measure $\ge m(E) - \varepsilon$ with the property that for trigonometric polynomials T_n of degree at most $n = 1, 2, \ldots$ we have

$$|T_n'(\theta)| \le nB(\varepsilon)\|T_n\|_E, \qquad \theta \in E'. \tag{2.12}$$

It is clear that (2.12) along with (2.9) is a much more precise result, e.g. it gives that in Privaloff's theorem one can put $B(\varepsilon) = 2\pi/\varepsilon$.

In a similar manner, one can derive from (2.6) and its sharpness the following

Corollary 2.3 *Let $K \subset \mathbf{R}$ be a compact subset of $\mathbf{R}$ and define for $x \in K$*

$$\widehat{\omega}_K(x) = \lim_{\delta \to 0} \omega_{K_\delta}(x).$$

If $x \in K$, then for any algebraic polynomial P_n of degree at most $n = 1, 2, \ldots$ we have

$$|P_n'(x)| \le n\pi\widehat{\omega}_K(x)\|P_n\|_K. \tag{2.13}$$

Conversely, if $\gamma < \widehat{\omega}_K(x)$, then there are algebraic polynomials $P_n \not\equiv 0$ of arbitrarily large degree n such that

$$|P_n'(x)| \ge n\pi\gamma\|P_n\|_K. \tag{2.14}$$

In the rest of this section we are going to prove Theorem 2.1 and Corollary 2.2. The proof of Corollary 2.3 follows the reasoning of Corollary 2.2, but using (2.6) and its sharpness instead of Theorem A.

Proof of Corollary 2.2. If $\theta \in E$, then θ lies in the interior of every E_δ, $\delta > 0$, so Theorem A gives

$$|T_n'(\theta)| \le n2\pi\omega_{\Gamma_{E_\delta}}(e^{i\theta})\|T_n\|_{E_\delta} \le n2\pi\tilde{\omega}_E(\theta)\|T_n\|_{E_\delta}.$$

Now if we make here $\delta \to 0$, then we obtain (2.10).

Conversely, if $\gamma < \tilde{\omega}_E(\theta)$, then there is a $\delta > 0$ such that

$$\omega_{\Gamma_{E_\delta}}(e^{i\theta}) > \gamma,$$

and we can apply Theorem B to conclude that there are trigonometric polynomials T_n of arbitrarily large degree n such that

$$|T_n'(\theta)| \geq n2\pi\gamma\|T_n\|_{E_\delta},$$

which is stronger than (2.11).

■

Proof of Theorem 2.1. We deduce the theorem from a result of N. I. Achieser and B. Ya. Levin [1], [2].

First of all, we may assume that Γ_E consists of finitely many arcs on the unit circle. Indeed, suppose (2.4) has been verified for such sets, and for an arbitrary E and for $\delta > 0$ let E_δ be the set of points lying of distance $\leq \delta$ from E. Then $E_\delta \cap [0, 2\pi]$ consists of finitely many intervals, and, as $\delta \to 0$, we have $\omega_{\Gamma_{E_\delta}}(e^{i\theta}) \to \omega_{\Gamma_E}(e^{i\theta})$ at any point θ in the interior of E (see for example [6, Lemma 3.2], apply it with $\rho_n = \delta_\infty$—the Dirac delta at the point infinity—, and use that the balayage of δ_∞ onto E is the equilibrium measure μ_E). Now if we apply (2.4) to this set E_δ and make $\delta \to 0$, we obtain (2.4).

We start with the function

$$\beta(\zeta) = \zeta(\mathrm{cap}(\Gamma_E))^2 \exp\left(-2\int \log(1-\bar{t}\zeta)d\mu_{\Gamma_E}(t)\right) \tag{2.15}$$

(recall that μ_{Γ_E} is the equilibrium measure of Γ_E). By [26, Proposition 9.15] this maps the unit disk Δ conformally onto a domain Δ^* that is obtained from the unit disk by finitely many radial cuts of the form $\{re^{i\tau} \,|\, r_\tau \leq r \leq 1\}$. Actually, the number of cuts equals the number m of arcs complementary to Γ_E (which is the same as the number of arcs in Γ_E). Furthermore, β is continuous on the closed unit disk, it maps Γ_E onto the unit circle C_1, and the complementary arcs to Γ_E are mapped into the radial cuts. Therefore,

$$\alpha(\zeta) = \frac{1}{\overline{\beta(1/\overline{\zeta})}}$$

is a conformal map from the exterior of the unit circle onto a domain which is obtained from the exterior domain $\overline{\mathbf{C}} \setminus \overline{\Delta}$ by the finitely many radial cuts $\{re^{i\tau} \,|\, 1 \leq r \leq 1/r_\tau\}$ (see [3, Sec. 4]). Hence, the function

$$\varphi(z) = i\log\alpha(e^{-iz}), \tag{2.16}$$

where log is any branch of the logarithm, maps the upper half plane $\mathbf{H}_+$ conformally onto a domain $\mathbf{H}_+^*$ which is obtained from $\mathbf{H}_+$ by vertical cuts of the form $\{a + iy \,|\, 0 \leq y \leq y_a\}$. On every interval $(A, A + 2\pi]$ there are m such a's, φ satisfies the property $\varphi(z + 2\pi) = \varphi(z) + 2\pi$, φ maps E onto the real line and it maps the intervals complementary to E into the cuts. Since $\beta(z) \sim \mathrm{const} \cdot z$ as $z \to 0$, it also follows that

$$\varphi(z) \sim z + \mathrm{const} \qquad \text{as } \Im z \to \infty. \tag{2.17}$$

The domain $\mathbf{H}_+^*$ is called Achieser's comb domain, and the mapping φ is called the Levin conformal map (with the proper normalization (2.17)), see [2]–[1], [3]. It is standard that φ can be extended to a continuous function on the closure $\overline{\mathbf{H}_+}$, and this extension is actually C^∞ on every open subinterval of E. The continuous extension of φ is just Caratheodory's theorem, and the C^∞ property can be seen as follows. Let I be a closed subinterval of the interior of E, and $J \subset I$ a closed subinterval of the interior of I. Attach a domain $G \subset \mathbf{H}_+$ to I in such a way that I lies on the boundary of G, and the boundary of G is a C^∞ Jordan curve. Let ψ be a conformal map from the unit disk onto G and let I', J' be the arcs of the unit circle that correspond to I, J, respectively under the map ψ. The function $\Im\varphi(\psi)$ is a positive harmonic function on Δ which is continuous on $\overline{\Delta}$ and which vanishes on I'. Hence, by Poisson's formula, it is a C^∞ function on any closed subarc of the interior of I', therefore so is its analytic conjugate. As a consequence, $\varphi(\psi)$ is a C^∞ function on J', and since ψ is C^∞, invertible function on C_1 with non-zero derivative by the Kellogg-Warschawskii theorem (see [26, Theorems 3.5, 3.6]) , the C^∞ property of φ on J follows.

Consider now that mapping

$$\Phi(z) = e^{-i\varphi(z)} = \alpha(e^{-iz}).$$

It was proved by Achieser and Levin [2, Theorem 3], [1, Theorem 2, Sec. 6] that if f is an entire function of exponential type σ such that $|f(x)| \le 1$ for $x \in E$, then

$$|f'(x)| \le \sigma|\Phi'(x)|, \qquad x \in \mathrm{Int}(E). \tag{2.18}$$

Next, we calculate the derivative on the right-hand side of (2.18).

It is clear that for $x \in \mathrm{Int}(E)$ we have

$$|\Phi'(x)| = |\alpha'(e^{-ix})| = |\beta'(e^{ix})|, \tag{2.19}$$

and here $\beta'(e^{ix})$ can be obtained by taking the limit $\beta'(re^{ix})$ as $r \nearrow 1$. Let $\zeta = re^{ix}$. From the form (2.15) of β we have

$$\beta'(\zeta) = \frac{\beta(\zeta)}{\zeta} + \beta(\zeta)\int_{\Gamma_E} \frac{2\bar{t}}{1-\bar{t}\zeta} d\mu_{\Gamma_E}(t) = \frac{\beta(\zeta)}{\zeta}\int_{\Gamma_E} \frac{1+\bar{t}\zeta}{1-\bar{t}\zeta} d\mu_{\Gamma_E}(t). \tag{2.20}$$

In calculating the limit of this as $r \nearrow 1$ we may assume $x = 0 (\in \mathrm{Int}(E))$. Then $\zeta = r$, and

$$\int_{\Gamma_E} \frac{1+\bar{t}\zeta}{1-\bar{t}\zeta} d\mu_{\Gamma_E}(t) = \int_{\Gamma_E} \frac{t+\zeta}{t-\zeta} d\mu_{\Gamma_E}(t) = \int_E \frac{e^{iu}+r}{e^{iu}-r}\omega_{\Gamma_E}(e^{iu})du,$$

where $\omega_{\Gamma_E}(e^{iu})$ is the density of the equilibrium measure μ_{Γ_E} with respect to arc measure on C_1 (recall that we assumed Γ_E to consist of finitely many arcs). It is easy to see that this $\omega_{\Gamma_E}(e^{iu})$ is a C^∞ function on $\mathrm{Int}(E)$ (indeed, the Green's function of $\overline{\mathbf{C}} \setminus \Gamma_E$ with pole at infinity is a C^∞ function on the interior of Γ_E by the argument made after (2.17), and ω_{Γ_E} is obtained from the Green's function by taking normal derivatives, see [23, II.(4.1)]). Now the last integral is

$$\int_E \left(\frac{1-r^2}{1-2r\cos u + r^2} - 2i\frac{r\sin u}{1-2r\cos u+r^2} \right) \omega_{\Gamma_E}(e^{iu})du,$$

and here the real part is 2π-times the Poisson integral of $\omega_{\Gamma_E}(e^{iu})$ at the point r, so it converges to $2\pi\omega_{\Gamma_E}(1)$ as $r \to 1$. The imaginary part equals

$$-2r\Im\frac{\partial}{\partial r}\int_E \log(r-e^{iu})\omega_{\Gamma_E}(e^{iu})du = -2r\frac{\partial}{\partial r}\Im\left(\int_E \log(r-e^{iu})\omega_{\Gamma_E}(e^{iu})du\right)$$

with any local branch of the logarithm. Therefore, by the Cauchy-Riemann equations, it equals

$$2r\frac{\partial}{\partial y}\Re\left(\int_E \log(r+iy-e^{iu})\omega_{\Gamma_E}(e^{iu})du\right)\Bigg|_{y=0}$$

$$= 2\frac{\partial}{\partial y}\left(\int_E \log|re^{iy}-e^{iu}|\omega_{\Gamma_E}(e^{iu})du\right)\Bigg|_{y=0}.$$

As $r \to 1$, this converges to

$$2\frac{\partial}{\partial y}\left(\int_E \log|e^{iy}-e^{iu}|\omega_{\Gamma_E}(e^{iu})du\right)\Bigg|_{y=0},$$

which is 0, since the logarithmic potential

$$-\int_E \log|e^{iy}-e^{iu}|\omega_{\Gamma_E}(e^{iu})du$$

of the equilibrium measure of Γ_E is constant on Γ_E (and $1 \in \mathrm{Int}(\Gamma_E)$ because we assumed that $0 \in \mathrm{Int}(E)$).

In view of the fact that $|\beta(r\zeta)|$ tends to 1 as $r \to 1$, $\zeta \in \Gamma_E$, the considerations from (2.20) give that

$$|\beta'(e^{ix})| = 2\pi\omega_{\Gamma_E}(e^{ix}),$$

and then (2.18) and (2.19) prove Theorem 2.1.

∎

3 T-sets

In what follows, we shall consider special 2π-periodic sets E for which $E\cap[0,2\pi]$ consists of finitely many intervals. The case $E = \mathbf{R}$ is trivial, and we may concentrate on $E \neq \mathbf{R}$, in which case we may assume $0 \notin E$. Then $E \cap [0,2\pi] = \cup_{j=1}^m [a_{2j-1}, a_{2j}]$, $a_j \in (0,2\pi)$. The special property we are referring to is that there is a real trigonometric polynomial U_N of some degree N such that $U_N(t)$ runs through the interval $[-1,1]$ $2N$-times as t runs through E. In other words,

$$E = \{t \,|\, U_N(t) \in [-1,1]\} \tag{3.1}$$

for some real trigonometric polynomial U_N of degree N which takes both the 1 and -1 values $2N$-times (recall that a trigonometric polynomial of degree N can take a given value at most $2N$-times). Clearly, in this case $|U_N(a_j)| = 1$ for all j.

These sets have been extensively investigated by F. Peherstorfer and R. Steinbauer, and after them let us call a set E with property (3.1) for some U_N a T-set. T-sets also appear as a special case of the sets in [19, Theorem 2] by A. Lukashov. It turns out that

$$\Gamma_E := \{e^{it} \,|\, t \in E\}$$

for T-sets E are also precisely the so-called rational compacts from the beautiful papers [16] and [17] by S. Khrushchev. The papers [16], [17], [24] and [25], proved the basic properties of T-sets (and their cousins Γ_E). The main emphasis in those papers were on orthogonal polynomials with periodic recurrence coefficients and on quadratic irrationalities, and the discussion in [16], [17], [24] and [25] were subject to this emphasis. The present section considers some of the properties of T-sets that we need in the next section to establish the Markov inequalities for trigonometric polynomials on several intervals. Not much originality is claimed here, rather we have a discussion that fits our needs. However, to have a concise treatment independent of orthogonal polynomials, we give full proofs.

Lemma 3.1 *Let E be such that there is a real trigonometric polynomial U_N of degree N such that $U_N(t)$ runs trough the interval $[-1, 1]$ $2N$-times as t runs through E. Then*

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi N} \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}}, \qquad t \in E. \tag{3.2}$$

Cf. [16, (25)].

Proof. There is a polynomial P_{2N} of degree $2N$ such that with $\theta = e^{it}$ we have $U_n(t) = \theta^{-N} P_{2N}(\theta)$. Then

$$\Gamma_E = \left\{ \zeta \,\middle|\, |\zeta| = 1,\ \zeta^{-N} P_{2N}(\zeta) \in [-1, 1] \right\}.$$

Let $E = \cup_{k=1}^{2N} E_k$, where E_k's are intervals, and $U_N(t)$ runs through $[-1, 1]$ precisely once as t runs through E_k. For a $t \in \mathrm{Int}(E_1)$ let $t_k \in E_k$ be the point where $U_N(t_k) = U_N(t)$, i.e. where $P_{2N}(e^{it_k}) = U_N(t)e^{iNt_k}$. Hence, $e^{it_1}, \ldots, e^{it_{2N}}$ are the zeros of the equation

$$P_{2N}(u) - U_N(t)u^N = 0. \tag{3.3}$$

Clearly, $t_k = t_k(t)$ is a differentiable and monotone function of $t \in \mathrm{Int}(E_1)$. Consider the integral

$$\int_E \log|z - e^{it}| \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}} dt = \sum_{k=1}^{2N} \int_{E_k} \log|z - e^{it_k}| \frac{|U_N'(t_k)|}{\sqrt{1 - U_N(t_k)^2}} dt_k.$$

Here the integral over E_k can be calculated with the substitution $t_k = t_k(t)$, $dt_k = t_k'(t)dt$, and it equals

$$\int_{E_1} \log|z - e^{it_k(t)}| \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}} dt,$$

where we used that $U_N(t_k(t)) = U_N(t)$. Hence, the full integral equals

$$\int_{E_1} \left(\sum_{k=1}^{2N} \log|z - e^{it_k(t)}| \right) \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}} dt.$$

Now if A_{2N} is the leading coefficient of P_{2N}, we have

$$\log A_{2N} + \sum_{k=1}^{2N} \log|z - e^{it_k(t)}| = \log|P_{2N}(z) - U_N(t)z^N|,$$

which is the same as $\log|U_N(x) - U_N(t)|$ if $z = e^{ix}$, $x \in \mathbf{R}$. Therefore, for $z = e^{ix} \in \Gamma_E$, i.e. for $x \in E$, or alternatively for $U_N(x) \in [-1, 1]$, we have

$$\begin{aligned}
&\int_{E} \log|z - e^{it}| \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}} dt \\
&\quad = \int_{E_1} \log|U_N(x) - U_N(t)| \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}} dt + \text{const} \\
&\quad = \int_{-1}^{1} \log|U_N(x) - u| \frac{1}{\sqrt{1 - u^2}} du = -\pi \log 2 + \text{const}.
\end{aligned}$$

What we have proven is that the logarithmic potential of the measure

$$d\nu(t) = \frac{1}{2\pi N} \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}} dt, \qquad t \in E, \tag{3.4}$$

is constant on Γ_E. The same calculation that we have just made shows that ν has total mass 1 on E, hence ν is the equilibrium measure of the set Γ_E (see e.g. [30, Theorem I.3.3]).

■

Lemma 3.2 *Let E, U_N as in Lemma 3.1, and for a $t \in E$ with $U_N(t) \in (-1, 1)$ let $t_1, \ldots, t_{2N}$ be those points in E which satisfy $U_N(t_k) = U_N(t)$. Then, if V_n is a trigonometric polynomial of degree at most n, there is an algebraic polynomial $S_{n/N}$ of degree at most n/N such that*

$$\sum_{k=1}^{2N} V_n(t_k) = S_{n/N}(U_N(t)). \tag{3.5}$$

Proof. We use the notations from the proof of Lemma 3.1. We can write

$$V_n(t) = \theta^{-n} Q_{2n}(\theta) = R_n(\theta) + R_n^*(1/\theta), \qquad \theta = e^{it},$$

with some polynomial Q_{2n} of degree at most $2n$ and with some polynomials R_n, R_n^* of degree at most n. With $\theta_k = e^{it_k}$ we have

$$\sum_{k=1}^{2N} V_n(t_j) = \sum_{k=1}^{2N} \theta_k^{-n} Q_{2n}(\theta_k) = \sum_{k=1}^{2N} R_n(\theta_k) + \sum_{k=1}^{2N} R_{2n}^*(1/\theta_k) = \Sigma_1 + \Sigma_2.$$

Since θ_k are the zeros of the equation (3.3), and Σ_1 is a symmetric polynomial of these θ_k, we get, using the fundamental theorem of symmetric polynomials, that Σ_1 can be written as a polynomial of the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_{2N}$ of $\theta_1, \ldots, \theta_{2N}$, and in this representation the exponent of σ_N does not exceed n/N. But, by (3.3) and Viéte's formulae, these σ_j's are constants, except for σ_N, which is $(-1)^N(\text{const} - U_N(t))/A_{2N}$, where A_{2N} is the leading coefficient of P_{2N}. Therefore, Σ_1 is a polynomial of U_N of degree at most $[n/N]$.

Now $1/\theta_j$ are the solutions of the reciprocal equation

$$u^{2N}\big(P_{2N}(1/u) - U_N(t)(1/u)^N\big) = u^{2N} P_{2N}(1/u) - U_N(t) u^N = 0,$$

so the preceding argument yields that Σ_2 is also a polynomial of U_N of degree at most $[n/N]$.

■

We need a characterization of T-sets due to Peherstorfer and Steinbauer, cf. [24, Theorem 4.2] (for a more general statement see [19, Theorem 2]).

Lemma 3.3 *The following are equivalent:*

(a) *There is a real trigonometric polynomial $U_N(t)$ of degree N such that U_N runs trough the interval $[-1, 1]$ $2N$-times as t runs through E.*

(b) *For all $j = 1, 2, \ldots, m$ the harmonic measure $\mu_{\Gamma_E}([e^{ia_{2j-1}}, e^{ia_{2j}}])$ is of the form $p_j/2N$ with some integer p_j.*

Proof of Lemma 3.3. The necessity is immediate from Lemma 3.1, since each interval $[a_{2j-1}, a_{2j}]$ is the union of some of the intervals E_k (see the proof of Lemma 3.1), hence the arcs $[e^{ia_{2j-1}}, e^{ia_{2j}}]$ are the unions of some of the Γ_{E_k}'s. As we have seen in the proof of Lemma 3.1, the integral of $\omega_E(e^{it})$ on any of the E_k is $1/2N$, i.e. $\mu_{\Gamma_E}(\Gamma_{E_k}) = 1/2N$.

Conversely, let us suppose that each $[e^{ia_{2j-1}}, e^{ia_{2j}}]$ carries a mass $p_j/2N$ of the equilibrium measure with some integers p_j:

$$\int_{a_{2j-1}}^{a_{2j}} \omega(e^{it})dt = \frac{p_j}{2N}.$$

Recall that we have assumed $1 \notin \Gamma_E$. Consider in $\mathbf{C} \setminus \Gamma_E$ the function

$$H(z) = \frac{1}{z^N} \exp\left(2N \int_0^{2\pi} \log(e^{it} - z)\omega_{\Gamma_E}(e^{it})dt - 2N \log \operatorname{cap}(\Gamma_E) - iN\gamma\right),$$

where

$$\gamma = \int_0^{2\pi} t\omega_{\Gamma_E}(e^{it})dt,$$

and where $\operatorname{cap}(\Gamma_E)$ denotes the logarithmic capacity of Γ_E. In this definition we used the main branch of the logarithm. As we circle once with z around the arc $[e^{ia_{2j-1}}, e^{ia_{2j}}]$, the argument changes by $\pm p_j 2\pi$, so H is a single-valued analytic function in $\mathbf{C} \setminus (\Gamma_E \cup \{0\})$. Since

$$\int_0^{2\pi} \log |e^{it} - z|\omega_{\Gamma_E}(e^{it})dt = \log \operatorname{cap}(\Gamma_E), \qquad z \in \Gamma_E,$$

the absolute value of H is 1 on both sides of Γ_E. The imaginary part of $\log(e^{it} - e^{ix})$ (taken on the outer part of the unit circle) is

$$= \begin{cases} \frac{t+x}{2} + \frac{\pi}{2} & \text{if } 0 \le x < t \le 2\pi \\ \frac{t+x}{2} - \frac{\pi}{2} & \text{if } 0 \le t \le x \le 2\pi. \end{cases}$$

When we integrate this against $\omega_{\Gamma_E}(e^{it})$ over $[0, 2\pi]$ and take into account that the argument of $z^N = e^{iNx}$ is Nx, we obtain that in the exponent defining $H(z)$ both iNx and $iN\gamma$ cancel, and the argument $A(x)$ of $H(e^{ix})$ (on the outer part of the unit circle) is

$$2N \left(\frac{\pi}{2}\mu_{\Gamma_E}([e^{ix}, e^{i2\pi}]) - \frac{\pi}{2}\mu_{\Gamma_E}([e^{i0}, e^{ix}])\right) = N\pi - 2N\pi\mu_{\Gamma_E}([1, e^{ix}]).$$

Hence, in view of $1 \notin \Gamma_E$ and of the assumption that $2N\mu_{\Gamma_E}([e^{ia_{2j-1}}, e^{ia_{2j}}]) = p_j$ is an integer for all j, we obtain that $H(e^{ix})$ is real on the complementary arcs $[e^{ia_{2j}}, e^{ia_{2j+1}}]$, and its argument changes by $p_j\pi$ as x runs through the interval $[a_{2j-1}, a_{2j}]$. Therefore, the function

$$G(z) := \frac{1}{2}\left(H(z) + \frac{1}{H(z)}\right)$$

is real-valued on the (outer part of the) unit circle, and for $x \in [a_{2j-1}, a_{2j}]$ the value $G(e^{ix})$ is $\cos A(x)$, so $U_N(x) := G(e^{ix})$ runs through $[-1, 1]$ precisely p_j-times as x runs through the interval $[a_{2j-1}, a_{2j}]$. Furthermore, since the logarithmic potential

$$\int_0^{2\pi} \log |e^{it} - z|\omega_{\Gamma_E}(e^{it})dt$$

is bigger than $\log \operatorname{cap}(\Gamma_E)$ outside Γ_E, it also follows that $|G(e^{ix})| > 1$ when $e^{ix} \notin \Gamma_E$. Hence, E is precisely the set of those points x for which $U_N(x) \in$

$[-1,1]$. Therefore, all what remains is to prove that U_N is a trigonometric polynomial of degree N.

To this end consider $\overline{H(1/\overline{z})}$. It is

$$\frac{1}{z^N} z^{2N} \exp\left(2N \int_0^{2\pi} \log(e^{-it} - 1/z)\omega_{\Gamma_E}(e^{it})dt - 2N \log \mathrm{cap}(\Gamma_E) + iN\gamma\right).$$

Now write z^{2N} in the form $\exp(2N \log z)$ and use that

$$\log z + \log(e^{-it} - 1/z) = \log(z - e^{it}) - it = \log(e^{it} - z) - it + \pi \qquad \mathrm{mod}(2\pi i),$$

which implies

$$z^{2N} \exp\left(2N \int_0^{2\pi} \log(e^{-it} - 1/z)\omega_{\Gamma_E}(e^{it})dt\right) = \exp\left(2N \int_0^{2\pi} \log(e^{it} - z)\omega_{\Gamma_E}(e^{it})dt - i2N\gamma\right).$$

As a result, it follows that $\overline{H(1/\overline{z})} = H(z)$. Therefore, $\overline{G(1/\overline{z})} = G(z)$, and since G is real on both sides of Γ_E, it follows from the extension principle that G can be continued analytically trough each arc $(e^{ia_{2j-1}}, e^{ia_{2j}})$. Around $e^{ia_{2j-1}}$ both H and $1/H$ are bounded because $\omega_{\Gamma_E}(e^{it})$ is $\leq C/\sqrt{|t - a_{2j-1}|}$ in a neighborhood of a_{2j-1} (note that

$$\omega_{\Gamma_E}(e^{it}) \leq \omega_{\Gamma_J}(e^{it}), \qquad J := [e^{ia_{2j-1}}, e^{ia_{2j}}],$$

and apply formula (1.3)), so G is analytic at every $e^{ia_{2j-1}}$. In a similar manner, G is analytic at every $e^{ia_{2j}}$. Hence, G is analytic on $\mathbf{C} \setminus \{0\}$. It is clear that G has a pole of order N both at 0 and at ∞, therefore, $G(z)$ is a rational function of the form $P_{2N}(z)/z^N$, which shows that, $U_N(x) = G(e^{ix}) = e^{-iNx}P_{2N}(e^{ix})$ is indeed a trigonometric polynomial of degree at most N.

■

Next, we need that an arbitrary E for which Γ_E consists of finitely many arcs can be approximated arbitrarily well by T-sets, c.f. also [17, Theorem 6.3]. Let E be a 2π-periodic set such that $E \cap [0, 2\pi] = \cup_{j=1}^m [a_{2j-1}, a_{2j}]$ with $a_j \in (0, 2\pi)$, and for some small $\mathbf{x} = (x_1, \ldots, x_m)$, let

$$E_{\mathbf{x}} = \bigcup_{j=1}^m [a_{2j-1}, a_{2j} + x_j]. \tag{3.6}$$

Let us write $\mathbf{x} < \varepsilon$ if all $x_j < \varepsilon$. Then for small $|x_j|$'s the set $E_{\mathbf{x}}$ consists of m intervals.

Lemma 3.4 *For every $\varepsilon > 0$ there are $0 < \mathbf{x}+, \mathbf{x}_- < \varepsilon$ such that $E_{\mathbf{x}+}$ and $E_{-\mathbf{x}_-}$ are T-sets.*

Note that $E_{-\mathbf{x}_-} \subseteq E \subseteq E_{\mathbf{x}_+}$, so we can approximate any such set E by T-sets both from the inside and from the outside. Actually, it will turn out that, under this approximation, besides fixing the left-endpoints as in (3.6), we can also fix one of the right-endpoints (any prescribed one), hence E and $E_{\mathbf{x}}$ can have a common interval $[a_{2j-1}, a_{2j}]$ (any one of these).

Proof. Let μ_{Γ_E} denote the equilibrium measure of the set Γ_E, and consider the functions

$$g_j(\mathbf{x}) = \mu_{\Gamma_{E_{\mathbf{x}}}}\left([e^{ia_{2j-1}}, e^{i(a_{2j}+x_j)}]\right), \qquad j = 1, \ldots, m,$$

i.e. $g_j(\mathbf{x})$ is the amount of mass that the equilibrium measure of $\Gamma_{E_{\mathbf{x}}}$ has on the j-th arc of $\Gamma_{E_{\mathbf{x}}}$. If we replace $\mathbf{x}$ by some smaller $\mathbf{x}'$, then $\mu_{\Gamma_{E_{\mathbf{x}'}}}$ is obtained from $\mu_{\Gamma_{E_{\mathbf{x}}}}$ by taking its balayage onto $\Gamma_{E_{\mathbf{x}'}}$ (see [30, Theorem IV.1.6,(e)]), hence the system $g_j(\mathbf{x})$ has the following properties (see [32, (2)]):

(A) The g_j's are continuous functions on some cube $(-a, a)^m$,

(B) g_j is strictly increasing in x_j and strictly decreases in every other variable x_k, $k \neq j$,

(C) $\sum_{j=1}^m g_j(\mathbf{x}) \equiv 1$.

Thus, $\{g_j\}_{j=1}^m$ is a monotone system in the sense of [32]. Now it was proved in [32, Lemma 12] and in the paragraph right after its proof that if one of the x_j's, say x_m, is fixed, say $x_m = 0$, then the mapping

$$\mathbf{x} \to (g_1(\mathbf{x}), \ldots, g_{m-1}(\mathbf{x}))$$

is a homeomorphism from $(-a, a)^{m-1}$ onto some open subset of $\mathbf{R}^{m-1}$. In particular, there are arbitrarily small $\mathbf{x} > 0$ for which all $g_j(\mathbf{x})$, $j = 1, \ldots, m-1$ are rational. But then $g_m(\mathbf{x})$ is also rational by property (C) above. However, in view of Lemma 3.3 this means that $E_{\mathbf{x}}$ is a T-set.

In a similar manner, there is an $\mathbf{x} < 0$ with $x_m = 0$ lying arbitrarily close to 0 for which $E_{\mathbf{x}}$ is a T-set.

∎

Next, we need an explicit form for the equilibrium measure of a set consisting of finitely many arcs on the unit circle. As before, we may assume that $E \not\equiv \mathbf{R}$, and then that $0 \notin E$. Let

$$E \cap [0, 2\pi] = \bigcup_{j=1}^{m} [a_{2j-1}, a_{2j}], \qquad a_j \in (0, 2\pi). \tag{3.7}$$

Then

$$\Gamma_E = \bigcup_{j=1}^{m} [e^{ia_{2j-1}}, e^{ia_{2j}}] \tag{3.8}$$

consists of the m arcs $I_j := [e^{ia_{2j-1}}, e^{ia_{2j}}]$, and $J_j := (e^{ia_{2j}}, e^{ia_{2j+1}})$, $j = 0, 1, \ldots, m-1$ (with $a_0 = a_{2m}$) are the complementary arcs to Γ_E. In what follows we use the main branch of $\sqrt{z}$. Then

$$h(z) = \sqrt{\prod_{j=1}^{2m}(z - e^{ia_j})} \tag{3.9}$$

is a single-valued analytic function on $\overline{\mathbf{C}} \setminus \Gamma_E$. We shall need the following form of the equilibrium density ω_{Γ_E} due to Peherstorfer and Steinbauer [24, Lemma 4.1]:

Lemma 3.5 *There are points $e^{i\beta_j} \in J_j$, $j = 0, 1, \ldots, m-1$, on the complementary arcs J_j with which*

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi} \frac{\prod_{j=0}^{m-1} |e^{it} - e^{i\beta_j}|}{\sqrt{\prod_{j=1}^{2m} |e^{it} - e^{ia_j}|}}, \qquad t \in E. \tag{3.10}$$

The $e^{i\beta_j}$ are the unique points on the unit circle for which

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{j=0}^{m}(e^{it} - e^{i\beta_j})}{\sqrt{\prod_{j=1}^{2m}(e^{it} - e^{ia_j})}} dt = 0, \qquad j = 0, 1, \ldots, m-1, \tag{3.11}$$

holds, where the denominator is considered as the value of the function (3.9).

Again, since the language here is somewhat different from that of [24], and since we also want to prove the unicity of the β_j's with property (3.11), we give a direct proof based on Lemma 3.1 and on the density of T-sets. Note that (3.11) is a linear system of equations for the coefficients of the polynomial $\prod(z - e^{i\beta_j})$, but it is not trivial that this system has a unique solution. We shall discuss how to find the β_j's after the proof.

Proof of Lemma 3.5, existence of the β_j's. First of all, it is enough to prove the lemma for T-sets. Indeed, suppose that $E = \cup_{j=1}^{m}[a_{2j-1}, a_{2j}]$, $a_j \in (0, 2\pi)$ is an arbitrary set and select T-sets of the form $E^{(s)} = \cup_{j=1}^{m}[a_{2j-1}^{(s)}, a_{2j}^{(s)}]$, $a_j^{(s)} \in (0, 2\pi)$ such that $a_{2j-1}^{(s)} = a_{2j-1}$ for all j and $a_{2j}^{(s)} \searrow a_{2j}$ as $s \to \infty$. Thus, if the lemma holds for T-sets then there are $\beta_j^{(s)}$ in the complementary intervals $(a_{2j}^{(s)}, a_{2j+1}^{(s)})$ (mod2π) such that

$$\omega_{\Gamma_{E^{(s)}}}(e^{it}) = \frac{1}{2\pi} \frac{\prod_{j=0}^{m} |e^{it} - e^{i\beta_j^{(s)}}|}{\sqrt{\prod_{j=1}^{2m} |e^{it} - e^{ia_j^{(s)}}|}}, \qquad t \in E^{(s)}. \tag{3.12}$$

It is standard that the equilibrium measure of $\Gamma_{E^{(s)}}$ converges in the weak* topology to the equilibrium measure of Γ_E as $s \to \infty$. Hence, the measures

$\omega_{E^{(s)}}(t)dt$ in (3.12) converge in the weak* topology, and is easy to see that then each sequence $\{\beta_j^{(s)}\}_{s=1}^\infty$ must converge, say $\beta_j^{(s)} \to \beta_j$ with some $\beta_j \in [a_{2j}, a_{2j+1}]$. Therefore, (3.11) is true, and the only problem may be that β_j does not belong to the open interval (a_{2j}, a_{2j+1}). But then β_j would be one of the endpoints, say $\beta_j = a_{2j}$, which would mean that $\omega_{\Gamma_E}(e^{it})$ has a zero at a_{2j}, which is not the case: Let $\zeta \in \mathrm{Int}(J)$, $J := [e^{ia_{2j-1}}, e^{ia_{2j}}]$ and let γ be a smooth closed curve in $\mathbf{C}\setminus\Gamma_E$ circling J once so that γ does not contain any other point of Γ_E in its interior.

Let $g_{\overline{\mathbf{C}}\setminus\Gamma_E}(\zeta,\infty)$ denote the Green's function of $\overline{\mathbf{C}}\setminus\Gamma_E$ with pole at infinity. The density $\omega_{\Gamma_E}(\zeta)$ is equal to

$$\frac{1}{2}\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_E}(\zeta,\infty)}{\partial \mathbf{n}_+} + \frac{\partial g_{\overline{\mathbf{C}}\setminus\Gamma_E}(\zeta,\infty)}{\partial \mathbf{n}_-}\right)$$

(see e.g. [23, II.(4.1)] or [30, Theorem VI.2.3]), and this is

$$\geq c\frac{1}{2}\left(\frac{\partial g_{\overline{\mathbf{C}}\setminus J}(\zeta,\infty)}{\partial \mathbf{n}_+} + \frac{\partial g_{\overline{\mathbf{C}}\setminus J}(\zeta,\infty)}{\partial \mathbf{n}_-}\right)$$

with any constant $c > 0$ for which

$$g_{\overline{\mathbf{C}}\setminus\Gamma_E}(z,\infty) \geq c g_{\overline{\mathbf{C}}\setminus J}(z,\infty), \qquad z \in \gamma,$$

because, by the maximum principle, for such a c we have the same inequality for all z that lies inside γ. Thus, a zero in $\omega_{\Gamma_E}(e^{it})$ at a_{2j} would yield a zero in $\omega_J(e^{it})$ at a_{2j}, which is not the case by (1.3). This verifies (3.10) pending its validity for T-sets.

Thus, let E be a T-set, $E \cap [0, 2\pi] = \cup_{j=1}^m [a_{2j-1}, a_{2j}]$, $a_j \in (0, 2\pi)$, and let U_N be the trigonometric polynomial appearing in the definition of T-sets. Let $E_1, \ldots, E_{2N}$ be those subintervals of E over which U_N runs through the interval $[-1, 1]$. Then their union is E, and suppose that $[a_{2j-1}, a_{2j}]$ contains p_j such subintervals. If two such E_k join each other at a point τ, then $U_N(\tau) = \pm 1$, and $U_N'(\tau) = 0$. There are $p_j - 1$ such τ inside $[a_{2j-1}, a_{2j}]$, so there are altogether $2N - m$ such τ's, let these be $\tau_1, \ldots, \tau_{2N-m}$. The trigonometric polynomial $1 - U_N^2(t)$ has a double zero at each τ_k, and, besides these, 1–1 zero at every a_j. These are altogether $(4N - 2m) + 2m = 4N$ zeros for $1 - U_N^2(t)$, which is of degree $2N$, hence these are all its zeros. As a consequence, U_N does not vanish on the complementary intervals (a_{2j}, a_{2j+1}) and it takes the same value (either 1 or -1) at both endpoints a_{2j} and a_{2j+1}, therefore U_N' must have a zero at some $\beta_j \in (a_{2j}, a_{2j+1})$, $j = 0, 1 \ldots, m-1$. These, together with the $N - m$ zeros of U_N' at the τ_k's give N zeros for U_N', and these are then all its zeros. Therefore, with some complex numbers a, b we can write

$$1 - U_N^2(t) = ae^{-i2Nt}\prod_{j=1}^{2m}\left(e^{it} - e^{ia_j}\right) \times \prod_{k=1}^{2N-m}\left(e^{it} - e^{i\tau_k}\right)^2, \tag{3.13}$$

$$U_N'(t) = be^{-iNt} \prod_{j=0}^{m-1} \left(e^{it} - e^{i\beta_j}\right) \times \prod_{k=1}^{2N-m} \left(e^{it} - e^{i\tau_k}\right). \tag{3.14}$$

A comparison of the coefficients of e^{i2Nt} in (3.13) and of e^{iNt} in (3.14) gives that $a = b^2/N^2$. Now substitute these forms into the right-hand side of (3.2), i.e. into

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi N} \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}} = \frac{1}{2\pi N} \frac{|U_N'(t)|}{\sqrt{|1 - U_N(t)^2|}}, \qquad t \in E.$$

The factors $|e^{it} - e^{i\tau_k}|$ cancel and so do b and N, and we get the form (3.10).

Note also that on the contiguous intervals (a_{2j}, a_{2j+1}) the expression $U_N^2 - 1$ is positive, and so we can write

$$\int_{a_{2j}}^{a_{2j+1}} \frac{U_N'(t)}{\sqrt{U_N(t)^2 - 1}} dt = \left(U_N(t) + \sqrt{U_N(t)^2 - 1} \right) \Bigg|_{t=a_{2j}}^{t=a_{2j+1}} = 0, \tag{3.15}$$

since U_N takes the same value (1 or -1) at a_{2j} and a_{2j+1}. Using the above substitutions based on (3.13)–(3.14), it is easy to see that (3.15) and (3.11) are the same.

■

Proof of Lemma 3.5, unicity of the β_j's. Note first of all, that

$$\exp(-\sum_{j=1}^{2m} a_j/2) e^{-imt} \prod_{j=1}^{2m} (e^{it} - e^{ia_j}) = \text{const} \prod_{j=1}^{2m} \sin((t - a_j)/2)$$

is a real trigonometric polynomial of degree m that vanishes precisely at the endpoints of the subintervals of E. In a similar manner,

$$T(t) := i^m \exp\left(-\sum_{j=0}^{m-1} \beta_j/2 \right) e^{-i(m/2)t} \prod_{j=0}^{m-1} (e^{it} - e^{i\beta_j})$$

is real. It is a trigonometric polynomial of degree at most $m/2$ if m is even, and it is a half-integer trigonometric polynomial of degree at most $m/2$ if m is odd. According to what we have just said, it follows that with it the system (3.11) takes the form

$$\int_{a_{2j}}^{a_{2j+1}} \frac{T(t)}{\sqrt{\prod_{j=1}^{2m} |\sin((t - a_j)/2)|}} dt = 0, \qquad j = 0, 1, \ldots, m-1. \tag{3.16}$$

Now if we have another system $\tilde{\beta}_j \in (a_{2j}, a_{2j+1})$ for which (3.11) is true, then we get another real

$$\tilde{T}(t) := i^m \exp\left(-\sum_{j=0}^{m-1} \tilde{\beta}_j/2 \right) e^{-i(m/2)t} \prod_{j=0}^{m-1} (e^{it} - e^{i\tilde{\beta}_j})$$

for which

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\tilde{T}(t)}{\sqrt{\prod_{j=1}^{2m} |\sin((t-a_j)/2)|}} dt = 0, \qquad j = 0, 1, \ldots, m-1. \tag{3.17}$$

Then the same is true of any linear combination $\cos\tau \cdot T(t) + \sin\tau \cdot \tilde{T}(t)$, hence any such linear combination has a zero on every interval (a_{2j}, a_{2j+1}), $j = 0, 1, \ldots, m-1$. On the other hand, any such linear combination is a trigonometric (or half-integer trigonometric) polynomial of degree at most $m/2$, therefore it can have at most m zeros unless it is identically zero, which is certainly not the case (use that non-trivial linear combinations of two polynomials with different zero sets cannot be the zero polynomial). Therefore, all $\cos\tau \cdot T(t) + \sin\tau \cdot \tilde{T}(t)$ have precisely one zero in every interval (a_{2j}, a_{2j+1}), $j = 0, 1, \ldots, m-1$. We shall show that this is enough to conclude that $\{\beta_j\} = \{\tilde{\beta}_j\}$.

Let us show e.g. that the $\beta_0 = \tilde{\beta}_0$. Suppose to the contrary that $\beta_0 \neq \tilde{\beta}_0$. Let x_τ be the unique zero of $\cos\tau \cdot T(t) + \sin\tau \cdot \tilde{T}(t)$ that lies in (a_0, a_1). Then $x_0 = x_\pi = \beta_0$, while $x_{\pi/2} = \tilde{\beta}_0$. As τ moves from 0 to $\pi/2$ the point x_τ moves continuously from β_0 to $\tilde{\beta}_0$, and when τ moves further on from $\pi/2$ to π, then x_τ moves continuously back from $\tilde{\beta}_0$ to β_0. Thus, there are a $\tau_1 \in (0, \pi/2)$ and a $\tau_2 \in (\pi/2, \pi)$ for which $x_{\tau_1} = x_{\tau_2} = (\beta_0 + \tilde{\beta}_0)/2$. This means that $(\beta_0 + \tilde{\beta}_0)/2$ is a common zero of $\cos\tau_1 \cdot T(t) + \sin\tau_1 \cdot \tilde{T}(t)$ and $\cos\tau_2 \cdot T(t) + \sin\tau_2 \cdot \tilde{T}(t)$. But then $(\beta_0 + \tilde{\beta}_0)/2$ must be also a common zero of $T(t)$ and of $\tilde{T}(t)$, meaning that $\beta_0 = (\beta_0 + \tilde{\beta}_0)/2 = \tilde{\beta}_0$, because β_0 is the only zero of $T(t)$, and $\tilde{\beta}_0$ is the only zero of $\tilde{T}(t)$ in (a_0, a_1).

■

Next, let us briefly discuss how to determine the numerator in (3.10). Perhaps the simplest is to note that, as we have just seen, (3.10) and (3.11) can be written in the alternate real form

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi} \frac{|S_{m/2}(t)|}{\sqrt{\prod_{j=1}^{2m} |\sin((t-a_j)/2)|}}, \qquad t \in E, \tag{3.18}$$

and

$$\int_{a_{2j}}^{a_{2j+1}} \frac{S_{m/2}(t)}{\sqrt{\prod_{j=1}^{2m} |\sin((t-a_j)/2)|}} dt = 0, \qquad j = 0, 1, \ldots, m-1, \tag{3.19}$$

with some real trigonometric (if m is even) or half-integer trigonometric (if m is odd) polynomial of the form

$$\begin{aligned} S_{m/2}(t) \quad &= \quad \left(\cos\delta\cos\left(\frac{m}{2}t\right) + \sin\delta\sin\left(\frac{m}{2}t\right)\right) \qquad (3.20) \\ &+ \quad \left(A_{m/2-1}\cos\left(\frac{m}{2}-1\right)t + B_{m/2-1}\sin\left(\frac{m}{2}-1\right)t\right) + \cdots \end{aligned}$$

of degree at most $m/2$ with some δ and some real coefficients

$$A_{m/2-1}, B_{m/2-1}, A_{m/2-2}, \ldots.$$

If m is even, then B_0 is missing ($\sin 0 \equiv 0$). Furthermore, the β_j's are the zeros of $S_{m/2}$.

For simplicity consider the case when m is even. Then (3.19) gives m equations for the m unknowns δ and $A_{m/2-1}, B_{m/2-1}, A_{m/2-2}, \ldots$. This is not a linear system, but we have seen in the preceding proof that the zeros of $S_{m/2}$ are uniquely determined, and this easily implies that the system (3.19) has a unique solution modulo a (-1) sign in the A_k's and B_k's and modulo replacing δ by $\delta+\pi$. Therefore, in principle one can determine the unknown quantities in the following way: consider δ as a parameter. Then (3.19) is an $m \times (m-1)$ linear system for the coefficients A_k, B_k, $k \le m/2-1$ (B_0 is missing), which has rank $m-1$. We can select an $(m-1)\times(m-1)$ subsystem which has non-zero determinant. Let these $m-1$ equations refer to the integrals over the intervals, $[a_{2j}, a_{2j+1}]$, $j \ne j_0$, i.e. we skip the j_0-th equation in (3.19). We solve this $(m-1)\times(m-1)$ system for the A_k's and B_k's and get unique expressions for them in terms of $\cos\delta$ and $\sin\delta$. This way we get two unique trigonometric polynomials $T_{1,\delta}(x)$ and $T_{2,\delta}(x)$ of degree at most $m/2-1$ such that

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\cos\delta\Big(\cos(m/2)t + T_{1,\delta}(t)\Big) + \sin\delta\Big(\sin(m/2)t + T_{2,\delta}(t)\Big)}{\sqrt{\prod_{j=1}^{2m} |\sin((t-a_j)/2)|}}\, dt = 0, \qquad j \ne j_0.$$

The same for $j = j_0$ can be achieved by selecting δ appropriately (this gives a unique value for $\tan\delta$, so δ is determined only up to modulo π).

We finish this section with the following observation that will be used in the next section. Let a_k be one of the endpoints of E, and U_N a trigonometric polynomial as in Lemma 3.1. Then we have (recall the values β_j from Lemma 3.5)

$$|U_N'(a_k)| = 2N^2 \frac{\prod_{j=0}^{m} |e^{ia_k} - e^{i\beta_j}|^2}{\prod_{j=1,\ j\ne k}^{2m} |e^{ia_k} - e^{ia_j}|}. \tag{3.21}$$

Indeed, we have the two forms (3.10) and (3.2) for ω_{Γ_E}, and in these let $t \to a_k$, $t \in E$. Since

$$\begin{aligned} 1 - U_N^2(t) = U_N^2(a_k) - U_N^2(t) &= (1+o(1))|t-a_k||2U_N(a_k)U_N'(a_k)| \\ &= (1+o(1))2|t-a_k||U_N'(a_k)|, \end{aligned}$$

we obtain from comparing what $1/\sqrt{|t-a_j|}$ is multiplied by in (3.10) and (3.2) that

$$\frac{1}{2\pi} \frac{\prod_{j=0}^{m} |e^{ia_k} - e^{i\beta_j}|}{\sqrt{\prod_{j=1,\ j\ne k}^{2m} |e^{ia_k} - e^{ia_j}|}} = \frac{1}{2\pi N} \frac{|U_N'(a_k)|}{\sqrt{2|U_N'(a_k)|}},$$

from which (3.21) follows.

4 Markov-type inequalities for trigonometric polynomials on finitely many intervals

Let E be a 2π-periodic subset of $\mathbf{R}$ such that $[0,2\pi]\cap E$ consists of finitely many intervals. The right-hand side in the estimate (2.1) given in Theorem A blows up if $\theta \in E$ approaches one of the endpoints, and in this case a Markov-type inequality should replace (2.1). For example, in the case of a single interval $E=[-a,a]$ Videnskii (see e.g. [9, Sec. 5.1, E19,c)]) proved that if $T_n(t)$ is a trigonometric polynomial of degree n, then

$$||T_n'||_{[-a,a]} \le (1+o(1))2n^2 \cot\frac{a}{2}\ ||T_n||_{[-a,a]}, \tag{4.1}$$

and this is sharp in the sense that one cannot write a constant smaller than $2\cot\frac{a}{2}$ on the right. This Markov-type behavior is typical around endpoints of E, but different endpoints play different roles, so we get different local Markov factors.

If $\rho>0$ is any fixed number and $E^\rho=\cup_j[a_{2j-1}+\rho, a_{2j}-\rho]$ is the set of points of E that are lying of distance $\ge\rho$ from the complementary arcs, then, in view of Theorem A, an estimate of the form

$$\|T_n'\|_{E^\rho} \le Cn\|T_n\|_E$$

holds with some constant C. Thus, in our Markov-type estimate we may restrict our attention to small neighborhoods $E\cap[a_k-\rho, a_k+\rho]$ of the endpoints a_k. Let us assume ρ so small that $E\cap[a_k-\rho,a_k+\rho]$ contains only the endpoint a_k and no other a_j. Let M_k be the smallest constant such that

$$\|T_n'\|_{E\cap[a_k-\rho,a_k+\rho]} \le (1+o(1))n^2M_k\|T_n\|_E \tag{4.2}$$

is true for all trigonometric polynomials T_n of degree at most n. This is asymptotically the best constant in the Markov-inequality around the endpoint a_k.

Theorem 4.1 *For all $k=1,2,\ldots,2m$ we have*

$$M_k = 2\frac{\prod_{j=0}^m |e^{ia_k}-e^{i\beta_j}|^2}{\prod_{j=1,\ j\neq k}^{2m} |e^{ia_k}-e^{ia_j}|}. \tag{4.3}$$

As an example, consider again the set $E=[-\beta,-\alpha]\cup[\alpha,\beta]$ from Section 2 with some $0\le\alpha<\beta\le\pi$, so that $a_1=-\beta$, $a_2=-\alpha$, $a_3=\alpha$ and $a_4=\beta$. In this case, by symmetry, $\beta_0=0$ and $\beta_1=\pi$, so (4.3) takes the form

$$M_2=M_3=2\frac{\sin\alpha}{\cos\alpha-\cos\beta},$$

while

$$M_1=M_4=2\frac{\sin\beta}{\cos\alpha-\cos\beta}.$$

Corollary 4.2 *Let E be a closed set such that $E \cap [0, 2\pi]$ consist of finitely many intervals, and let M_E be the maximum of the M_k's defined in (4.3) for all endpoints of the subintervals of E. Then for arbitrary trigonometric polynomials of degree at most $n = 1, 2, \dots$ we have*

$$\|T_n'\|_E \le (1 + o(1))n^2 M_E \|T_n\|_E, \tag{4.4}$$

and this is sharp, for there are trigonometric polynomials $T_n \not\equiv 0$ of degree at most $n = 1, 2, \dots$ for which

$$\|T_n'\|_E \ge (1 - o(1))n^2 M_E \|T_n\|_E. \tag{4.5}$$

This is the global Markov inequality for trigonometric polynomials on several intervals.

As an immediate corollary of Theorem 4.1 we get the following for algebraic polynomials. Let $\Gamma = \cup_{j=1}^m [e^{ia_{2j-1}}, e^{ia_{2j}}]$ be a closed set on the unit circle consisting of finitely many arcs, and let β_j be the numbers from (3.10) for this Γ. Let e^{ia_k} be one of the endpoints of Γ, and let H be a closed neighborhood of e^{ia_k} which does not contain any other endpoint of Γ.

Corollary 4.3 *If P_n is a polynomial of degree at most n, then*

$$\|P_n'\|_{H\cap\Gamma} \le (1 + o(1))n^2 \frac{\prod_{j=0}^m |e^{ia_k} - e^{i\beta_j}|^2}{2\prod_{j=1,\ j\ne k}^{2m} |e^{ia_k} - e^{ia_j}|} \|P_n\|_\Gamma, \tag{4.6}$$

and this is sharp, for there is a sequence of polynomials $P_n \not\equiv 0$ of degree at most $n = 1, 2, \dots$ such that

$$|P_n'(e^{ia_k})| \ge (1 - o(1))n^2 \frac{\prod_{j=0}^m |e^{ia_k} - e^{i\beta_j}|^2}{2\prod_{j=1,\ j\ne k}^{2m} |e^{ia_k} - e^{ia_j}|} \|P_n\|_\Gamma, \tag{4.7}$$

Indeed, since $(n+1)^2/n^2 = 1 + o(1)$, we may assume that n is even. Then $T_{n/2}(t) = e^{-itn/2}P_n(e^{it})$ is a trigonometric polynomial of degree at most $n/2$. Now if we apply Theorem 4.1 to this $T_{n/2}$, then we obtain (4.6). The proof of the converse (4.7) is similar, for in the proof of Theorem 4.1 we are going to verify that there are trigonometric polynomials T_n for which

$$|T_n'(a_k)| \ge (1 - o(1))n^2 M_k \|T_n\|_E.$$

As another corollary we obtain Markov-type inequalities for algebraic polynomials on a system of intervals on $\mathbf{R}$. Let $K = \cup_{j=1}^m [A_{2j-1}, A_{2j}]$. Then (see e.g. [31, (2.4)]) there are points $\xi_j \in (A_{2j}, A_{2j+1})$ in the contiguous intervals such that the density of the equilibrium measure of K has the form

$$\omega_K(x) = \frac{1}{\pi} \frac{\prod_{j=1}^{m-1} |x - \xi_j|}{\sqrt{\prod_{j=1}^{2m} |x - A_j|}}, \tag{4.8}$$

and ξ_j are the unique points that satisfy the system of equations

$$\int_{A_{2j}}^{A_{2j+1}} \frac{\prod_{j=1}^{m-1}(u-\xi_j)}{\sqrt{\prod_{j=1}^{2m}|u-A_j|}}\,du = 0, \quad j = 1, \ldots, m-1. \tag{4.9}$$

Theorem 4.4 *Let A_k be one of the endpoints of K and let $\eta > 0$ be so small that $[A_k - \eta, A_k + \eta]$ does not contain any other endpoint A_j. Then*

$$\|P_n'\|_{K\cap[A_k-\eta,A_k+\eta]} \le (1+o(1))2n^2 \frac{\prod_{j=0}^{m}|A_k-\xi_j|^2}{\prod_{j=1,\ j\neq k}^{2m}|A_k-A_j|}\|P_n\|_K \tag{4.10}$$

holds for all algebraic polynomials P_n of degree at most n.

Furthermore, this estimate is sharp in the sense that no smaller constant can be written on the right-hand side of (4.10).

This is Theorem 4.1 from [31]. There are two ways to deduce Theorem 4.4 from Theorem 4.1: either assume that $K \subseteq [-1, 1]$ and use the substitution $x = \cos t$ to go to trigonometric polynomials, or place a huge circle C_R over the real line touching it at the origin, project the set K up onto that circle, apply Theorem 4.1 for the projected arcs and let the radius R of C_R tend to infinity. In the limit we get Theorem 4.4.

Proof of Theorem 4.1. Let $M_k^* = M_k^*(E)$ be the expression on the right of (4.3), so our task is to show $M_k = M_k^*$. Without loss of generality we may assume that a_k belongs to the last interval, i.e. $k = 2m-1$ or $k = 2m$ (the ordering of the intervals is arbitrary).

First we prove that there are $T_n \not\equiv 0$ such that

$$|T_n'(a_k)| \ge (1-o(1))n^2 M_k^* \|T_n\|_E, \tag{4.11}$$

which then proves $M_k \ge M_k^*$. Indeed, by the proof of Lemma 3.4 there is a nonnegative sequence $\{\mathbf{x}_s = (x_1^{(s)}, \ldots, x_m^{(s)})\}_{s=1}^\infty$ of non-zero vectors converging to the 0 vector such that $x_m^{(s)} = 0$ and $E_{\mathbf{x}_s}$ is a T-set for all s, i.e. there are N_s and trigonometric polynomials U_{N_s} of degree N_s such that Lemma 3.1 holds for $E_{\mathbf{x}_s}$ and U_{N_s}. It is clear that we must have $N_s \to \infty$ as $s \to \infty$. Thus, in view of the fact that a_k is an endpoint of $E_{\mathbf{x}_s}$ (note that the last interval of E did not change when we moved to $E_{\mathbf{x}}$), we can apply (3.21), according to which

$$U_{N_s}'(a_k) = 2N_s^2 \frac{\prod_{j=0}^{m}|e^{ia_k} - e^{i\beta_j^{(s)}}|^2}{\prod_{j=1,\ j\neq k}^{2m}|e^{ia_k} - e^{i(a_j + x_j^{(s)})}|}, \tag{4.12}$$

where $e^{i\beta_j^{(s)}}$ are the points from Lemma 3.1 for the set $E_{\mathbf{x}_s}$, and where we set $x_{2j-1}^{(s)} = 0$ for all j. As $s \to \infty$ we have $x_j^{(s)} \to 0$ for all $j = 1, 2, \ldots, 2m$, and it is then easy to show that $\beta_j^{(s)} \to \beta_j$ for all $j = 0, 1, \ldots, m-1$ (this follows from the fact that $\mu_{\Gamma_{E_{\mathbf{x}_s}}}$ converges to μ_{Γ_E} in the weak*-topology and that, in view of (3.10), the equilibrium densities $\omega_{\Gamma_{E_{\mathbf{x}_s}}}$ are uniformly equicontinuous on every closed subinterval of the interior of Γ_E). From all these and from the fact that $|U_{N_s}| \le 1$ on E, we obtain (4.11) along the subsequence $n = N_s$, $s = 1, 2, \ldots$ of the natural numbers. Now note that if $\mathcal{T}_l = \cos(l \arccos x)$ are the classical Chebyshev polynomials, then $\mathcal{T}_l(U_{N_s})$ is a trigonometric polynomial of degree lN_s for which

$$E_{\mathbf{x}_s} = \{x \,|\, \mathcal{T}_l(U_{N_s}(x)) \in [-1, 1]\}$$

and

$$(\mathcal{T}_l(U_{N_s}))'(a_k) = 2(lN_s)^2 \frac{\prod_{j=0}^{m} |e^{ia_k} - e^{i\beta_j^{(s)}}|^2}{\prod_{j=1,\ j\neq k}^{2m} |e^{ia_k} - e^{i(a_j + x_j^{(s)})}|},$$

i.e. whenever N_s, U_{N_s} is suitable for $E_{\mathbf{x}_s}$ in the T-set definition, so is $lN_s, \mathcal{T}_l(U_{N_s})$ for any $l = 1, 2, \ldots$. Hence, in the preceding reasoning we can replace N_s by any lN_s, $l = 1, 2, \ldots$. For $n \to \infty$ we can select numbers of the form lN_s with $s \to \infty$ and $n/lN_s \to 1$, hence (4.11) holds.

Thus, to complete the theorem, we need to show that $M_k \le M_k^*$. As before, we may assume that $k = 2m-1$ or $k = 2m$, say, for definiteness, that $a_k = a_{2m-1}$. Let $\gamma > M_{2m-1}^*$, and select a small $\mathbf{x}_- \ge 0$, $x_m = 0$, such that $\tilde{E} := E_{-\mathbf{x}_-}$ is a T-set and $M_{2m-1}^*(\tilde{E}) < \gamma$, where $M_{2m-1}^*(\tilde{E})$ is the expression on the right of (4.3) for the set $\tilde{E}$. This is clearly possible, since $M_{2m-1}^*(E)$ changes continuously with the endpoints of the set E. Recall also that $\tilde{E} \subset E$ because $-\mathbf{x}_- \le 0$.

Let U_N be a real trigonometric polynomial of some degree N such that $U_N(t)$ runs through the interval $[-1, 1]$ $2N$-times as t runs through $\tilde{E}$, and let $\tilde{E}_1, \ldots, \tilde{E}_{2N}$ be those intervals on which $U_N(t)$ runs through $[-1, 1]$. Then the union of these $\tilde{E}_s$'s is $\tilde{E}$, and a_{2m-1} is the left-endpoint of one of these, say of $\tilde{E}_1$. Since a_{2m-1} is also a left-endpoint of a subinterval of $\tilde{E}$, there is no $\tilde{E}_s$ attached to $\tilde{E}_1$ from the left. By (3.21) we have $U_N'(a_{2m-1}) = N^2 M_{2m-1}^*(\tilde{E})$, and select a small $\eta > 0$ so that

$$\|U_N'\|_{[a_{2m-1}, a_{2m-1}+\eta]} < N^2\gamma. \tag{4.13}$$

We may assume η so small that $[a_{2m-1}, a_{2m-1}+\eta]$ lies of positive distance from $\tilde{E} \setminus \tilde{E}_1$. Next, we need the following lemmas that we verify after completing the proof.

Let $\tau_1, \ldots, \tau_{2N+m}$ be those $x \in \tilde{E}$ for which $|U_N(x)| = 1$ (cf. the proof of Lemma 3.5). Without loss of generality we may assume that $\tau_1 = a_{2m-1}$.

Lemma 4.5 *For any l there are real trigonometric polynomials Q_l of degree at most l such that for large l we have $0 \le Q_l(x) \le 1$ for all x, $0 \le Q_l(x) \le l^{-4} \prod_{k>1} \sin^2((x - \tau_k)/2)$ for $x \in \tilde{E} \setminus \tilde{E}_1$ and $1 - l^{-4} \le Q_l(x) \le 1$ for $x \in [a_{2m-1}, a_{2m-1}+\eta]$. Furthermore, $|Q_l'(x)| \le l^{-2}$ on $\tilde{E}\setminus\tilde{E}_1$ and on $[a_{2m-1}, a_{2m-1}+\eta]$.*

Lemma 4.6 *If I is an interval then there is a constant C_I such that*

$$\|T_n'\|_I \le C_I n^2 \|T_n\|_I$$

for all trigonometric polynomials T_n of degree at most $n = 1, 2, \ldots$.

Now let n be a large number and T_n a trigonometric polynomial of degree at most n. We may assume $\|T_n\|_E \le 1$. With $l = \sqrt{n}$ consider the trigonometric polynomials Q_l from Lemma 4.5, and set $V_n(x) = T_n(x)Q_{\sqrt{n}}(x)$. Then this is a trigonometric polynomial of degree at most $n + \sqrt{n}$ and in it $Q_{\sqrt{n}}(x)$ is

- smaller than n^{-2} on $\tilde{E} \setminus \tilde{E}_1$,
- closer than n^{-2} to 1 on the interval $[a_{2m-1}, a_{2m-1} + \eta]$,
- in absolute value its derivative is smaller than $1/n$ on $\tilde{E} \setminus \tilde{E}_1$ and on $[a_{2m-1}, a_{2m-1} + \eta]$.

For a $t \in \tilde{E}_1$ let $t_1, \ldots, t_{2N}$, $t_1 = t$, be those points in $\tilde{E}$ for which $U_N(t_j) = U_N(t)$, and consider the sum of the values $V_n(t_j)$. According to Lemma 3.2 there is an algebraic polynomial $S_{(n+\sqrt{n})/N}$ of degree at most $(n + \sqrt{n})/N$ such that

$$\sum_{j=1}^{2N} V_n(t_j) = S_{(n+\sqrt{n})/N}(U_N(t)). \tag{4.14}$$

On $\tilde{E}$ the absolute value on the left-hand side is at most $1 + 2N/n^2$, so we have $\|S_{(n+\sqrt{n})/N}\|_{[-1,1]} \le 1 + 2N/n^2$, since the image of $\tilde{E}$ under U_N is the interval $[-1, 1]$. Therefore, by the classical Markov inequality on $[-1, 1]$, we have that

$$\left|\Big(S_{(n+\sqrt{n})/N}(U_N(x))\Big)'(x)\right| \le \left(\frac{n+\sqrt{n}}{N}\right)^2 \left(1 + \frac{2N}{n^2}\right) |U_N'(x)|,$$

which is smaller than

$$\left(\frac{n+\sqrt{n}}{N}\right)^2 \left(1 + \frac{2N}{n^2}\right) N^2\gamma = (n + \sqrt{n})^2 (1 + 2N/n^2)\gamma$$

on $[a_{2m-1}, a_{2m-1} + \eta]$ by (4.13). Hence, we get from (4.14) on the interval $[a_{2m-1}, a_{2m-1} + \eta]$

$$\left| T_n'(t)Q_{\sqrt{n}}(t) + \sum_{j=2}^{2N} \frac{dT_n(t_j(t))}{dt} Q_{\sqrt{n}}(t_j) + \sum_{j=1}^{2N} T_n(t_j)Q_{\sqrt{n}}'(t_j) \right| \tag{4.15}$$
$$\le (n + \sqrt{n})^2 (1 + 2N/n^2)\gamma.$$

Now we use that for $t \in [a_{2m-1}, a_{2m-1} + \eta]$ we have for $j = 2, 3, \ldots, 2N$ the relation $t_j(t) \in \tilde{E} \setminus \tilde{E}_1$. Therefore, on the left $|Q_{\sqrt{n}}'(t_j)| \le 1/n$ for all j

and $|Q_{\sqrt{n}}(t_j)| \le 1/n^2$ for all $j \ge 2$. Furthermore, in view of Lemma 4.6, the estimate

$$|T_n'(t_j)| \le C_0 n^2 \tag{4.16}$$

also holds with some C_0. However,

$$\frac{dT_n(t_j(t))}{dt} = T_n'(t_j)\frac{dt_j(t)}{dt}$$

involves also $dt_j(t)/dt$, and this derivative needs special attention. As before, let $\tau_1, \ldots, \tau_{2N+m}$ be those $x \in \tilde{E}$ for which $|U_N(x)| = 1$, and suppose that $\tau_1 = a_{2m-1}$, is the point around which we are considering the Markov inequality. For $t \in [a_{2m-1}, a_{2m-1} + \eta]$ the point $t_j(t)$ belongs to some interval determined by the τ_k's, say $t_j(t) \in [\tau_s, \tau_{s+1}]$, and if $\eta > 0$ is small enough, then for all $t \in [a_{2m-1}, a_{2m-1} + \eta]$ the point $t_j(t)$ belongs either to the left-half or to the right-half of that interval, say $t_j(t) \in [\tau_s, (\tau_s + \tau_{s+1})/2]$. Now there are two possibilities:

I τ_s is an endpoint of one of the subintervals making $\tilde{E}$. In that case $U_N'(\tau_s) \ne 0$, and, as a consequence (use the mean value theorem), $|t_j(t) - \tau_s| \sim |t - a_{2m-1}|$ (meaning that the ratio of the two sides lies in between two positive constants),

$$\left|\frac{dt_j(t)}{dt}\right| \le C \qquad \text{on } [a_{2m-1}, a_{2m-1} + \eta].$$

II τ_s belongs to the interior of $\tilde{E}$. Then $U_N'(\tau_s) = 0$, and τ_s is a double (and not higher order) zero of $1 - U_N^2$. Hence, in this case (use Taylor's formula), $|t_j(t) - \tau_s| \sim |t - a_{2m-1}|^{1/2}$,

$$\left|\frac{dt_j(t)}{dt}\right| \le \frac{C_1'}{|t - a_{2m-1}|^{1/2}} \le \frac{C_1}{|t_j(t) - \tau_s|} \qquad \text{on } [a_{2m-1}, a_{2m-1} + \eta].$$

These imply (cf. also (4.16))

$$\left|\frac{dT_n(t_j(t))}{dt}\right| \le C_0 n^2 \frac{C_1}{|t_j(t) - \tau_s|},$$

so by Lemma 4.5

$$\left|\frac{dT_n(t_j(t))}{dt}\right| Q_{\sqrt{n}}(t_j(t)) \le C_0 n^2 \frac{C_1}{|t_j(t) - \tau_s|}\frac{1}{n^2}\prod_{k=2}^{2N+m} \sin^2((t_j - \tau_k)/2) \le L,$$

where L is the maximum of all

$$\max_{x \in E} \frac{\sin^2((x - \tau_s)/2)}{|x - \tau_s|} \prod_{k=2,\ k \ne s}^{2N+m} \sin^2((x - \tau_k)/2), \qquad s = 2, \ldots, 2N + m.$$

Finally, since $|Q_{\sqrt{n}}(t_1) - 1| \leq 1/n^2$ is also true, we can infer from (4.15) that for all $t \in [a_{2m+1}, a_{2m-1} + \eta]$

$$|T_n'(t)| \leq (n + \sqrt{n})^2 (1 + N/n^2)\gamma + 2C_0 NL + 2N\frac{1}{n} = (1 + o(1))n^2\gamma.$$

Since $\gamma > M_{2m-1}^*$ was arbitrary, $M_{2m-1} \leq M_{2m-1}^*$ follows, and this is what we needed to prove.

■

Proof of Lemmas 4.5 and 4.6. Let f be the function that is equal to $1/\prod_{k>1} \sin^2((x - \tau_k)/2)$ on the interval $[a_{2m-1}, a_{2m-1} + \eta]$ and is zero on $\tilde{E} \setminus \tilde{E}_1$, and extend this f to a 6 times continuously differentiable 2π periodic f so that $0 \leq f(x) \leq 1/\prod_{k>1} \sin^2((x - \tau_k)/2)$ holds for the extended function. By Jackson's theorem [10, Corollary 7.2.4] there are trigonometric polynomials Q_l^* of degree at most $l/2$ such that $|f - Q_l^*| \leq Cl^{-6}$ is true with some constant C. Then it is easy to see that

$$Q_l(x) = \frac{Q_l^*(x) + Cl^{-6}}{1 + 2CMl^{-6}} \prod_{k=2}^{2N+m} \sin^2((x - \tau_k)/2),$$

where M is the maximum of $\prod_{k>1} \sin^2((x - \tau_k)/2)$ on $\tilde{E}$, satisfies the requirements not considering those for the derivatives. For example, for $x \in [a_{2m-1}, a_{2m-1} + \eta]$ we have

$$\left| Q_l^*(x) - \frac{1}{\prod_{k>1} \sin^2((x - \tau_k)/2)} \right| \leq Cl^{-6},$$

therefore

$$\begin{aligned} |1 - Q_l(x)| &= \\ = &\left| \frac{1 + CMl^{-6} - Q_l^*(x) \prod_{k>1} \sin^2((x - \tau_k)/2) + Cl^{-6} \prod_{k>1} \sin^2((x - \tau_k)/2)}{1 + CMl^{-6}} \right| \\ &\leq \quad C'l^{-6} \leq l^{-4} \end{aligned}$$

for all large l.

However, on each interval $\tilde{E}_j$, $j = 2, \ldots, 2N$ we have $|Q_l| \leq C_0 l^{-6}$ while on $[a_{2m-1}, a_{2m-1} + \eta]$ we have $|1 - Q_l| \leq C_0/l^6$ with some C_0 by our construction, hence Lemma 4.6 gives on all these intervals $|Q_l'| \leq C_1 l^2 l^{-6} = C_1 l^{-4}$ with some C_1 that may depend on N, $\tilde{E}$ and $\eta > 0$. Therefore, for large l we have $|Q_l'| < l^{-2}$ on $(\tilde{E} \setminus \tilde{E}_1) \cup [a_{2m-1}, a_{2m-1} + \eta]$.

Lemma 4.6 is due to D. Jackson [14], and it is also an immediate consequence of Videnskii's inequality (4.1).

■

5 Appendix An elementary proof for Theorem A

Having T-sets at our disposal, we are in the position to give a relatively simple elementary proof of Theorem A. This is justified by the fact that the original proof given by A. Lukashov in [19] is based on the Schottky-Burnside theory of automorphic forms, while the proof given in Section 2 is based on the Achiezer-Levin theory of conformal maps.

First of all, by Theorem 2.1 of [33] it is enough to prove that for fixed θ

$$|T_n'(\theta)| \leq (1+o(1))n2\pi\omega_{\Gamma_E}(e^{i\theta})\|T_n\|_E. \tag{5.1}$$

with $o(1)$ tending to 0 uniformly in T_n as $n\to\infty$.

The proof follows the scheme of [31, Theorem 3.1].

Case 1: E is a T-set and T_n is a polynomial of U_N

Let U_N be a real trigonometric polynomial of some degree N such that $U_N(t)$ runs through the interval $[-1,1]$ $2N$-times as t runs through E, and assume that $T_n(t)=P_m(U_N)$ with some polynomial P_m. Then $n=mN$ and $\|P_m\|_{[-1,1]}=\|T_n\|_E$. Using Bernstein's inequality

$$|P_m'(x)| \leq \frac{m}{\sqrt{1-x^2}}\|P_m\|_{[-1,1]}, \qquad x\in[-1,1],$$

for P_m, we obtain

$$\begin{aligned} |T_n'(\theta)| = |P_m'(U_N(\theta))U_N'(\theta)| &\leq \frac{m}{\sqrt{1-U_N(\theta)^2}}\|P_m\|_{[-1,1]}|U_N'(\theta)| \\ &\leq Nm2\pi\omega_{\Gamma_E}(\theta)\|T_n\|_E, \end{aligned}$$

where, in the last step, we used (3.2).

Case 2: E is a T-set and T_n is arbitrary

Let U_N be a real trigonometric polynomial of some degree N such that $U_N(t)$ runs through the interval $[-1,1]$ $2N$-times as t runs through E, and let $E_1,\ldots,E_{2N}$ be those intervals on which $U_N(t)$ runs through $[-1,1]$. Actually, we need this case only when the fixed θ lies in the *interior* of one of the E_j's, say $\theta\in\mathrm{Int}(E_{i_0})$. With

$$Q_{\sqrt{n}}(t)=\left(\frac{1-\cos(t-\theta)}{2}\right)^{2[\sqrt{n}/2]}$$

we set $V_n(t)=T_n(t)Q_{\sqrt{n}}(t)$. We have again, as in (4.14),

$$\sum_{j=1}^{2N} V_n(t_j) = S_{(n+\sqrt{n})/N}(U_N(t)), \tag{5.2}$$

where, for a t in E_{i_0}, the numbers $t_1, \ldots, t_{2N}$, $t_{i_0} = t$, are those points in E for which $U_N(t_j) = U_N(t)$. Since both $Q_{\sqrt{n}}$ and its derivative tend to zero uniformly outside any neighborhood (mod 2π) of θ as $n \to \infty$, furthermore $Q'_{\sqrt{n}}(\theta) = 0$, it follows that, as $n \to \infty$,

$$\|S_{(n+\sqrt{n})/N}(U_N)\|_E \le (1 + o(1))\|T_n\|_E,$$

$$\frac{d}{dt} V_n(t_j)\Big|_{t\,=\,\theta} = o(1)\|T_n\|_E, \qquad j \ne j_0,$$

(see also Lemma 4.6) and

$$V_n'(\theta) = T_n'(\theta) + o(1)\|T_n\|_E,$$

hence

$$\begin{aligned} \frac{d}{dt} S_{(n+\sqrt{n})/N}(U_N(t))\Big|_{t\,=\,\theta} &= \frac{d}{dt}\sum_{j=1}^{2N} V_n(t_j)\Big|_{t\,=\,\theta} = V_n'(\theta) + o(1)\|T_n\|_E \\ &= T_n'(\theta) + o(1)\|T_n\|_E, \end{aligned}$$

where the $o(1)$ depends only on the set E and on the location of θ *inside* E_{i_0}. Therefore, the just proven Case 1 gives

$$\begin{aligned} |T_n'(\theta)| &\le \left| \frac{d}{dt} S_{(n+\sqrt{n})/N}(U_N(t))\Big|_{t\,=\,\theta} \right| + o(1)\|T_n\|_E \\ &\le (1 + o(1))(n + \sqrt{n})2\pi\omega_{\Gamma_E}(\theta)\|S_{(n+\sqrt{n})/N}(U_N)\|_E + o(1)\|T_n\|_E \\ &\le (1 + o(1))n2\pi\omega_{\Gamma_E}(\theta)\|T_n\|_E. \end{aligned}$$

Completion of the proof

We may assume E to consist of finitely many intervals, see the remark after Theorem A.

Let θ lie in the interior of E, and for an $\varepsilon > 0$ choose a T-set $\tilde{E} \subset \mathrm{Int}(E)$ such that θ lies inside $\tilde{E}$ and $\omega_{\Gamma_{\tilde{E}}}(\theta) \le (1+\varepsilon)\omega_{\Gamma_E}(\theta)$. By Lemma 3.4 this is possible (see also Lemma 3.5 and its proof). Let U_N be a real trigonometric polynomial of some degree N such that $U_N(t)$ runs through the interval $[-1, 1]$ $2N$-times as t runs through $\tilde{E}$, and let $\tilde{E}_1, \ldots, \tilde{E}_{2N}$ be those intervals on which $U_N(t)$ runs through $[-1, 1]$. By slightly shifting $\tilde{E}$ if necessary, we may assume that θ lies in the interior of one of the E_j's, so θ and $\tilde{E}$ have the properties that were used in Case 2 above. Therefore, by Case 2,

$$|T_n'(\theta)| \le (1 + o(1))n2\pi\omega_{\Gamma_{\tilde{E}}}(\theta)\|T_n\|_{\tilde{E}} \le (1 + o(1))n2\pi(1+\varepsilon)\omega_{\Gamma_E}(\theta)\|T_n\|_E,$$

where we also used that $\tilde{E} \subset E$. Since $\varepsilon > 0$ is arbitrary, we are done with the proof of (5.1).

■

The author thanks the referee for correcting several misprints and for calling his attention to the papers [14] and [25].

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Bolyai Institute
MTA-SZTE Analysis and Stochastics Research Group
University of Szeged
Szeged
Aradi v. tere 1, 6720, Hungary

and

Department of Mathematics and Statistics
University of South Florida
4202 E. Fowler Ave, CMC342
Tampa, FL 33620-5700, USA

totik@mail.usf.edu