Some problems of A. Kroó on multiple Chebyshev polynomials*

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Abstract

Three problems of A. Kroó on multiple Chebyshev polynomials are solved using the Borsuk-Ulam antipodal theorem.

Multiple Chebyshev polynomials have been introduced in the paper [4] by András Kroó. Their definition is as follows. Let $w_1, \ldots, w_m$ be nonnegative continuous weight functions on an interval $[a, b] \subset \mathbf{R}$, neither of which vanishes identically, and let $n_1, \ldots, n_m$ be positive integers. An $(n_1, \ldots, n_m)$-Chebyshev polynomial associated with $(w_1, \ldots, w_m)$ is a polynomial $P(x) = x^k + \cdots$ of some degree $k \leq n_1 + \cdots + n_m$ such that for each $j = 1, \ldots, m$, zero is its best w_j-approximant among all polynomials of degree at most $n_j - 1$, i.e. for every polynomial q of degree at most $n_j - 1$ we have

$$\|w_j P\|_{[a,b]} \leq \|w_j(P + q)\|_{[a,b]},$$

where $\|\cdot\|_{[a,b]}$ denotes the supremum norm on $[a, b]$. This is an analogue of multiple orthogonal polynomials, see [4]. We also refer to [2, Secs. 3.5, 3.6] for the classical case and for discussions of Chebyshev alternations/equioscillations that we shall use below.

The paper [4] proves the existence of any $(n_1, \ldots, n_m)$-Chebyshev polynomial if the system $(w_1, \ldots, w_m)$ satisfies a certain weak-Chebyshev property. In particular, it was proven that all $(n_1, \ldots, n_m)$-Chebyshev polynomials exist for exponential weights $e^{i\lambda_1 x}, \ldots, e^{i\lambda_m x}$, $\lambda_i \neq \lambda_j$. These results were obtained in [4] as the $p \to \infty$ case of similar L^p statements. In connection with these several questions have been asked in [4]:

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- Are there weights different from exponential ones for which multiple Chebyshev polynomials exist?
- When multiple Chebyshev polynomials exist, then is there one with maximal degree (i.e. of degree $n = n_1 + \cdots + n_m$)?
- Are multiple Chebyshev polynomials unique?

The aim of this paper is to answer these questions, namely we show that

- Multiple Chebyshev polynomials exist for all $(w_1, \ldots, w_m)$ and all $(n_1, \ldots, n_m)$.
- There may not exist one of maximal degree.
- In general, multiple Chebyshev polynomials are not unique.

We begin with

Theorem 1 *For any weights $(w_1, w_2, \ldots, w_m)$ a multiple Chebyshev polynomial exists for any degrees $(n_1, n_2, \ldots, n_m)$.*

Note however, that, in view of Proposition 2 below, the degree may be smaller than n. In the extreme case when all w_j's are even functions and $[a, b]$ is an interval symmetric with respect to the origin, $f(x) = x$ is clearly a $(1, 1, \cdots, 1)$ multiple multiple Chebyshev polynomial, and so is any odd power x^{2k+1}, $2k + 1 \leq m$. This shows that, in general, multiple Chebyshev polynomials are not unique.

Proof. First we show that a multiple Chebyshev polynomial of any degree $(n_1, \ldots, n_m)$ exists in L^{2k}-norms, $k = 1, 2, \ldots$ (see below what exactly that means):

$$\|f\|_{L^{2k}(w_j)} = \left\{ \int_a^b f^{2k} w_j^{2k} \right\}^{1/2k}.$$

Set $n = n_1 + \cdots + n_m$, let S^n be the unit sphere in $\mathbf{R}^{n+1}$, and for $\xi = (\xi_0, \ldots, \xi_n) \in S^n$ set

$$f_\xi(x) = \xi_0 + \xi_1 x + \cdots + \xi_n x^n.$$

Then $\|f_\xi\|_{L^{2k}(w_j)}^{2k}$ is a homogenous polynomial of degree $2k$ of the variables $\xi_0, \ldots, \xi_n$ whenever k is a positive integer, so the partial derivatives below exist.

Define the vector $(\eta_1, \ldots, \eta_n)$ as

$$\Big(\xi_0, \xi_1, \ldots, \xi_{n_1-1}, \xi_0, \xi_1, \ldots, \xi_{n_2-1}, \xi_0, \xi_1, \ldots, \xi_{n_3-1}, \ldots, \xi_0, \xi_1, \ldots, \xi_{n_m-1}\Big),$$

and let $i_s = j$ if $n_1 + \cdots + n_{j-1} < s \le n_1 + \cdots + n_j$, $s = 1, \ldots, n$, where we set $n_0 = 0$. The function

$$F_k(\xi) = \left(\frac{\partial \|f_\xi\|_{L^{2k}(w_{i_s})}}{\partial \eta_s} \right)_{s=1}^{n}$$

is a continuous odd function on S^n that maps S^n into $\mathbf{R}^n$, hence, by the Borsuk-Ulam antipodal theorem [1, p. 241], there is a $\xi^{(k)}$ such that $F_k(\xi^{(k)}) = (0, \ldots, 0)$. If we look at the definition of the vector η then we can see that this means that

$$\left. \frac{\partial \|f_\xi\|_{L^{2k}(w_j)}}{\partial \xi_s} \right|_{\xi^{(k)}} = 0$$

for all $0 \le s < n_j$, $j = 1, \ldots, m$. Then for any vector $v = (c_0, \ldots, c_{n_j-1})$ the directional derivative in the direction of v also vanishes:

$$\left. \frac{d\|f_{\xi+tv}\|_{L^{2k}(w_j)}}{dt} \right|_{t=0} =: \left. \frac{\partial \|f_\xi\|_{L^{2k}(w_j)}}{\partial v} \right|_{\xi^{(k)}} = 0 \tag{1}$$

because this directional derivative is

$$\sum_{s=0}^{n_j-1} c_s \left. \frac{\partial \|f_\xi\|_{L^{2k}(w_j)}}{\partial \xi_s} \right|_{\xi^{(k)}}.$$

We claim that this $f_{\xi^{(k)}}$ has the extremality property that for any $j = 1, \ldots, m$

$$\|f_{\xi^{(k)}}\|_{L^{2k}(w_j)} \le \|f_{\xi^{(k)}} + p\|_{L^{2k}(w_j)} \tag{2}$$

for any polynomial p of degree $< n_j$. Indeed, suppose that is not true, and for some $p(x) = c_0 + c_1 x + \cdots + c_{n_j-1} x^{n_j-1}$ we have

$$\|f_{\xi^{(k)}}\|_{L^{2k}(w_j)} \ge \|f_{\xi^{(k)}} + p\|_{L^{2k}(w_j)} + \varepsilon$$

with some $\varepsilon > 0$. Then for small $\lambda > 0$

$$\begin{aligned} \|f_{\xi^{(k)}} + \lambda p\|_{L^{2k}(w_j)} &= \|(1-\lambda) f_{\xi^{(k)}} + \lambda(f_{\xi^{(k)}} + p)\|_{L^{2k}(w_j)} \\ &\le \|(1-\lambda) f_{\xi^{(k)}}\|_{L^{2k}(w_j)} + \|\lambda(f_{\xi^{(k)}} + p)\|_{L^{2k}(w_j)} \\ &\le (1-\lambda)\|f_{\xi^{(k)}}\|_{L^{2k}(w_j)} + \lambda\left(\|f_{\xi^{(k)}}\|_{L^{2k}(w_j)} - \varepsilon\right) \\ &= \|f_{\xi^{(k)}}\|_{L^{2k}(w_j)} - \lambda\varepsilon, \end{aligned}$$

which shows that with $v = (c_0, \ldots, c_{n_j-1})$

$$\lim_{\lambda \to 0+0} \frac{\|f_{\xi^{(k)}} + \lambda p\|_{L^{2k}(w_j)} - \|f_{\xi^{(k)}}\|_{L^{2k}(w_j)}}{\lambda} = \left. \frac{\partial \|f_\xi\|_{L^{2k}(w_j)}}{\partial v} \right|_{\xi^{(k)}}$$

cannot be zero, which contradicts (1). Hence, (2) is true for all j and p.

Let now $\xi^* \in S^n$ be a limit point of $\{\xi^{(k)}\}_{k=1}^{\infty}$, say $\xi^{(k)} \to \xi^*$ as $k \to \infty$, $k \in \mathcal{N}$. We claim that, modulo a multiplicative constant, f_{ξ^*} is an $(n_1, \ldots, n_m)$ multiple Chebyshev polynomial for $(w_1, \ldots, w_m)$. Suppose to the contrary that this is not the case, and for some $j = 1, \ldots, m$ and for some polynomial p of degree $< n_j$ we have with some $\varepsilon > 0$

$$\|(f_{\xi^*} + p)w_j\| < (1-\varepsilon)^4 \|f_{\xi^*} w_j\|,$$

where $\|\cdot\| = \|\cdot\|_{[a,b]}$. Then for all large $k \in \mathcal{N}$ we also have

$$\|(f_{\xi^{(k)}} + p)w_j\| < (1-\varepsilon)^3 \|f_{\xi^{(k)}} w_j\|,$$

which implies

$$\|f_{\xi^{(k)}} + p\|_{L^{2k}(w_j)} \le \|(f_{\xi^{(k)}} + p)w_j\|(b-a)^{1/2k} < (1-\varepsilon)^2 \|f_{\xi^{(k)}} w_j\|, \qquad (3)$$

provided k is so large that $(b-a)^{1/2k} < 1/(1-\varepsilon)$. On the other hand, the family of functions

$$\{f_\xi w_j, \mid \xi \in S^n,\ 1 \le j \le m\}$$

is uniformly equicontinuous on $[a,b]$, hence there is a $\theta > 0$ such that

$$\left|\{x \in [a,b] \,\Big|\, |f_\xi(x) w_j(x)| > (1-\varepsilon)\|f_\xi w_j\|\}\right| \ge \theta, \qquad \xi \in S^n,\ 1 \le j \le m,$$

where $|\cdot|$ stands for the Lebesgue-measure. But then for all k

$$\|f_{\xi^{(k)}}\|_{L^{2k}(w_j)} \ge (1-\varepsilon)\|f_{\xi^{(k)}} w_j\| \theta^{1/2k} > (1-\varepsilon)^2 \|f_{\xi^{(k)}} w_j\| \qquad (4)$$

if k is so large that $\theta^{1/2k} > 1-\varepsilon$. Now for sufficiently large $k \in \mathcal{N}$ both (3) and (4) must be true. However, that contradicts (2), and this contradiction proves the claim that f_{ξ^*} becomes, after proper normalization (to have leading coefficient 1), an $(n_1, \ldots, n_m)$ multiple Chebyshev polynomial for the weights $(w_1, \ldots, w_m)$.

■

Next, we show that multiple Chebyshev polynomials of maximal $n_1 + \cdots + n_m$ degree may not exist.

Proposition 2 *There are two continuous weights w_1, w_2 such that both of them are positive on $(-3,3)$ and vanish outside that interval, and there is no $(1,1)$ multiple Chebyshev polynomial of degree 2 for the pair (w_1, w_2).*

Naturally, $[-3,3]$ could be replaced by any interval $[a,b]$.

Proof. *Part 1.* For some small $\varepsilon > 0$ ($\varepsilon < 1/1000$ certainly suffices) consider the intervals

$$I_{-2} = [-2, -2+\varepsilon], \quad I_{-1} = [-1, -1+\varepsilon], \quad I_1 = [1-\varepsilon, 1], I_2 = [2-\varepsilon, 2], \tag{5}$$

the sets $K_1 = I_{-1} \cup I_1$ and $K_2 = I_{-2} \cup I_2$, and let W_1 be equal to 1 on K_1 and W_2 equal to 1 on K_2 and both of them be zero elsewhere. We claim that there is no $(1,1)$-multiple Chebyshev polynomial of degree 2 for these weights.

Suppose to the contrary that $f(x) = x^2 + \alpha x + \beta$ is a $(1,1)$ multiple Chebyshev polynomial. Then it has a 2-point Chebyshev equioscillation system $x_1^{(j)} < x_2^{(j)}$ for the weight W_j, i.e. for $j = 1, 2$

- $x_1^{(j)}, x_2^{(j)} \in K_j$ and $f(x_1^{(j)}) = -f(x_2^{(j)})$,
- $|f(x_1^{(j)})| = \max_{x \in K_j} |f(x)|$.

Now we need to distinguish three cases.

Case I. $x_1^{(1)} \in I_{-1}$, $x_2^{(1)} \in I_1$. If $\alpha > 5$ then f is strictly increasing on $[-2, 2]$, so we must have $x_1^{(1)} = -1$ and $x_2^{(1)} = 1$. If $\alpha < -5$ then f is strictly decreasing on $[-2, 2]$, and we must have again $x_1^{(1)} = -1$ and $x_2^{(1)} = 1$. On the other hand, if $-5 \le \alpha \le 5$, then $f(-1) = f(x_1^{(1)}) + O(\varepsilon)$ and $f(1) = f(x_2^{(1)}) + O(\varepsilon)$, so in any case $f(-1) = -f(1) + O(\varepsilon)$, i.e. $1 - \alpha + \beta = -(1 + \alpha + \beta) + O(\varepsilon)$, which gives

$$\beta = -1 + O(\varepsilon). \tag{6}$$

In a similar manner, if $x_1^{(2)} \in I_{-2}$, $x_2^{(2)} \in I_2$, then $f(-2) = -f(2) + O(\varepsilon)$, i.e. $4 - 2\alpha + \beta = -(4 + 2\alpha + \beta) + O(\varepsilon)$ follows, and so

$$\beta = -4 + O(\varepsilon). \tag{7}$$

Since for small ε (6) and (7) contradict one another, we must have in the case considered that either $x_1^{(2)}, x_2^{(2)} \in I_{-2}$ or $x_1^{(2)}, x_2^{(2)} \in I_2$. If $x_1^{(2)}, x_2^{(2)} \in I_{-2}$, then f must have a zero in I_{-2}, and then to match (6), it must be of the form $f(x) = (x + 2 + O(\varepsilon))(x - \frac{1}{2} + O(\varepsilon))$. In this case $|f(x_1^{(2)})| = O(\varepsilon)$ while $f(2) = 6 + O(\varepsilon)$, so $x_1^{(2)}$ cannot be a point where $|f| = |f|W_2$ takes its maximum on K_2, which contradicts the definition of $x_1^{(2)}$.

In a similar manner, if $x_1^{(2)}, x_2^{(2)} \in I_2$ then f must have a zero in I_2, and then to match (6), it must be of the form $f(x) = (x - 2 + O(\varepsilon))(x + \frac{1}{2} + O(\varepsilon))$. Then again $|f(x_1^{(2)})| = O(\varepsilon)$ while $f(-2) = 6 + O(\varepsilon)$, which again contradicts the definition of $x_1^{(2)}$.

Case II. $x_1^{(2)} \in I_{-2}$, $x_2^{(2)} \in I_2$ and Case I does not hold. As we have seen above, in this case (7) is true, and we must have either $x_1^{(1)}, x_2^{(1)} \in I_{-1}$ or $x_1^{(1)}, x_2^{(1)} \in I_1$.

In the first case f must have a zero in I_{-1}, and then to match (7), it must be of the form $f(x) = (x + 1 + O(\varepsilon))(x - 4 + O(\varepsilon))$, which gives $|f(x_1^{(1)})| = O(\varepsilon)$ while $f(1) = -6 + O(\varepsilon)$, a contradiction. If $x_1^{(1)}, x_2^{(1)} \in I_1$ then f is of the form $f(x) = (x - 1 + O(\varepsilon))(x + 4 + O(\varepsilon))$, which gives $|f(x_1^{(1)})| = O(\varepsilon)$ while $f(-1) = -6 + O(\varepsilon)$, again a contradiction.

Thus, neither of the cases I or II is possible, so we must have

Case III. $x_1^{(2)}, x_2^{(2)}$ both belong either to I_{-2} or to I_2, and at the same time $x_1^{(1)}, x_2^{(1)}$ both belong either to I_{-1} or to I_1. However, this is also impossible:

- If $x_1^{(2)}, x_2^{(2)} \in I_{-2}$ and $x_1^{(1)}, x_2^{(1)} \in I_{-1}$, then $f(x) = (x + 1 + O(\varepsilon))(x + 2 + O(\varepsilon))$, which implies $|f(x_1^{(1)})| = O(\varepsilon)$, $f(1) = 6 + O(\varepsilon)$, a contradiction.
- If $x_1^{(2)}, x_2^{(2)} \in I_2$ and $x_1^{(1)}, x_2^{(1)} \in I_{-1}$, then $f(x) = (x + 1 + O(\varepsilon))(x - 2 + O(\varepsilon))$, which implies $|f(x_1^{(1)})| = O(\varepsilon)$, $f(1) = -2 + O(\varepsilon)$, a contradiction.
- If $x_1^{(2)}, x_2^{(2)} \in I_{-2}$ and $x_1^{(1)}, x_2^{(1)} \in I_1$, then $f(x) = (x - 1 + O(\varepsilon))(x + 2 + O(\varepsilon))$, which implies $|f(x_1^{(1)})| = O(\varepsilon)$, $f(-1) = -2 + O(\varepsilon)$, a contradiction.
- If $x_1^{(2)}, x_2^{(2)} \in I_2$ and $x_1^{(1)}, x_2^{(1)} \in I_1$, then $f(x) = (x - 1 + O(\varepsilon))(x - 2 + O(\varepsilon))$, which implies $|f(x_1^{(1)})| = O(\varepsilon)$, $f(-1) = 6 + O(\varepsilon)$, a contradiction.

This proves the claim of Part 1 that no $(1, 1)$ multiple Chebyshev polynomial of degree 2 exists for (W_1, W_2).

Part 2. Next, we extend W_1, W_2 from the sets K_1 and K_2 to continuous weights w_1, w_2 that are positive on $(-3, 3)$ and vanish outside that interval, in such a way that for any polynomial $f(x) = x^2 + \alpha x + \beta$ the norms $\|fw_1\|_{[-3,3]}$ and $\|fw_2\|_{[-3,3]}$ can be attained only on K_1, resp. K_2. That is easy, e.g. if $\|f\|_{K_1} = \|fW_1\|_{K_1} = M$, then, by Markov's inequality (see [2]) applied to the interval I_1, we get $|f'(x)| = |2x + \alpha| \le 8M/\varepsilon$ on I_1, so $|\alpha| \le 8M/\varepsilon + 2$, and $|f'(x)| \le 8M/\varepsilon + 11$ for all $x \in [-3, 3]$. As a consequence, for $x \in [-3, 3] \setminus I_1$ we have $|f(x)| \le M + (8M/\varepsilon + 11)\text{dist}(x, K_1)|$, and so if

$$w_1(x) < \frac{M}{M + (8M/\varepsilon + 11)\text{dist}(x, K_1)} \tag{8}$$

on $[-3, 3] \setminus I_1$ and $w_1(x) = 0$ outside $(-3, 3)$, then $|f(x)|w_1(x)$ attains its maximum M only on K_1. Now, by V. A. Markov's inequality (see [2]) for the second derivative on $I_s = I_1$ or $I_s = I_{-1}$ (depending where the maximum of $|f|$ occurs on K_1), we get that $2 = \|f''\|_{I_s} \le (4/\varepsilon^2)(4 \cdot 3/3)M$, i.e. $M \ge \varepsilon^2/8$. Since the right-hand side in (8) is monotone increasing in M, the inequality (8) certainly holds if

$$w_1(x) < \frac{\varepsilon^2/8}{\varepsilon^2/8 + (\varepsilon + 11)\text{dist}(x, K_1)}, \qquad x \in [-3, 3] \setminus I_1, \tag{9}$$

which can be easily achieved fulfilling at the same time the relations $w_1(x) > 0$ for $x \in (-3, 3)$ and $w_1(x) = 0$ for $x \notin (-3, 3)$. The extension of W_2 is similar.

Now since $|f|w_1$ can attain its maximal value only on K_1 and $|f|w_2$ can attain its maximal value only on K_2, a multiple $(1, 1)$ Chebyshev polynomial $f(x) = x^2 + \alpha x + \beta$ for the pair (W_1, W_2) would also be a multiple $(1, 1)$ Chebyshev polynomial for the pair (w_1, w_2), which is not the case as we have seen in Part 1.

■

The discussion so far shows that non-unicity of multiple Chebyshev polynomials and non-existence with maximal degree can happen when the smallest intervals containing the support of the different w_j's overlap. On the other hand, when the weights $w_1, \ldots, w_m$ are supported on disjoint intervals, then unicity easily follows. Indeed, suppose that $w_1, \ldots, w_m$ are zero outside some closed intervals $I_1, \ldots, I_m \subseteq [a, b]$ with pairwise disjoint interior. If P and Q are two $(n_1, \ldots, n_m)$-Chebyshev polynomials, then w_jP and w_jQ must have $n_j + 1$ Chebyshev equioscillations (of possibly different amplitudes for w_jP and for w_jQ) on I_j, therefore both P and Q must have n_j zeros inside I_j. Thus, P and Q both must be of maximal $n = n_1 + \cdots + n_m$ degree, which implies that $P - Q$ is of degree $< n$ (the highest terms cancel). Next, note that w_j must vanish at both endpoints of I_j, with the exception of a or b, i.e. if a or b belongs to I_j then w_j does not need to vanish at a or b. As a consequence, the points of equioscillations cannot include the endpoints of I_j except perhaps for a or b. To simplify the language below let us agree that when we say "inside I_j" then this means the interior of I_j except that if a or b belongs to I_j then we also include them in the interior. Now $P - Q$ also has n_j zeros "inside I_j". Indeed, this is clear if the amplitudes of equioscillations on I_j for w_jP and for w_jQ are different, and in these cases one gets n_j different zeros in the interior of I_j. When the amplitudes in question are the same, then, by the same argument, for any $\lambda < 1$ the polynomial $P - \lambda Q$ has n_j distinct zeros lying in the interior of I_j, and for $\lambda \to 1$ we get that $P - Q$ also has n_j (not necessarily distinct) zeros "inside I_j" counting multiplicity. This is true for all j and we get altogether $n_1 + \cdots + n_m = n$ zeros for $P - Q$. But $P - Q$, being of degree smaller than n, can have n zeros only if $P - Q \equiv 0$, which proves the unicity. We note that the disjoint interval case has also been settled by [4, Corollaries 3,4].

Finally, we prove that in the case just discussed ($w_1, \ldots, w_m$ are zero outside some closed intervals $I_1, \ldots, I_m$ with pairwise disjoint interior) also the existence of a multiple Chebyshev polynomial of maximal degree follows rather easily from Brower's fixed point theorem (note that this statement also follows from Theorem 1 and from the unicity proof just given, however the following direct and simple proof is rather instructive).

Set, as before, $n = n_1 + \cdots + n_m$. If $X_j = (x_1^{(j)}, \ldots, x_{n_j}^{(j)}) \in I_j^{n_j}$, $j = 1, \ldots, m$, then let

$$X := (X_1, \dots, X_m) = (x_1^{(1)}, \dots, x_{n_1}^{(1)}, x_1^{(2)}, \dots, x_{n_2}^{(2)}, \dots, x_1^{(m)}, \dots, x_{n_m}^{(m)})$$

be the vector in $\prod_{j=1}^m I_j^{n_j}$ which is obtained by listing the coordinates of $X_1, X_2, \dots, X_m$ one after the other in this order. Conversely, if $X = (x_1, \dots, x_n) \in \prod_{j=1}^m I_j^{n_j}$, then let $X_1 = (x_1, \dots, x_{n_1})$, $X_2 = (x_{n_1+1}, x_{n_1+2}, \dots, x_{n_1+n_2})$, etc., so that $X = (X_1, \dots, X_m)$. Also, for a vector $Y = (y_1, \dots, y_l)$ define

$$P_Y(x) = \prod_{s=1}^{l} (x - y_s).$$

For an $X \in \prod_{j=1}^m I_j^{n_j}$ consider the point $X' = (X_1', \dots, X_m') \in \prod_{j=1}^m I_j^{n_j}$, where X_j', $j = 1, \dots, m$, has, as its coordinates, the zeros—in increasing order—of the n_j-th classical weighted Chebyshev polynomial for the weight

$$W_j(x) = w_j(x) \prod_{s \neq j} |P_{X_s}(x)|.$$

This W_j is a nonnegative and not identically zero function on I_j, so, by the classical Chebyshev argument (which is valid for weights like W_j that may have zeros), there exists a polynomial $U_{n_j}(x) = x^{n_j} + \cdots$ which minimizes the weighted norm $\|W_j U_{n_j}\|_{I_j}$ among all polynomials $x^{n_j} + \cdots$. Again by the classical argument, this $W_j U_{n_j}$ must have a set of $n_j + 1$ Chebyshev equioscillations on I_j, which implies that U_{n_j} is unique. Thus, the X_j' consists of the zeros of U_{n_j} listed in increasing order. The unicity of U_{n_j} also implies its continuity: if W_j changes continuously, then so does U_{n_j} (this continuity claim is easy to prove, or see [3]). As a consequence, X_j' depends continuously on X.

In other words, $X \to X'$ is a continuous mapping of $\prod_{j=1}^m I_j^{n_j}$ into itself, therefore, by the Brower fixed point theorem, it has a fixed point: $X = X'$. But that means that each P_{X_j} is the n_j-th Chebyshev polynomial for the weight W_j. Now on I_j we have $W_j P_{X_j} \equiv w_j P_X$ or $W_j P_{X_j} \equiv -w_j P_X$ (all sign changes of $\prod_{s \neq j} P_{X_s}(x)$ are outside I_j), i.e., by the construction of the mapping $X \to X'$, the weighted polynomial $w_j P_X$ has an $(n_j + 1)$-equioscillation set on I_j, say

$$w_j P_X(x_s^{(n_j)}) = (-1)^{n_j+1-s} A, \qquad x_1^{(n_j)} < x_2^{(n_j)} < \ldots < x_{n_j+1}^{(n_j)}, \quad x_s^{(n_j)} \in I_j$$

with $A = \|w_j P_X\|_{[a,b]}$. Now if we had for some $1 \leq j \leq m$ and for some polynomial q of degree $< n_j$ the relation $\|w_j(P_X + q)\|_{[a,b]} < A$, then for $s = 1, \dots, n_j + 1$ the equality

$$\begin{aligned} \operatorname{sign}(w_j q(x_s^{(n_j)})) &= \operatorname{sign}\Big(w_j(P_X + q)(x_s^{(n_j)}) - w_j P_X(x_s^{(n_j)})\Big) \\ &= \operatorname{sign}\Big(-w_j P_X(x_s^{(n_j)})\Big) = (-1)^{n_j - s} \end{aligned}$$

would be true, which is not possible for a polynomial $q \not\equiv 0$ of degree $< n_j$. Hence, P_X is a multiple Chebyshev polynomial for $(w_1, \ldots, w_m)$ and $(n_1, \ldots, n_m)$ of maximal degree $n = n_1 + \cdots + n_m$.

■

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