FUNDAMENTAL PROPERTIES OF DIFFERENTIAL EQUATIONS WITH DYNAMICALLY DEFINED DELAYED FEEDBACK

Dedicated to Professor László Hatvani on the occasion of his 70th birthday.

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Abstract. We consider an initial value problem for nonautonomous functional differential equations where the delay term is defined via the solution of another system of differential equations. We obtain the general existence and uniqueness result for the initial value problem by showing a Lipschitz property of the dynamically defined delayed feedback function and give conditions for the nonnegativity of solutions. We determine the steady-state solution of the autonomous system and obtain the linearized equation about the equilibria. Our work was motivated by biological applications where the model setup leads to a system of differential equations with such dynamically defined delay terms. We present a simple model from population dynamics with fixed period of temporary separation and an epidemic model with long distance travel and entry screening.

Keywords: functional differential equations, dynamically defined delay, Lipschitz property, biological applications

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1 Introduction

The topic of delay differential equations concerns with a type of differential equations in which the derivative of the unknown function at a certain time is determined by the values of the function at previous times. Many studies focus on well known model equations where the delayed feedback function is given explicitly (Mackey-Glass equation, Nicholson's blowflies, Wright's equation etc.), while others only require some important properties for the feedback, for example monotonicity or unimodularity. However, the study of some models from population dynamics and epidemiology leads to differential equations where the delay terms arise as the solution of another system of differential equations. In this work, we consider initial value problems for differential equations with such dynamically defined delayed feedback. Our goal is to obtain fundamental properties of the system to ensure that the model equations coming from biological applications are meaningful.
The paper is organized as follows. In Section 2 we introduce the general form of systems of functional differential equations with dynamically defined delay term, and we prove the existence and uniqueness of the solution in Section 3. Due to possible biological interpretations, we give conditions so that the solution preserves nonnegativity. In Section 4 we detail the autonomous case, determine equilibria of the system and formulate the linearized equation, while Section 5 concerns with two examples of applications in population dynamics and the spread of infectious diseases with travel delay.

2 A system of differential equations with dynamically defined delayed feedback

Consider the initial-value problem for the nonautonomous functional differential equation

$$
\begin{aligned}
x'(t) &= \mathcal{F}(t, x_t), \\
x_\sigma &= \varphi,
\end{aligned}
\tag{1}
$$

where $x : \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{Z}_+$, $t, \sigma \in \mathbb{R}$ and $t \geq \sigma$. For $\tau > 0$, we define our phase space $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ as the Banach space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$, equipped with the usual supremum norm $||\cdot||$. Let $\varphi \in \mathcal{C}$ be the state of the system at σ. We use the notation $x_t \in \mathcal{C}$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$. Let $\mathcal{F} : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ and let $\mathcal{F}$ have the special form $\mathcal{F}(t, \phi) = f(t, \phi(0)) + W(t, \phi(-\tau))$ for $\phi \in \mathcal{C}$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.
In the sequel we use the notation $|v|_k$ for the Euclidean norm of any vector $v \in \mathbb{R}^k$ for $k \in \mathbb{Z}_+$. For $k = 1$ we omit lower index 1 for simplicity. We define a Lipschitz condition as follows. For $k, l \in \mathbb{Z}_+$, we say that a function $F : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^l$ satisfies the Lipschitz condition (*Lip*) on each bounded subset of $\mathbb{R} \times \mathbb{R}^k$ if:

(*Lip*) For all a, $b \in \mathbb{R}$ and $M > 0$, there is a $K(a, b, M) > 0$ such that:

$$|F(t, x_1) - F(t, x_2)|_l \leq K|x_1 - x_2|_k, \;\; a \leq t \leq b, \;\; |x_1|_k, |x_2|_k \leq M.$$

We assume that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies (*Lip*) on each bounded subset of $\mathbb{R} \times \mathbb{R}^n$. For the definition of W, we make the following preparations. For any $s_0 \in \mathbb{R}$ and $y_* \in \mathbb{R}^m$, $m \in \mathbb{Z}_+$, we consider the initial value problem

$$\begin{aligned} y'(s) &= g(s, y(s)), \\ y(s_0) &= y_*, \end{aligned} \tag{2}$$

where $y : \mathbb{R} \to \mathbb{R}^m$, $s, s_0 \in \mathbb{R}$, $s \geq s_0$, $g : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$, g is continuous on $\mathbb{R}\times\mathbb{R}^m$ and satisfies the Lipschitz condition (Lip) on each bounded subset of $\mathbb{R}\times\mathbb{R}^m$. The Picard-Lindelöf theorem (see Chapter II, Theorem 1.1 and Chapter V, Theorem 2.1 in [2]) states that as g is continuous on a parallelepiped $R : s_0 \leq s \leq s_0 + c, |y - y_*|_m \leq d$ with the bound B for $|g|_m$ on R and g possesses the Lipschitz property (Lip), there exists a unique solution of (2) $y(s; s_0, y_*)$ on the interval $[s_0, s_0 + \alpha]$ for $\alpha = \alpha_{s_0,y_*,c,d} := \min\{c, \frac{d}{B}\}$, and the solution continuously depends on the initial data. We make the following additional assumption:

($\star$) For every s_0 and y_*, the solution $y(s; s_0, y_*)$ of (2) exists at least for τ units of time, i.e. on $[s_0, s_0 + \tau]$.

Remark 2.1. *The reader may notice that ($\star$) is equivalent to the following assumption:*
For every s_0 and y_*, solution $y(s; s_0, y_*)$ exists for all $s \geq s_0$.

Remark 2.2. *With various conditions on g, we can guarantee that assumption ($\star$) is fulfilled. For instance, for any $s_0 \in \mathbb{R}$ and $L \in \mathbb{R}_+$, we define the constant $L_g = L_g(s_0, L)$ as the maximum of $|g|_m$ on the set $[s_0, s_0 + \tau] \times \{v \in \mathbb{R}^m : |v|_m \leq 2L\}$ (continuous functions attain their maximum on every compact set). Then, the condition that for every $s_0 \in \mathbb{R}$ and $L \in \mathbb{R}_+$ the inequailty*

$$\tau \leq \frac{L}{L_g} \tag{3}$$

holds immediately implies that ($\star$) is satisfied.
Indeed, for any s_0 and y_*, choose $c = \tau$, $d = |y_*|_m$. Then the Picard-Lindelöf theorem guarantees the existence and uniqueness of solution $y(s; s_0, y_*)$ on $[s_0, s_0 + \alpha]$ for $\alpha = \min\{\tau, \frac{|y_*|_m}{B}\}$, where B is the bound for $|g|_m$ on the parallelepiped $s_0 \leq s \leq s_0 + \tau, |y - y_*|_m \leq |y_*|_m$. Choosing $L = |y_*|_m$, it follows from the definition of $L_g(s_0, L)$ that $B \leq L_g$ is satisfied, and using (3) we get $\tau \leq \frac{|y_*|_m}{L_g} \leq \frac{|y_*|_m}{B}$. We conclude that $\alpha = \tau$, hence solution $y(s; s_0, y_*)$ exists on $[s_0, s_0 + \tau]$, ($\star$) is satisfied.

The less restrictive condition $\kappa := \inf_{s_0,L} \frac{L}{L_g} > 0$ implies the existence of the solution of (2) on $[s_0, s_0 + \kappa]$ for any s_0. Then it follows that for any s_0 the solution exists for all $s \geq s_0$, which is equivalent to ($\star$). If we assume that a global Lipschitz condition (gLip) holds for g, that is, the Lipschitz constant for g in (Lip) can be chosen independently of a, b and M, then for any s_0 and y_* the solution of (2) exists for all $s \geq s_0$, thus also for τ units of time.

Now we are ready for the definition of W. For $h : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$, $k : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^n$, let us assume that h and k are continuous and satisfy the Lipschitz condition (Lip). For simplicity, we use the notation $y_{s_0,v}(s) = y(s; s_0, h(s_0, v))$ for the unique solution of system (2) in the case $y_* = h(s_0, v)$, $v \in \mathbb{R}^n$. We define $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$W(s, v) := k(s, y_{s-\tau,v}(s)) = k(s, y(s; s - \tau, h(s - \tau, v))). \tag{4}$$

3 Basic properties

Our goal is to prove the usual existence and uniqueness theorem for (1). First we obtain the following simple results.

Proposition 3.1. *$\mathcal{F}$ is continuous on $\mathbb{R} \times \mathcal{C}$.*

Proof. The Picard-Lindelöf theorem and ($\star$) guarantee that for every s_0, y_*, there exists a unique solution of system (2) on the interval $[s_0, s_0 + \tau]$ and the solution $y(s; s_0, y_*)$ continuously depends on the initial data. Moreover, h and k are continuous which implies the continuity of W. The function f is also continuous, hence we conclude that $\mathcal{F}$ is continuous on $\mathbb{R} \times \mathcal{C}$. □

Proposition 3.2. *For any $c, d \in \mathbb{R}$ such that $c < d$ and for any $L \in \mathbb{R}_+$, there exists a bound $J = J(c, d, L)$ such that for any $s_0 \in [c, d]$ and for any $y_* \in \mathbb{R}^m$ such that $|y_*|_m \leq L$, the inequality*

$$|y(s; s_0, y_*)|_m \leq J$$

holds for $s \in [s_0, s_0 + \tau]$.

Proof. The Picard-Lindelöf theorem and ($\star$) guarantee that for every $s_0 \in \mathbb{R}$ and $y_* \in \mathbb{R}^m$, there exists a unique solution $y(s; s_0, y_*)$ of system (2) on the interval $[s_0, s_0 + \tau]$, and the solution continuously depends on the initial data. Thus, for any $c, d \in \mathbb{R}$ where $c < d$ and for any $L \in \mathbb{R}_+$, the solution $y(s; s_0, y_*)$ as an $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m$ variable function is continuous on the set $\{(s_1, s_2, v) : s_1 \in [c, d + \tau], s_2 \in [c, d], s_1 \geq s_2, |v|_m \leq L\}$. Continuous functions reach their maximum on any compact set, i.e. there exists a constant $J(c, d, L)$ such that $|y(s; s_0, y_*)|_m \leq J$. The proof is complete. □

Now we show that besides continuity, $\mathcal{F}$ also satisfies a Lipschitz condition on each bounded subset of $\mathbb{R} \times \mathcal{C}$:

($Lip^{\mathcal{C}}$) For all a, $b \in \mathbb{R}$ and $M > 0$, there is a $K(a, b, M) > 0$ such that:

$$|f(t, \phi) - f(t, \psi)|_n \leq K||\phi - \psi||, \; a \leq t \leq b, \; ||\phi||, ||\psi|| \leq M.$$

Lemma 3.3. *$\mathcal{F}$ satisfies the Lipschitz condition ($Lip^{\mathcal{C}}$) on each bounded subset of $\mathbb{R} \times \mathcal{C}$.*

Proof. Fix constants a, b and M, $a < b$, $M > 0$. Our aim is to find $K(a, b, M)$. Due to the continuity of h, there exists a constant $L_h(a, b, M)$ such that for any $||\psi|| \leq M$ and $s_0 \in [a - \tau, b - \tau]$, the inequality $|h(s_0, \psi(-\tau))|_m \leq L_h$ holds. By choosing $c = a - \tau$, $d = b - \tau$, $L = L_h$ and $y_* = h(s_0, \psi(-\tau))$ it follows from Proposition 3.2 that for any $s_0 \in [a - \tau, b - \tau]$, the inequality $|y_{s_0,\psi(-\tau)}(s)|_m \leq J(a, b, L_h)$ holds for $s \in [s_0, s_0 + \tau]$.

Let $K_h = K_h(a, b, M)$ be the Lipschitz constant of h on the set $[a-\tau, b-\tau] \times \{v \in \mathbb{R}^n : |v|_n \leq M\}$, let $K_g = K_g(a, b, M)$ be the Lipschitz constant of g on the set $[a - \tau, b] \times \{v \in \mathbb{R}^m : |v|_m \leq J\}$ (note that J is determined by a, b and M). For any

$||\phi||, ||\psi|| \leq M$ it holds that $|\phi(-\tau)|_n, |\psi(-\tau)|_n \leq M$. Since the solution of (2) can be expressed as $y(s; s_0, y_*) = y_* + \int_{s_0}^{s} g(r, y(r; s_0, y_*))dr$, for any $s_0 \in [a-\tau, b-\tau]$ we have

$$
\begin{aligned}
|y_{s_0,\phi(-\tau)}(s) - y_{s_0,\psi(-\tau)}(s)|_m &= \bigg| h(s_0, \phi(-\tau)) + \int_{s_0}^{s} g(r, y_{s_0,\phi(-\tau)}(r))dr \\
&\quad - \left(h(s_0, \psi(-\tau)) + \int_{s_0}^{s} g(r, y_{s_0,\psi(-\tau)}(r))dr \right) \bigg|_m \\
&\leq |h(s_0, \phi(-\tau)) - h(s_0, \psi(-\tau))|_m \\
&\quad + \int_{s_0}^{s} |g(r, y_{s_0,\phi(-\tau)}(r)) - g(r, y_{s_0,\psi(-\tau)}(r))|_m dr \\
&\leq K_h ||\phi - \psi|| \\
&\quad + \int_{s_0}^{s} K_g |y_{s_0,\phi(-\tau)}(r) - y_{s_0,\psi(-\tau)}(r)|_m dr
\end{aligned}
\tag{5}
$$

for $s \in [s_0, s_0 + \tau]$. For a given $s_0 \in [a-\tau, b-\tau]$ we define

$$\Gamma(s) = |y_{s_0,\phi(-\tau)}(s) - y_{s_0,\psi(-\tau)}(s)|_m$$

for $s \in [s_0, s_0 + \tau]$. Then (5) gives

$$\Gamma(s) \leq K_h ||\phi - \psi|| + K_g \int_{s_0}^{s} \Gamma(r)dr,$$

and from Gronwall's inequality we have that for any $s_0 \in [a-\tau, b-\tau]$

$$\Gamma(s) \leq K_h ||\phi - \psi|| e^{K_g(s-s_0)} \tag{6}$$

holds for $s \in [s_0, s_0 + \tau]$.
For any $t \in [a, b]$, it is satisfied that $t - \tau \in [a-\tau, b-\tau]$, hence for $s \in [t-\tau, t]$ we obtain

$$|y_{t-\tau,\phi(-\tau)}(s) - y_{t-\tau,\psi(-\tau)}(s)|_m \leq K_h ||\phi - \psi|| e^{K_g(s-(t-\tau))} \tag{7}$$

as a special case of (6) with $s_0 = t - \tau$. The constant $J = J(a, b, L_h)$ was defined as the bound for $|y_{s_0,\psi(-\tau)}(s)|_m$ for any $s_0 \in [a-\tau, b-\tau]$, $||\psi|| \leq M$, $s \in [s_0, s_0+\tau]$. For any $t \in [a, b]$, it follows that $t-\tau \in [a-\tau, b-\tau]$, hence the inequality $|y_{t-\tau,\psi(-\tau)}(t)|_m \leq J$ holds for any $||\psi|| \leq M$. Let $K_k = K_k(a, b, M)$ be the Lipschitz constant of k on the set $[a, b] \times \{v \in \mathbb{R}^m : |v|_m \leq J\}$. Then for any $t \in [a, b]$, $||\phi||, ||\psi|| \leq M$ it holds that $|\phi(-\tau)|_n, |\psi(-\tau)|_n \leq M$, so we arrive to the following inequality:

$$
\begin{aligned}
|W(t, \phi(-\tau)) - W(t, \psi(-\tau))|_n &= |k(t, y_{t-\tau,\phi(-\tau)}(t)) - k(t, y_{t-\tau,\psi(-\tau)}(t))|_n \\
&\leq K_k |y_{t-\tau,\phi(-\tau)}(t) - y_{t-\tau,\psi(-\tau)}(t)|_m \\
&\leq K_k K_h ||\phi - \psi|| e^{K_g \tau},
\end{aligned}
$$

where we used (4) and (7).

Finally, let $K_f(a, b, M)$ be the Lipschitz constant of f on the set $[a, b] \times \{v \in \mathbb{R}^n : |v|_n \leq M\}$. Then for any $t \in [a, b]$ and for any $||\phi||, ||\psi|| \leq M$, $|\phi(0)|_n$, $|\psi(0)|_n$, $|\phi(-\tau)|_n$, $|\psi(-\tau)|_n \leq M$ holds and thus

$$
\begin{aligned}
|\mathcal{F}(t, \phi) - \mathcal{F}(t, \psi)|_n &\leq |f(t, \phi(0)) - f(t, \psi(0))|_n + |W(t, \phi(-\tau)) - W(t, \psi(-\tau))|_n \\
&\leq K_f ||\phi - \psi|| + K_k K_h ||\phi - \psi|| e^{K_g \tau},
\end{aligned}
$$

hence it is clear that $K_f(a,b,M) + K_k(a,b,M)K_h(a,b,M)e^{\tau K_g(a,b,M)}$ is a suitable choice for $K(a,b,M)$, the Lipschitz constant of $\mathcal{F}$ for the set $[a,b] \times \{\psi \in \mathcal{C} : ||\psi|| \leq M\}$. □

We state the following simple remark.

Remark 3.4. *If f, g, h and k satisfy a global Lipschitz condition (gLip), that is, if K_f, K_g, K_h and K_k in the definition of the Lipschitz condition (Lip) can be chosen independent of a, b and M, then a global Lipschitz condition ($gLip^{\mathcal{C}}$) for $\mathcal{F}$ arises, i.e. there exists a Lipschitz constant K of $\mathcal{F}$ which is independent of a, b and M.*

Now, as we have proved that $\mathcal{F}$ is continuous and satisfies the Lipschitz condition ($Lip^{\mathcal{C}}$), all conditions of Theorem 3.7 in [4] are satisfied. We arrive to the following result.

Theorem 3.5. *Let $\sigma \in \mathbb{R}$, $M > 0$. There exists $A > 0$, depending only on M such that if $\phi \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ satisfies $||\phi|| \leq M$, then there exists a unique solution $x(t) = x(t;\sigma,\phi)$ of (1), defined on $[\sigma-\tau, \sigma+A]$. In addition, if K is the Lipschitz constant for $\mathcal{F}$ corresponding to $[\sigma, \sigma+A]$ and M, then*

$$\max_{\sigma-\tau\leq\eta\leq\sigma+A} |x(\eta;\sigma,\phi) - x(\eta;\sigma,\psi)|_n \leq ||\phi-\psi||e^{KA} \text{ for any } ||\phi||, ||\psi|| \leq M.$$

Assuming stronger conditions on f, g, h and k, we arrive to a more general existence result. We follow Remark 3.8 in [4].

Remark 3.6. *If f, g, h and k satisfy condition (gLip), then condition ($gLip^{\mathcal{C}}$) arises for $\mathcal{F}$ and we do not need to make any restrictions on A in Theorem 3.5. More precisely, its statements hold for all $A > 0$. In this case, the solution exists for all $t \geq \sigma$ and the inequality*

$$||x_t(\phi) - x_t(\psi)|| \leq ||\phi - \psi||e^{K(t-\sigma)}$$

holds for all $t \geq \sigma$.

Most functional differential equations that arise in population dynamics or epidemiology deal only with nonnegative quantities. Therefore it is important to see what conditions ensure that nonnegative initial data give rise to nonnegative solution.

We reformulate (1) using the definition of $\mathcal{F}$. Since $\mathcal{F}(t, x_t) = f(t, x(t)) + W(t, x(t-\tau))$, we consider the following differential equation system, which is equivalent to (1):

$$\begin{aligned} x'(t) &= f(t, x(t)) + W(t, x(t-\tau)), \\ x_\sigma &= \varphi. \end{aligned} \tag{8}$$

We claim that under reasonable assumptions the solution of system (8) preserves nonnegativity for nonnegative initial data. Let us suppose that for each $t \in \mathbb{R}$, h and k map nonnegative vectors to nonnegative vectors. We also assume that for every $i \in \{1, \ldots n\}$, $j \in \{1, \ldots m\}$, $u \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m_+$ and $t, s \in \mathbb{R}$, $u_i = 0$ implies

$f_i(t,u) \geq 0$ and $w_j = 0$ implies $g_j(s,w) \geq 0$. Then for nonnegative initial value the solution of system (2) is nonnegative, which implies that for every $i \in \{1, \dots n\}$, $v \in \mathbb{R}_+^n$ and $t \in \mathbb{R}$, the inequality $(k(t, y(t; t-\tau, h(t-\tau, v))))_i = W_i(t,v) \geq 0$ holds. Hence $f_i(t,u) + W_i(t,v) \geq 0$ is satisfied for $u, v \in \mathbb{R}_+^n$, $u_i = 0$, $t \in \mathbb{R}$, all conditions of Theorem 3.4 in [4] hold and we conclude that nonnegative initial data give rise to nonnegative solution of system (8). Clearly, systems (8) and (1) are equivalent, which implies that the result automatically holds for system (1). We summarize our assumptions and consequence.

Proposition 3.7. *Suppose that $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and $k : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ map nonnegative vectors to nonnegative vectors for each $t \in \mathbb{R}$, moreover assume that*

$$\forall i, t, \forall u \in \mathbb{R}_+^n : u_i = 0 \Rightarrow f_i(t,u) \geq 0,$$
$$\forall j, s, \forall w \in \mathbb{R}_+^m : w_j = 0 \Rightarrow g_j(s,w) \geq 0.$$

Then for nonnegative initial data the solution of system (1) preserves nonnegativity i.e. $x(t) \geq 0$ for $t \geq \sigma$ where it is defined.

4 The autonomous case

4.1 Fundamental properties

As a special case of system (1), we may derive similar results for the autonomous system. Let $x : \mathbb{R} \to \mathbb{R}^n$, $y : \mathbb{R} \to \mathbb{R}^m$, $t, s \in \mathbb{R}$, let $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $k : \mathbb{R}^m \to \mathbb{R}^n$. Let us assume that f, g, h and k satisfy the Lipschitz condition (Lip), which can be stated as follows. For $k, l \in \mathbb{Z}_+$, we say that a function $F : \mathbb{R}^k \to \mathbb{R}^l$ satisfies the Lipschitz condition (Lip) if for all $M > 0$ there is a $K(M) > 0$ such that for $|x_1|_k, |x_2|_k \leq M$ the inequality $|F(x_1) - F(x_2)|_l \leq K|x_1 - x_2|_k$ holds. There is no need to assume the continuity for f, g, h and k since these functions are independent of t and hence this property follows from the Lipschitz condition (Lip). For $\tau > 0$, let $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ be the phase space as we defined it in Section 2. Then system (1) has the form

$$\begin{aligned} x'(t) &= \mathcal{F}(x_t), \\ x_0 &= \varphi, \end{aligned} \tag{9}$$

where $t \geq 0$, $\varphi \in \mathcal{C}$ is the state of the system at $t = 0$, $\mathcal{F} : \mathcal{C} \to \mathbb{R}^n$ and $\mathcal{F}$ has the special form $\mathcal{F}(\phi) = f(\phi(0)) + W(\phi(-\tau))$, $\phi \in \mathcal{C}$. For any $y_* \in \mathbb{R}^m$, system (2) turns into

$$\begin{aligned} y'(s) &= g(y(s)), \\ y(0) &= y_*, \end{aligned} \tag{10}$$

where $s \geq 0$. Similarly as in Section 2, the Picard-Lindelöf theorem guarantees the existence and uniqueness of the solution of system (10) on $[0, \alpha]$ for some $\alpha > 0$. We make the following additional assumption:

($\star\star$) For every y_*, solution $y(s; 0, y_*)$ of (10) exists at least for τ units of time.

This is equivalent to the assumption that $y(s; 0, y_*)$ exists on $[0, \infty)$ for every y_*, which holds if g satisfies $(gLip)$ (see Remark 2.2). We use the notation $y_{0,v}(s) = y(s; 0, h(v))$ for the unique solution of system (10) in the case $y_* = h(v)$, and we define $W : \mathbb{R}^n \to \mathbb{R}^n$ by

$$W(v) := k(y_{0,v}(\tau)) = k(y(\tau; 0, h(v))), \tag{11}$$

where $v \in \mathbb{R}^n$. It is straightforward that the Lipschitz condition $(Lip^{\mathcal{C}})$ and therefore the continuity hold for $\mathcal{F}$, since this function is only a special case of the $\mathcal{F}$ defined in Section 2. Moreover if we assume that f, g, h and k satisfy the global Lipschitz condition $(gLip)$, then we obtain that condition $(gLip^{\mathcal{C}})$ holds for $\mathcal{F}$ (for the definitions of $(gLip)$ and $(gLip^{\mathcal{C}})$ see Remark 3.4). As an immediate consequence of Theorem 3.5, we state the following corollary.

Corollary 4.1. *Suppose that $M > 0$. There exists $A > 0$, depending only on M such that if $\phi \in \mathcal{C}$ satisfies $||\phi|| \leq M$, then there exists a unique solution $x(t) = x(t; 0, \phi)$ of (9), defined on $[-\tau, A]$. In addition, if K is the Lipschitz constant for $\mathcal{F}$ corresponding to M, then*

$$\max_{-\tau \leq \eta \leq A} |x(\eta; 0, \phi) - x(\eta; 0, \psi)|_n \leq ||\phi - \psi|| e^{KA} \text{ for any } ||\phi||, ||\psi|| \leq M.$$

The following remark rises automatically as the autonomous case of Remark 3.6.

Remark 4.2. *If f, g, h and k satisfy the global Lipschitz condition (gLip), then we do not need to make any restrictions on A in Corollary 4.1. More precisely, its statements hold for all $A > 0$. In this case, the solution exists for all $t \geq 0$ and the inequality*

$$||x_t(\phi) - x_t(\psi)|| \leq ||\phi - \psi|| e^{Kt}$$

holds for all $t \geq 0$.

Clearly, we can adapt Proposition 3.7 to the autonomous system with similar conditions.

Corollary 4.3. *Suppose that $h : \mathbb{R}^n \to \mathbb{R}^m$ and $k : \mathbb{R}^m \to \mathbb{R}^n$ map nonnegative vectors to nonnegative vectors, moreover assume that*

$$\forall i, \forall u \in \mathbb{R}^n_+ : u_i = 0 \Rightarrow f_i(u) \geq 0,$$
$$\forall j, \forall w \in \mathbb{R}^m_+ : w_j = 0 \Rightarrow g_j(w) \geq 0.$$

Then for nonnegative initial data the solution of system (9) preserves nonnegativity i.e. $x(t) \geq 0$ for $t \geq 0$ where it is defined.

4.2 Equilibria and linearization

Consider the nonlinear functional differential equation system (9):

$$x'(t) = \mathcal{F}(x_t),$$

where $\mathcal{F}(\phi) = f(\phi(0)) + W(\phi(-\tau))$ for $\phi \in \mathcal{C}$. Then $x(t) = \bar{x} \in \mathbb{R}^n$ is a steady-state solution of (9) if and only if $\mathcal{F}(\hat{\bar{x}}) = 0$, where $\hat{\bar{x}} \in \mathcal{C}$ is the constant function equal to

$\bar{x}$. Suppose there exists such an equilibrium. We formulate the linearized equation system about the equilibrium $\bar{x}$. We obtain the linear system

$$z'(t) = D\mathcal{F}(\hat{\bar{x}})z_t, \tag{12}$$

where $D\mathcal{F}(\hat{\bar{x}}) : \mathcal{C} \to \mathbb{R}^n$ is a bounded linear operator and $z : \mathbb{R} \to \mathbb{R}^n$. Due to the special form of $\mathcal{F}$, (12) can be written as

$$z'(t) = A_1 z(t) + A_2 z(t - \tau),$$

where $A_1 = Df(\bar{x}) \in \mathbb{R}^{n\times n}$ and $A_2 = DW(\bar{x}) \in \mathbb{R}^{n\times n}$.

Proposition 4.4. *Let us suppose that g, h and k are continuously differentiable. Then the matrix $DW(\bar{x})$ can be represented with g, h and k as follows:*

$$DW(\bar{x}) = Dk(y(\tau; 0, h(\bar{x})))e^{\int_0^\tau Dg(y(r;0,h(\bar{x})))dr} Dh(\bar{x}).$$

Proof. Theorem 3.3 in Chapter I in [1] states that as g has continuous first derivative, the solution $y(s; 0, y_*)$ of system (10) is continuously differentiable with respect to s and y_* on its domain of definition. The matrix $\frac{\partial y(s;0,y_*)}{\partial y_*} \in \mathbb{R}^{m\times m}$ satisfies the linear variational equation

$$Y'(s) = Dg(y(s; 0, y_*))Y(s) \tag{13}$$

where $Y : \mathbb{R} \to \mathbb{R}^{m\times m}$ (we use slightly different notations as [1]) and $\frac{\partial y(0;0,y_*)}{\partial y_*} = I$, where I denotes the identity. As from (13) it follows that $Y(s) = e^{\int_0^s Dg(y(r;0,y_*))dr} Y(0)$, for $Y(0) = I$ we conclude that for $s = \tau$

$$\frac{\partial y(\tau; 0, y_*)}{\partial y_*} = e^{\int_0^\tau Dg(y(r;0,y_*))dr} \tag{14}$$

holds. From (11) it follows that for $v \in \mathbb{R}^n$

$$\begin{aligned} DW(v) &= Dk(y(\tau; 0, h(v)))\frac{\partial y(\tau; 0, h(v))}{\partial v} \\ &= Dk(y(\tau; 0, h(v)))\frac{\partial y(\tau; 0, h(v))}{\partial y_*} Dh(v), \end{aligned} \tag{15}$$

hence from (14) and (15) we have

$$DW(v) = Dk(y(\tau; 0, h(v)))e^{\int_0^\tau Dg(y(r;0,h(v)))dr} Dh(v). \tag{16}$$

Finally, setting $v = \bar{x}$ in (16) we arrive to the equality

$$DW(\bar{x}) = Dk(y(\tau; 0, h(\bar{x})))e^{\int_0^\tau Dg(y(r;0,h(\bar{x})))dr} Dh(\bar{x}).$$

Note that $Dk(y(\tau; 0, h(\bar{x}))) \in \mathbb{R}^{n\times m}$, $Dg(y(r; 0, h(\bar{x}))) \in \mathbb{R}^{m\times m}$ and $Dh(\bar{x}) \in \mathbb{R}^{m\times n}$, hence the result of the matrix multiplication is indeed $DW(\bar{x}) \in \mathbb{R}^{n\times n}$. The proof is complete. □

It follows from (11) that $\bar{x}$ satisfies the equation $-f(\bar{x}) = k(y(\tau; 0, h(\bar{x})))$. However, $\bar{x}$ being a steady-state solution of (9) does not necessarily imply that $y(s, 0; h(\bar{x})) = h(\bar{x})$ for $s \in [0, \tau]$ i.e. $h(\bar{x})$ is an equilibrium of (10). It is easy to construct examples for f, g, h and k such that such situation occurs.
We say that $\bar{x} \in \mathbb{R}^n$ is a total equilibrium of systems (9) and (10) if $x(t) = \bar{x}$ is a steady-state solution of (9) and $y(s) = h(\bar{x})$ is a steady-state solution of (10). The equilibrium solution $y(s) = \bar{y}$, $\bar{y} \in \mathbb{R}^m$ of (10) satisfies the equation $g(\bar{y}) = 0$, and since $h(\bar{x}) = \bar{y}$ and $-f(\bar{x}) = k(y(\tau; 0, h(\bar{x})))$ should hold for the total equilibrium, we conclude that $\bar{x}$ arises as the solution of the system

$$\begin{aligned} -f(\bar{x}) &= k(h(\bar{x})), \\ g(h(\bar{x})) &= 0. \end{aligned} \tag{17}$$

It follows from (17) that in the special case when f and g are invertable functions, the total equilibrium can be expressed by $\bar{x} = f^{-1}(-k(g^{-1}(0)))$, and we also obtain that $\bar{y} = h(f^{-1}(-k(g^{-1}(0))))$.
We remark that if functions g, h and k are continuously differentiable and $\bar{x}$ is the total equilibrium of systems (9) and (10), then it follows from Proposition 4.4 that the matrix $DW(\bar{x})$ has the form $DW(\bar{x}) = Dk(h(\bar{x}))e^{\tau Dg(h(\bar{x}))}Dh(\bar{x})$.

5 Applications

5.1 A basic model from population dynamics

A simple model describing the growth of a single population with fixed period of temporary separation is given by

$$\begin{aligned} n'(t) &= b(n(t)) - d(n(t)) - q(n(t)) + V(n(t-\tau)), \\ n_0 &= \varphi, \end{aligned} \tag{18}$$

where t denotes time and functions b, d and q stand for recruitment, mortality and temporary separation (e.g. migration). Let $\tau > 0$ be the fixed period of separation. We define the phase space $\mathcal{C}_+$ as the nonnegative cone of $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R})$, let $\varphi \in \mathcal{C}_+$. We assume that $b, d, q : \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition (Lip) on each bounded subset of $\mathbb{R}$, which implies their continuity on $\mathbb{R}$. Since b, d and q denote the recruitment, mortality and separation functions, it should hold that they map nonnegative values to nonnegative values. Function V expresses the inflow of individuals arriving to the population at time t after τ units of time of separation. For the thorough definition of V, the growth of the separated population needs describing. We assume that individuals who left the population due to separation in different times do not make contact to each other. Hence for each time t_*, the evolution of the density of the separated population with respect to the time elapsed since the beginning of separation is given by the following differential equation, when separation started at time t_*:

$$\begin{aligned} \frac{d}{ds}m(s; t_*) &= b^S(m(s; t_*)) - d^S(m(s; t_*)), \\ m(0; t_*) &= q(n(t_*)), \end{aligned} \tag{19}$$

where s denotes the time elapsed since the beginning of separation and functions b^S and d^S stand for recruitment and mortality during separation. At $s = 0$, the density of the separated population is determined by the number of individuals who start separation at time t_*, hence the initial value for system (19) is given by $m(0; t_*) = q(n(t_*))$. We assume that b^S,$d^S : \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition (Lip) on each bounded subset of $\mathbb{R}$, this also means they are continuous on $\mathbb{R}$. The Picard-Lindelöf theorem ensures that for any initial value m_* there exists a unique solution $y(s; 0, m_*)$ of system (19) on $[0, \alpha]$ for some $\alpha > 0$. We make the additional assumption that the unique solution exists at least for τ units of time for every m_*, we have seen in Section 2 that this assumption can be fulfilled with some conditions on b^S and d^S. In order to guarantee that nonnegative initial data give rise to nonnegative solution of (19), we assume that the inequality $b^S(0) - d^S(0) \geq 0$ holds, this condition can be satisfied with many reasonable choices of the recruitment and mortality functions. We assumed that separation lasts exactly for τ units of time, i.e. the feedback in (18) at $t_* + \tau$ is determined by the solution of (19) at $s = \tau$. We define the feedback function $V : \mathbb{R} \to \mathbb{R}$ as $V(v) := y(\tau; 0, q(v))$.

For $x : [0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, let $x(t) = n(t)$, $f(x) = b(x) - d(x) - q(x)$. For a given t_*, define $y(s) = m(s; t_*)$ and let $g(y) = b^S(y) - d^S(y)$, where $y : [0, \tau] \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$. Furthermore, for $h, k : \mathbb{R} \to \mathbb{R}$ let $h(v) = q(v)$, $k(v) = v$. Then system (18) can be written in a closed form as (9) and for each t_* (10) is a compact form of (19).
Clearly, functions f, g, h and k satisfy the Lipschitz condition (Lip) on each bounded subset of $\mathbb{R}$, moreover $(\star\star)$ also holds by assumption. Hence $\mathcal{F}$, defined by $\mathcal{F}(\phi) = f(\phi(0)) + W(\phi(-\tau))$ for $\phi \in \mathcal{C}_+$ satisfies the Lipschitz condition $(Lip^{\mathcal{C}})$, so Corollary 4.1 states that system (18) has a unique solution defined on $[-\tau, A]$ for some $A > 0$. By assuming that condition $(gLip)$ holds for b, d, q, b^S and d^S, we get that f, g, h and k satisfy the global Lipschitz condition $(gLip)$ and $A = \infty$. We have assumed that $q = h$ maps nonnegative values to nonnegative values, which obviously holds for k as well, moreover we gave the condition $b^S(0) - d^S(0) \geq 0$. In addition, if we suppose that $b(0) - d(0) - q(0) \geq 0$ is satisfied (e.g. $b(0) = d(0) = q(0) = 0$ holds in many models), then Corollary 4.3 implies that for nonnegative initial data the solution of system (18) preserves nonnegativity, that is, $\mathcal{C}_+$ is invariant.

5.2 Epidemic model with travel delay and entry screening

We formulate a dynamic model describing the spread of an infectious disease in two regions, and also during travel from one region to the other. We assume that the time required to complete travel between the regions is not neglectable, this leads to delay differential equations in the model setup. Several recent works (see e.g. [3] and [5]) considered SIS type transportation models where the delay terms arise explicitely. Here we present an epidemic model where delay is defined via the solution of another system of differential equations.
We divide the entire populations of the two regions into the disjoint classes S_1, I_1, R_1, J_1, S_2, I_2, R_2 and J_2. Lower index denotes the current region, letters S and R represent the compartments of susceptible and recovered individuals. We assume that individuals are traveling between the regions and travelers are requested to undergo an entry screening procedure before entering a region after travel. The purpose of the examination is to detect travelers who are infected with the disease and isolate them in order to minimize the chances of an infected agent spreading the

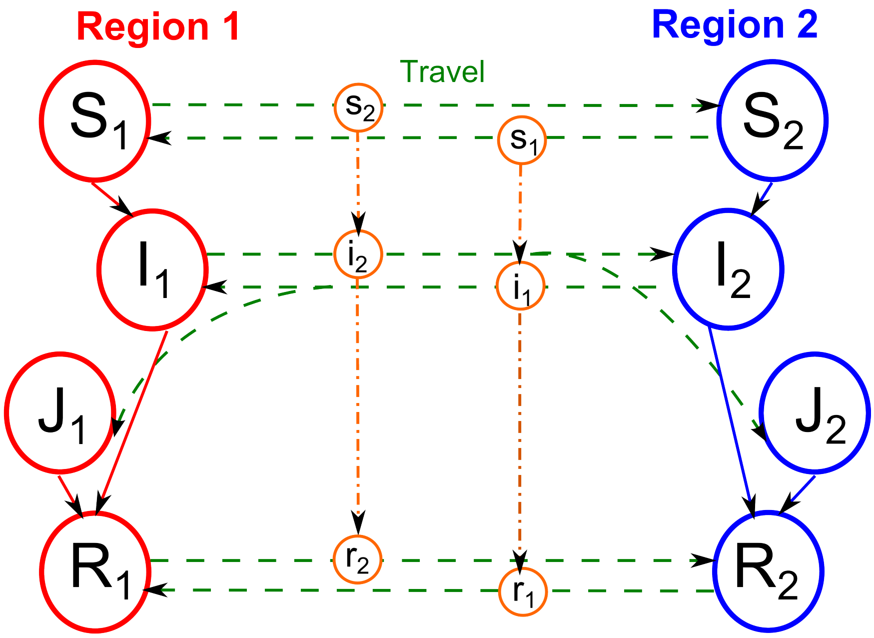

Figure 1: Flow chart of disease transmission and travel dynamics. The disease transmission in the two regions is shown in two different columns, the disease progresses vertically from the top to the bottom (solid arrows). Dashed arrows represent that individuals are traveling, dot-dashed arrows show the dynamics of the disease spread during the course of the travel.

infection in a disease free region. Such interventions were proven to have significant effect in mitigating the severity of epidemic outbreaks. Some individuals who have been infected with the disease get screened out by the arrival to a region and they become isolated and belong to class J. Others whose infection remained hidden by the examination or those who are sick but do not travel are in class I and we simply call them infecteds. Let $S_1(t)$, $I_1(t)$, $R_1(t)$, $J_1(t)$, $S_2(t)$, $I_2(t)$, $R_2(t)$ and $J_2(t)$ be the number of individuals belonging to S_1, I_1, R_1, J_1, S_2, I_2, R_2 and J_2 at time t, respectively. Susceptible, infected and recovered individuals of region 1 travel to region 2 by travel rate μ_1. The travel rate of individuals in classes S_2, I_2 and R_2 from region 2 to region 1 is denoted by μ_2. Isolated individuals are not allowed to travel, moreover we assume that they do not make contact with individuals of other classes until they recover. Model parameters γ_1 and γ_2 represent the recovery rate of infected and isolated individuals in region 1 and region 2, we denote the transmission rates in region 1 and region 2 by β_1 and β_2.

Let s_1, i_1, r_1, s_2, i_2 and r_2 be the classes of susceptible, infected and recovered individuals during travel to region 1 and to region 2. We denote the recovery rate of traveling infecteds by γ^T, they can transmit the disease by rate β^T during travel. Let $\tau > 0$ denote the time required to complete a one-way travel, which is assumed to be fixed. To describe the disease dynamics during travel, for each t_* we define $s_1(s;t_*)$, $i_1(s;t_*)$, $r_1(s;t_*)$, $s_2(s;t_*)$, $i_2(s;t_*)$ and $r_2(s;t_*)$ as the density of individuals with respect to s who started travel at time t_* and belong to classes s_1, i_1, r_1, s_2, i_2 and r_2, where $s \in [0, \tau]$ denotes the time elapsed since the beginning of the travel. The total density of traveling individuals is constant during the travel started at t_*,

Key model parameters	
β_1, β_2	transmission rate in region 1 and in region 2
μ_1, μ_2	traveling rate of individuals in region 1 and in region 2
γ_1, γ_2	recovery rate of infected and isolated individuals in region 1 and in region 2
p_1, p_2	probability of screening out infected travelers arriving to region 1 and to region 2
τ	duration of travel between the regions
β^T	transmission rate during travel
γ^T	recovery rate during travel

Table 1: Key model parameters

that is,

$$\begin{aligned} s_1(s;t_*) + i_1(s;t_*) + r_1(s;t_*) &= s_1(0;t_*) + i_1(0;t_*) + r_1(0;t_*), \\ s_2(s;t_*) + i_2(s;t_*) + r_2(s;t_*) &= s_2(0;t_*) + i_2(0;t_*) + r_2(0;t_*) \end{aligned}$$

for all $s \in [0, \tau]$. Choosing $s = \tau$, $t_* = t - \tau$, terms $s_1(\tau; t-\tau)$, $r_1(\tau; t-\tau)$ and $s_2(\tau; t-\tau)$, $r_2(\tau; t-\tau)$ express the inflow of susceptible and recovered individuals arriving to region 1 to compartments S_1, R_1 and region 2 to compartments S_2, R_2 at time t, respectively. We assume that travelers undergo an examination by the arrival to region 1 and 2, which detects infection by infecteds with probability $0 < p_1, p_2 < 1$. This implies that the densities $p_1 i_1(\tau; t-\tau)$ and $p_2 i_2(\tau; t-\tau)$ determine individuals who enter J_1 and J_2 at time t since p_1 and p_2 are the probabilities that infected travelers get screened out by the arrival. However, infected travelers enter classes I_1 and I_2 with probabilities $1 - p_1$ and $1 - p_2$ by the arrival, hence $(1 - p_1)i_1(\tau; t-\tau)$ and $(1 - p_2)i_2(\tau; t-\tau)$ give the inflow to classes I_1 and I_2 at time t.

The flow chart of the model is depicted in Figure 1, see Table 1 for the key model parameters. We obtain the following system of differential equations for the disease spread in the regions, where disease transmission is modeled by standard incidence:

$$\begin{aligned} S_1'(t) &= -\beta_1 \frac{S_1(t)I_1(t)}{S_1(t)+I_1(t)+R_1(t)} - \mu_1 S_1(t) + s_1(\tau; t-\tau), \\ I_1'(t) &= \beta_1 \frac{S_1(t)I_1(t)}{S_1(t)+I_1(t)+R_1(t)} - \gamma_1 I_1(t) - \mu_1 I_1(t) + (1-p_1)i_1(\tau; t-\tau), \\ R_1'(t) &= \gamma_1(I_1(t)+J_1(t)) - \mu_1 R_1(t) + r_1(\tau; t-\tau), \\ J_1'(t) &= -\gamma_1 J_1(t) + p_1 i_1(\tau; t-\tau), \\ S_2'(t) &= -\beta_2 \frac{S_2(t)I_2(t)}{S_2(t)+I_2(t)+R_2(t)} - \mu_2 S_2(t) + s_2(\tau; t-\tau), \\ I_2'(t) &= \beta_2 \frac{S_2(t)I_2(t)}{S_2(t)+I_2(t)+R_2(t)} - \gamma_2 I_2(t) - \mu_2 I_2(t) + (1-p_2)i_2(\tau; t-\tau), \\ R_2'(t) &= \gamma_2(I_2(t)+J_2(t)) - \mu_2 R_2(t) + r_2(\tau; t-\tau), \\ J_2'(t) &= -\gamma_2 J_2(t) + p_2 i_2(\tau; t-\tau). \end{aligned} \tag{20}$$

For each t_*, the following system describes the evolution of the densities during the travel which started at time t_*:

$$
\begin{aligned}
&\frac{d}{ds}s_1(s;t_*) = -\beta^T \frac{s_1(s;t_*)i_1(s;t_*)}{s_1(s;t_*)+i_1(s;t_*)+r_1(s;t_*)},\\
&\frac{d}{ds}i_1(s;t_*) = \beta^T \frac{s_1(s;t_*)i_1(s;t_*)}{s_1(s;t_*)+i_1(s;t_*)+r_1(s;t_*)} - \gamma^T i_1(s;t_*),\\
&\frac{d}{ds}r_1(s;t_*) = \gamma^T i_1(s;t_*),\\
&\frac{d}{ds}s_2(s;t_*) = -\beta^T \frac{s_2(s;t_*)i_2(s;t_*)}{s_2(s;t_*)+i_2(s;t_*)+r_2(s;t_*)},\\
&\frac{d}{ds}i_2(s;t_*) = \beta^T \frac{s_2(s;t_*)i_2(s;t_*)}{s_2(s;t_*)+i_2(s;t_*)+r_2(s;t_*)} - \gamma^T i_2(s;t_*),\\
&\frac{d}{ds}r_2(s;t_*) = \gamma^T i_2(s;t_*),
\end{aligned}
\tag{21}
$$

where again we assume standard incidence for the disease transmission. Note that the dimensions of systems (20) and (21) are different.

For $s=0$, the densities are determined by the rates individuals start their travels from one region to the other at time t_*. Hence, the initial values for system (21) at $s=0$ are given by

$$
\begin{cases}
s_1(0;t_*) = \mu_2 S_2(t_*), & s_2(0;t_*) = \mu_1 S_1(t_*),\\
i_1(0;t_*) = \mu_2 I_2(t_*), & i_2(0;t_*) = \mu_1 I_1(t_*),\\
r_1(0;t_*) = \mu_2 R_2(t_*), & r_2(0;t_*) = \mu_1 R_1(t_*).
\end{cases}
\tag{22}
$$

Now we turn our attention to the terms $s_1(\tau;t-\tau)$, $(1-p_1)i_1(\tau;t-\tau)$, $r_1(\tau;t-\tau)$, $p_1 i_1(\tau;t-\tau)$, $s_2(\tau;t-\tau)$, $(1-p_2)i_2(\tau;t-\tau)$, $r_2(\tau;t-\tau)$ and $p_2 i_2(\tau;t-\tau)$ in system (20), which are the densities of individuals arriving to classes S_1, I_1, R_1, J_1, S_2, I_2, R_2 and J_2 at time t, respectively. At time t, these terms are determined by the solution of system (21) with initial values (22) for $t_* = t-\tau$ at $s=\tau$. An individual may move to a different compartment during travel, for example a susceptible individual who travels from region 1 may arrive as infected to region 2, as given by the dynamics of system (21).

Next we specify initial values for system (20) at $t=0$. Since travel takes τ units of time to complete, arrivals to region 1 are determined by the state of classes S_2, I_2 and R_2 at $t-\tau$ and vice versa, via the solution of systems (21) and (22). Thus, we set up the initial functions as follows:

$$
\begin{cases}
S_1(\theta) = \varphi_{S,1}(\theta), & S_2(\theta) = \varphi_{S,2}(\theta),\\
I_1(\theta) = \varphi_{I,1}(\theta), & I_2(\theta) = \varphi_{I,2}(\theta),\\
R_1(\theta) = \varphi_{R,1}(\theta), & R_2(\theta) = \varphi_{R,2}(\theta),\\
J_1(0) = \varphi_{J,1}(\theta), & J_2(0) = \varphi_{J,2}(\theta),
\end{cases}
\tag{23}
$$

where $\theta \in [-\tau, 0]$ and for each $j \in \{1,2\}, K \in \{S,I,R,J\}$, $\varphi_{K,j}$ is continuous.

For $x : [0,\infty) \to \mathbb{R}^8$ and $f : \mathbb{R}^8 \to \mathbb{R}^8$, define $x(t) = (S_1(t), I_1(t), R_1(t), J_1(t), S_2(t), I_2(t), R_2(t), J_2(t))^T$ and define $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T$. For a given t_*, let $y(s) = (s_1(s;t_*), i_1(s;t_*), r_1(s;t_*), s_2(s;t_*), i_2(s;t_*), r_2(s;t_*))^T$ and let $g =$

$(g_1, g_2, g_3, g_4, g_5, g_6)^T$, where $y : [0, \tau] \to \mathbb{R}^6$ and $g : \mathbb{R}^6 \to \mathbb{R}^6$. Define g as $g_i(y)$ equals the right hand side of the equation for y_i in system (21). For instance,

$$g_5(y) = \beta^T \frac{y_4 y_5}{y_4 + y_5 + y_6} - \gamma^T y_5.$$

Then for each t_*, (10) is a compact form of (21) with the initial value y_* in (22) for $m = 6$. Define $h = (h_1, h_2, h_3, h_4, h_5, h_6) : \mathbb{R}^8 \to \mathbb{R}^6$ as

$$\begin{cases} h_1(v) = \mu_2 v_5, & h_4(v) = \mu_1 v_1, \\ h_2(v) = \mu_2 v_6, & h_5(v) = \mu_1 v_2, \\ h_3(v) = \mu_2 v_7, & h_6(v) = \mu_1 v_3, \end{cases}$$

let $k = (k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8) : \mathbb{R}^6 \to \mathbb{R}^8$ be given as

$$\begin{cases} k_1(v) = v_1, & k_5(v) = v_4, \\ k_2(v) = (1 - p_1)v_2, & k_6(v) = (1 - p_2)v_5, \\ k_3(v) = v_3, & k_7(v) = v_6, \\ k_4(v) = p_1 v_2, & k_8(v) = p_2 v_5. \end{cases}$$

The feasible phase space is the nonnegative cone $\mathcal{C}_+$ of $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^8)$. Define $f_i(x)$ to be the right hand side of the equation of x_i in (20) without the inflow from travel. For instance,

$$f_1(x) = -\beta_1 \frac{x_1 x_2}{x_1 + x_2 + x_3} - \mu_1 x_1.$$

Clearly our system (20) with initial conditions (23) can be written in a closed form as (9) for $n = 8$.
Our aim is to show that there exists a unique positive solution of system (20), moreover nonnegative initial data give rise to nonnegative solution. We showed in the previous sections that these results can be obtained by assuming certain conditions on f, g, h and k. Now we check if these conditions hold for the f, g, h and k defined for the SIRJ model. It is not hard to see that h and k possess the global Lipschitz condition (*gLip*), now we prove that it holds for f and g as well.

Proposition 5.1. *Functions f and g, as defined for the SIRJ model, satisfy the global Lipschitz condition (gLip) on each bounded subset of $\mathbb{R}^8_+$ and $\mathbb{R}^6_+$.*

Proof. Due to the similarities in the definitions of f and g, it is sufficient to prove the condition only for one of them, e.g. for f. The function $f : \mathbb{R}^8 \to \mathbb{R}^8$ possesses the global Lipschitz condition (*gLip*) if there exists a Lipschitz constant $K > 0$ such that $|f(\mathbf{p}) - f(\mathbf{q})|_8 \leq K|\mathbf{p} - \mathbf{q}|_8$ holds for any $\mathbf{p}, \mathbf{q} \in \mathbb{R}^8_+$. First, we show that there exists a $K_1 > 0$ such that $|f_1(\mathbf{p}) - f_1(\mathbf{q})| \leq K_1|\mathbf{p} - \mathbf{q}|_8$. For $\mathbf{p}, \mathbf{q} \in \mathbb{R}^8_+$, it holds that

$$\begin{aligned}
|f_1(\mathbf{p}) - f_1(\mathbf{q})| &= \left| -\beta_1 \frac{p_1 p_2}{p_1 + p_2 + p_3} - \mu_1 p_1 + \beta_1 \frac{q_1 q_2}{q_1 + q_2 + q_3} + \mu_1 q_1 \right| \\
&\leq \mu_1 |q_1 - p_1| + \beta_1 \left| \frac{q_1 q_2}{q_1 + q_2 + q_3} - \frac{p_1 p_2}{p_1 + p_2 + p_3} \right| \\
&= \mu_1 |q_1 - p_1| + \beta_1 \left| \frac{q_1 q_2}{q_1 + q_2 + q_3} - \frac{q_1 p_2}{q_1 + q_2 + q_3} \right. \\
&+ \frac{q_1 p_2}{q_1 + q_2 + q_3} - \frac{q_1 p_2}{q_1 + p_2 + q_3} + \frac{q_1 p_2}{q_1 + p_2 + q_3} - \frac{q_1 p_2}{q_1 + p_2 + p_3} \\
&\left. + \frac{q_1 p_2}{q_1 + p_2 + p_3} - \frac{q_1 p_2}{p_1 + p_2 + p_3} + \frac{q_1 p_2}{p_1 + p_2 + p_3} - \frac{p_1 p_2}{p_1 + p_2 + p_3} \right| \\
&\leq \mu_1 |q_1 - p_1| + \beta_1 \left(\left| \frac{q_1 q_2}{q_1 + q_2 + q_3} - \frac{q_1 p_2}{q_1 + q_2 + q_3} \right| \right. \\
&+ \left| \frac{q_1 p_2}{q_1 + q_2 + q_3} - \frac{q_1 p_2}{q_1 + p_2 + q_3} \right| + \left| \frac{q_1 p_2}{q_1 + p_2 + q_3} - \frac{q_1 p_2}{q_1 + p_2 + p_3} \right| \\
&\left. + \left| \frac{q_1 p_2}{q_1 + p_2 + p_3} - \frac{q_1 p_2}{p_1 + p_2 + p_3} \right| + \left| \frac{q_1 p_2}{p_1 + p_2 + p_3} - \frac{p_1 p_2}{p_1 + p_2 + p_3} \right| \right) \\
&= \mu_1 |q_1 - p_1| + \beta_1 \left(|q_2 - p_2| \left| \frac{q_1}{q_1 + q_2 + q_3} \right| + |p_2 - q_2| \cdot \right. \\
&\left| \frac{q_1 p_2}{(q_1 + q_2 + q_3)(q_1 + p_2 + q_3)} \right| + |p_3 - q_3| \left| \frac{q_1 p_2}{(q_1 + p_2 + q_3)(q_1 + p_2 + p_3)} \right| \\
&\left. + |p_1 - q_1| \left| \frac{q_1 p_2}{(q_1 + p_2 + p_3)(p_1 + p_2 + p_3)} \right| + |q_1 - p_1| \left| \frac{p_2}{p_1 + p_2 + p_3} \right| \right) \\
&\leq \mu_1 |q_1 - p_1| + \beta_1 \left(2|q_2 - p_2| + |p_3 - q_3| + 2|p_1 - q_1| \right) \\
&\leq (\mu_1 + 5\beta_1)|\mathbf{q} - \mathbf{p}|_8 \\
&= K_1 |\mathbf{q} - \mathbf{p}|_8,
\end{aligned}$$

where we used that the inequality $\frac{a}{a+b+c} \leq 1$ holds for any $a, b, c \in \mathbb{R}_+$. We define $K_1 = \mu_1 + 5\beta_1$, similarly we obtain that

$$\begin{aligned}
|f_5(\mathbf{p}) - f_5(\mathbf{q})| &= \left| -\beta_2 \frac{p_5 p_6}{p_5 + p_6 + p_7} - \mu_2 p_5 + \beta_2 \frac{q_5 q_6}{q_5 + q_6 + q_7} + \mu_2 q_5 \right| \\
&\leq K_5 |\mathbf{q} - \mathbf{p}|_8
\end{aligned}$$

for $K_5 = \mu_2 + 5\beta_2$. Furthermore,

$$\begin{aligned}
|f_2(\mathbf{p}) - f_2(\mathbf{q})| &= \left| \beta_1 \frac{p_1 p_2}{p_1 + p_2 + p_3} - \gamma_1 p_2 - \mu_1 p_2 - \beta_1 \frac{q_1 q_2}{q_1 + q_2 + q_3} + \gamma_1 q_2 + \mu_1 q_2 \right| \\
&\leq (\mu_1 + \gamma_1)|q_2 - p_2| + \beta_1 \left(2|q_2 - p_2| + |p_3 - q_3| + 2|q_1 - p_1| \right) \\
&\leq (\mu_1 + \gamma_1 + 5\beta_1)|\mathbf{q} - \mathbf{p}|_8 \\
&= K_2 |\mathbf{q} - \mathbf{p}|_8,
\end{aligned}$$

where $\mu_1 + \gamma_1 + 5\beta_1$ is a suitable choice for K_2. Clearly by choosing $K_6 = \mu_2 + \gamma_2 + 5\beta_2$, we can derive the inequality

$$|f_6(\mathbf{p}) - f_6(\mathbf{q})| = \left|\beta_2 \frac{p_5 p_6}{p_5 + p_6 + p_7} - \gamma_2 p_6 - \mu_2 p_6 - \beta_2 \frac{q_5 q_6}{q_5 + q_6 + q_7} + \gamma_2 q_6 + \mu_2 q_6\right|$$
$$\leq K_6 |\mathbf{q} - \mathbf{p}|_8.$$

Moreover,

$$\begin{aligned}|f_3(\mathbf{p}) - f_3(\mathbf{q})| &= |\gamma_1(p_2 + p_4) - \mu_1 p_3 - \gamma_1(q_2 + q_4) + \mu_1 q_3| \\ &\leq (2\gamma_1 + \mu_1)|\mathbf{q} - \mathbf{p}|_8 \\ &= K_3 |\mathbf{q} - \mathbf{p}|_8\end{aligned}$$

for $K_3 = 2\gamma_1 + \mu_1$ and

$$\begin{aligned}|f_7(\mathbf{p}) - f_7(\mathbf{q})| &= |\gamma_2(p_6 + p_8) - \mu_2 p_7 - \gamma_2(q_6 + q_8) + \mu_2 q_7| \\ &\leq K_7 |\mathbf{q} - \mathbf{p}|_8\end{aligned}$$

with the choice of $K_7 = 2\gamma_2 + \mu_2$, and finally the inequalities

$$\begin{aligned}|f_4(\mathbf{p}) - f_4(\mathbf{q})| &= | - \gamma_1 p_4 + \gamma_1 q_4| \\ &\leq \gamma_1 |\mathbf{q} - \mathbf{p}|_8 \\ &= K_4 |\mathbf{q} - \mathbf{p}|_8,\end{aligned}$$

$$\begin{aligned}|f_8(\mathbf{p}) - f_8(\mathbf{q})| &= | - \gamma_2 p_8 + \gamma_2 q_8| \\ &\leq \gamma_2 |\mathbf{q} - \mathbf{p}|_8 \\ &= K_8 |\mathbf{q} - \mathbf{p}|_8\end{aligned}$$

hold for $K_4 = \gamma_1$, $K_8 = \gamma_2$. To obtain the global Lipschitz constant for f, we simply choose $K = \sqrt{K_1^2 + K_2^2 + K_3^2 + K_4^2 + K_5^2 + K_6^2 + K_7^2 + K_8^2}$. The proof is complete. □

Function g, defined for the SIRJ model, possesses the global Lipschitz condition ($gLip$) on $\mathbb{R}^6_+$, it is also obvious that for nonnegative initial data system (21) preserves nonnegativity. We conclude that for nonnegative initial data there exists a unique nonnegative solution of system (21) on $[0, \infty)$. The global Lipschitz property ($gLip$) of f, h and k are also satisfied, hence Corollary 4.1 and Remark 4.2 state that there exists a unique solution of (20) on $[-\tau, \infty)$ with initial conditions (23). Clearly h and k map nonnegative vectors to nonnegative vectors and the nonnegativity conditions of Corollary 4.3 also hold, thus nonnegative initial data give rise to a nonnegative solution of (20), which means that $\mathcal{C}_+$ is invariant.

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