Backward bifurcation in SIVS model with immigration of non-infectives

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***Abstract*—This paper investigates a simple SIVS (susceptible–infected–vaccinated–susceptible) disease transmission model with immigration of susceptible and vaccinated individuals. We show global stability results for the model, and give an explicit condition for the existence of backward bifurcation and multiple endemic equilibria. We examine in detail how the structure of the bifurcation diagram depends on the immigration.**

***Index Terms*—vaccination model with immigration; backward bifurcation; stability analysis**

I. Introduction

The basic reproduction number $\mathcal{R}_0$ is a central quantity in epidemiology as it determines the average number of secondary infections caused by a typical infected individual introduced into a wholly susceptible population. In epidemic models describing the spread of infectious diseases, the reproduction number works as a threshold quantity for the stability of the disease-free equilibrium. The usual situation is that for $\mathcal{R}_0 < 1$ the DFE is the only equilibrium and it is asymptotically stable, but it loses its stability as $\mathcal{R}_0$ increases through 1, where a stable endemic equilibrium emerges, which depends continuously on $\mathcal{R}_0$. Such a transition of stability between the disease-free equilibrium and the endemic equilibrium is called *forward* bifurcation. However, it is possible to have a very different situation at $\mathcal{R}_0 = 1$, as there might exist positive equilibria also for values of $\mathcal{R}_0$ less than 1. In this case we say that the model undergoes a *backward* bifurcation at $\mathcal{R}_0 = 1$, when for values of $\mathcal{R}_0$ in an interval to the left of 1, multiple positive equilibria coexist, typically one unstable and one stable. The behavior in the change of stability is of particular interest from the perspective of controlling the epidemic: considering $\mathcal{R}_0 > 1$, in order to eradicate the disease it is sufficient to decrease $\mathcal{R}_0$ to 1 if there is a forward bifurcation at $\mathcal{R}_0 = 1$, nevertheless it is necessary to bring $\mathcal{R}_0$ well below 1 to eliminate the infection in case of a backward bifurcation. This also implies that the qualitative behavior of a model with backward bifurcation is more complicated than that of a model which undergoes forward bifurcation at $\mathcal{R}_0 = 1$, since in the latter case the infection usually does not persist if $\mathcal{R}_0 < 1$, although with backward bifurcation the presence of a stable endemic equilibrium for $\mathcal{R}_0 < 1$ implies that, even for values of $\mathcal{R}_0$ less than 1, the epidemic can sustain itself if enough infected individuals are present.

Backward bifurcation has been observed in several studies in the recent literature. The well known works [4], [6], [7] consider multi-group epidemic models with asymmetry between groups or multiple interaction mechanisms. Some simple epidemic models of disease transmission in a single population with vaccination of susceptible individuals are presented and analyzed in [1], [2], [8], [9]. A basic model can be described by the following system of ordinary differential equations:

$$
\begin{aligned}
S'(t) =& \Lambda(N(t)) - \beta(N(t))S(t)I(t) \\
&- (\mu+\phi)S(t) + \gamma I(t) + \theta V(t), \\
I'(t) =& \beta(N(t))S(t)I(t) + \sigma\beta(N(t))V(t)I(t) \\
&- (\mu+\gamma)I(t), \\
V'(t) =& \phi S(t) - \sigma\beta(N(t))V(t)I(t) \\
&- (\mu+\theta)V(t),
\end{aligned}
\tag{1}
$$

where $S(t)$, $I(t)$, $V(t)$ and $N(t)$ denote the number of susceptible, infected, vaccinated individuals and the total population, respectively, at time t. Λ represents the birth function into the susceptible class and μ is the natural death rate in each class. Disease transmission is modeled by the infection term $\beta(N)SI$, ϕ and γ stand for the vaccination rate of susceptible individuals and the recovery rate of infected individuals. It is assumed that vaccination loses effect at rate θ, moreover $0 \leq \sigma \leq 1$ is introduced to model the phenomenon that vaccination may reduce but not completely eliminate susceptibility to infection. With certain conditions on the birth function Λ, system (1) can be reduced to a two-dimensional system, of which a complete qualitative analysis including a condition for the existence of backward bifurcation has been derived in [1].

The aim of this paper is to describe and analyze an epidemic model in which demographic effects, such as immigration of non-infected individuals are included into a single population. The model we study generalizes the above presented vaccination model (1) by incorporating the possibility of immigration, and we investigate how immigration changes the bifurcation behavior.

The paper is organized as follows. A three-dimensional ODE model is given in section II, which we reduce to two dimensions by means of the theory of asymptotically autonomous systems. Some fundamental properties of the two-dimensional system –as positivity and boundedness of solutions and stability of the disease-free equilibrium– are discussed in section III, then section IV concerns with the existence of endemic equilibria and conditions for the forward / backward bifurcation. We obtain our results by algebraic means, without using center manifold theory and normal forms. In section V a complete qualitative analysis has been carried out for the two-dimensional system, furthermore we analyze how immigration deforms the bifurcation curve in section VI. Finally, in section VII we return to the original three-dimensional model, then discuss our findings in the last section.

II. SIVS MODEL WITH IMMIGRATION

A general vaccination model with immigration of non-infected individuals can be described by the system

$$\begin{aligned} S'(t) =&\Lambda(N(t)) - \beta(N(t))S(t)I(t) \\ &- (\mu+\phi)S(t) + \gamma I(t) + \theta V(t) + \eta, \\ I'(t) =&\beta(N(t))S(t)I(t) + \sigma\beta(N(t))V(t)I(t) \\ &- (\mu+\gamma)I(t), \\ V'(t) =&\phi S(t) - \sigma\beta(N(t))V(t)I(t) \\ &- (\mu+\theta)V(t) + \omega, \end{aligned} \tag{2}$$

where we assume that immigration of susceptible and vaccinated individuals occurs with constant rate η and ω, respectively. The other parameters of the model have been described in section I, and for the total population $N(t)$ we obtain

$$N'(t) = \Lambda(N(t)) - \mu N(t) + \eta + \omega. \tag{3}$$

The proof of the following proposition is obvious and thus omitted.

Proposition II.1. *If for the birth function Λ it holds that $\Lambda(0) = 0$, $\Lambda'(0) > \mu$ and there exists an $x_* > 0$ such that $\Lambda'(x_*) < \mu$, moreover $\Lambda'(x) > 0$ and $\Lambda''(x) < 0$ for all $x > 0$, then for any $\eta, \omega \geq 0$ there exists a unique positive solution of $\Lambda(x) = \mu x - \eta - \omega$.*

We define the population carrying capacity $K = K(\Lambda, \mu, \eta, \omega)$ as the unique solution of $\Lambda(x) = \mu x - \eta - \omega$. Note that from $\Lambda(K) = \mu K - \eta - \omega$ it follows that $\mu K - \eta - \omega > 0$. We can rewrite equations $(2)_2$ and $(2)_3$ in terms of $N(t)$, $I(t)$ and $V(t)$ using $S(t) = N(t) - I(t) - V(t)$ and consider this system as a system of non-autonomous differential equations with non-autonomous term $N(t)$, which is governed by system (3). Then, by $\lim_{t\to\infty} N(t) = K$ we find that system (2) is asymptotically autonomous with the limiting system

$$\begin{aligned} I'(t) =&\beta(K - I(t) - (1-\sigma)V(t))I(t) \\ &- (\mu+\gamma)I(t), \\ V'(t) =&\phi(K - I(t)) - \sigma\beta V(t)I(t) \\ &- (\mu+\theta+\phi)V(t) + \omega, \end{aligned} \tag{4}$$

where $\beta = \beta(K)$. In what follows we focus on the mathematical analysis of system (4), then we use the theory of asymptotically autonomous systems [10], [11], [12] to obtain information on the long-term behavior of solutions of (2).

III. FUNDAMENTAL PROPERTIES OF THE SYSTEM

The existence and uniqueness of solutions of system (4) follows from fundamental results for ODEs. Since K was defined as the carrying capacity of the population, it is biologically meaningful to assume that for the initial conditions of system (4) it is satisfied that $0 \leq I(0), V(0), I(0) + V(0) \leq K$.

Proposition III.1. *If $0 \leq I(0), V(0), I(0) + V(0) \leq K$, then $0 \leq I(t), V(t), I(t) + V(t) \leq K$ is satisfied for all $t > 0$.*

Proof: If $I(t) = 0$ then $I'(t) = 0$, which yields that for nonnegative initial conditions I never goes negative.

If $V(t) = 0$ when $0 \leq I(t) \leq K$, then $V'(t) \geq \omega \geq 0$, thus solutions never cross the line $V = 0$ from the inside of the region $R : 0 \leq I, V, I + V \leq K$. If $I(t) + V(t) = K$ when $I(t), V(t) \geq 0$, then summing $(4)_1$ and $(4)_2$ gives

$$I'(t) + V'(t) = -\mu K - \gamma I(t) - \theta V(t) + \omega,$$

which is negative since $\omega - \mu K$ is non-positive, thus $I(t) + V(t) > K$ is impossible. ■

The disease-free equilibrium of system (4) can be obtained as

$$\bar{V} = \frac{\phi K + \omega}{\mu + \theta + \phi}.$$

In the initial stage of the epidemic, we can assume that system (4) is near the equilibrium $(0, \bar{V})$ and approximate the equation of class I with the linear equation

$$y'(t) = (\beta(K - (1 - \sigma)\bar{V}) - (\mu + \gamma))y(t), \tag{5}$$

where $y : \mathbb{R} \to \mathbb{R}$. The term $\beta(K - (1 - \sigma)\bar{V})$ describes the production of new infections, and $\mu + \gamma$ is the transition term describing changes in state, hence with the formula for the disease-free equilibrium $\bar{V}$ we can define the basic reproduction number as

$$\begin{aligned} \mathcal{R}_0 &= \frac{\beta(K - (1 - \sigma)\bar{V})}{\mu + \gamma} \\ &= \frac{\beta}{\mu + \gamma}\left(\frac{K(\mu + \theta + \sigma\phi)}{\mu + \theta + \phi} - \frac{(1 - \sigma)\omega}{\mu + \theta + \phi}\right). \end{aligned} \tag{6}$$

The following proposition shows that $\mathcal{R}_0$ works as a threshold quantity for the stability of the disease-free equilibrium of system (4).

Proposition III.2. *The disease-free equilibrium of system (4) is asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof: The stability of the zero steady-state of system (5) is determined by the sign of $\beta(K - (1 - \sigma)\bar{V}) - (\mu + \gamma)$, which coincides with the sign of $\mathcal{R}_0 - 1$. This means that the zero solution of (5) is asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$. This statement extends to the nonlinear system (4) by the principle of linearized stability. ■

IV. Endemic equilibrium

The problem of finding equilibrium $(\hat{I}, \hat{V})$ for system (4) yields the two dimensional system

$$\begin{aligned} 0 &= \beta(K - \hat{I} - (1 - \sigma)\hat{V})\hat{I} - (\mu + \gamma)\hat{I}, \\ 0 &= \phi(K - \hat{I}) - \sigma\beta\hat{V}\hat{I} - (\mu + \theta + \phi)\hat{V} + \omega. \end{aligned} \tag{7}$$

The existence of a unique disease-free equilibrium has been proved, so now we focus on finding endemic equilibria $(\hat{I}, \hat{V})$ with $\hat{I} > 0$. From $(7)_1$ we obtain the formula

$$\hat{V} = \frac{\beta(K - \hat{I}) - (\mu + \gamma)}{\beta(1 - \sigma)}, \tag{8}$$

then by substituting $\hat{V}$ into $(7)_2$ it follows from straightforward computations that

$$A\hat{I}^2 + B\hat{I} + C = 0 \tag{9}$$

should hold for $\hat{I}$, where

$$\begin{aligned} A =& \sigma\beta, \\ B =& (\mu + \theta + \sigma\phi) + \sigma(\mu + \gamma) - \sigma\beta K, \\ C =& \frac{(\mu + \gamma)(\mu + \theta + \phi)}{\beta} \\ & - (\mu + \theta + \sigma\phi)K + (1 - \sigma)\omega. \end{aligned} \tag{10}$$

We note that $\beta C = (1 - \mathcal{R}_0)(\gamma + \mu)(\mu + \phi + \theta)$ and we characterize the number of solutions of the equilibrium condition (9).

Proposition IV.1. *If $\mathcal{R}_0 > 1$ then there exists a unique positive equilibrium $\hat{I} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$.*

Proof: If $C < 0$, or equivalently, $\mathcal{R}_0 > 1$, then the equilibrium condition (9) has a unique positive solution, which can be obtained as $\hat{I} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$. ■

At $\mathcal{R}_0 = 1$ it holds that $A > 0$ and $C = 0$, so there exists a unique nonzero solution $\hat{I} = -B/A$ of (9), which is positive (and thus, biologically relevant) if and only if $B < 0$. Let us now assume that B is negative at $\mathcal{R}_0 = 1$, which also implies that $B^2 - 4AC = B^2 > 0$. Then there is a positive root of the equilibrium condition at $\mathcal{R}_0 = 1$, and due to the continuous dependence of the coefficients A, B and C on β there must be an interval to the left of $\mathcal{R}_0 = 1$ where $B < 0$ and $B^2 - 4AC > 0$ still hold. Since $C > 0$ whenever $\mathcal{R}_0 < 1$, it follows that on this interval there exist exactly two positive solutions of (9) and thus, two endemic equilibria of system (4). We denote these equilibria by

$$\begin{aligned} \breve{I}_1 &= \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \\ \breve{I}_2 &= \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \end{aligned}$$

and with the aid of formula (8) we can derive the $\hat{V}$-components to get the equilibria $(\breve{I}_1, \breve{V}_1)$ and $(\breve{I}_2, \breve{V}_2)$. With other words, if $B < 0$ when $\mathcal{R}_0 = 1$ then system (4) has a backward bifurcation at $\mathcal{R}_0 = 1$ since besides the zero equilibrium and the positive equilibrium $\breve{I}_2 = \frac{-B+\sqrt{B^2-4AC}}{2A}$, which also exist for $\mathcal{R}_0 > 1$, another positive equilibrium emerges when $\mathcal{R}_0$ passing through 1 from the right to the left.

Theorem IV.2. *If the condition*

$$\frac{(1-\sigma)\omega}{K} > \frac{(\theta+\mu+\sigma\phi)^2 - \sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)} \tag{11}$$

holds then there is a backward bifurcation at $\mathcal{R}_0 = 1$.

Proof: The condition for the backward bifurcation is that $B < 0$ when β satisfies $\mathcal{R}_0 = 1$. This can be obtained as an explicit criterion of the parameters: as $B < 0$ yields

$$\sigma\beta K > (\mu+\theta+\sigma\phi) + \sigma(\mu+\gamma),$$

moreover from $C = 0$ we derive

$$\beta K = \frac{(\mu+\gamma)(\mu+\theta+\phi)}{(\theta+\mu+\sigma\phi) - \frac{(1-\sigma)\omega}{K}},$$

we get

$$\begin{aligned}
\frac{\sigma(\mu+\gamma)(\mu+\theta+\phi)}{(\theta+\mu+\sigma\phi)-\frac{(1-\sigma)\omega}{K}} &> (\mu+\theta+\sigma\phi)+\sigma(\mu+\gamma),\\
\frac{\sigma(\mu+\gamma)(\mu+\theta+\phi)}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)} &> (\theta+\mu+\sigma\phi) - \frac{(1-\sigma)\omega}{K},\\
\frac{(1-\sigma)\omega}{K} &> (\theta+\mu+\sigma\phi),\\
&\quad - \frac{\sigma(\mu+\gamma)(\mu+\theta+\phi)}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)},\\
\frac{(1-\sigma)\omega}{K} &> \frac{(\theta+\mu+\sigma\phi)^2}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)},\\
&\quad - \frac{\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)},
\end{aligned}$$

where we used that $\mu K - \omega > 0$.

■

Theorem IV.3. *If condition (11) does not hold, then system (4) undergoes a forward bifurcation at $\mathcal{R}_0 = 1$. In this case there is no endemic equilibrium for $\mathcal{R}_0 \in [0,1]$.*

Proof: We proceed similarly as in the proof of Theorem IV.2 to find that if

$$\frac{(1-\sigma)\omega}{K} \leq \frac{(\theta+\mu+\sigma\phi)^2 - \sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)},$$

then $B \geq 0$ when $C = 0$, or equivalently, when β is set to satisfy $\mathcal{R}_0 = 1$. For $\mathcal{R}_0 < 1$ it holds that $A, C > 0$, moreover B is also positive because B is decreasing in β, these imply that there is no endemic equilibrium on $\mathcal{R}_0 \in [0,1)$. At $\mathcal{R}_0 = 1$ the equilibrium condition (9) becomes $A\hat{I}^2 + B\hat{I} = 0$, and $A > 0$, $B \geq 0$ give that (9) has only non-positive solutions. However, we know from Proposition IV.1 that there is a positive solution of (9) for $\mathcal{R}_0 > 1$, thus we conclude that if the condition (11) does not hold then system (4) undergoes a forward bifurcation at $\mathcal{R}_0 = 1$, where a single endemic equilibrium emerges when $\mathcal{R}_0$ exceeds 1. ■

If (11) is satisfied, then there is an interval to the left of $\mathcal{R}_0 = 1$ where there exist positive equilibria. In what follows we determine the left endpoint of this interval. Let us assume that there is a backward bifurcation at $\mathcal{R}_0 = 1$. We define

$$\begin{aligned}
U &= (\theta+\mu+\sigma\phi) - \frac{(1-\sigma)\omega}{K},\\
x &= \frac{(1-\sigma)\omega}{K} + \sigma(\mu+\gamma),\\
W &= -x + \sigma\frac{(\gamma+\mu)(\mu+\phi+\theta)}{U}.
\end{aligned} \tag{12}$$

Note that x and U are positive since $\mu K - \omega > 0$ by assumption. The condition for the backward bifurcation can be obtained as

$$W > U, \tag{13}$$

which also yields the positivity of W. We let

$$\mathcal{R}_c = \frac{x - U + 2\sqrt{UW}}{(\mu+\gamma)\sigma} \cdot \frac{U}{\mu+\theta+\phi} \tag{14}$$

and claim that it defines the critical value of the reproduction number for which there exist endemic equilibria on the interval $[\mathcal{R}_c, 1]$.

Proposition IV.4. *Let us assume that there is a backward bifurcation at $\mathcal{R}_0 = 1$. With $\mathcal{R}_c$ defined in (14) only the disease-free equilibrium exists if $\mathcal{R}_0 < \mathcal{R}_c$, a positive equilibrium emerges at $\mathcal{R}_0 = \mathcal{R}_c$, and on $(\mathcal{R}_c, 1)$ there exist two distinct endemic equilibria. There also exists a positive equilibrium at $\mathcal{R}_0 = 1$.*

Proof: The last statement follows from the fact that at $\mathcal{R}_0 = 1$ ($C = 0$) the single non-zero solution $\hat{I} = \frac{-B}{A}$ of (9) of is positive since $B < 0$. The necessary and sufficient conditions $B < 0$ and $B^2 - 4AC > 0$ for the existence of two positive distinct equilibria hold on an interval to the left of $\mathcal{R}_0 = 1$. $B = 0$ automatically yields $B^2 - 4AC < 0$ if $\mathcal{R}_0 < 1$, hence it is clear that the condition $B^2 - 4AC = 0$ determines the value of $\mathcal{R}_0$ for which the positive equilibria disappear. First, we derive the critical value β_c of the transmission rate from this equation, then substitute $\beta = \beta_c$ into the formula of $\mathcal{R}_0$ (6) to give the critical value of the reproduction number. Using notations U, x and W introduced in (12), we reformulate B as $B = U + x - \sigma\beta K$ and C as $C = \frac{(\mu+\gamma)(\mu+\theta+\phi)}{\beta} - UK$. The condition $B^2 - 4AC = 0$ becomes

$$\begin{aligned}
&U^2 + 2U(x - \beta K\sigma) + (x - \beta K\sigma)^2 \\
&\quad - 4\sigma(\mu+\gamma)(\mu+\theta+\phi) + 4\sigma\beta KU \\
=&U^2 - 2U(x - \beta K\sigma) + (x - \beta K\sigma)^2 + 4Ux \\
&\quad - 4\sigma(\mu+\gamma)(\mu+\theta+\phi) \\
=&U^2 - 2U(x - \beta K\sigma) + (x - \beta K\sigma)^2 - 4UW = 0,
\end{aligned}$$

so we obtain the roots

$$\begin{aligned}
(x - \beta K\sigma)_{1,2} &= \frac{2U \pm \sqrt{4U^2 - 4U^2 + 16UW}}{2} \\
&= U \pm 2\sqrt{UW}.
\end{aligned}$$

For the positive root $(x - \beta K\sigma)_2$ we get $B = U + (x - \beta K\sigma)_2 > 0$, but we require $B < 0$ thus we derive from $x - \beta K\sigma = U - 2\sqrt{UW}$ that

$$\beta_c = \frac{x - U + 2\sqrt{UW}}{K\sigma}. \tag{15}$$

Substituting β_c into (6) gives

$$\begin{aligned}
\mathcal{R}_0(\beta_c) &= \frac{\beta_c}{\mu+\gamma}\left(\frac{K(\mu+\theta+\sigma\phi)}{\mu+\theta+\phi} - \frac{(1-\sigma)\omega}{\mu+\theta+\phi}\right) \\
&= \frac{x - U + 2\sqrt{UW}}{(\mu+\gamma)\sigma} \cdot \frac{U}{\mu+\theta+\phi},
\end{aligned}$$

which is indeed equal to $\mathcal{R}_c$ defined in (14). ■

The condition $\mathcal{R}_0 = 1$ reformulates as $\sigma\beta K = W + x$, so with the aid of (13) and the computations

$$\begin{aligned}
0 &< \left(\sqrt{U} - \sqrt{W}\right)^2, \\
2\sqrt{UW} &< U + W, \\
x - U + 2\sqrt{UW} &< W + x,
\end{aligned}$$

it is easy to verify that $\mathcal{R}_c < 1$. The positivity of β_c, and hence, the positivity of $\mathcal{R}_c$ follows from the fact that at $\beta = \beta_c$ it should hold that $B < 0$, which is only possible if $\beta > 0$.

We wish to draw the graph of $\hat{I}$ as a function of β to obtain the bifurcation curve. By implicitly differentiating the equilibrium condition (9) with respect to β we get

$$\begin{aligned}
(2A\hat{I} + B)\frac{d\hat{I}}{d\beta} &= -\left(\frac{dA}{d\beta}\hat{I}^2 + \frac{dB}{d\beta}\hat{I} + \frac{dC}{d\beta}\right), \\
(2A\hat{I} + B)\frac{d\hat{I}}{d\beta} &= \sigma\hat{I}(K - \hat{I}) \\
&\quad + \frac{(\gamma+\mu)(\mu+\phi+\theta)}{\beta^2}.
\end{aligned}$$

The positivity of the right hand side follows from $K \geq \hat{I}$, which implies that the term $2A\hat{I} + B$ has the same sign as $\frac{d\hat{I}}{d\beta}$. If $\mathcal{R}_0 > 1$ then there exists the equilibrium $\breve{I}_2 = \frac{-B+\sqrt{B^2-4AC}}{2A}$, and we obtain that $2A\breve{I}_2 + B > 0$ hence for $\mathcal{R}_0 > 1$ the curve has positive slope. If there is a backward bifurcation at $\mathcal{R}_0 = 1$, then on $(\mathcal{R}_c, 1)$ there exists two positive equilibria $\breve{I}_2$ and $\breve{I}_1 = \frac{-B-\sqrt{B^2-4AC}}{2A}$ with $\breve{I}_2 > \breve{I}_1$, and since it holds that $2A\breve{I}_1 + B < 0$, we conclude that on $(\mathcal{R}_c, 1)$ the bifurcation curve has negative slope for the smaller endemic equilibrium and positive slope for the larger one. As a matter of fact, the unstable equilibrium is a saddle point, and thus the system experiences a saddle-node bifurcation.

V. Stability and Global Behavior

The stability of the disease-free equilibrium has been examined in section III, so now we derive local stability analysis of endemic equilibria. The Jacobian of the linearization of system (4) at $(\hat{I}, \hat{V})$ gives

$$J = \begin{pmatrix} -\beta\hat{I} & -(1-\sigma)\beta\hat{I} \\ -(\phi + \sigma\beta\hat{V}) & -(\mu+\theta+\phi+\sigma\beta\hat{I}) \end{pmatrix},$$

where we used the identity $\beta(K - \hat{I} - (1-\sigma)\hat{V}) = \mu + \gamma$ from (7), hence the characteristic equation has the form

$$a_2\lambda^2 + a_1\lambda + a_0 = 0$$

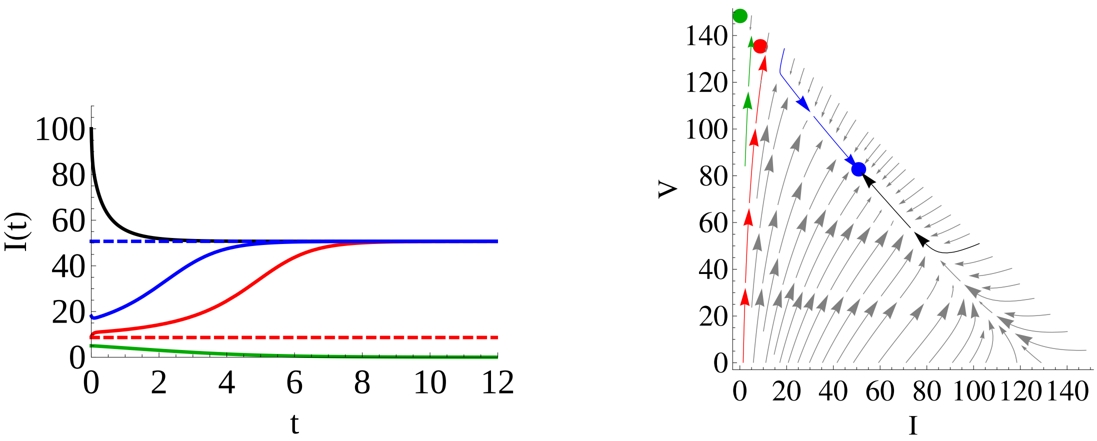
with

$$\begin{aligned}
a_2 &= 1, \\
a_1 &= \beta\hat{I} + (\mu+\theta+\phi+\sigma\beta\hat{I}), \\
a_0 &= \beta\hat{I}(\mu+\theta+\phi+\sigma\beta\hat{I}) - (1-\sigma)\beta\hat{I}(\phi+\sigma\beta\hat{V}).
\end{aligned}$$

Theorem V.1. *The endemic equilibrium $(\hat{I}, \hat{V})$ for which $\hat{I} = \breve{I}_2$ is locally asymptotically stable where it exists: on $\mathcal{R}_0 \in (1, \infty)$, and also on $\mathcal{R}_0 \in (\mathcal{R}_c, 1]$ in case there is a backward bifurcation at $\mathcal{R}_0 = 1$. The endemic equilibrium $(\hat{I}, \hat{V})$ for which $\hat{I} = \breve{I}_1$ is unstable where it exists: on $\mathcal{R}_0 \in (\mathcal{R}_c, 1)$ in case there is a backward bifurcation at $\mathcal{R}_0 = 1$.*

Proof: The Routh-Hurwitz stability criterion (for a reference see, for example, [5]) states that for all the solutions of the characteristic equation to have negative real parts, all coefficients must have the same sign. a_2 and a_1 are positive, hence the sign of a_0 determines the stability. For that it holds that

$$\begin{aligned}
a_0 =& \beta\hat{I}(\mu+\theta+\phi+\sigma\beta\hat{I}) - (1-\sigma)\beta\hat{I}(\phi+\sigma\beta\hat{V}) \\
=& \beta\hat{I}(\mu+\theta+\sigma\phi+2\sigma\beta\hat{I} - \sigma\beta(\hat{I} + (1-\sigma)\hat{V}),
\end{aligned}$$



(a) Solutions of system (4).

(b) Stream plot of system (4) on $R : 0 \leq I, V, I+V \leq K$.

Fig. 1: Solutions of system (4) in case there is a backward bifurcation at $\mathcal{R}_0 = 1$ and $\mathcal{R}_c < \mathcal{R}_0 < 1$. We let $\Lambda(x) = \frac{x}{c+dx}$ and choose parameter values as $\mu = 0.1$, $\gamma = 12$, $\theta = 0.5$, $\sigma = 0.2$, $\phi = 16$, $c = 1$, $d = 1.8$, $\beta = 0.33$, $\eta = 5$, $\omega = 5$, which makes $K = 153.6$ and $\mathcal{R}_0 = 0.95$. Endemic equilibria $(\breve{I}_1, \breve{V}_1) = (8.6, 135.4)$ and $(\breve{I}_2, \breve{V}_2) = (50.7, 82.8)$ are represented as (a) red-dashed and blue-dashed lines, (b) red and blue points, respectively. On (b) the green point denotes the unique disease-free equilibrium $(0, 148.4)$. Solutions with initial values $(I(0), V(0)) = (9, 120)$ – red curve, $(18, 130)$ – blue curve and $(100, 50)$ – black curve converge to $(\breve{I}_2, \breve{V}_2)$, however for $(I(0), V(0)) = (5, 140)$ the curve of I – here, green – approaches the DFE.

so using $-\beta(\hat{I} + (1-\sigma)\hat{V}) = \mu + \gamma - \beta K$ we derive

$$\begin{aligned} a_0 =& \beta\hat{I}(\mu + \theta + \sigma\phi + 2\sigma\beta\hat{I} + \sigma(\mu + \gamma - \beta K)) \\ =& \beta\hat{I}(2A\hat{I} + B). \end{aligned}$$

For $\mathcal{R}_0 > 1$ the only endemic equilibrium is $\breve{I}_2 = \frac{-B+\sqrt{B^2-4AC}}{2A}$, for which $2A\breve{I}_2 + B > 0$ holds and thus $a_0 > 0$ yields its stability. If there is a backward bifurcation at $\mathcal{R}_0 = 1$, then endemic equilibria exists on $(\mathcal{R}_c, 1]$ as well; here $\breve{I}_2$ is again stable for the same reason as above, however $\breve{I}_1 = \frac{-B-\sqrt{B^2-4AC}}{2A}$ is unstable since $a_0 = \beta\breve{I}_1(A\breve{I}_1 + B) < 0$. ∎

With the next theorem we describe the global behavior of solutions of system (4).

Theorem V.2. *If there exists no endemic equilibrium, that is, if $\mathcal{R}_0 < 1$ in case of a forward bifurcation and if $\mathcal{R}_0 < \mathcal{R}_c$ in case of a backward bifurcation, then every solution converges to the disease-free equilibrium. For $\mathcal{R}_0 > 1$, the unique endemic equilibrium is globally attracting. If there is a backward bifurcation at $\mathcal{R}_0 = 1$ then on $(\mathcal{R}_c, 1)$ there is no globally attracting equilibrium, though every solution approaches an equilibrium.*

Proof: We first show that every solution of system (4) converges to an equilibrium. In section III we have proved that the region $R : 0 \leq I, V, I+V \leq K$ is positively invariant for the solutions of system (4). We take the C^1 function $\varphi(I, V) = 1/I$, which does not change sign on R to show that system (4) has no periodic solutions lying entirely within the region R. The computation

$$\begin{aligned} &\frac{\partial}{\partial I}\frac{\beta(K - I - (1-\sigma)V)I - (\mu+\gamma)I}{I} \\ &+ \frac{\partial}{\partial V}\frac{\phi(K - I) - \sigma\beta VI - (\mu+\theta+\phi)V + \omega}{I} \\ =& -\beta - \sigma\beta - \frac{\mu+\theta+\phi}{I} < 0 \end{aligned}$$

yields the result by means of the Dulac criterion [3]. We use the well known Poincaré-Bendixson theorem to conclude that every solution of (4) approaches an equilibrium.

The first statement of the theorem immediately follows from the fact that every solution of (4) approaches an equilibrium. If $\mathcal{R}_0 > 1$, then besides the disease-free equilibrium, which is unstable according to Theorem V.1, there exists a single locally stable endemic equilibrium $\breve{I}_2$. We show that no solution can converge to the disease-free equilibrium.
If $\lim_{t\to\infty} I(t) = 0$ when $I(0) > 0$, then it follows from $(4)_2$ that $\lim_{t\to\infty} V(t) = \frac{\phi K + \omega}{\mu+\theta+\phi}$. Then for every $\epsilon >$

0 there exists a $t_*(\epsilon)$ such that $I(t) < \epsilon$ and $V(t) < \frac{\phi K+\omega}{\mu+\theta+\phi} + \epsilon$ for $t > t_*$. Using $(4)_1$ we get

$$\begin{aligned} I'(t) \geq & \beta \left(K - \epsilon - (1-\sigma)\left(\frac{\phi K + \omega}{\mu+\theta+\phi} + \epsilon \right) \right) I(t) \\ & - (\mu+\gamma) I(t) \\ = & \beta \left(\frac{K(\mu+\theta+\sigma\phi)}{\mu+\theta+\phi} - \frac{(1-\sigma)\omega}{\mu+\theta+\phi} \right) I(t) \\ & + (-2\epsilon + \sigma\epsilon - (\mu+\gamma)) I(t) \end{aligned} \tag{16}$$

for $t > t_*$, moreover $\mathcal{R}_0 = \frac{\beta}{\mu+\gamma}\left(\frac{K(\mu+\theta+\sigma\phi)}{\mu+\theta+\phi} - \frac{(1-\sigma)\omega}{\mu+\theta+\phi}\right) > 1$ implies that there exists an ϵ_1 small enough such that

$$\begin{aligned} \beta & \left(\frac{K(\mu+\theta+\sigma\phi)}{\mu+\theta+\phi} - \frac{(1-\sigma)\omega}{\mu+\theta+\phi} \right) \\ & + (-2\epsilon_1 + \sigma\epsilon_1 - (\mu+\gamma)) > 0. \end{aligned}$$

With the choice of $\epsilon = \epsilon_1$ the right hand side of (16) is linear in $I(t)$ with positive multiplier, which implies that $I(t)$ increases for $t_*(\epsilon_1) > t$ and thus, cannot converge to 0. We conclude that no solution of (4) with positive initial conditions converges to the disease-free equilibrium, so the endemic equilibrium indeed attracts every solution.

If there is a backward bifurcation at $\mathcal{R}_0 = 1$ then besides the disease-free equilibrium there exist two endemic equilibra on $(\mathcal{R}_c, 1)$, one locally stable and one unstable (see again Theorem V.1). As the DFE is locally stable when $\mathcal{R}_0 < 1$, we experience bistability on $(\mathcal{R}_c, 1)$, which implies the third statement of the theorem. ■

We present Figure 1 to illustrate the statements of this section. The values of the model parameters were set to ensure that system (4) undergoes a backward bifurcation at $\mathcal{R}_0 = 1$, moreover we chose the value of β such that there exist two endemic equilibria. The plots of the figure support our results about the long-term behavior of solutions and the local stability of equilibria; solutions starting near the unstable saddle point $(\breve{I}_1, \breve{V}_1)$ approach another equilibrium, however $(\breve{I}_2, \breve{V}_2)$ seems to attract every solution with $I(0) \geq \breve{I}_1$ for the particular set of parameter values indicated in the caption of the figure.

VI. The influence of immigration on the backward bifurcation

In this section, we would like to investigate the effect of parameters η and ω on the bifurcation curve. In section IV we gave the condition (11)

$$\frac{(1-\sigma)\omega}{K} > \frac{(\theta+\mu+\sigma\phi)^2 - \sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi) + \sigma(\mu+\gamma)}$$

for the existence of backward bifurcation at $\mathcal{R}_0 = 1$; in what follows we analyze this inequality in terms of the immigration parameters. We keep in mind that if there is no backward bifurcation at $\mathcal{R}_0 = 1$ then there is forward bifurcation, i.e., there always exists an endemic equilibrium for $\mathcal{R}_0 > 1$.

First we present results about how the existence of backward bifurcation depends on η and ω. The non-negativity of ω and K immediately yields the following proposition.

Proposition VI.1. *If $(\theta+\mu+\sigma\phi)^2 < \sigma(\mu+\gamma)(1-\sigma)\phi$, then for all η and ω there is a backward bifurcation at $\mathcal{R}_0 = 1$.*

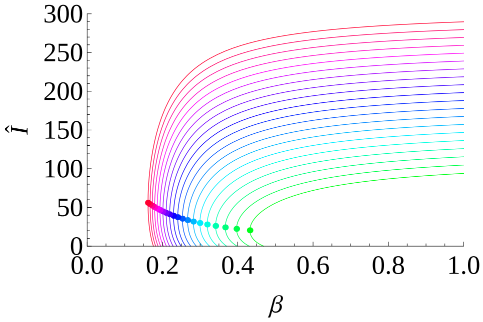
The special case of $\omega = 0$ automatically makes the left hand side of inequality (11) zero, hence in this case there is a backward bifurcation *if and only if* the right hand side is negative; note that the right hand side is independent of η.

Proposition VI.2. *If $\omega = 0$, then there is a backward bifurcation at $\mathcal{R}_0 = 1$ if and only if $(\theta+\mu+\sigma\phi)^2 < \sigma(\mu+\gamma)(1-\sigma)\phi$. This also means that in this case η has absolutely no effect on the existence of a backward bifurcation.*

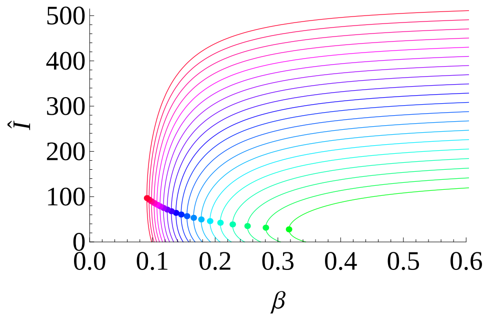
Figure 2 shows how the bifurcation curve deforms as we increase (a) ω and (b) η. Parameter values $\mu = 0.1$, $\gamma = 12$, $\theta = 0.5$, $\sigma = 0.2$, $\phi = 16$ were chosen so that the condition $(\theta+\mu+\sigma\phi)^2 < \sigma(\mu+\gamma)(1-\sigma)\phi$ holds $(14.44 < 30.976)$.

After all this, the following question arises naturally: is it possible to have backward bifurcation at $\mathcal{R}_0 = 1$ *when* $(\theta+\mu+\sigma\phi)^2 \geq \sigma(\mu+\gamma)(1-\sigma)\phi$, i.e., when the right hand side of condition (11) is nonnegative? Recall that if $\omega = 0$ then $(\theta+\mu+\sigma\phi)^2 \geq \sigma(\mu+\gamma)(1-\sigma)\phi$ means forward bifurcation.

Note that the right hand side of (11) is independent of η and ω; however, K depends on both of these parameters, μ and the birth function Λ. As we did not define Λ explicitly (in section II, we only gave conditions to ensure that for each $\eta, \omega \geq 0$ the population carrying capacity $K > 0$ can be defined uniquely), it is not clear how the left hand side of (11) depends on the immigration parameters. In the sequel, we use the



(a) $\eta = 5, \omega = 0, 1, \ldots 19.$.



(b) $\omega = 0, \eta = 10, 12, \ldots 48$.

Fig. 2: Bifurcation diagrams for 20 different values of (a) ω and (b) η in the case when $(\theta + \mu + \sigma\phi)^2 < \sigma(\mu + \gamma)(1 - \sigma)\phi$. Proposition VI.1 implies that for all η and ω there is a backward bifurcation at $\mathcal{R}_0 = 1$. The curves move to the left as the immigration parameter increases. We let $\Lambda(x) = \frac{x}{c+dx}$ and choose parameter values as $\mu = 0.1$, $\gamma = 12$, $\theta = 0.5$, $\sigma = 0.2$, $\phi = 16$, $c = 1$, $d = 1.8$.

general form

$$\Lambda(x) = \frac{x}{c + dx} \tag{17}$$

for the birth function with parameters $0 < c < 1/\mu$ and $d > 0$; it is not hard to see that with this definition all the conditions made in section II for Λ are satisfied. The carrying capacity $K(\mu, \eta, \omega)$ then arises as the solution of

$$\Lambda(x) = \mu x - \eta - \omega,$$

which with our above definition (17) gives the second-order equation

$$x^2 \mu d + x(-1 + c\mu - d(\eta + \omega)) - c(\eta + \omega) = 0.$$

The unique positive root yields K as

$$\begin{aligned} K(\mu, \eta, \omega) = &\frac{1 - c\mu + d(\eta + \omega)}{2\mu d} \\ &+ \frac{\sqrt{(1 - c\mu + d(\eta + \omega))^2 + 4\mu dc(\eta + \omega)}}{2\mu d}. \end{aligned} \tag{18}$$

Our assumption $c < 1/\mu$ implies $1 - c\mu > 0$, hence

$$\begin{aligned} \frac{K}{\omega} = &\frac{1}{2\mu d} \left(\frac{1 - c\mu + d\eta}{\omega} + d \right. \\ &\left. + \sqrt{\left(\frac{1 - c\mu + d\eta}{\omega} + d \right)^2 + \frac{4\mu dc\eta}{\omega^2} + \frac{4\mu dc}{\omega}} \right) \\ > &\frac{1}{2\mu d} \left(\frac{1 - c\mu + d\eta}{\omega} + d + \frac{1 - c\mu + d\eta}{\omega} + d \right) \\ > &\frac{1}{2\mu d} 2d = \frac{1}{\mu} \end{aligned}$$

and thus

$$\frac{(1 - \sigma)\omega}{K} < (1 - \sigma)\mu. \tag{19}$$

It also follows from the above computations that $\lim_{\omega \to \infty} \frac{(1-\sigma)\omega}{K} = (1 - \sigma)\mu$, i.e., although the left hand side of (11) is always less than $(1 - \sigma)\mu$, the expression gets arbitrary close to this limit as ω approaches ∞.

Next we fix every model parameter but η and ω and obtain two propositions as follows.

Proposition VI.3. *Let us assume that $(\theta + \mu + \sigma\phi)^2 \geq \sigma(\mu + \gamma)(1 - \sigma)\phi$ holds. If the condition*

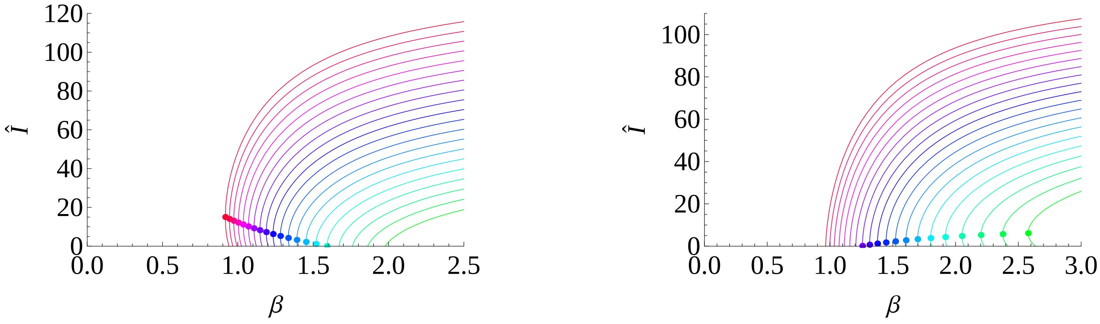
$$(\theta + \mu + \sigma\phi)(\theta + \sigma\mu + \sigma\phi) < \sigma(1 - \sigma)(\mu + \gamma)(\mu + \phi)$$

is satisfied, then for any η there is an ω_c such that for any $\omega \in (\omega_c, \infty)$ there is a backward bifurcation at $\mathcal{R}_0 = 1$, and for any $\omega \in [0, \omega_c]$ there is a forward bifurcation at $\mathcal{R}_0 = 1$. In case the above condition does not hold, then for any η and ω there is a forward bifurcation at $\mathcal{R}_0 = 1$.

With other words, for parameter values satisfying the assumption and condition of Proposition VI.3, the ω_c defined in (23) works as a threshold value of ω for the backward bifurcation: there is no backward bifurcation if $\omega \leq \omega_c$, and once ω is large enough so that a backward bifurcation is established at $\mathcal{R}_0 = 1$, it can not happen that for any larger values of ω the system undergoes forward bifurcation again. With certain conditions, such threshold also exists for η as we show it in the following proposition.

Proposition VI.4. *We assume that $(\theta + \mu + \sigma\phi)^2 \geq \sigma(\mu + \gamma)(1 - \sigma)\phi$ holds, and fix ω. If ω is such that*

$$\frac{(1 - \sigma)\omega}{K(\mu, 0, \omega)} > \frac{(\theta + \mu + \sigma\phi)^2 - \sigma(\mu + \gamma)(1 - \sigma)\phi}{(\theta + \mu + \sigma\phi) + \sigma(\mu + \gamma)}$$



(a) $\eta = 10, \omega = 1, 6, \dots 96$.

(b) $\omega = 60, \eta = 1, 6, \dots 96$.

Fig. 3: Bifurcation diagrams for 20 different values of (a) ω and (b) η in the case when $(\theta + \mu + \sigma\phi)^2 \geq \sigma(\mu + \gamma)(1 - \sigma)\phi$. The curves move to the left as the immigration parameter increases. We let $\Lambda(x) = \frac{x}{c+dx}$ and choose parameter values as (a) $\mu = 1$, $\gamma = 7.5$, $\theta = 0.5$, $\sigma = 0.02$, $\phi = 16$, $c = 0.1$, $d = 0.03$, (b) $\mu = 1.5$, $\gamma = 11$, $\theta = 0.5$, $\sigma = 0.02$, $\phi = 16$, $c = 1/15$, $d = 9/300$.

then there exists $\eta_c > 0$ such that there is a backward bifurcation at $\mathcal{R}_0 = 1$ for $\eta < \eta_c$, and the system undergoes a forward bifurcation for $\eta \geq \eta_c$. If the above inequality does not hold then there is a forward bifurcation at $\mathcal{R}_0 = 1$.

We illustrate Propositions VI.3 and VI.4 with Figure 3. With parameter values $\mu = 1$, $\gamma = 7.5$, $\theta = 0.5$, $\sigma = 0.02$, $\phi = 16$, $c = 0.1$, $d = 0.03$ and $\eta = 10$ used for Figure 3 (a), the condition in Proposition VI.3 becomes $1.5288 < 2.8322$. In case of Figure 3 (b), the parameters $\mu = 1.5$, $\gamma = 11$, $\theta = 0.5$, $\sigma = 0.02$, $\phi = 16$, $c = 1/15$, $d = 9/300$ and $\omega = 60$ give $\frac{(1-\sigma)\omega}{K(\mu,0,\omega)} = 0.956928$ and $\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)} = 0.569027$, so the condition in Proposition VI.4 is satisfied. It is easy to check that the assumption $(\theta + \mu + \sigma\phi)^2 \geq \sigma(\mu + \gamma)(1 - \sigma)\phi$ holds in both cases since (a) $3.3124 \geq 2.6656$ and (b) $5.3824 \geq 3.92$.

Proposition VI.1 states that for any values of η and ω the condition $(\theta + \mu + \sigma\phi)^2 < \sigma(\mu + \gamma)(1 - \sigma)\phi$ is sufficient for the existence of a backward bifurcation at $\mathcal{R}_0 = 1$; moreover we know from Proposition VI.2 that it is also necessary in the special case of $\omega = 0$. We remark that backward bifurcation is possible for any $\eta \geq 0$ and $\omega > 0$, even if $(\theta+\mu+\sigma\phi)^2 \geq \sigma(\mu+\gamma)(1-\sigma)\phi$. Let us choose $\eta \geq 0$ and $\omega > 0$ arbitrary, fix parameters μ, σ, ϕ, and choose θ and γ such that $(\theta + \mu + \sigma\phi)^2 = \sigma(\mu + \gamma)(1 - \sigma)\phi$ holds. As now the right hand side of condition (11) is 0 and $\omega, K > 0$, there is a backward bifurcation, moreover it is easy to see that the right hand side is increasing in θ. Thus, due to the continuous dependence of the right hand side on θ, there is an interval for θ (with all the other parameters fixed) where condition (11) still holds, though $(\theta + \mu + \sigma\phi)^2 > \sigma(\mu + \gamma)(1 - \sigma)\phi$ since the quadratic term increases in θ.

Next, we investigate how immigration deforms the bifurcation curve. Let us denote by β_0 the value of the transmission rate for which $\mathcal{R}_0 = 1$ is satisfied, using (6) it arises as

$$\beta_0 = \frac{(\mu + \theta + \phi)(\mu + \gamma)}{K(\mu + \theta + \sigma\phi) - (1 - \sigma)\omega}. \tag{20}$$

Proposition VI.5. *It holds that β_0 decreases in both ω and η.*

We recall that endemic equilibria $\breve{I}_1$ and $\breve{I}_2$ were defined as

$$\breve{I}_1 = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$
$$\breve{I}_2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

with A, B and C given in (10). Obviously $-B - \sqrt{B^2 - 4AC} > 0$ where $\breve{I}_1$ exists and $-B + \sqrt{B^2 - 4AC} > 0$ where $\breve{I}_2$ exists.

Proposition VI.6. *For the endemic equilibrium $\breve{I}_2$ it holds that $\frac{\partial}{\partial\omega}\breve{I}_2, \frac{\partial}{\partial\eta}\breve{I}_2 > 0$, the inequalities $\frac{\partial}{\partial\omega}\breve{I}_1, \frac{\partial}{\partial\eta}\breve{I}_1 < 0$ are satisfied for the endemic equilibrium $\breve{I}_1$. The equilibrium $\breve{I}_1 = \breve{I}_2 = \frac{-B}{2A}$ increases in both ω and η.*

These results give us information about how the bifurcation curve changes when the immigration parameters

$$
\begin{aligned}
\beta_c K\sigma =& x - U + 2\sqrt{UW} \\
=& \sigma(\mu+\gamma) - (\theta+\mu+\sigma\phi) + 2\sqrt{-(\theta+\mu+\sigma\phi)\sigma(\mu+\gamma)+\sigma(\gamma+\mu)(\mu+\phi+\theta)} \\
=& \sigma(\mu+\gamma) - (\theta+\mu+\sigma\phi) + 2\sqrt{\sigma(\mu+\gamma)\phi(1-\sigma)}
\end{aligned} \tag{21}
$$

increase. If there is a forward bifurcation at $\mathcal{R}_0 = 1$, the curve moves to the left since β_0 decreases in η and ω, and the curve expands because $\frac{\partial}{\partial\omega}\breve{I}_2, \frac{\partial}{\partial\eta}\breve{I}_2 > 0$. In case there is a backward bifurcation at $\mathcal{R}_0 = 1$, β_0 again moves to the left, and $\frac{\partial}{\partial\omega}\breve{I}_1, \frac{\partial}{\partial\eta}\breve{I}_1 < 0$ and $\frac{\partial}{\partial\omega}\breve{I}_2, \frac{\partial}{\partial\eta}\breve{I}_2 > 0$ imply that for each fixed β the two equilibria move away from each other in the region where they coexist, moreover $\breve{I}_2$ increases when it is the only endemic equilibrium. The singular point of the bifurcation curve, where the equilibrium is $-B/2A$, moves upward as η and ω increase, this together with the above described behavior of $\breve{I}_1$ and $\breve{I}_2$ imply that the left-most equilibrium cannot move to the right, or equivalently, the corresponding value of the transmission rate β_c decreases if we increase η and ω. We give the last statement of the above discussion in the form of a proposition. See Figures 2 and 3 for visual proof of the results of this section.

Proposition VI.7. *In case there is a backward bifurcation at $\mathcal{R}_0 = 1$, β_c decreases in both ω and η.*

Actually, using (20) it is easy to see that β_0 converges to 0 as any of the immigration parameters approaches infinity: for any fixed ω (η), the carrying capacity K reaches arbitrary large values if we increase η (ω), moreover $\mu K - \omega$ is positive by assumption, hence

$$
\begin{aligned}
&\lim_{\omega\to\infty}(K(\mu+\theta+\sigma\phi) - (1-\sigma)\omega) = \\
&= \lim_{\omega\to\infty}(K(\theta+\sigma\phi) + \sigma\omega + \mu K - \omega) = \infty.
\end{aligned}
$$

$\beta_c < \beta_0$ implies that β_c also goes to 0 as $\omega \to \infty$ or $\eta \to \infty$. We can also show that in the special case of $\omega = 0$, increasing η decreases the region where two endemic equilibria exist. The equation (15) for β_c then reformulates as (21), thus for $\beta_0 - \beta_c$ we have

$$
\begin{aligned}
(\beta_0 - \beta_c)K\sigma =& \frac{\sigma(\mu+\theta+\phi)(\mu+\gamma)}{(\mu+\theta+\sigma\phi)} - \sigma(\mu+\gamma) \\
& - \left((\theta+\mu+\sigma\phi) + 2\sqrt{\sigma(\mu+\gamma)\phi(1-\sigma)}\right).
\end{aligned}
$$

The right hand side is independent of η and K increases monotonically as η increases, so the length of the interval (β_c, β_0) decreases as η increases.

In the light of the results of this section we conclude that, although SIVS models without immigration can also exhibit backward bifurcation [1], incorporating the possibility of the inflow of non-infectives may significantly influence the dynamics: under certain conditions on the model parameters, increasing ω just as decreasing η can drive a system with forward bifurcation into backward bifurcation and the existence of multiple endemic equilibria. Nevertheless we showed that including immigration moves the left-most point of the bifurcation curve to the left, which means that the larger the values of the immigration parameters the smaller the threshold for the emergence of endemic equilibria.

VII. Revisiting the three-dimensional system

Based on our results for system (4), we draw some conclusions on the global behavior of the original model (2). Given that $N(t)$ converges, and substituting $S(t) = N(t) - I(t) - V(t)$, $(2)_2$ and $(2)_3$ together can be considered as an asymptotically autonomous system with limiting system (4). We use the theory from [12].

Theorem VII.1. *All nonnegative solutions of* (2) *converge to an equilibrium. In particular, if $\mathcal{R}_0 > 1$, then the endemic equilibrium is globally asymptotically stable. If there is a forward bifurcation for* (4) *and $\mathcal{R}_0 \leq 1$, or there is a backward bifurcation for* (4) *and $\mathcal{R}_0 < \mathcal{R}_c$, then the disease free equilibrium is globally asymptotically stable.*

Proof: Theorem V.2 excluded periodic orbits in the limit system by a Dulac-function, hence we can apply Corollary 2.2. of [12] and conclude that all solutions of $(2)_2 - (2)_3$ converge. As $I(t)$,$V(t)$ and $N(t)$ converge, $S(t)$ converges as well for system (2).

Now consider the case $\mathcal{R}_0 > 1$. Then the endemic equilibrium is globally asymptotically stable for (2) (see Theorem V.2), and its basin of attraction is the whole phase space except the disease-free equilibrium. We can proceed analogously as in (16) to show that no positive solutions of $(2)_2 - (2)_3$ can converge to $(0, \bar{V})$ when $\mathcal{R}_0 > 1$, since $N(t) > K - \epsilon$ holds for sufficiently large t. Thus, the ω-limit set of any positive solution of $(2)_2 - (2)_3$ intersects the basin of attraction of the endemic equilibrium in the limit system, and then by Theorem 2.3 of [12] we conclude that the positive solutions of $(2)_2-(2)_3$ converge to the endemic equilibrium.

When the disease-free is the unique equilibrium of (4), (i.e., when $\mathcal{R}_0 \leq 1$ in the case of forward, or $\mathcal{R}_0 < \mathcal{R}_c$ in the case of backward bifurcation), then it is globally asymptotically stable for (4) (see Theorem V.2) with the basin of attraction being the whole space, thus Theorem 2.3 of [12] ensures that the DFE is globally asymptotically stable for $(2)_2-(2)_3$ as well.

■

VIII. Conclusion

We have examined a dynamic model which describes the spread of an infectious disease in a population divided into the classes of susceptible, infected and vaccinated individuals, and took the possibility of immigration of non-infectives into account. Such an assumption is reasonable if there is an entry screening of infected individuals, or if the disease is so severe that it inhibits traveling. After obtaining some fundamental, but biologically relevant properties of the model, we investigated the possible equilibria and gave an explicit condition for the existence of backward bifurcation at $\mathcal{R}_0 = 1$ in terms of the model parameters. Our analysis showed that besides the disease-free equilibrium – which always exists – there is a unique positive fixed point for $\mathcal{R}_0 > 1$, moreover in case of a backward bifurcation there exist two endemic equilibria on an interval to the left of $\mathcal{R}_0 = 1$. An equilibrium is locally asymptotically stable if and only if it corresponds to a point on the bifurcation curve where the curve is increasing, moreover it is also globally attracting if $\mathcal{R}_0 > 1$.

We investigated how the structure of the bifurcation curve depends on η and ω (the immigration parameter for susceptible and vaccinated individuals, respectively), when other model parameters are fixed. As discussed in Propositions VI.1 and VI.3, two regions can be characterized in the parameter space where for any values of the immigration parameters the system experiences a backward or forward bifurcation, respectively. Nevertheless, under certain conditions described in Propositions VI.3 and VI.4, modifying the value of ω and η has a significant effect on the dynamics: critical values ω_c and η_c can be defined such that the bifurcation behavior at $\mathcal{R}_0 = 1$ changes from forward to backward when we increase ω through ω_c and/or we decrease η through η_c. However, Propositions VI.2 and VI.4 yield that in some cases ω can be chosen such that, independently from the value of η, backward bifurcation is impossible.

We also showed that immigration decreases the value of the transmission rate for which endemic equilibria emerge, furthermore increasing ω and/or η moves the branches of the bifurcation curve apart which implies that the stability region of the disease-free equilibrium shrinks (see Figures 2 and 3). Last, we wish to point out that, as it follows from the discussion after Proposition VI.4, backward bifurcation is possible for any values of ω and η, so when one's aim is to mitigate the severity of an outbreak it is desirable to control the values of other model parameters, for example, the vaccination rate in a way that such scenario is never realized.

Appendix

For readers' convenience here we recall Propositions VI.3, VI.4, VI.5 and VI.6, and state their proofs.

Proposition VI.3. *Let us assume that $(\theta+\mu+\sigma\phi)^2 \geq \sigma(\mu+\gamma)(1-\sigma)\phi$ holds. If the condition*

$$(\theta+\mu+\sigma\phi)(\theta+\sigma\mu+\sigma\phi) < \sigma(1-\sigma)(\mu+\gamma)(\mu+\phi)$$

is satisfied, then for any η there is an ω_c such that for any $\omega \in (\omega_c, \infty)$ there is a backward bifurcation at $\mathcal{R}_0 = 1$, and for any $\omega \in [0, \omega_c]$ there is a forward bifurcation at $\mathcal{R}_0 = 1$. In case the above condition does not hold, then for any η and ω there is a forward bifurcation at $\mathcal{R}_0 = 1$.

Proof of Proposition VI.3: If

$$\begin{aligned}
(\theta+\mu+\sigma\phi)(\theta+\sigma\mu+\sigma\phi) \geq& \sigma(1-\sigma)\cdot \\
&\cdot(\mu+\gamma)(\mu+\phi), \\
(\theta+\mu+\sigma\phi)\left(\frac{\theta+\mu+\sigma\phi}{1-\sigma}-\mu\right) \geq& \sigma(\mu+\gamma)(\mu+\phi), \\
\frac{(\theta+\mu+\sigma\phi)^2}{1-\sigma}-\sigma(\mu+\gamma)\phi \geq& \mu(\theta+\mu+\sigma\phi) \\
&+\mu\sigma(\mu+\gamma)), \\
\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)} \geq& (1-\sigma)\mu,
\end{aligned}$$

then it follows from (19) that backward bifurcation is not possible at $\mathcal{R}_0 = 1$ since the right hand side of condition

$$
\begin{aligned}
K-\omega\frac{\partial K}{\partial\omega} =&\frac{1-c\mu+d(\eta+\omega)+\sqrt{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}}{2\mu d}\\
&-\omega d\frac{1}{2\mu d}\left(1+\frac{1-c\mu+d(\eta+\omega)+2\mu c}{\sqrt{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}}\right)\\
=&\frac{1-c\mu+d\eta}{2\mu d}+\frac{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}{2\mu d\sqrt{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}}\\
&-\frac{\omega d(1-c\mu+d(\eta+\omega)+2\mu c)}{2\mu d\sqrt{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}}\\
=&\frac{1-c\mu+d\eta}{2\mu d}+\frac{(1-c\mu+d(\eta+\omega))(1-c\mu+d\eta)+4\mu dc\eta+2\mu dc\omega}{2\mu d\sqrt{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}}>0
\end{aligned}
\tag{22}
$$

(11) is always greater than or equal to the left hand side. Next let us consider the case when

$$
\begin{aligned}
(\theta+\mu+\sigma\phi)\left(\theta+\sigma\mu+\sigma\phi\right) <&\sigma(1-\sigma)\cdot\\
&\cdot(\mu+\gamma)(\mu+\phi),\\
\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)} <&(1-\sigma)\mu.
\end{aligned}
$$

We show that $\frac{(1-\sigma)\omega}{K}$ is monotone increasing in ω; if so, then, following relation (19) and the discussion afterwards, the formulas $\frac{(1-\sigma)\cdot 0}{K(\mu,\eta,0)}=0$ and $\lim_{\omega\to\infty}\frac{(1-\sigma)\omega}{K(\mu,\eta,\omega)}=(1-\sigma)\mu$ imply that ω_c can be defined uniquely by

$$
\frac{(1-\sigma)\omega_c}{K(\mu,\eta,\omega_c)}=\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)}, \tag{23}
$$

and from the monotonicity it follows that the condition for the backward bifurcation (11) is satisfied if and only if $\omega>\omega_c$.

We obtain the derivative

$$
\frac{\partial}{\partial\omega}\left(\frac{\omega}{K}\right)=\frac{K-\omega\frac{\partial K}{\partial\omega}}{K^2},
$$

which implies that $\frac{(1-\sigma)\omega}{K}$ increases in ω if and only if $K-\omega\frac{\partial K}{\partial\omega}$ is positive. With our assumption $1-c\mu>0$ the computations in (22) yield the result. ∎

Proposition VI.4. *We assume that $(\theta+\mu+\sigma\phi)^2\geq\sigma(\mu+\gamma)(1-\sigma)\phi$ holds, and fix ω. If ω is such that*

$$
\frac{(1-\sigma)\omega}{K(\mu,0,\omega)}>\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)}
$$

then there exists $\eta_c>0$ such that there is a backward bifurcation at $\mathcal{R}_0=1$ for $\eta<\eta_c$, and the system undergoes a forward bifurcation for $\eta\geq\eta_c$. If the above inequality does not hold then there is a forward bifurcation at $\mathcal{R}_0=1$.

Proof of Proposition VI.4: First we note that $K(\mu,\eta,\omega)$ (defined in (18)) is an increasing function of η and it attains its minimum at $\eta=0$. This implies that

$$
\frac{(1-\sigma)\omega}{K(\mu,\eta,\omega)}\leq\frac{(1-\sigma)\omega}{K(\mu,0,\omega)}
$$

for all η, hence the condition for the backward bifurcation (11) cannot be satisfied if

$$
\frac{(1-\sigma)\omega}{K(\mu,0,\omega)}\leq\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)}.
$$

On the other hand, $K(\mu,\eta,\omega)$ takes arbitrary large values, and hence $\frac{(1-\sigma)\omega}{K(\mu,\eta,\omega)}$ converges to zero monotonically as η increases, so if

$$
\frac{(1-\sigma)\omega}{K(\mu,0,\omega)}>\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)},
$$

then there is a unique $\eta_c>0$ which satisfies

$$
\frac{(1-\sigma)\omega}{K(\mu,\eta_c,\omega)}=\frac{(\theta+\mu+\sigma\phi)^2-\sigma(\mu+\gamma)(1-\sigma)\phi}{(\theta+\mu+\sigma\phi)+\sigma(\mu+\gamma)},
$$

and the monotonicity of K in η yields that for $\eta<\eta_c$ ($\eta\geq\eta_c$) the condition for the backward bifurcation (11) holds (does not hold). Thus it is clear that η_c is a threshold for the existence of backward bifurcation. Note that if $(\theta+\mu+\sigma\phi)^2=\sigma(\mu+\gamma)(1-\sigma)\phi$ then $\eta_c=\infty$, i.e., for each value of η there is a backward bifurcation if $\omega>0$. The proof is complete. ∎

Proposition VI.5. *It holds that β_0 decreases in both ω and η.*

$$
\begin{aligned}
\frac{\partial}{\partial\omega}\left(\sqrt{B^2-4AC}-B\right) &= \frac{2B\frac{\partial B}{\partial\omega}-4(-\sigma\beta(\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\omega}+\sigma\beta(1-\sigma))}{2\sqrt{B^2-4AC}}-\frac{\partial B}{\partial\omega},\\
&= \frac{\frac{\partial B}{\partial\omega}\left(B-\sqrt{B^2-4AC}\right)}{\sqrt{B^2-4AC}}+\frac{2\sigma\beta((\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\omega}-(1-\sigma))}{\sqrt{B^2-4AC}},\\
\frac{\partial}{\partial\eta}\left(\sqrt{B^2-4AC}-B\right) &= \frac{2B\frac{\partial B}{\partial\eta}-4(-\sigma\beta(\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\omega})}{2\sqrt{B^2-4AC}}-\frac{\partial B}{\partial\eta},\\
&= \frac{\frac{\partial B}{\partial\eta}(B-\sqrt{B^2-4AC})}{\sqrt{B^2-4AC}}+\frac{2\sigma\beta(\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\eta}}{\sqrt{B^2-4AC}}.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\omega}\left(\sqrt{B^2-4AC}+B\right) &= \frac{\frac{\partial B}{\partial\omega}\left(B+\sqrt{B^2-4AC}\right)}{\sqrt{B^2-4AC}}+\frac{2\sigma\beta((\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\omega}-(1-\sigma))}{\sqrt{B^2-4AC}}>0,\\
\frac{\partial}{\partial\eta}\left(\sqrt{B^2-4AC}+B\right) &= \frac{\frac{\partial B}{\partial\eta}\left(B+\sqrt{B^2-4AC}\right)}{\sqrt{B^2-4AC}}+\frac{2\sigma\beta(\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\eta}}{\sqrt{B^2-4AC}}>0.
\end{aligned}
\tag{25}
$$

Proof of Proposition VI.5: Using (20) we see that β_0 decreases as η increases since

$$
\begin{aligned}
&\frac{\partial}{\partial\eta}(K(\mu+\theta+\sigma\phi)-(1-\sigma)\omega)\\
=&\frac{\partial K}{\partial\eta}(\mu+\theta+\sigma\phi)>0.
\end{aligned}
$$

On the other hand, β_0 decreases in ω if and only if

$$
\begin{aligned}
&\frac{\partial}{\partial\omega}(K(\mu+\theta+\sigma\phi)-(1-\sigma)\omega)\\
=&\frac{\partial K}{\partial\omega}(\mu+\theta+\sigma\phi)-(1-\sigma)>0.
\end{aligned}
$$

First, $\frac{\partial K}{\partial\omega}>\frac{1}{\mu}$ since

$$
\frac{1-c\mu+d(\eta+\omega)+2\mu c}{\sqrt{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}}>1
$$

$$
\begin{aligned}
\frac{\partial K}{\partial\omega}&=\frac{1}{2\mu}\left(1+\frac{1-c\mu+d(\eta+\omega)+2\mu c}{\sqrt{(1-c\mu+d(\eta+\omega))^2+4\mu dc(\eta+\omega)}}\right)\\
&>\frac{1}{\mu},
\end{aligned}
$$

second, from

$$
\theta+\sigma\phi>-\mu\sigma
$$
$$
\mu+\theta+\sigma\phi>\mu(1-\sigma)
$$

we have $\frac{1}{\mu}>\frac{1-\sigma}{\mu+\theta+\sigma\phi}$. We conclude that

$$
\frac{\partial K}{\partial\omega}>\frac{1}{\mu}>\frac{1-\sigma}{\mu+\theta+\sigma\phi}\tag{26}
$$

and hence β_0 decreases as ω increases. ∎

Proposition VI.6. *For the endemic equilibrium $\breve{I}_2$ it holds that $\frac{\partial}{\partial\omega}\breve{I}_2, \frac{\partial}{\partial\eta}\breve{I}_2>0$, the inequalities $\frac{\partial}{\partial\omega}\breve{I}_1, \frac{\partial}{\partial\eta}\breve{I}_1<0$ are satisfied for the endemic equilibrium $\breve{I}_1$. The equilibrium $\breve{I}_1=\breve{I}_2=\frac{-B}{2A}$ increases in both ω and η.*

Proof of Proposition VI.6: As

$$
\begin{aligned}
\frac{\partial AC}{\partial\omega}&=-\sigma\beta(\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\omega}+\sigma\beta(1-\sigma),\\
\frac{\partial AC}{\partial\eta}&=-\sigma\beta(\mu+\theta+\sigma\phi)\frac{\partial K}{\partial\eta},
\end{aligned}
$$

we derive (24), moreover it follows from (26), $\frac{\partial B}{\partial\omega}=-\sigma\beta\frac{\partial K}{\partial\omega}<0$, $\frac{\partial B}{\partial\eta}=-\sigma\beta\frac{\partial K}{\partial\eta}<0$ and $B-\sqrt{B^2-4AC}<0$ that

$$
\begin{aligned}
\frac{\partial}{\partial\omega}\left(\sqrt{B^2-4AC}-B\right)>0,\\
\frac{\partial}{\partial\eta}\left(\sqrt{B^2-4AC}-B\right)>0.
\end{aligned}
$$

Similarly, using $B+\sqrt{B^2-4AC}<0$ we get (25) and thus

$$
\begin{aligned}
\frac{\partial}{\partial\omega}\breve{I}_1&=\frac{-\frac{\partial}{\partial\omega}\left(\sqrt{B^2-4AC}+B\right)}{2A}<0,\\
\frac{\partial}{\partial\eta}\breve{I}_1&=\frac{-\frac{\partial}{\partial\eta}\left(\sqrt{B^2-4AC}+B\right)}{2A}<0,
\end{aligned}
$$

moreover

$$\frac{\partial}{\partial \omega}\breve{I}_2 = \frac{\frac{\partial}{\partial \omega}\left(\sqrt{B^2-4AC}-B\right)}{2A} > 0,$$

$$\frac{\partial}{\partial \eta}\breve{I}_2 = \frac{\frac{\partial}{\partial \eta}\left(\sqrt{B^2-4AC}-B\right)}{2A} > 0.$$

The equilibrium $\breve{I}_1 = \breve{I}_2 = \frac{-B}{2A}$ is increasing in both ω and η since A is independent of these parameters and $\frac{\partial B}{\partial \omega} < 0$, $\frac{\partial B}{\partial \eta} < 0$. ∎

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