Asymptotics of Christoffel functions on arcs and curves

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Contents

Abstract

For a system of smooth Jordan curves and arcs asymptotics for Christoffel functions is established. A separate new method is developed to handle the upper and lower estimates. In the course to the upper bound a theorem of Widom on the norm of Chebyshev polynomials is generalized.

1 Results

Let ν be a finite Borel measure on the plane with compact support consisting of infinitely many points. The Christoffel functions associated with ν are defined as

$$\lambda_n(z,\nu) = \inf_{P_n(z)=1} \int |P_n|^2 d\nu,$$

where the infimum is taken for all polynomials of degree at most n that take the value 1 at z.

Christoffel functions are closely related to orthogonal polynomials (for a survey see [13] by P. Nevai and [21] by B. Simon), to statistical physics (see e.g. [15] by L. Pastur), to universality in random matrix theory (see e.g. the recent breakthrough [10] by D. Lubinsky, as well as [2],[22],[24]), to spectral theory (see e.g. [20], [21] by B. Simon and [1] by Breuer, Last and Simon) and to several other fields in mathematics. For the role and various use of Christoffel functions see [4], [6], [20], and particularly [13] by P. Nevai and [21] by B. Simon.

In this paper we consider asymptotics of Christoffel functions on smooth ($C^{1+\alpha}$) Jordan curves and arcs. Recall that a Jordan curve is the homeomorphic image of the unit circle while a Jordan arc is the homeomorphic image of $[-1,1]$. Thus, a Jordan arc has two endpoints. The asymptotics of Christoffel functions on C^2-Jordan curves was established in [25] with a systematic use of polynomial inverse images of the unit circle (lemniscates). The idea of that paper was that many things can be carried over to lemniscates from the unit circle, and a system of C^2 Jordan curves can be well approximated (in a very specific sense) by lemniscates both from the inside and from the outside. This method does not work for arcs, and, in fact, except for the case when the set is a subset of the real line, no result has been known regarding Christoffel function asymptotics for arcs. In this paper we develop a method that handles both Jordan curves and arcs. We emphasize that we need a new method (actually very different ones) for both the upper and lower estimates, for previous methods do not work in either cases.

Thus, let Γ be the union of finitely many $C^{1+\alpha}$, $\alpha > 0$, smooth Jordan curves and arcs lying exterior to one another, and let $s_\Gamma = s$ be the arc measure on Γ. Let Γ_k, $k = 0,1,\ldots,k_0$ be the disjoint components of Γ: $\Gamma = \cup_{k=0}^{k_0}\Gamma_k$. We call those Γ_k that are Jordan arcs the arc-components of Γ. Since we need $C^{1+\alpha}$ smoothness just to have higher smoothness than C^1, we may and shall always assume $0 < \alpha < 1$.

Our main theorem is

Theorem 1.1 *Let Γ be a system of $C^{1+\alpha}$-smooth Jordan arcs and curves lying exterior to one another, let $z_0 \in \Gamma$ be a point on Γ that is different from the endpoints of the arc components of Γ, and assume that Γ is C^2-smooth in a neighborhood of z_0. Assume that $d\nu = w ds_\Gamma$ is a measure on Γ with density w (with respect to the arc measure s_Γ) which is continuous on Γ and positive*

s_Γ-almost everywhere. Then

$$\lim_{n\to\infty} n\lambda_n(z_0,\nu) = \frac{d\nu(z_0)}{d\mu_\Gamma}, \tag{1.1}$$

where μ_Γ denotes the equilibrium measure of Γ, and on the right-hand side $d\nu(z)/d\mu_\Gamma$ is the Radon-Nikodym derivative of ν with respect to μ_Γ.

For the concepts from potential theory (like equilibrium measure, logarithmic capacity, Greens' function etc.) see e.g. [5], [9], [17], or [19].

In the case that we are considering the equilibrium measure μ_Γ is absolutely continuous with respect to the arc measure s_Γ on Γ: $d\mu_\Gamma(t) = \omega_\Gamma(t)ds_\Gamma(t)$ with a C^α-continuous density function ω_Γ (see Proposition 2.2), and with it (1.1) takes the form

$$\lim_{n\to\infty} n\lambda_n(z_0,\nu) = \frac{w(z_0)}{\omega_\Gamma(z_0)}. \tag{1.2}$$

The $C^{1+\alpha}$-smoothness could be replaced by piecewise $C^{1+\alpha}$-smoothness without cusps, in which case Γ could have corners, and then the result is claimed for z_0 which is not an endpoint or a corner (at endpoints and at corners the order of the Christoffel function is no longer $1/n$, see [28]).

The global positivity and continuity of w was assumed only to have an easy formulation, the proof actually gives a much more general result. To this end we recall the class **Reg** from [23]: a measure ν with support Γ is said to be in the **Reg** class if

$$\lim_{n\to\infty}\left(\sup_{P_n}\frac{\|P_n\|_\Gamma}{\|P_n\|_{L^2(\nu)}}\right)^{1/n} = 1, \tag{1.3}$$

where the supremum is taken for all polynomials of degree at most n, and where $\|P_n\|_\Gamma$ stands for the supremum norm of P_n on Γ. This is not the standard definition of the **Reg** class (which is in terms of the leading coefficients of orthogonal polynomials), but it is equivalent to it, see [23, Theorem 3.4.3,(v)]. See [23] for several other equivalent formulations and for general criteria implying $\nu \in$ **Reg**. We only mention here that $\nu \in$ **Reg** is a very weak global assumption on ν, e.g. it holds if $d\nu(t)/ds_\Gamma(t) > 0$ s_Γ-almost everywhere. Therefore, Theorem 1.1 is a special case of

Theorem 1.2 *Let Γ be a system of $C^{1+\alpha}$-smooth Jordan arcs and curves lying exterior to one another, let $z_0 \in \Gamma$ be different from the endpoints of the arc components of Γ and assume that Γ is C^2-smooth in a neighborhood of z_0. Assume that $d\nu = w ds_\Gamma + d\nu_{\rm sing}$ is a measure on Γ with density w and with singular part $\nu_{\rm sing}$ (with respect to the arc measure s_Γ) which is in the* **Reg** *class. Then, if w is continuous at z_0 and z_0 is a Lebesgue-point for $\nu_{\rm sing}$, we have*

$$\lim_{n\to\infty} n\lambda_n(z_0,\nu) = \frac{w(z_0)}{\omega_\Gamma(z_0)}, \tag{1.4}$$

where ω_Γ denotes the density of the equilibrium measure μ_Γ (with respect to s_Γ).

The Lebesgue-point property of $\nu_{\rm sing}$ mentioned in the statement is

$$\nu_{\rm sing}(\{\zeta \mid |\zeta - z_0| \le \tau\}) = o(\tau) \qquad \text{as } \tau \to 0. \tag{1.5}$$

Let us mention that some kind of global condition like $\nu \in \mathbf{Reg}$ is needed, e.g. if ν vanishes on a subarc of Γ, then (1.4) is necessarily false because then

$$\liminf_{n\to\infty} n\lambda_n(z_0, \nu) > \frac{w(z_0)}{\omega_\Gamma(z_0)}. \tag{1.6}$$

We shall give a detailed proof for Theorem 1.1, the proof of Theorem 1.2 follows by simple changes. During the proof of the upper estimate in Theorem 1.1 we shall also verify (see Proposition 2.4)

Theorem 1.3 *Let Γ be a finite union of disjoint $C^{1+\alpha}$ Jordan curves and arcs. Then there is a constant C and for every $n = 1, 2, \dots$ there are monic polynomials $P_n(z) = z^n + \cdots$ of degree n such that*

$$\|P_n\|_\Gamma \le C \mathrm{cap}(\Gamma)^n,$$

where $\mathrm{cap}(\Gamma)$ *denotes the logarithmic capacity of* Γ.

This should be compared to the fact (see e.g. [17, Theorem 5.5.4]) that for any n and monic polynomial $P_n(z) = z^n + \cdots$ we have

$$\|P_n\|_\Gamma \ge \mathrm{cap}(\Gamma)^n.$$

Thus, the theorem says that on unions of smooth curves and arcs this theoretical lower bound can be achieved for every n disregarding a constant factor. For $C^{2+\alpha}$ curves and arcs this follows from deep results of Widom [30]. Let us also mention that if there are at least two components, or Γ is a single smooth arc, then the better estimate

$$\|P_n\|_\Gamma = (1 + o(1))\mathrm{cap}(\Gamma)^n$$

is impossible for all n ([27], [26]). It is a delicate problem (connected with simultaneous Diphantine approximation of the harmonic measures of the components of Γ) how close one can get by the norm of monic polynomials of degree n to the theoretical lower bound $\mathrm{cap}(\Gamma)^n$, see the papers [26] and [30].

First we shall deal with Theorem 1.1 in the special case when w is continuous and positive on Γ. The general case of Theorem 1.1 will follow from this via a simple argument. The proof of the upper and lower estimates are distinctively different. The upper estimate will be obtained by a careful discretization of the equilibrium measure. That part of the proof holds at every Lebesgue-point of ν (Lebesgue-point with respect to arc-measure) and the local C^2 property is not needed there. The lower estimate will be reduced to the case when there are no arc-components of Γ.

2 Upper estimate for Christoffel functions

In this section we establish that

$$\limsup_{n\to\infty} n\lambda_n(z_0,\nu) \le \frac{d\nu(z_0)}{d\mu_\Gamma}. \tag{2.1}$$

We need the concept of Lebesgue-point of a measure on Γ. Thus, let ν be a Borel-measure on Γ and $d\nu(t) = w(t)ds(t) + d\nu_{\rm sing}$ its decomposition into its absolutely continuous and singular parts with respect to arc measure $s = s_\Gamma$. We say that $z_0 \in \Gamma$, which is not an endpoint of an arc-component of Γ, is a Lebesgue-point for ν (with respect to arc measure) if for every $\varepsilon > 0$ there is a $\rho > 0$ such that if $0 \le \tau \le \rho$ then

$$\int_{|\zeta - z_0|\le\tau} |w(\zeta) - w(z_0)|ds(\zeta) \le \varepsilon\tau \tag{2.2}$$

and

$$\nu_{\rm sing}(\{\zeta\,|\,|\zeta - z_0| \le \tau\}) \le \varepsilon\tau. \tag{2.3}$$

Since the derivative of $\nu_{\rm sing}$ with respect to s_Γ is 0 s_Γ-almost everywhere (see [18, Theorem 7.13]), standard proof shows that s_Γ-almost every point is a Lebesgue-point for ν.

The main theorem of this part of the paper is

Theorem 2.1 *Let Γ be a finite union of disjoint $C^{1+\alpha}$ Jordan curves or arcs lying exterior to one another, and ν a Borel measure on Γ. If $z_0 \in \Gamma$ is not an endpoint of an arc-component of Γ and z_0 is a Lebesgue-point (with respect to arc measure s_Γ) of ν, then*

$$\limsup_{n\to\infty} n\lambda_n(\nu, z_0) \le \frac{d\nu(z_0)}{d\mu_\Gamma}.$$

For the proof of Theorem 2.1, let, as before, Γ_k be the disjoint components of Γ with Γ_0 being the one containing z_0. There is a change in the argument when Γ_0 is a Jordan arc as opposed to the case when it is a Jordan curve. First we consider the latter case, and return to the arc case after we have presented the proof for curves.

2.1 Part I: Γ_0 is a Jordan curve

Without loss of generality we may assume that $z_0 = 0$ and that the real line is the tangent line to Γ at 0. Then in a neighborhood of 0 the curve Γ_0 has a parametrization $t + i\gamma(t)$ with $\gamma' \in C^\alpha$ and $\gamma(0) = 0$, $\gamma'(0) = 0$; hence $|\gamma'(t)| \le C|t|^\alpha$, $|\gamma(t)| \le C|t|^{1+\alpha}$.

Let $\theta_k = \mu_\Gamma(\Gamma_k)$, and for an n consider the integers $n_k = [\theta_k n]$. Divide each Γ_k into n_k arcs I_j^k, (for each k the number of such j's is n_k), each having equal weight θ_k/n_k with respect to μ_Γ, i.e. $\mu_\Gamma(I_j^k) = \theta_k/n_k$. Then

$$\left|\frac{\theta_k}{n_k} - \frac{1}{n}\right| = |\mu_\Gamma(I_j^k) - 1/n| \leq C/n^2. \tag{2.4}$$

Let

$$\xi_j^k = \frac{1}{\mu_\Gamma(I_j^k)} \int_{I_j^k} u \, d\mu_\Gamma(u) \tag{2.5}$$

be the center of mass with respect to μ_Γ. Simple argument shows that on Γ_0 we can choose the I_j^0's so that the real part of one of the ξ_j^0's lying close to 0 is 0, say $\Re\xi_0^0 = 0$. Indeed, since Γ_0 is a closed curve, the aforementioned subdivision can be started from any point on Γ_0, i.e. if $P \in \Gamma_0$ is any point then there is a unique subdivision σ_P such that P is one of the division points. Take now any subdivision σ, and in that subdivision let 0 lie in the subarc $\widehat{bc}$, with, say, $\Re b \leq 0$, $\Re c \geq 0$ (recall that at 0 the x-axis is tangent to Γ), and let the two neighboring arcs of that subdivision be $\widehat{ab}$ and $\widehat{cd}$ with $\Re a < 0$, $\Re c > 0$. Call a the left endpoint of $\widehat{ab}$. Now if P is moving on Γ_0 from a to c in a continuous manner, then the subarc $I(P)$ in σ_P for which P is its left endpoint moves from $\widehat{ab}$ to $\widehat{cd}$. Since the first one lies in the negative half-plane $\Re z \leq 0$, while the latter lies in the positive half-plane $\Re z \geq 0$, in the first case the center of mass lies in $\Re z < 0$, while in the second case it lies in $\Re z > 0$. Therefore, there will be a moment for which the center of mass of $I(P)$ lies on the imaginary axis, and then σ_P is the required subdivision, and we select $I(P)$ as I_0^0. It then easily follows that ξ_0^0 lies closest to 0 among the ξ_j^k's.

Consider now the polynomial

$$R_n(z) = \prod_{j,k}(z - \xi_j^k) \tag{2.6}$$

of degree at most n. We claim that the polynomial

$$P_n(z) = R_n(z)/(z - \xi_0^0) \tag{2.7}$$

verifies Theorem 2.1. We prove this via a series of propositions.

In what follows $A \sim B$ means that the ratio A/B is bounded away from zero and infinity.

Proposition 2.2 *$d\mu_\Gamma(t) = \omega_\Gamma(t)ds(t)$ with a positive density function ω_Γ which is C^α-smooth away from the endpoints of the arc-components of Γ. If E is an endpoint of an arc-component of Γ, then $\omega_\Gamma(z) \sim 1/|z - E|^{1/2}$ around E.*

This is a standard result. When Γ consists of one component which is a Jordan curve it immediately follows from the Kellogg-Warschawski theorem (see [16,

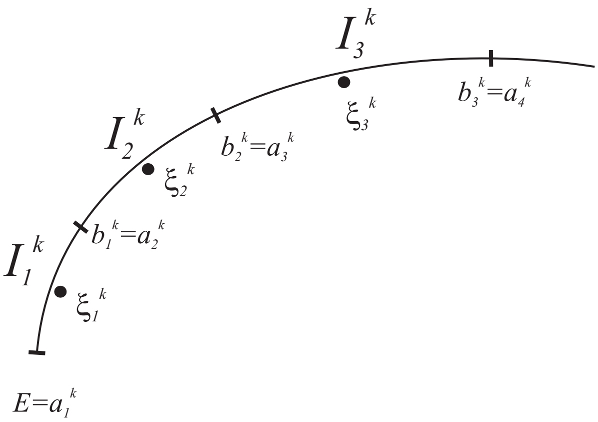

Figure 1: The choice of the intervals I_j^k and of the points ξ_j^k

Theorem 3.6]). When Γ consists of several components, we could not find it in the appropriate form in the literature, hence we will present a proof in the Appendix to this paper. Actually, the proof gives that around an endpoint E of an arc-component of Γ the function $\omega_\Gamma(t)|t-E|^{1/2}$ is a positive Lip α function.

Since away from endpoints of arc-components of Γ the density ω_Γ is bounded away from 0 and infinity, it follows that away from the endpoints we have $s(I_j^k) \sim 1/n$, and if a_j^k, b_j^k are the endpoints of the arc I_j^k, then in this case $|\xi_j^k - a_j^k| \sim 1/n$, $|\xi_j^k - b_j^k| \sim 1/n$ and $|\xi_j^k - \xi_i^k| \sim |j-i|/n$.

Proposition 2.3 *If E is an endpoint of an arc-component of Γ, say $E \in I_1^k$ and $I_1^k, I_2^k, \dots$ follow one another in this order on Γ, then $|\xi_j^k - E| \sim (j/n)^2$ and $s(I_j^k) \sim j/n^2$ in a neighborhood of E. Furthermore, if the endpoints of the arc I_j^k are a_j^k, b_j^k then*

$$|\xi_j^k - a_j^k| \sim |\xi_j^k - b_j^k| \sim s(I_j^k) \sim j/n^2, \tag{2.8}$$

and

$$|\xi_j^k - \xi_i^k| \sim \frac{|j^2 - i^2|}{n^2}. \tag{2.9}$$

See Figure 1.

Proof. Let I_j^k be the arc $\widehat{a_j^k b_j^k}$ with a_j^k lying closer to E. Then, by Proposition 2.2,

$$j\frac{\theta_k}{n_k} = \int_{\widehat{Eb_j^k}} \omega_\Gamma(t)ds(t) \sim \int_{\widehat{Eb_j^k}} |t-E|^{-1/2}ds(t),$$

and since $|t-E|\sim s(\widehat{Et})$ we can continue this as

$$\int_{\widehat{Eb_j^k}} s(\widehat{Et})^{-1/2}ds(t)\sim s(\widehat{Eb_j^k})^{1/2}\sim |E-b_j^k|^{1/2}.$$

Therefore, $|E-b_j^k|\sim (j/n)^2$ and $s(I_1^k)\sim 1/n^2$ follow because $\theta_k/n_k\sim 1/n$. Since $a_j^k=b_{j-1}^k$, we also get for $j\geq 2$ the relation $|E-a_j^k|\sim (j/n)^2$. Therefore, for $j\geq 2$

$$\frac{\theta_k}{n_k}=\int_{\widehat{a_j^k b_j^k}}\omega_\Gamma(t)ds(t)\sim \int_{\widehat{a_j^k b_j^k}}((j/n)^2)^{-1/2}ds(t)\sim s(I_j^k)(n/j),$$

which, in view again of $\theta_k/n_k\sim 1/n$, gives $s(I_j^k)\sim j/n^2$.

Since ξ_j^k lies close to I_j^k, $|\xi_j^k-E|\sim (j/n)^2$ is immediate for $j\geq 2$. To prove it for $j=1$ we may assume temporarily (i.e. just for the proof of this relation) that $E=0$ and $\mathbf{R}_+$ is the half-tangent to the arc Γ_k of Γ. Let the orthogonal projection of the arc I_1^k onto the real line be $[0,d]$. Then, as we have just seen, $d\sim 1/n^2$, and $\Re\xi_1^k$ is the center of mass of a measure $\rho(t)dt$ on $[0,d]$ for which $\rho(t)\sim t^{-1/2}$. Elementary estimate shows then that $\Re\xi_1^k/d$ is bounded away from 0 and infinity (no matter how small d is), which combined with $\mathrm{diam}(I_1^k)\sim 1/n^2$ yields the desired estimate $|\xi_1^k|\sim (1/n)^2$.

The same argument verifies (2.8), while (2.9) follows from the other statements in the proposition: for example if $i<j\leq 2i$, $i\neq j$ then

$$|\xi_j^k-\xi_i^k|\sim \widehat{s}(a_i^k b_j^k)=\sum_{\tau=i}^{j}s(I_\tau^k)\sim\sum_{\tau=i}^{j}(\tau/n^2)\sim (j^2-i^2)/n^2,$$

while if $j>2i$ then (use also the preceding relation with $j=2i$)

$$|\xi_j^k-\xi_i^k|\sim |E-\xi_j^k|\sim j^2/n^2\sim (j^2-i^2)/n^2.$$

∎

Proposition 2.4 *For the polynomials* (2.6) *we have*

$$\|R_n\|_\Gamma\leq C\mathrm{cap}(\Gamma)^n \tag{2.10}$$

with some C independent of n.

This almost proves Theorem 1.3, the only problem is that the degree of R_n is $\sum_k[\theta_k n]$, which may be smaller than n but at most by k_0. To have exact degree n one should divide some of the Γ_k's into not $[\theta_k n]$ but $[\theta_k n]+1$ parts so as to get totally n arcs, and proceed as below.

Proof. By Frostman's theorem (see [17, Theorem 3.3.4])

$$\int \log|z-t|d\mu_\Gamma(t) = \log \operatorname{cap}(\Gamma), \qquad z \in \Gamma. \tag{2.11}$$

Note that (with $\log^+ = \max(0, \log)$)

$$\int \log^+|z-t|d\mu_\Gamma(t) \le \log^+ \operatorname{diam}(\Gamma),$$

hence

$$\int |\log|z-t||d\mu_\Gamma(t) \le 2\log^+ \operatorname{diam}(\Gamma) - \log \operatorname{cap}(\Gamma). \tag{2.12}$$

Now we write in view of (2.11)

$$\begin{aligned} n \log \operatorname{cap}(\Gamma) &= \sum_{j,k} \left(n - \frac{1}{\mu_\Gamma(I_j^k)} \right) \int_{I_j^k} \log|z-t|d\mu_\Gamma(t) \\ &+ \sum_{j,k} \frac{1}{\mu_\Gamma(I_j^k)} \int_{I_j^k} \log|z-t|d\mu_\Gamma(t) = \Sigma_1 + \Sigma_2. \end{aligned} \tag{2.13}$$

Here, by (2.4) and (2.12),

$$|\Sigma_1| \le \sum_{j,k} O(1) \left| \int_{I_j^k} \log|z-t|\mu_\Gamma(t) \right| = O(1). \tag{2.14}$$

Therefore, to prove the claim we have to show that on Γ

$$\log|R_n(z)| - \Sigma_2 = \sum_{j,k} \frac{1}{\mu_\Gamma(I_j^k)} \int_{I_j^k} \log \left| \frac{z-\xi_j^k}{z-t} \right| \omega_\Gamma(t)ds(t) \le C. \tag{2.15}$$

The proof uses the idea of [19, Theorem VI.4.2]. It is more involved around endpoints of arc-components of Γ, so we give it only there. Thus, let z lie in an arc $I_{j_0}^l$ that lies around an endpoint E of an arc-component Γ_l of Γ, on which, say, the arcs I_j^l are following each other in the order $I_1^l, \ldots, I_{j_0}^l, \ldots$ with I_1^l containing E. z and (j_0, l) will always have this meaning below. We consider the sum

$$\sum_{(j,k)\neq(j_0,l)} \frac{1}{\mu_\Gamma(I_j^k)} \int_{I_j^k} \log \left| \frac{z-\xi_j^k}{z-t} \right| \omega_\Gamma(t)ds(t) =: \sum_{(j,k)\neq(j_0,l)} L_{j,k}(z), \tag{2.16}$$

and prove that it is uniformly bounded (both from below and above). Note that this sum differs from the one on the right of (2.15) in one term (the term with

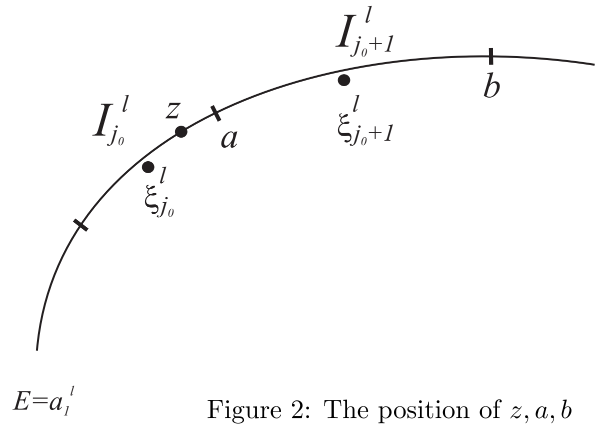

Figure 2: The position of z, a, b

integral over $I^l_{j_0}$ is missing), and we shall actually show that not just the sum, but also the sum consisting of the absolute values $|L_{j,k}|$ is uniformly bounded, i.e.

$$\sum_{(j,k)\neq(j_0,l)} |L_{j,k}(z)| = O(1). \tag{2.17}$$

First we verify that the individual terms $L_{j,k}(z)$ in (2.16) are uniformly bounded. This is clear for $k \neq l$ (i.e. when I^k_j is on a different component of Γ than z) or for $k = l$ but $j \neq j_0 \pm 1$ (the $j = j_0$ term is not in the sum), for then in the integrand

$$|z - \xi^k_j| \sim \operatorname{dist}\{I^l_{j_0}, I^k_j\} \sim |z - t| \qquad \text{for all } t \in I^k_j.$$

So let $j = j_0 \pm 1$, say $j = j_0 + 1$. Then we know from Proposition 2.3 that $|z - \xi^l_{j_0+1}| \sim s(I^l_{j_0+1}) \sim j_0/n^2$, and from Propositions 2.2 and 2.3 that $\omega_\Gamma(t) \leq Cn/j_0$ on $I^l_{j_0+1}$. Let $I^l_{j_0+1}$ be the arc $\widehat{ab}$, see Figure 2. Clearly

$$\begin{aligned} L_{j_0+1,l}(z) &= \frac{1}{\mu_\Gamma(I^l_{j_0+1})} \int_{I^l_{j_0+1}} \log \left| \frac{z - \xi^l_{j_0+1}}{z - t} \right| \omega_\Gamma(t) ds(t) \\ &\leq Cn\frac{n}{j_0} \int_{I^l_{j_0+1}} \left(\log |z - \xi^l_{j_0+1}| + \log \frac{1}{|a - t|} \right) ds(t). \end{aligned} \tag{2.18}$$

Here

$$\int_{I^l_{j_0+1}} \log \frac{1}{|a-t|} ds(t) \leq \int_{I^l_{j_0+1}} \log \frac{C_0}{s(\widehat{at})} ds(t) = s(I^l_{j_0+1})(\log C_0 + 1 - \log s(I^l_{j_0+1})).$$

Therefore, the integral on the right of (2.18) equals

$$s(I^l_{j_0+1}) \log \frac{|z - \xi^l_{j_0+1}|}{s(I^l_{j_0+1})} + O\left(s(I^l_{j_0+1})\right) \leq Cs(I^l_{j_0+1}) \leq C\frac{j_0}{n^2}.$$

If we substitute this into (2.18) then we obtain the boundedness of $L_{j_0+1,l}(z)$ from above. Its boundedness from below is clear since for $z \in I^l_{j_0}$, $t \in I^l_{j_0+1}$ we have

$$\left| \frac{z - \xi^l_{j_0+1}}{z - t} \right| \geq c > 0 \tag{2.19}$$

by (2.8).

The case $j = j_0 - 1$ is similar provided $j_0 - 1 > 1$, but for $j_0 - 1 = 1$, we must proceed somewhat differently, for then $\omega_\Gamma(t) \leq Cn/j_0$ is no longer true on I^l_1. In this case (i.e. when $I^l_{j_0-1} = I^l_1$) we have $\mu(I^l_1) \sim 1/n \sim s(I^l_1)^{1/2}$, $|z - t| \sim s(\widehat{zt})$, so

$$L_{1,l} \leq \frac{C}{s(\widehat{ab})^{1/\alpha}} \int_{\widehat{ab}} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{at})^{-1/2} ds(t),$$

and the right-hand side will be shown to be bounded from above in the proof of (2.24)–(2.25) (the boundedness from below of $L_{1,l}$ follows again from (2.19)).

These prove the uniform boundedness of the individual terms $L_{j,k}$, $(j,k) \neq (j_0, l)$.

It follows from Proposition 2.3 that there is an M such that if either $k \neq l$ or $k = l$ but $|j - j_0| \geq M$ then for $z \in I^l_{j_0}$ and $t \in I^k_j$ we have

$$\left| \frac{\xi^k_j - t}{z - \xi^k_j} \right| \leq \frac{1}{2}$$

(a closer look at the proof of Propositions 2.2 and 2.3 reveals that $M = 4$ suffices for large n, but we do not need the best value of M).

Thus, in this case for the integrands in $L_{j,k}(z)$ we get (use that $\log|1-u| = \Re \log(1-u)$ with any local branch of the log)

$$\log \left| \frac{z - \xi^k_j}{z - t} \right| = -\log \left| 1 - \frac{\xi^k_j - t}{z - \xi^k_j} \right| = \Re \frac{\xi^k_j - t}{z - \xi^k_j} + O\left(\left| \frac{\xi^k_j - t}{z - \xi^k_j} \right|^2 \right).$$

Therefore, for such j and k we have

$$|L_{j,k}(z)| = \frac{1}{\mu_\Gamma(I^k_j)} \int_{I^k_j} O\left(\left| \frac{\xi^k_j - t}{z - \xi^k_j} \right|^2 \right) d\mu_\Gamma(t) = O\left(\frac{s(I^k_j)^2}{|\xi^k_j - \xi^l_{j_0}|^2} \right), \tag{2.20}$$

because the integral

$$\int_{I^k_j} \Re \frac{\xi^k_j - t}{z - \xi^k_j} d\mu_\Gamma(t) = \Re \int_{I^k_j} \frac{\xi^k_j - t}{z - \xi^k_j} d\mu_\Gamma(t)$$

vanishes by the choice of ξ_j^k.

The expression on the right of (2.20) is bounded by a constant times $s(I_j^k)^2$ when $k \neq l$ or $k = l$ but I_j^l is far from E (say farther than a fixed constant $\delta > 0$), and for $k = l$ and I_j^l close to E (say for $|\xi_j^k - E| \leq \delta$) it is at most (see Proposition 2.3) a constant times

$$\frac{s(I_j)^2}{|(j/n)^2 - (j_0/n)^2|^2} \sim \frac{(j/n^2)^2}{|(j/n)^2 - (j_0/n)^2|^2} = \frac{j^2}{|j^2 - j_0^2|^2}.$$

All in all, if we take into account the uniform boundedness of the terms $L_{j,k}$ we obtain that the sum in (2.17) is at most

$$\sum_{|j-j_0|\leq M,\ j\neq j_0} |L_{j,l}| \quad + \sum_{|j-j_0|>M} |L_{j,l}| + \sum_{j,k,\ k\neq l} |L_{j,k}|$$

$$\leq \quad (2M)C + C \sum_{|j-j_0|>M} \frac{j^2}{|j^2 - j_0^2|^2} + C\sum_{j,k} s(I_j^k)^2 \leq C.$$

To complete the proof of the proposition we have to show that the additional term

$$\frac{1}{\mu_\Gamma(I_{j_0}^l)} \int_{I_{j_0}^l} \log\left|\frac{z - \xi_{j_0}^l}{z - t}\right| \omega_\Gamma(t)ds(t) \tag{2.21}$$

in (2.15) is also bounded from above (from below we cannot claim boundedness for z can be very close to $\xi_{j_0}^l$). As before, we get from Proposition 2.3 that for $j_0 > 1$ this term is at most

$$Cn \int_{I_{j_0}^l} \left(\log \frac{Cs(I_{j_0}^l)}{s(\widehat{zt})}\right)\left(\frac{j_0^2}{n^2}\right)^{-1/2} ds(t),$$

which, with $I_{j_0}^l =: \widehat{ab}$, equals

$$C\frac{n^2}{j_0}\left(s(\widehat{ab})\log(Cs(\widehat{ab})) - s(\widehat{zb})\log s(\widehat{zb}) - s(\widehat{az})\log s(\widehat{az}) + s(\widehat{ab})\right). \tag{2.22}$$

Now we use for $0 \leq x \leq y \leq 1$ the inequality

$$-\frac{2}{e}(x+y) \leq x\log x + y\log y - (x+y)\log(x+y) \leq 0, \tag{2.23}$$

which is immediate from the concavity of log and from the fact that on the interval $(0,1)$ the minimum of $t\log t$ is $-1/e$. Apply (2.23) with $s(\widehat{zb})$, $s(\widehat{az})$ in place of x, y (in which case $x + y = s(\widehat{ab})$) to continue (2.22) as

$$\leq C\frac{n^2}{j_0}\left(s(\widehat{ab})\log(Cs(\widehat{ab})) - s(\widehat{ab})\log s(\widehat{ab}) + O(s(\widehat{ab}))\right) \leq C\frac{n^2}{j_0}s(\widehat{ab}) \leq C,$$

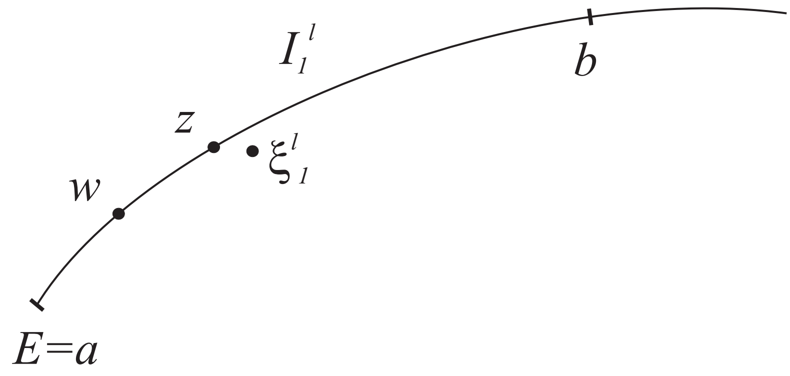

Figure 3: The choice of w

where, in the last step we used that by Proposition 2.3, $s(\widehat{ab}) = s(I_{j_0}^l) \sim j_0/n^2$. This gives the required estimate for (2.21) when $j_0 > 1$.

When $j_0 = 1$ then E is an endpoint of the arc $I_{j_0}^l$, e.g. $E = a$. In that case ω_Γ is not bounded on $I_{j_0}^l$, so we have to proceed differently than before. Similarly as above, now we have with $s(\widehat{ab}) = s(I_{j_0}^l) \sim 1/n^2$, $\mu_\Gamma(I_1^l) \sim 1/n \sim s(\widehat{ab})^{1/2}$ the bound

$$\frac{C}{s(\widehat{ab})^{1/2}} \int_{\widehat{ab}} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{at})^{-1/2} ds(t) =: I \tag{2.24}$$

for the expression in (2.21). Recall that z lies on the arc $\widehat{ab} = \widehat{Eb}$, and let w be the midpoint on the arc $\widehat{Ez}$ in the sense that $s(\widehat{Ew}) = s(\widehat{wz})$, see Figure 3. Now we split the integral in (2.24) over $\widehat{ab}$ into three parts: the integrals over $\widehat{zb}$, $\widehat{wz}$ and $\widehat{Ew}$. For the first we have (use that the antiderivative of $t^{-1/2}\log t$ is $2t^{1/2}\log t - 4t^{1/2}$)

$$\int_{\widehat{zb}} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{Et})^{-1/2} ds(t) \leq \int_{\widehat{zb}} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{zt})^{-1/2} ds(t)$$

$$= 2\log(Cs(\widehat{ab}))s(\widehat{zb})^{1/2} - 2s(\widehat{zb})^{1/2}\log s(\widehat{zb}) + 4s(\widehat{zb})^{1/2} \leq Cs(\widehat{ab})^{1/2}$$

because, for any $C_0 > e^2$ (by the monotonicity of $x^{1/2}\log 1/x$ on $(0, e^{-2})$),

$$2s(\widehat{zb})^{1/2} \log \frac{C_0 s(\widehat{ab})}{s(\widehat{zb})} \leq 2s(\widehat{ab})^{1/2} \log \frac{C_0 s(\widehat{ab})}{s(\widehat{ab})} = 2s(\widehat{ab})^{1/2} \log C_0.$$

The integral over $\widehat{wz}$ can be similarly handled. Finally, for the integral over $\widehat{Ew}$ we have the bound

$$\int_{\widehat{Ew}} \log \frac{Cs(\widehat{ab})}{s(\widehat{Ew})} s(\widehat{Et})^{-1/2} ds(t) \leq \log \frac{Cs(\widehat{ab})}{s(\widehat{Ew})} 2s(\widehat{Ew})^{1/2} \leq \log \frac{Cs(\widehat{ab})}{s(\widehat{ab})} 2s(\widehat{ab})^{1/2}$$

$$= 2(\log C)s(\widehat{ab})^{1/2}.$$

Substituting all these into (2.24) we get

$$I \leq C, \tag{2.25}$$

and with this the upper boundedness of (2.21) for $j_0 = 1$, as well.

■

Proposition 2.5 *For the polynomials from* (2.7) *we have*

$$|P_n(z)| \sim n \ \mathrm{cap}(\Gamma)^n \tag{2.26}$$

uniformly in n *and* $z \in I_0^0$*, in particular*

$$|P_n(0)| \sim n \ \mathrm{cap}(\Gamma)^n. \tag{2.27}$$

For $z \in \Gamma \setminus I_0^0$

$$|P_n(z)| \leq C \mathrm{cap}(\Gamma)^n \frac{1}{|z|} \tag{2.28}$$

with some C *independent of* n*.*

Proof. Let $z \in I_0^0$, i.e. with the notations of the preceding proof we have $l = i_0 = 0$. By the proof of Proposition 2.4 (see in particular (2.11)–(2.14)) and (2.17)) we have uniformly in n and $z \in I_0^0$

$$\log |P_n(z)| - n \log \mathrm{cap}(\Gamma) + \frac{1}{\mu_\Gamma(I_0^0)} \int_{I_0^0} \log |z - t| \omega_\Gamma(t) ds(t) = O(1). \tag{2.29}$$

Now use that $|z - t| = (1 + o(1)) s(\widehat{zt})$ (for $z - t \sim 0$) to get with $I_0^0 =: \widehat{ab}$ for the last term in (2.29)

$$\frac{1}{\mu_\Gamma(I_0^0)} \int_{I_0^0} \log |z - t| \omega_\Gamma(t) ds(t) = \frac{1}{\mu_\Gamma(I_0^0)} \int_{I_0^0} \log \Big((1 + o(1)) s(\widehat{zt}) \Big) \omega_\Gamma(t) ds(t)$$

$$= \quad o(1) + \frac{1}{\mu_\Gamma(I_0^0)} \int_{I_0^0} \log s(\widehat{zt}) \omega_\Gamma(t) ds(t).$$

Here we need that for $t \in I_0^0$ Proposition 2.2 yields

$$|\omega_\Gamma(t) - \omega_\Gamma(0)| \leq C|t|^\alpha \leq Cn^{-\alpha}$$

to continue the preceding estimates as

$$= \quad o(1) + \frac{\omega_\Gamma(0)(1 + O(n^{-\alpha}))}{\mu_\Gamma(I_0^0)} \int_{I_0^0} \log s(\widehat{zt}) ds(t)$$

$$= \quad o(1) + \frac{\omega_\Gamma(0)(1+O(n^{-\alpha}))}{\mu_\Gamma(I_0^0)}(s(\widehat{az})\log s(\widehat{az}) + s(\widehat{zb})\log s(\widehat{zb}) - s(\widehat{ab}))$$

$$= \quad o(1) + \frac{\omega_\Gamma(0)(1+O(n^{-\alpha}))}{\mu_\Gamma(I_0^0)}\Big(s(\widehat{ab})\log s(\widehat{ab}) + O(s(\widehat{ab}))\Big),$$

where, in the last step we used again (2.23) with $s(\widehat{az})$ and $s(\widehat{zb})$ playing the role of x, y (note that then $x + y = s(\widehat{ab})$).

Since

$$\frac{\omega_\Gamma(0)}{\mu_\Gamma(I_0^0)} s(\widehat{ab}) = 1 + O(n^{-\alpha})$$

and

$$\log s(\widehat{ab}) = O(1) - \log n$$

are also true (the latter one follows from the first one in view of $\mu_\Gamma(I_0^0) = (1 + o(1))/n$), finally we can conclude

$$\frac{1}{\mu_\Gamma(I_0^0)} \int_{I_0^0} \log|z - t|\omega_\Gamma(t)ds(t) = O(1) - \log n.$$

This and (2.29) prove (2.26).

The claim (2.28) follows immediately from Proposition 2.4, for $|z - \xi_0^0| \sim |z|$ when $z \in \Gamma \setminus I_0^0$.

■

Label the points ξ_j^0 around 0 in such a way that, as their real part increases, they follow each other in the order

$$\cdots < \Re\xi_{-2}^0 < \Re\xi_{-1}^0 < 0 = \Re\xi_0^0 < \Re\xi_1^0 < \Re\xi_2^0 < \cdots.$$

We may also assume that this labeling is such that the range of j includes all integers in $[-\tau n, \tau n]$ for some $\tau > 0$.

Proposition 2.6 *For all j we have*

$$\left|\xi_j^0 - \frac{j}{n\omega_\Gamma(0)}\right| \le C\left(\frac{|j|}{n}\right)^{1+\alpha}. \tag{2.30}$$

Proof. Enough to prove this for $|\xi_j^0| \le \delta$ with some small $\delta > 0$ (otherwise the discussion below gives $|j| \ge c_\delta n$ and then the statement is obvious).

Recall that the real line is the tangent line to Γ at 0 and in a neighborhood of 0 the curve Γ has a parametrization $t + i\gamma(t)$ with $\gamma' \in C^\alpha$ and $\gamma(0) = 0$, $\gamma'(0) = 0$, $|\gamma'(t)| \le C|t|^\alpha$, $|\gamma(t)| \le C|t|^{1+\alpha}$.

Let $u = t + i\gamma(t) \in \Gamma$. Then

$$ds(u) = \sqrt{1 + (\gamma'(t))^2} dt = dt + O(|u|^{2\alpha}) dt. \tag{2.31}$$

Let a, b be the endpoints of I_0^0, $\Re a < \Re b$. We know that $|a|, |b| \sim 1/n$ (this is immediate from the facts that $\Re \xi_0^0 = 0$, the equilibrium density ω_Γ is continuous and positive at 0, and $ds(u) \sim dt$ by (2.31)). We can write

$$\begin{aligned} 0 &= \Re \xi_0^0 = \frac{n_0}{\theta_0} \int_{I_0^0} \Re u \, \omega_\Gamma(u) ds(u) = \frac{n_0}{\theta_0} \int_{I_0^0} \Re u \, \omega_\Gamma(0) ds(u) + O\left(n \frac{1}{n} n^{-\alpha} \frac{1}{n} \right) \\ &= \frac{n_0}{\theta_0} \int_{\Re a}^{\Re b} t \omega_\Gamma(0) dt + O(n^{-1-\alpha}) \\ &= \frac{n_0 \omega_\Gamma(0)}{\theta_0} \frac{1}{2} ((\Re b)^2 - (\Re a)^2) + O(n^{-1-\alpha}), \end{aligned}$$

from which it follows that (note $n \sim n_0$, $\Re b - \Re a \sim 1/n$)

$$|\Re b + \Re a| = O(n^{-1-\alpha}).$$

Now let $t_0 = \Re(a+b)/2 + i\gamma(\Re(a+b)/2)$. (2.31) implies

$$s(\widehat{t_0 b}) = \Re b - \Re t_0 + O(n^{-1-\alpha}); \qquad s(\widehat{a t_0}) = \Re t_0 - \Re a + O(n^{-1-\alpha})$$

and hence

$$s(\widehat{t_0 b}) - s(\widehat{a t_0}) = O(n^{-1-\alpha}).$$

Therefore, if $\overline{\xi}_j^0$ is the midpoint of the arc I_j^0 with respect to arc length, then $|t_0 - \overline{\xi}_0^0| \le C n^{-1-\alpha}$. Since

$$|t_0 - \xi_0^0| \le |t_0| + |\xi_0^0| \le C n^{-1-\alpha}$$

is also true, finally we obtain $|\xi_0^0 - \overline{\xi}_0^0| \le C n^{-1-\alpha}$. Note that by the definition of $\overline{\xi}_j^0$ and the C^α-smoothness of ω_Γ we also have

$$\begin{aligned} \mu_\Gamma(\widehat{a \overline{\xi}_0^0}) &= \frac{1}{2} \mu_\Gamma(\widehat{ab}) + O(n^{-\alpha}) \\ \mu_\Gamma(\widehat{\overline{\xi}_0^0 b}) &= \frac{1}{2} \mu_\Gamma(\widehat{ab}) + O(n^{-\alpha}) \end{aligned}$$

Since ξ_j^0, $\overline{\xi}_j^0$ are geometric quantities defined in terms of ω_Γ and s_Γ, the same argument can be given for all j and we obtain

$$|\xi_j^0 - \overline{\xi}_j^0| \le C n^{-1-\alpha}, \qquad |\xi_j| \le \delta, \tag{2.32}$$

and (with a_j, b_j being the endpoints of I_j^0)

$$\begin{aligned}\mu_\Gamma(\widehat{a_j\overline{\xi}_j^0}) &= \frac{1}{2}\mu_\Gamma(\widehat{a_jb_j}) + O(n^{-\alpha})\\ \mu_\Gamma(\widehat{\overline{\xi}_j^0 b_j}) &= \frac{1}{2}\mu_\Gamma(\widehat{a_jb_j}) + O(n^{-\alpha}).\end{aligned}$$

These latter imply for $j \neq 0$, say for $j > 0$,

$$\begin{aligned} j\frac{\theta_0}{n_0} &= \mu\left(\bigcup_{l=0}^{j-1} I_l^0\right) = \mu_\Gamma(\widehat{\overline{\xi}_0^0\overline{\xi}_j^0}) + O(n^{-1-\alpha})\\ &= \int_{\overline{\xi}_0^0}^{\overline{\xi}_j^0} \omega_\Gamma(0)ds(u) + O(|\overline{\xi}_j^0|^{1+\alpha}) + O(n^{-1-\alpha})\\ &= \int_{\Re\overline{\xi}_0^0}^{\Re\overline{\xi}_j^0} \omega_\Gamma(0)dt + O(|\overline{\xi}_j^0|^{1+\alpha}) = (\Re\overline{\xi}_j^0 - \Re\overline{\xi}_0^0)\omega_\Gamma(0) + O(|\overline{\xi}_j^0|^{1+\alpha})\\ &= (\overline{\xi}_j^0 - \overline{\xi}_0^0)\omega_\Gamma(0) + O(|\overline{\xi}_j^0|^{1+\alpha}),\end{aligned}$$

which, in view of $|\overline{\xi}_0^0| \le Cn^{-1-\alpha}$, $|\overline{\xi}_j^0| \le Cj/n$ and (2.4) implies

$$\left|\overline{\xi}_j^0 - \frac{j}{n\omega_\Gamma(0)}\right| \le C\left(\frac{|j|}{n}\right)^{1+\alpha}.$$

The argument for negative j is just the same. Finally, this inequality combined with (2.32) gives (2.30).

■

Fix a large integer number M and a small $\rho > 0$, so small that even $\rho^\alpha M$ is small. Let

$$Q_n(z) = \prod_{M^3 < |j| \le \rho n} (z - \xi_j^0). \tag{2.33}$$

Proposition 2.7 . *For $z \in \Gamma$, $|z| \le M/n$ we have*

$$\left|\left|\frac{Q_n(z)}{Q_n(0)}\right| - 1\right| \le \left(CM\rho^\alpha + \frac{1}{M}\right) \tag{2.34}$$

with a C that depends only on Γ.

Proof.

$$\log\left|\frac{Q_n(z)}{Q_n(0)}\right| = \sum_{M^3 < |j| \le \rho n} \log\left|1 - \frac{z}{\xi_j^0}\right|,$$

and on applying (2.30) this can be written as

$$\sum_{M^3<j\leq \rho n} \log \left| \left(1-\frac{z}{j/n\omega_\Gamma(0)+O((j/n)^{1+\alpha})}\right)\left(1-\frac{z}{-j/n\omega_\Gamma(0)+O((j/n)^{1+\alpha})}\right)\right|$$

$$= \sum_{M^3<j\leq \rho n} \log \left|1+\frac{O(|z|(j/n)^{1+\alpha})+O(|z|^2)}{(j/n)^2}\right|$$

$$= \sum_{M^3<j\leq \rho n} O\left(|z|(j/n)^{\alpha-1}+\frac{|z|^2}{(|j|/n)^2}\right) = O\Big((M/n)n^{1-\alpha}(\rho n)^\alpha + \frac{(M/n)^2}{M^3/n^2}\Big),$$

from which the claim follows.

■

The key statement in the proof of Theorem 2.1 is

Proposition 2.8 *Let Γ_δ be the part of Γ that lies of distance $\geq \delta$ from the origin. Then*

$$\lim_{\delta\to 0} \Re \int_{\Gamma_\delta} \frac{d\mu_\Gamma(u)}{u} = 0. \tag{2.35}$$

The statement is that the real part of the principal value integral

$$\mathrm{PV}\int_\Gamma \frac{d\mu_\Gamma(u)}{u} \tag{2.36}$$

is zero at 0. This is due to the fact that the tangent line to Γ at 0 is horizontal.

Proof. Let $\overline{\Gamma}_\delta$ be the complementary arc, i.e. the set of points on Γ which are closer to 0 than δ. With some local branch of log we have to show that

$$\lim_{\delta\to 0} \Re \int_{\Gamma_\delta} (\log(z-u))'\Big|_{z\,=\,0} d\mu_\Gamma(u) = 0.$$

Here, with $z = x+i\gamma(x) \in \Gamma$, $u = t+i\gamma(t) \in \Gamma$ (with some global $t+i\gamma(t)$ parametrization of Γ that extends the local parametrization $t+i\gamma(t)$ around the origin, see the discussion before (2.4))

$$\Re \int_{\Gamma_\delta} (\log(z-u))'\Big|_{z\,=\,0} d\mu_\Gamma(u) =$$

$$= \lim_{x\to 0} \frac{1}{x+i\gamma(x)} \int_{t+i\gamma(t)\in\Gamma_\delta} \log\frac{|(x+i\gamma(x))-(t+i\gamma(t))|}{|t+i\gamma(t)|} d\mu_\Gamma(t+i\gamma(t)).$$

Since the whole integral

$$\int_\Gamma \log|z-u|d\mu_\Gamma(u) \tag{2.37}$$

is constant on Γ (see (2.11)), what we need to show is that the previous expression with Γ_δ replaced by $\overline{\Gamma}_\delta$ tends to 0 as $\delta \to 0$ (in this case the existence of the limit/derivative follows from what we have just done and from the constancy of (2.37)).

Since $x/(x+i\gamma(x)) \to 1$ as $x \to 0$, we need to show that

$$\frac{1}{x}\int_{t+i\gamma(t)\in\overline{\Gamma}_\delta} \log\frac{|(x+i\gamma(x))-(t+i\gamma(t))|}{|t+i\gamma(t)|}d\mu_\Gamma(t+i\gamma(t)) =: \frac{1}{x}I \tag{2.38}$$

is as small in absolute value as we wish for small $|x|$ and small, but fixed $\delta > 0$. Without loss of generality assume $x > 0$. Let the endpoints of $\overline{\Gamma}_\delta$ be $-\delta_1 + i\gamma(-\delta_1)$ and $\delta_2 + i\gamma(\delta_2)$, $\delta_1, \delta_2 > 0$. Then $\delta_j^2 + \gamma(\delta_j)^2 = \delta^2$, and hence, in view of $\gamma(\delta_j) = O(\delta_j^{1+\alpha})$, we have

$$\delta_j = \delta + O(\delta^{1+2\alpha}), \qquad j = 1, 2. \tag{2.39}$$

With some large N

$$\begin{aligned} I &= \int_{-\delta_1}^{\delta_2} \log\frac{|(x+i\gamma(x))-(t+i\gamma(t))|}{|t+i\gamma(t)|}\omega_\Gamma(t+i\gamma(t))\sqrt{1+(\gamma'(t))^2}dt \\ &= \int_{-\delta_1}^{-Nx} + \int_{-Nx}^{Nx} + \int_{Nx}^{\delta_2} = I_1 + I_2 + I_3. \end{aligned}$$

First we deal with I_2. It is the sum of

$$I_{21} = \int_{-Nx}^{Nx} \log\frac{|(x+i\gamma(x))-(t+i\gamma(t))|}{|x-t|}\omega_\Gamma(t+i\gamma(t))\sqrt{1+(\gamma'(t))^2}dt,$$

$$I_{22} = -\int_{-Nx}^{Nx} \log\frac{|t+i\gamma(t)|}{|t|}\omega_\Gamma(t+i\gamma(t))\sqrt{1+(\gamma'(t))^2}dt$$

and

$$I_{23} = \int_{-Nx}^{Nx} \log\frac{|x-t|}{|t|}\omega_\Gamma(t+i\gamma(t))\sqrt{1+(\gamma'(t))^2}dt.$$

In I_{21} the log term is $\log(1 + O((Nx)^\alpha) = O((Nx)^\alpha))$ because, with some $\zeta \in [-Nx, Nx]$,

$$|\gamma(x) - \gamma(t)| = |x-t||\gamma'(\zeta)| \le |x-t|C(Nx)^\alpha,$$

so $I_{21} = O(N^{1+\alpha}x^{1+\alpha}) = o(x)$ as $x \to 0$. Similarly, $I_{22} = o(x)$. Finally,

$$\begin{aligned} I_{23} &= \int_{-Nx}^{Nx} \log\frac{|x-t|}{|t|}\Big(\omega_\Gamma(t+i\gamma(t))\sqrt{1+(\gamma'(t))^2} - \omega_\Gamma(0)\Big)dt \\ &+ \omega_\Gamma(0)\int_{-Nx}^{Nx}\log\frac{|x-t|}{|t|}dt = I_{231} + I_{232}. \end{aligned}$$

The factor after the log term in I_{231} is in absolute value $\le C|t|^\alpha \le C(Nx)^\alpha$ and

$$\int_{-Nx}^{Nx}\left|\log\frac{|x-t|}{|t|}\right|dt = x\int_{-N}^{N}\left|\log\frac{|1-t|}{|t|}\right|dt \le Cx\log N,$$

hence $I_{231} = O(N^\alpha(\log N)x^{1+\alpha}) = o(x)$. For I_{232}, as simple calculation shows, we can write

$$\frac{|I_{232}|}{\omega_\Gamma(0)} = \int_{(N-1)x}^{Nx}\log\frac{x+t}{t}dt \le \int_{(N-1)x}^{Nx}\frac{x}{t}dt \le \frac{x}{N-1}.$$

So $|I_2| \le \omega_\Gamma(0)x/(N-1) + o(x)$.

For $I_1 + I_3$ we set $J = [-\delta_1, -Nx] \cup [Nx, \delta_2]$ and note that the log term in the integrals in I_1 and I_3 is

$$\begin{aligned} \Re\log\left(1 - \frac{x+i\gamma(x)}{t+i\gamma(t)}\right) &= -\Re\frac{x+i\gamma(x)}{t+i\gamma(t)} + O\left(\left(\frac{x}{t}\right)^2\right) \\ &= -\frac{xt+\gamma(x)\gamma(t)}{t^2+\gamma(t)^2} + O\left(\left(\frac{x}{t}\right)^2\right) \\ &= -\frac{x}{t} + xO\left(\frac{\gamma(t)^2}{t^3}\right) + O\left(\frac{\gamma(x)\gamma(t)}{t^2}\right) + O\left(\left(\frac{x}{t}\right)^2\right) \end{aligned}$$

Here on the right $\gamma(t)^2/t^3$ and $\gamma(t)/t^2$ are integrable, so the contribution to the integral over $\overline{\Gamma}_\delta$ of the corresponding terms is $xo_\delta(1)$ and $\gamma(x)o_\delta(1) = xo_\delta(1)$, respectively, where $o_\delta(1)$ means a quantity tending to 0 as $\delta \to 0$. The contribution of the term $O(x^2/t^2)$ is

$$\int_J O\left(\left(\frac{x}{t}\right)^2\right)dt = O\left(\frac{x^2}{Nx}\right) = O\left(\frac{x}{N}\right).$$

Finally, the integral over J of the term $-x/t$ is equal to

$$-\int_J \frac{x}{t}\Big(\omega_\Gamma(t+i\gamma(t))\sqrt{1+(\gamma'(t))^2} - \omega_\Gamma(0)\Big)dt + \omega_\Gamma(0)\int_J\frac{x}{t}dt = I_4 + I_5.$$

In I_4 we have

$$|\omega_\Gamma(t+i\gamma(t))\sqrt{1+(\gamma'(t))^2}-\omega_\Gamma(0)|\le C|t|^\alpha,$$

so exactly as before $I_4 = xo_\delta(1)$. Finally,

$$|I_5| = \omega_\Gamma(0)x\int_{\min(\delta_1,\delta_2)}^{\max(\delta_1,\delta_2)}\frac{1}{t}dt \le Cx\frac{\delta^{1+2\alpha}}{\delta} = xo_\delta(1)$$

where we used (2.39). All in all, we have $|I| \le xo_\delta(1)+o(x)+O(x/N)$ which shows that the term in (2.38) is as small as we wish if we select N large and then $\delta > 0$ small (and also x sufficiently small after these selections).

■

Proposition 2.9 *Let*

$$S_n(z) = \prod_{|\xi_j^k|\ge\delta}(z-\xi_j^k). \tag{2.40}$$

Then, for fixed M and $z\in\Gamma$, $|z|\le M/n$, we have $|S_n(z)/S_n(0)| = 1+o_\delta(1)$ uniformly in n.

Proof. As always, we set $z = x+i\gamma(x)$.

$$\frac{1}{n}\log\left|\frac{S_n(z)}{S_n(0)}\right| = \sum_{|\xi_j^k|\ge\delta}\frac{1}{n}\log\left|1-\frac{z}{\xi_j^k}\right| \tag{2.41}$$

is easily seen to converge to

$$\int_{\Gamma_\delta}\log\left|1-\frac{z}{u}\right|d\mu_\Gamma(u) \tag{2.42}$$

(recall, that Γ_δ is the part of Γ that lies of distance $\ge\delta$ from the origin). Indeed, the same sum on the right of (2.41) with $1/n$ replaced by $\mu_\Gamma(I_j^k)$ and ξ_j^k replaced by $\overline{\xi}_j^k$ (that was the midpoint of I_j^k with respect to arc length) is essentially a Riemannian sum for the integral (2.42), and the sums with ξ_j^k, $1/n$ and with $\overline{\xi}_j^k$, $\mu_\Gamma(I_j^k)$ are very close because of (2.4) and (2.32) and its analogue for other intervals. The integral in (2.42) is

$$\int_{\Gamma_\delta}\left(\Re\left(-\frac{z}{u}\right)+O\left(\left(\frac{|z|}{u}\right)^2\right)\right)d\mu_\Gamma(u),$$

and here

$$\int_{\Gamma_\delta}O\left(\left(\frac{|z|}{u}\right)^2\right)d\mu_\Gamma(u) = O\left(\frac{|z|^2}{\delta}\right),$$

while

$$\int_{\Gamma_\delta} \Re\left(-\frac{z}{u}\right) d\mu_\Gamma(u) = -x\Re\int_{\Gamma_\delta} \frac{d\mu_\Gamma(u)}{u} + \gamma(x)\Im\int_{\Gamma_\delta} \frac{d\mu_\Gamma(u)}{u}.$$

Here the second term is $O(\gamma(x)/\delta) = o(|x|) = o(|z|)$ as $z \to 0$, and for the first term Proposition 2.8 gives that it is $xo_\delta(1)$.

Therefore,

$$\log\left|\frac{S_n(z)}{S_n(0)}\right| = n(1+o(1))\Big[|z|o_\delta(1) + O(|z|^2/\delta) + o(|z|)\Big] \le 2Mo_\delta(1)$$

for large n and $|z| \le M/n$, $z \in \Gamma$. This proves the claim.

■

In the polynomial Q_n in (2.33) we put all factors $(z-\xi_j^0)$ with $M^3 < |j| \le \rho n$, while S_n in (2.40) contained the factors $(z-\xi_j^k)$ with $|\xi_j^k| \ge \delta$. For sufficiently small δ these latter include all ξ_j^k with $k > 0$ (i.e. which are created for the components Γ_k, $k > 0$). Furthermore, for small $\delta > 0$ if, with a sufficiently large fixed L we select $\rho = \omega_\Gamma(0)\delta - L\delta^{1+\alpha}$, then (2.30) shows that Q_n and S_n have no common factors. On the other hand, if we selected $\rho = \omega_\Gamma(0)\delta + L\delta^{1+\alpha}$, then (2.30) shows that all the factors $(z-\xi_j^k)$ except for $(z-\xi_j^0)$ with $|j| \le M^3$ appear either in Q_n or in S_n. We make the former selection, i.e. we set $\rho = \omega_\Gamma(0)\delta - L\delta^{1+\alpha}$, and let $\tilde{S}_n$ be the product of all factors $(z-\xi_j^0)$ for which $|j| > \rho n$ but $|\xi_j^0| < \delta$ (these are the ones with $|j| > M^3$ that appear neither in Q_n nor in S_n). According to what we have just said, their number is at most $4L\delta^{1+\alpha}n$.

Proposition 2.10 *We have $|\tilde{S}_n(z)/\tilde{S}_n(0)| = 1+o_\delta(1)$ uniformly in n and $|z| \le M/n$, $z \in \Gamma$.*

Proof. Let H be the set of j's for which $|j| > \rho n$ but $|\xi_j^0| < \delta$. Note that all such ξ_j^0's satisfy $|\xi_j^0| \ge \delta/2$ (see (2.30) and the definition of ρ). Now

$$\log\left|\frac{\tilde{S}_n(z)}{\tilde{S}_n(0)}\right| = \sum_{j\in H} \log\left|1 - \frac{z}{\xi_j^0}\right| = \sum_{j\in H} O\left(\frac{|z|}{|\xi_j^0|}\right) = O\left(\frac{M}{n}\frac{4L\delta^{1+\alpha}n}{\delta}\right),$$

from which the claim follows.

■

After these preparations we turn to the proof of Theorem 2.1.

From the definition of our polynomials it follows that

$$P_n(z) = Q_n(z)S_n(z)\tilde{S}_n(z)\prod_{-M^3\le j\le M^3,\ j\ne 0}(z-\xi_j^0) =: Q_n(z)S_n(z)\tilde{S}_n(z)V_n(z),$$

and Propositions 2.7, 2.9 and 2.10 show that here the first three factors change little (i.e. (1+o(1))) as z varies on the arc of Γ with $|z| \le M/n$. The idea of the proof is to compare the remaining factor $V_n(z)$ to something the behavior of which we already know. This something is the unit circle and the polynomial $1 + z + \cdots + z^{m-1} = (z^m - 1)/(z - 1)$, but with $m = [2\pi\omega_\Gamma(0)n]$ (sic!). Apply the transformation $T(z) = -i(z - 1)$ to the unit circle under which 1 gets into the point 0 and the real line becomes the tangent line to the transformed circle at 0. Let $\Gamma^* : \{z^* \mid |z^* - i| = 1\}$ denote this rotated/translated circle, and let $\xi_j^* = -i(e^{ij2\pi/m} - 1)$, $j = -[m/2], \ldots, [(m+1)/2]$, $j \ne 0$ be the images under T of the m-th roots of unity different from 1. Their enumeration is such that

$$\left|\xi_j^* - \frac{2\pi j}{m}\right| \le C\left(\frac{|j|}{m}\right)^2 \tag{2.43}$$

for all j.

To a $z \in \Gamma$, $|z| \le \delta$ we associate the point $z^* \in \Gamma^*$ via $\Re z = \Re z^*$ (and of course of the two possibilities for z^* we take the one lying closer to the real line $\mathbf{R}$). If ν is a measure on Γ, then we define a measure ν^* on Γ^* in a neighborhood of the origin by stipulating $d\nu^*(z^*) = d\nu(z)$; in other words, ν^* is the pull-back of the measure ν under the mapping $z^* \to z$. Away from the origin let ν^* be the arc measure on Γ^*. Assume that 0 is a Lebesgue-point for ν (with respect to s_Γ). Then 0 is also a Lebesgue-point for ν^* (with respect to s_{Γ^*}), see (2.31). Let $d\nu = w ds_\Gamma + d\nu_{\text{sing}}$ be the decomposition of ν into its absolutely continuous and singular part with respect to s_Γ, and let $d\nu^* = w^* ds_{\Gamma^*} + d\nu^*_{\text{sing}}$ be the similar decomposition of ν^*. Then, using the just mentioned Lebesgue-point property, we obtain $w(0) = w^*(0)$.

Let $P_m^*(z^*) = \prod_j (z^* - \xi_j^*)$ be the transform of the polynomial

$$1 + z + \cdots + z^{m-1} = (z^m - 1)/(z - 1)$$

under the transformation T. The expression

$$\frac{1}{m}|1 + z + \cdots + z^{m-1}|^2$$

is the m-th Fejér kernel on the unit circle, and it is known (see [11, Lemma 2] and make the transformation T) that

$$\int_{\Gamma^*} \left|\frac{P_m^*}{P_m^*(0)}\right|^2 d\nu^* \le (1 + o(1))\frac{2\pi w^*(0)}{m}. \tag{2.44}$$

Now the idea of the proof is that for $|z| \le M/n$ the ratio $|P_m^*(z^*)/P_m^*(0)|$ looks just like $|P_n(z)/P_n(0)|$. To show that we write

$$P_m^*(z^*) =: U_m^*(z^*) \prod_{-M^3 \le j \le M^3,\ j \ne 0} (z^* - \xi_j^*) =: U_m^*(z^*) V_m^*(z^*).$$

Note that here $U_m^*(z^*)$ corresponds to the factor $Q_n(z)S_n(z)\tilde{S}_n(z)$ in P_n. For that factor we proved in Propositions 2.7, 2.9 and 2.10 that

$$\left|\frac{Q_n(z)S_n(z)\tilde{S}_n(z)}{Q_n(0)S_n(0)\tilde{S}_n(0)}\right| = 1 + o(1) \tag{2.45}$$

as $n \to \infty$ uniformly in $z \in \Gamma$, $|z| \leq M/n$. Since the ξ_j^* have the same property as the ξ_j^0 had, notably (2.30), the same proof (or direct verification) shows that

$$\left|\frac{U_m^*(z^*)}{U_m^*(0)}\right| = 1 + o(1) \tag{2.46}$$

as $m \to \infty$ uniformly in $|z| \leq M^*/m$ for any fixed M^*. The choice $m = [2\pi\omega_\Gamma(0)n]$ and formulae (2.30) and (2.43) show that for $j \in [-M^3, M^3]$, $j \neq 0$ we have

$$|\xi_j^0 - \xi_j^*| \leq C\left(\frac{M^3}{n}\right)^{1+\alpha}. \tag{2.47}$$

Also, for $z \in \Gamma$, $|z| \leq M/n$ we have $|z - z^*| \leq C(M/n)^{1+\alpha}$. These imply that if J_n is the set of those $\Re z \in \Gamma$ for which $|z| \leq M/n$ but $\mathrm{dist}(\Re z, j/n\omega_\Gamma(0)) \geq n^{-1-\alpha/2}$ for all $j = -M^3, \ldots, M^3$, $j \neq 0$, then

$$\begin{aligned}
\left|\frac{V_n(z)}{V_m^*(z^*)}\right| &= \prod_{-M^3 \leq j \leq M^3,\ j\neq 0} \left|1 + \frac{z - z^*}{z^* - \xi_j^*} - \frac{\xi_j^0 - \xi_j^*}{z^* - \xi_j^*}\right| \\
&\leq \left(1 + O\left(\frac{(M^3)^{1+\alpha}}{n^{\alpha/2}}\right)\right)^{2M^3} = 1 + o(1), \qquad z \in J_n
\end{aligned}$$

as $n \to \infty$. This implies

$$\left|\frac{V_n(z)}{V_n(0)}\right| = (1 + o(1))\left|\frac{V_m^*(z^*)}{V_m^*(0)}\right|$$

for all such z, and hence (see (2.45) and (2.46))

$$\left|\frac{P_n(z)}{P_n(0)}\right| = (1 + o(1))\left|\frac{P_m^*(z^*)}{P_m^*(0)}\right|.$$

Therefore, the integral of $|P_n(z)/P_n(0)|^2$ against ν over J_n is

$$\begin{aligned}
\int_{J_n\cap\Gamma} \left|\frac{P_n(z)}{P_n(0)}\right|^2 d\nu(z) &\leq (1 + o(1)) \int_{J_n\cap\Gamma^*} \left|\frac{P_m^*(z^*)}{P_m^*(0)}\right|^2 d\nu^*(z^*) \qquad (2.48) \\
&\leq (1 + o(1))\frac{2\pi w^*(0)}{m} = (1 + o(1))\frac{w(0)}{n\omega_\Gamma(0)},
\end{aligned}$$

where, in the second inequality, we used (2.44).

For the integrals over the sets

$$\left\{ z \,\middle|\, \left| \Re z - \frac{j}{n\omega_\Gamma(0)} \right| \leq n^{-1-\alpha/2} \right\}$$

with $j = -M^3 \cdots, M^3$, $j \neq 0$ we get from the Lebesgue-point property at 0 and from the fact that $|P_n(z)/P_n(0)|$ is uniformly bounded (see Proposition 2.5)

$$\int_{|\Re z - j/n\omega_\Gamma(0)| \leq n^{-1-\alpha/2}} \left| \frac{P_n(z)}{P_n(0)} \right|^2 w(z) ds_\Gamma(z)$$

$$\leq C \int_{|\Re z - j/n\omega_\Gamma(0)| \leq n^{-1-\alpha/2}} |w(z) - w(0)| ds_\Gamma(z)$$

$$+ C \int_{|\Re z - j/n\omega_\Gamma(0)| \leq n^{-1-\alpha/2}} |w(0)| ds_\Gamma(z)$$

$$= o(|j|/n) + O(n^{-1-\alpha/2}) = o(1/n).$$

There are $\leq CM$ such sets intersecting $\{|z| \leq M/n\}$, so their contribution to the whole integral of $|P_n/P_n(0)|^2$ against $w ds_\Gamma$ over $\Gamma \cap \{|z| \leq M/n\}$ is $o(1/n)$. The same can be done for the singular part, and with this and (2.48) we have verified

$$\int_{|z| \leq M/n} \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq (1 + o(1)) \frac{w(0)}{n\omega_\Gamma(0)}. \tag{2.49}$$

As for the integral over $|z| > M/n$, we use that there $|P_n(z)/P_n(0)|^2 \leq C/n^2|z|^2$ (see Proposition 2.5), as well as the fact that by the Lebesgue-point property

$$\int_{2^{k-1}M/n \leq |z| \leq 2^k M/n} d\nu(z) \leq C \frac{2^k M}{n}.$$

Therefore, we can write

$$\int_{|z| \geq M/n} \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq \sum_{k \geq 1} \int_{2^{k-1}M/n \leq |z| \leq 2^k M/n} \frac{C}{n^2|z|^2} d\nu(z)$$

$$\leq \sum_{k \geq 1} \frac{C}{n^2(2^{k-1}M/n)^2} \frac{2^k M}{n} \leq \frac{C}{Mn}.$$

This, together with (2.49), gives

$$\limsup_{n \to \infty} n \int \left| \frac{P_n(z)}{P_n(0)} \right|^2 d\nu(z) \leq \frac{w(0)}{\omega_\Gamma(0)} + \frac{C}{M},$$

and since here M is arbitrary, finally we obtain

$$\limsup_{n\to\infty} n \int \left|\frac{P_n(z)}{P_n(0)}\right|^2 d\nu(z) \le \frac{w(0)}{\omega_\Gamma(0)},$$

as was to be proved.

■

2.2 Part II: Γ_0 is a Jordan arc

Let again $\theta_k = \mu_\Gamma(\Gamma_k)$, and consider the integers $n_k = [\theta_k n]$, and divide again each Γ_k, $k > 0$, into n_j arcs I_j^k each having equal weight θ_k/n_k with respect to μ_Γ, i.e. $\mu_\Gamma(I_j^k) = \theta_k/n_k$. If we do the same division on Γ_0, then, unfortunately, we cannot guarantee any more that we can achieve that one of the ξ_j^0's has zero real part, i.e. in this case we cannot guarantee (with the previous notations) $\Re\xi_0^0 = 0$, which was crucial in the proof in Part I. So, when Γ_0 is a Jordan arc, we make the division of Γ_0 in such a way that this property hold: let I_0^0 be the unique arc (at least for large n it is unique) with the property that $0 \in I_0^0$, $\mu_\Gamma(I_0^0) = \theta_0/n_0$, and if ξ_0^0 is the center of mass of μ_Γ on I_0^0, then we have $\Re\xi_0^0 = 0$. (The unicity follows, since for $a \in \Gamma_0$, $\Re a < 0$ lying sufficiently close to 0 there is a unique $b \in \Gamma_0$, $\Re b > 0$ such that the arc $\widehat{ab}$ has μ_Γ-mass equal to θ_0/n_0, and, by the $C^{1+\alpha}$-smoothness of Γ_0, the real part of the center of mass of the arc $\widehat{ab}$ is strictly increasing as $\Re a$ does so). Now to the "left" resp. to the "right" of I_0^0 (in the direction of the two endpoints of Γ_0) consider the arcs $I_{-1}^0, I_{-2}^0, \ldots$ resp. $I_1^0, I_2^0, \ldots$ that continuously fill Γ_0 and have the property $\mu_\Gamma(I_j^0) = \theta_0/n_0$. We can select (including I_0^0) at least $n_0 - 1$ such arcs (we get stuck in the selection only when the remaining part around one of the endpoints of Γ_0 has μ_Γ-mass smaller than θ_0/n_0), however, it may happen that with this selection around the endpoints of Γ_0 there still remain two "little" arcs, say $I_{-l_0}^0$ and $I_{l_1}^0$ with $0 < \mu_\Gamma(I_{-l_0}^0) < \theta_0/n_0$ and $0 < \mu_\Gamma(I_{l_1}^0) < \theta_0/n_0$. We include these two small arcs also into our subdivision of Γ_0, so in this case we divide Γ_0 into $n_0 + 1$ arcs I_j^0, $j = -l_0, \ldots, l_1$.

Then $|\mu_\Gamma(I_j^k) - 1/n| \le C/n^2$ except for $k = 0$ and $j = -l_0$ or $j = l_1$, in which case $\mu_\Gamma(I_j^k)$ can be very small compared to $1/n$. Let ξ_j^k be the center of mass from (2.5) of the arc I_j^k with respect to μ_Γ, and consider the polynomial

$$R_n(z) = \prod_{j,k} (z - \xi_j^k) \tag{2.50}$$

of degree at most $n + 1$ (note that now the degree is not necessarily at most n since from Γ_0 we may get $n_0 + 1$ zeros namely the ξ_j^0's). Since $\Re\xi_0^0 = 0$, it is still true that ξ_0^0 lies closest to 0 among the ξ_j^k's. We claim that the polynomial

$$P_n(z) = R_n(z)/(z - \xi_0^0) \tag{2.51}$$

verifies Theorem 2.1.

Most of the proof remains the same, except for the proof of Proposition 2.4, which needs modifications when z belongs to the intervals $I^0_{-l_0}$, $I^0_{-l_0+1}$, $I^0_{l_1-1}$, $I^0_{l_1}$ – these require some substantial modifications because $I^0_{-l_0}$ or $I^0_{l_1}$ can be very short. Thus, let again $z \in I^l_{j_0}$ and we need to prove that

$$|P_n(z)| \le C\text{cap}(\Gamma)^n.$$

First of all now (2.13) contains the terms

$$\left(n - \frac{1}{\mu_\Gamma(I^0_j)}\right) \int_{I^0_j} \log|z-t|\mu_\Gamma(t) \tag{2.52}$$

with $j = -l_0, l_1$, and for these terms the coefficient

$$n - \frac{1}{\mu_\Gamma(I^0_j)}$$

is not bounded due to the fact that $\mu_\Gamma(I^0_j)$ can be very small. However, this coefficient is bounded from above, and for $\text{dist}(z, I^0_j) \le 1/2$ the integrand in (2.52) is negative, hence in this case

$$\left(n - \frac{1}{\mu_\Gamma(I^0_j)}\right) \int_{I^0_j} \log|z-t|d\mu_\Gamma(t) \ge -C \int_{I^0_j} |\log|z-t||d\mu_\Gamma(t) \ge -C,$$

while for $\text{dist}(z, I^0_j) \ge 1/2$ we just have the bound

$$\left|\left(n - \frac{1}{\mu_\Gamma(I^0_j)}\right) \int_{I^0_j} \log|z-t|d\mu_\Gamma(t)\right| \le C \left|n - \frac{1}{\mu_\Gamma(I^0_j)}\right| \int_{I^0_j} d\mu_\Gamma(t) \le C$$

since $\mu_\Gamma(I^0_j) \le 2/n$. Therefore, Σ_1 in (2.13) is bounded from below: $\Sigma_1 \ge -C$, and then (see (2.11))

$$\log|R_n(z)| - n\log\text{cap}(\Gamma) = \log|R_n(z)| - \Sigma_1 - \Sigma_2 \le C + \log|R_n(z)| - \Sigma_2,$$

so it is sufficient to prove again (2.15).

The boundedness of the individual terms $L_{j,k}$ follows as before except for $L_{-l_0,0}$, $L_{-l_0+1,0}$ when $z \in I^0_{-l_0} \cup I^0_{-l_0+1}$ or for $L_{l_1,0}$, $L_{l_1+1,0}$ when $z \in I^0_{l_1-1} \cup I^0_{l_1}$, in which cases it may not be true. But, as we shall show below, we can still claim the boundedness of these terms from above. We shall show this for $L_{-l_0,0}$, $L_{-l_0+1,0}$ when $z \in I^0_{-l_0} \cup I^0_{-l_0+1}$, the other case is similar.

For simpler notation let $J_1 = I^0_{-l_0}$, $\zeta_1 = \xi^0_{-l_0}$ and $J_2 := I^0_{-l_0+1}$, $\zeta_2 = \xi^0_{-l_0+1}$, hence E is an endpoint of the "short" arc J_1, and J_2 is the neighboring arc in the subdivision. The arc $\widehat{ab}$ plays different roles in different parts of the proof below;

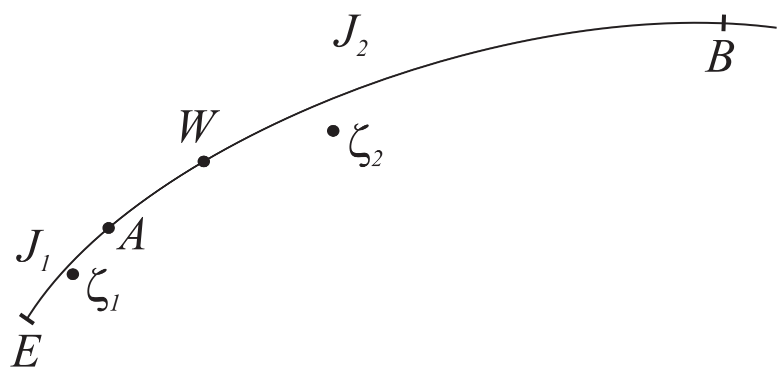

Figure 4: The choice of W

we shall always indicate its meaning. Let first $z \in J_1$, $\widehat{ab} := J_1 \cup J_2$. Then (since $a = E$ is an endpoint of the arc Γ_0 around which we have $\omega_\Gamma(t) \sim |t-a|^{-1/2}$ by Proposition 2.2)

$$\begin{aligned}
L_{-l_0+1,0} &= \frac{1}{\mu_\Gamma(J_2)} \int_{J_2} \log\left|\frac{z-\zeta_2}{z-t}\right| \omega_\Gamma(t) ds(t) \\
&\le \frac{C}{s(\widehat{ab})^{1/2}} \int_{J_2} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{at})^{-1/2} ds(t) \\
&\le \frac{C}{s(\widehat{ab})^{1/2}} \int_{\widehat{ab}} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{at})^{-1/2} ds(t),
\end{aligned}$$

and the expression on the right was treated in (2.24)–(2.25), so the same argument gives $L_{-l_0+1,0} \le C$. Next consider with $\widehat{ab} = J_1$ (z still being on J_1)

$$\begin{aligned}
L_{-l_0,0} &\le \frac{C}{\mu_\Gamma(J_1)} \int_{J_1} \log \frac{Cs(J_1)}{s(\widehat{zt})} s(\widehat{at})^{-1/2} ds(t) \\
&\le \frac{C}{s(\widehat{ab})^{1/2}} \int_{\widehat{ab}} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{at})^{-1/2} ds(t),
\end{aligned}$$

which is again what we handled in (2.24)–(2.25) (that argument works for short arcs like $\widehat{ab} = J_1$, as well) and we can conclude $L_{-l_0,0} \le C$.

When $z \in J_2$, the reasoning is similar for $L_{-l_0+1,0}$. Finally, let $z \in J_2$ and consider $L_{-l_0,0}$. Let W be the point on the arc $J_2 =: \widehat{AB}$ for which $s(\widehat{AW}) = s(J_1) = s(\widehat{EA})$ (there is such a W since $s(J_1) \le s(J_2) = \theta_0/n_0$), see Figure 4. If $z \in J_2$ but $z \notin \widehat{AW}$ then in

$$L_{-l_0,0} = \frac{1}{\mu_\Gamma(J_1)} \int_{J_1} \log\left|\frac{z-\zeta_1}{z-t}\right| \omega_\Gamma(t) ds(t)$$

we have

$$\frac{1}{3} \le \left| \frac{z-\zeta_1}{z-t} \right| \le 3,$$

so in this case $L_{-l_0,0} \le C$ is obvious. However, if $z \in \widehat{AW}$ then with $\widehat{ab} = \widehat{EW}$ we get

$$L_{-l_0,0} \le \frac{C}{s(\widehat{ab})^{1/2}} \int_{\widehat{ab}} \log \frac{Cs(\widehat{ab})}{s(\widehat{zt})} s(\widehat{at})^{-1/2} ds(t),$$

and $L_{-l_0,0} \le C$ follows again from the bound (2.25) for (2.24).

Once we have established the upper boundedness of the individual terms $L_{j,k}$ in (2.15), the rest of the argument in Proposition 2.4 remains the same.

Note also that there is no problem whatsoever with the lower and upper boundedness of the sum in (2.15)–(2.17) when we are not close to the endpoints of arc-components of Γ, and certainly this is the case for $z \in I_0^0$. Hence, the proof of Proposition 2.5 is unchanged, and then so is the rest of the proof of Theorem 2.1.

■

3 The lower estimate for the Christoffel functions in Theorem 1.1 for positive weights

The following theorem together with Theorem 2.1 completes the proof of Theorem 1.1 in the case when w is strictly positive on Γ.

Theorem 3.1 *Let Γ be a system of $C^{1+\alpha}$-smooth Jordan arcs and curves lying exterior to one another, $z_0 \in \Gamma$ not an endpoint of an arc-component of Γ and assume that Γ is C^2-smooth in a neighborhood of z_0. Assume that $d\nu = wds_\Gamma$ is a measure on Γ with continuous and positive density w. Then*

$$\liminf_{n\to\infty} n\lambda_n(z_0, \nu) \ge \frac{d\nu(z_0)}{d\mu_\Gamma}. \tag{3.1}$$

Indeed, the definition of the Christoffel functions shows that $\nu_1 \ge \nu_2$ implies $\lambda_n(z, \nu_1) \ge \lambda_n(z, \nu_2)$, so if the w in Theorem 1.1 is strictly positive, then we can just drop the singular part ν_{sing} from $\nu = \nu_a + \nu_{\text{sing}}$ and apply (3.1) to the absolutely continuous part $d\nu_a(t) = w(t)ds_\Gamma(t)$ to conclude from (3.1)

$$\liminf_{n\to\infty} n\lambda_n(z_0, \nu) \ge \liminf_{n\to\infty} n\lambda_n(z_0, \nu_a) \ge \frac{d\nu_a(z_0)}{d\mu_\Gamma} = \frac{d\nu(z_0)}{d\mu_\Gamma}.$$

In the proof of Theorem 3.1 let Ω be the unbounded component of $\mathbf{C} \setminus \Gamma$, and we denote by $g_{\mathbf{C}\setminus\Gamma}$ the Green's function of Ω with respect to the pole at infinity (see e.g. [17, Sec. 4.4]).

Proof of of Theorem 3.1. Without loss of generality assume $z_0 = 0$. Assume to the contrary that there are infinitely many n and for each n a polynomial Q_n of degree at most n such that $Q_n(0) = 1$ and

$$n \int |Q_n|^2 d\nu < (1-\beta) \frac{d\nu(0)}{d\mu_\Gamma} \tag{3.2}$$

with some $\beta > 0$. Our aim will be to show that this implies the following: there exists another system Γ^* of Jordan curves (no arcs!) such that $\Gamma \subseteq \Gamma^*$, in a neighborhood Δ_0 of 0 we have $\Gamma \cap \Delta_0 = \Gamma^* \cap \Delta_0$, and there is a measure ν^* on Γ^* with positive and continuous density which coincides with ν on Γ for which, at 0, we have

$$\liminf_{n\to\infty} n\lambda_n(0, \nu^*) < \frac{d\nu^*(0)}{d\mu_{\Gamma^*}}. \tag{3.3}$$

Since this contradicts [25, Theorem 1.1], it follows that (3.2) cannot be true, i.e. (3.1) holds.

Let $\Gamma_0, \ldots, \Gamma_{k_0}$ be the connected components of Γ, Γ_0 being the one that contains 0. First we deal with the case when Γ_0 is a Jordan arc—after that we shall indicate what changes are necessary when Γ_0 is a Jordan curve. Let $\mathbf{n}_\pm$ be the two normals to Γ_0 at 0, and let $A_\pm = \partial g_{\mathbf{C}\setminus\Gamma}(0)/\partial \mathbf{n}_\pm$ be the corresponding normal derivatives of the Green's function of $\mathbf{C}\setminus\Gamma$ with pole at infinity. Assume, for example, that $A_+ \geq A_-$. Note that necessarily $A_- > 0$. In fact, there is a small closed disk D containing 0 that lies on the side of Γ (i.e. lies outside except for the point 0) which is determined by the direction of the normal $\mathbf{n}_-$. For simplicity assume that D is the disk $\{z \mid |z-1| = 1\}$. Then $g_{\mathbf{C}\setminus\Gamma}(z+1)$ is harmonic in the unit disk and continuous on its boundary (this follows from the C^2-property of Γ_0), hence from Poisson's formula we easily get

$$g_{\mathbf{C}\setminus\Gamma}(z+1) \geq (1-|z|) g_{\mathbf{C}\setminus\Gamma}(1), \qquad |z| < 1.$$

Therefore, with $z = -1 + t\mathbf{n}_-$, for small $t > 0$ we have $g_{\mathbf{C}\setminus\Gamma}(t\mathbf{n}_-) \geq t g_{\mathbf{C}\setminus\Gamma}(1)$, from which $A_- \geq g_{\mathbf{C}\setminus\Gamma}(1)$ follows.

Let $\varepsilon > 0$ be an arbitrarily small number. For each Γ_j that is a Jordan arc (i.e. NOT a Jordan curve), connect the two endpoints of Γ_j by another Jordan arc Γ_j' that lies close to Γ_j so that we obtain a system Γ' of k_0+1 Jordan curves with boundary $(\cup_j \Gamma_j) \bigcup (\cup_j \Gamma_j')$. Assume also that Γ_0' is selected so that $\mathbf{n}_+$ is the outer normal to Γ' at 0. This can be done in such a way that

$$\frac{\partial g_{\mathbf{C}\setminus\Gamma'}(0)}{\partial \mathbf{n}_+} > \frac{1}{1+\varepsilon} \frac{\partial g_{\mathbf{C}\setminus\Gamma}(0)}{\partial \mathbf{n}_+} \tag{3.4}$$

(note that since the unbounded component of $\mathbf{C} \setminus \Gamma'$ is part of the unbounded component of $\mathbf{C} \setminus \Gamma$, we necessarily have $\partial g_{\mathbf{C}\setminus\Gamma'}(0)/\partial \mathbf{n}_+ \leq \partial g_{\mathbf{C}\setminus\Gamma}(0)/\partial \mathbf{n}_+$). Indeed, to see (3.4) if Γ' is sufficiently close to Γ, we can apply [12, Lemma 7.1] since, as Γ' tends to Γ, we have $g_{\mathbf{C}\setminus\Gamma'}(z) \to g_{\mathbf{C}\setminus\Gamma'}(z)$ locally uniformly on compact subsets of the unbounded component Ω of $\mathbf{C} \setminus \Gamma$.

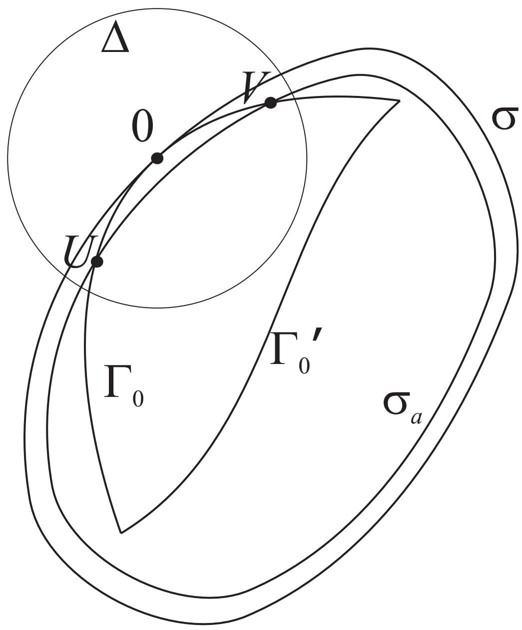

Figure 5: Illustration of the case of a single Jordan arc

Select a small disk Δ_0 about 0 for which $\Gamma' \cap \Delta_0 = \Gamma \cap \Delta_0$. By [12, Theorem 1.2] we can choose a lemniscate $\sigma = \{z \mid |T_N(z)| = 1\}$ (with some polynomial T_N of degree equal to some integer N) such that Γ' lies in the interior of σ (i.e. in the union of the bounded components of $\mathbf{C} \setminus \sigma$) except for the point 0, where σ and Γ' touch each other, and

$$\frac{\partial g_{\mathbf{C}\setminus\sigma}(0)}{\partial \mathbf{n}_+} > \frac{1}{1+\varepsilon} \frac{\partial g_{\mathbf{C}\setminus\Gamma}(0)}{\partial \mathbf{n}_+}. \tag{3.5}$$

By [12, Theorem 1.2] this σ can be chosen so that it has precisely $k_0 + 1$ components each containing one-one component of Γ', and if τ_0' denotes the signed curvature of Γ' at 0 seen from the outside, then in a neighborhood of 0 the signed curvature τ_0 of σ is smaller than τ_0'. Since the Green's function $g_{\mathbf{C}\setminus\sigma}(z)$ is just $(\log |T_N(z)|)/N$, simple computation shows (see formula [25, (2.2)]) that

$$\frac{\partial g_{\mathbf{C}\setminus\sigma}(0)}{\partial \mathbf{n}_+} = \frac{|T_N'(0)|}{N}. \tag{3.6}$$

Let, for a small a to be determined later, σ_a be the lemniscate $\sigma_a := \{z \mid |T_N(z)| = e^{-a}\}$. If $\Delta \subset \Delta_0$ is a fixed small neighborhood of 0, then for sufficiently small a this σ_a contains $\Gamma \setminus \Delta$ in its interior (i.e. in the interior of its components), while in Δ the two curves Γ_0 and σ_a intersect in two points U, V; see Figure 5. In fact, this is due to the fact that for small Δ and a the maximal signed curvature of σ_a in Δ, which is close to τ_0, is smaller than the minimal curvature of Γ' in Δ, which is close to $\tau_0' > \tau_0$ (and recall also that Γ and Γ' coincide in Δ_0). Now the points U and V are connected by the arc $\widehat{UV}_{\Gamma_0}$ on Γ_0 (which is the same as on Γ) and also by the arc $\widehat{UV}_{\sigma_a}$ on σ_a (there are

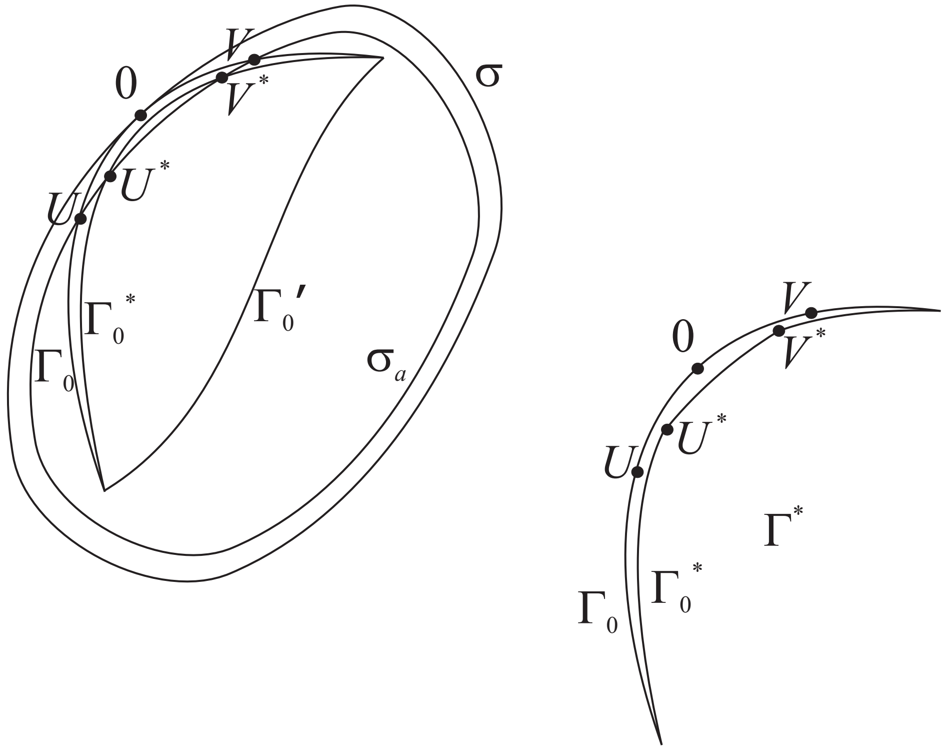

Figure 6: The case of one Jordan curve and the formation of Γ^*

actually two such arcs on σ_a, we take the one lying in Δ). For each Γ_j which is a Jordan arc connect the two endpoints of Γ_j by a new C^2 Jordan arc Γ_j^* going inside Γ' so that on Γ_j^* we have

$$g_{C\setminus\Gamma}(z) \le a^2, \qquad z \in \Gamma_j^*. \tag{3.7}$$

In addition, Γ_0^* can be selected so that in Δ it intersects σ_a in two points U^*, V^*; see Figure 6. Then $\widehat{U^*V^*}_{\sigma_a}$ is a subarc of $\widehat{UV}_{\sigma_a}$. Let now Γ^* be the union of Γ, of the Γ_j^*'s with $j > 0$, of $\Gamma_0^* \setminus \widehat{U^*V^*}_{\Gamma_0^*}$ and of $\widehat{U^*V^*}_{\sigma_a}$; see Figure 6. This Γ^* is the union of $k_0 + 1$ Jordan curves, and it is contained in σ_a and its interior except for the arc $\widehat{UV}_{\Gamma_0}$. Furthermore, Γ^* lies within σ and contains Γ, so

$$\frac{\partial g_{\mathbf{C}\setminus\sigma}(0)}{\partial \mathbf{n}_+} \le \frac{\partial g_{\mathbf{C}\setminus\Gamma^*}(0)}{\partial \mathbf{n}_+} \le \frac{\partial g_{\mathbf{C}\setminus\Gamma}(0)}{\partial \mathbf{n}_+}. \tag{3.8}$$

Clearly, for any $m = 1, 2, \ldots$

$$|T_N(z)|^m \le \begin{cases} e^{-am} & z \in \Gamma^* \setminus \widehat{UV}_{\Gamma_0} \\ 1 & z \in \widehat{UV}_{\Gamma_0}. \end{cases} \tag{3.9}$$

For the Q_n from (3.2) we get

$$\int_\Gamma |Q_n|^2 ds_\Gamma \le C_0/n$$

with some C_0 (recall that w is continuous and positive), and hence the inequality from Lemma 3.2 below gives that for its supremum norm $\|Q_n\|_\Gamma$ we have

$$\|Q_n\|_\Gamma \le C_1 n^{1/2} \tag{3.10}$$

with some C_1 independent of n. Therefore, by the Bernstein-Walsh inequality [29, p. 77] the estimate

$$|Q_n(z)| \le C_1 n^{1/2} e^{n g_{\mathbf{C}\setminus\Gamma}(z)} \tag{3.11}$$

follows everywhere on the complex plane. In particular, in view of (3.7)

$$|Q_n(z)| \le C_1 n^{1/2} e^{na^2}, \qquad z \in \Gamma^* \setminus \widehat{U^*V^*}_{\sigma_a} \tag{3.12}$$

(note that the part $\widehat{U^*V^*}_{\sigma_a}$ of Γ^* may lie outside $\Gamma_0 \cup \Gamma_0^*$, so there (3.7) is not applicable).

We shall also need to estimate $g_{\mathbf{C}\setminus\Gamma}$ on $\widehat{U^*V^*}_{\sigma_a}$ to get a bound for the polynomials Q_n there ((3.12) is not applicable there). We shall actually do the estimate on $\widehat{UV}_{\sigma_a}$, which contains $\widehat{U^*V^*}_{\sigma_a}$. The lens-shaped region enclosed by $\widehat{UV}_{\Gamma_0} \cup \widehat{UV}_{\sigma_a}$ is contained in a neighborhood Δ_a of 0 where this Δ_a shrinks to 0 as $a \to 0$ (here a is not the radius of Δ_a, just signals that Δ_a depends on a). For small a we have uniformly in $z \in \Delta_a \cap \Gamma_0$

$$\frac{\partial g_{\mathbf{C}\setminus\Gamma}(z)}{\partial \mathbf{n}_-} \le (1+\varepsilon)A_-,$$

which easily implies (note also that $g_{\mathbf{C}\setminus\Gamma}$ is a $C^{1+\alpha}$ smooth function—see the reasoning in the Appendix below) that for small a

$$g_{\mathbf{C}\setminus\Gamma}(z) \le (1+\varepsilon)^2 b A_-, \qquad z \in \widehat{UV}_{\sigma_a} \tag{3.13}$$

where b is the largest distance from a point $z \in \widehat{UV}_{\sigma_a}$ to Γ_0. This b is at most as large as the largest distance b' from a point $z \in \widehat{UV}_{\sigma_a}$ to σ. Next, we estimate this b'. Since for small a

$$|T_N'(t) - T_N'(0)| = O(|t|) \le \varepsilon |T_N'(0)|$$

in Δ_a, it follows that

$$b \le b' \le \frac{1}{1-\varepsilon} \frac{1-e^{-a}}{|T_N'(0)|} \le (1+\varepsilon)^2 \frac{a}{|T_N'(0)|}. \tag{3.14}$$

Indeed, for a $z \in \widehat{UV}_{\sigma_a}$ let Z be the closest point on σ such that (modulo 2π) $\arg(T_N(z))$ $=\arg(T_N(Z))$. Then

$$1 - e^{-a} \quad = \quad |T_N(Z) - T_N(z)| = \left| \int_z^Z T_N'(t)\,dt \right|$$

$$\geq \left|\int_z^Z T_N'(0)dt\right| - \left|\int_z^Z |T_N'(t) - T_N'(0)|dt\right|$$
$$\geq (1-\varepsilon)|T_N'(0)||z-Z|,$$

from which we get for small a and appropriate $z \in \widehat{UV}_{\sigma_a}$

$$b' \leq |z-Z| \leq \frac{1}{1-\varepsilon}\frac{1-e^{-a}}{|T_N'(0)|}.$$

In view of (3.13) and (3.14) we have on $\widehat{UV}_{\sigma_a}$ the estimate

$$g_{\mathbf{C}\setminus\Gamma}(z) \leq \frac{(1+\varepsilon)^4 aA_-}{|T_N'(0)|},$$

and hence, by (3.11),

$$|Q_n(z)| \leq C_1 n^{1/2} \exp\left(n(1+\varepsilon)^4 aA_-/|T_N'(0)|\right), \qquad z \in \widehat{UV}_{\sigma_a}. \tag{3.15}$$

Now consider with

$$m = \left[(1+\varepsilon)^7 A_- n/NA_+\right] \tag{3.16}$$

the polynomial

$$P_{n+mN}(z) = Q_n(z)T_N(z)^m \tag{3.17}$$

on Γ^*, and let the measure ν^* be equal to ν on Γ and equal to the arc measure s_{Γ^*} on $\Gamma^* \setminus \Gamma$. The density w^* of ν^* with respect to σ_{Γ^*} may not be continuous at the endpoints of those components of Γ that are Jordan arcs, but this will not bother us below (alternatively, one could easily choose a continuous w^*). For this polynomial we have on $\Gamma^* \setminus \left(\widehat{UV}_{\Gamma_0} \cup \widehat{U^*V^*}_{\sigma_a}\right)$ (see (3.9) and (3.12))

$$|P_{n+mN}(z)| \leq C_1 n^{1/2} e^{na^2 - ma}, \tag{3.18}$$

on $\widehat{UV}_{\Gamma_0}$ the bound

$$|P_{n+mN}(z)| \leq |Q_n(z)| \tag{3.19}$$

and on $\widehat{U^*V^*}_{\sigma_a}$ the estimate

$$|P_{n+mN}(z)| \leq C_1 n^{1/2} \exp\left(n(1+\varepsilon)^4 aA_-/|T_N'(0)| - ma\right) \tag{3.20}$$

(see (3.15) and (3.9)). Here, by the choice of m in (3.16) and by (3.5) and (3.6) the quantity in the exponent is at most

$$n\left(\frac{(1+\varepsilon)^5 aA_-}{A_+N} - \frac{(1+\varepsilon)^6 aA_-}{NA_+}\right) = -\varepsilon n \frac{(1+\varepsilon)^5 aA_-}{NA_+}.$$

Fix a so small that we have $a^2 - aA_-/NA_+ < 0$. Then the estimates (3.18)–(3.20) yield

$$\lambda_{n+mN}(0,\nu^*) \le \int |P_{n+mN}|^2 w^* ds_{\Gamma^*} \le \int |Q_n|^2 w ds_\Gamma + O(n^{-2}).$$

Hence, by (3.2), if $\omega_\Gamma := d\mu_\Gamma/ds_\Gamma$ is the density of the equilibrium measure μ_Γ of Γ with respect to arc measure, then for infinitely many n

$$(n+mN)\lambda_{n+mN}(0,\nu^*) \le \frac{n+mN}{n}(1-\beta)\frac{w(0)}{\omega_\Gamma(0)} + o(1). \tag{3.21}$$

It is well known (see e.g. (5.3) below) that

$$\omega_\Gamma(0) = \frac{1}{2\pi}\left(\frac{\partial g_{\mathbf{C}\setminus\Gamma}}{\partial \mathbf{n}_+} + \frac{\partial g_{\mathbf{C}\setminus\Gamma}}{\partial \mathbf{n}_-}\right) = \frac{1}{2\pi}(A_+ + A_-) \tag{3.22}$$

and

$$\omega_{\Gamma^*}(0) = \frac{1}{2\pi}\frac{\partial g_{\mathbf{C}\setminus\Gamma^*}(0)}{\partial \mathbf{n}_+} \le \frac{1}{2\pi}\frac{\partial g_{\mathbf{C}\setminus\Gamma}(0)}{\partial \mathbf{n}_+} = \frac{1}{2\pi}A_+, \tag{3.23}$$

so

$$\begin{aligned}\frac{n+mN}{n}(1-\beta)\frac{w(0)}{\omega_\Gamma(0)} &\le \left(1+(1+\varepsilon)^7\frac{A_-}{A_+}\right)(1-\beta)\frac{w(0)}{\omega^*_\Gamma(0)}\frac{A_+}{A_+ + A_-}\\ &\le \left(1-\frac{\beta}{2}\right)\frac{w(0)}{\omega^*_\Gamma(0)}\end{aligned}$$

if ε is sufficiently small. Therefore, (3.21) implies

$$\liminf_{n\to\infty}(n+mN)\lambda_{n+mN}(0,\nu^*) \le \left(1-\frac{\beta}{2}\right)\frac{w(0)}{\omega^*_\Gamma(0)},$$

which is impossible, since, according to [25, Theorm 1.1] (applicable to the family Γ^* of finitely many Jordan curves and to the measure ν^* on it)

$$\lim_{n\to\infty}(n+mN)\lambda_{n+mN}(0,\nu^*) = \frac{w(0)}{\omega^*_\Gamma(0)}.$$

This contradiction emerged since we assumed (3.2), and so (3.1) has been proven.

■

Next, consider the proof of Theorem 3.1 in the case when Γ_0 is a Jordan curve. In that case $A_- = 0$. We construct Γ' as before, and select again a lemniscate $\sigma = \{z \mid |T_N(z)| = 1\}$ that contains Γ' in its interior except for the point 0 where it touches Γ', and for which (3.5) is true. Now construct Γ^* inside

Γ' as before satisfying (3.7), and let ν^* agree with ν on Γ and with s_{Γ^*} on $\Gamma^* \setminus \Gamma$. There is an $a > 0$ such that $|T_N(z)| \le e^{-a}$ for $z \in \Gamma^* \setminus \Gamma$ (recall that σ contains Γ' in its interior except for the point 0, and now there is no Γ_0^* because the Γ_j^*'s were constructed only for those j for which Γ_j is a Jordan arc).

With $m = [n\beta/N]$ (recall that β is from 3.2) consider the polynomial P_{n+mN} from (3.17). For it we have on Γ the inequality $|P_{n+mN}(z)| \le |Q_n(z)|$, while on $\Gamma^* \setminus \Gamma$ we have

$$|P_{n+mN}(z)| \le C_1 n^{1/2} e^{na^2 - ma},$$

which implies for small a just as before

$$\lambda_{n+mN}(0, \nu^*) \le \int |P_{n+mN}|^2 w^* ds_{\Gamma^*} \le \int |Q_n|^2 w ds_\Gamma + O(n^{-2}).$$

Hence

$$\liminf_{n\to\infty} (n + mN)\lambda_{n+mN}(0, \nu^*) \le (1 + \beta)(1 - \beta)\frac{w(0)}{\omega_\Gamma(0)} \le (1 - \beta^2)\frac{w^*(0)}{\omega_{\Gamma^*}(0)},$$

since $w^*(0) = w(0)$, and for the density $\omega_\Gamma = d\mu_\Gamma/ds_\Gamma$ we have (see (3.22) and (3.23))

$$\omega_\Gamma(0) = \frac{1}{2\pi}\frac{\partial g_{\mathbf{C}\setminus\Gamma}(0)}{\partial \mathbf{n}_+} \ge \frac{1}{2\pi}\frac{\partial g_{\mathbf{C}\setminus\Gamma^*}(0)}{\partial \mathbf{n}_+} = \omega_{\Gamma^*}(0).$$

This again contradicts [25, Theorem 1.1], and the proof is complete.

■

The proof above used the following lemma.

Lemma 3.2 *With the assumptions of Theorem 3.1, there is a constant C such that if Q_n is a polynomial of degree at most n then*

$$\|Q_n\|_\Gamma \le Cn\|Q_n\|_{L^2(s_\Gamma)}, \tag{3.24}$$

where, on the left-hand side, the norm is the supremum norm on Γ.

Proof. Let M be the maximum of $|Q_n(z)|$ on Γ. By [25, Corollary 7.2] (applied to each one of the components of Γ) we have

$$|Q_n'(z)| \le C_1 M n^2, \qquad \text{for } \operatorname{dist}(z, \Gamma) \le 1/n^2$$

with some constant $C_1 \ge 1$. Therefore, if $z_0 \in \Gamma$ is a place with $|Q_n(z_0)| = M$, then for $|z - z_0| \le 1/2C_1 n^2$ we have $|Q_n(z)| \ge M/2$. The s_Γ-measure of the set of these z's is at least $1/2C_1 n^2$, hence

$$\int |Q_n|^2 ds_\Gamma \ge (M/2)^2/2C_1 n^2,$$

from which the claim follows.

■

4 Proof of Theorems 1.1 and 1.2

So far we have established Theorem 1.1 for the case when w is strictly positive. Now we can easily complete the

Proof of Theorem 1.1. That

$$\limsup_{n\to\infty} n\lambda_n(z,\nu) \le \frac{w(z_0)}{\omega_\Gamma(z_0)}, \tag{4.1}$$

was proven in Theorem 2.1.

In particular, if $w(z_0)=0$, then (1.4) is true, so in establishing the matching lower bound to (4.1) we may assume that $w(z_0)>0$. Let Σ be the set of zeros of w, and for a small $\tau>0$ let Σ_τ be the τ-neighborhood of Σ. The set $\Gamma_\tau := \Gamma \setminus \Sigma_\tau$ consists of finitely many Jordan curves and arcs, some of which may be degenerated (may consist of a single point), which we discard from Γ_τ. If τ is sufficiently small, then z_0 is a point on Γ_τ which is not an endpoint of any of Γ_τ's components. Now on Γ_τ the measure $\nu_\tau := \nu_{|\Gamma_\tau}$ has already a strictly positive density w with respect to the arc measure, so we can apply the already proven case to it to conclude that

$$\liminf_{n\to\infty} n\lambda_n(z_0,\nu) \ge \liminf_{n\to\infty} n\lambda_n(z_0,\nu_\tau) \ge \frac{w(z_0)}{\omega_{\Gamma_\tau}(z_0)},$$

so it has remained to show that on the right-hand side $\omega_{\Gamma_\tau}(z_0)$ tends to $\omega_\Gamma(z_0)$ as $\tau\to 0$.

To this end first we prove that the logarithmic capacity $\mathrm{cap}(\Gamma_\tau)$ tends to the capacity $\mathrm{cap}(\Gamma)$ of Γ, and in doing so we may assume that Γ lies inside the disk $\{|z|\le 1/2\}$ (apply a homothetic transformation). The equilibrium measure μ_Γ is absolutely continuous with respect to the arc measure s_Γ (see Proposition 2.2), hence $\alpha_n := \mu_\Gamma(\Gamma_\tau)$ tends to 1 as $\tau\to 0$. Now the measure

$$\mu_n := \frac{1}{\alpha_n}\mu_{\Gamma|\Gamma_\tau}$$

is a positive unit measure on Γ_τ for which the logarithmic energy

$$\begin{aligned} I(\mu_n) := \int\int \log\frac{1}{|z-t|}d\mu_n(z)d\mu_n(t) &\le \frac{1}{\alpha_n^2}\int\int \log\frac{1}{|z-t|}d\mu_\Gamma(z)d\mu_\Gamma(t) \\ &= \frac{1}{\alpha_n^2}I(\mu_\Gamma). \end{aligned}$$

Therefore,

$$I(\mu_{\Gamma_\tau}) \leq \frac{1}{\alpha_n^2} I(\mu_\Gamma)$$

because μ_{Γ_τ} minimizes the logarithmic energy. Hence

$$\mathrm{cap}(\Gamma_\tau) = \exp(-I(\mu_{\Gamma_\tau})) \geq \exp(-I(\mu_\Gamma))^{1/\alpha_n^2} = \mathrm{cap}(\Gamma)^{1/\alpha_n^2},$$

from which $\mathrm{cap}(\Gamma_\tau) \to \mathrm{cap}(\Gamma)$ follows (note that $\Gamma_\tau \subseteq \Gamma$ implies $\mathrm{cap}(\Gamma_\tau) \leq \mathrm{cap}(\Gamma)$).

The function $g_{\Gamma_\tau}(z) - g_\Gamma(z)$ is nonnegative and harmonic in Ω (the exterior of Γ) including infinity, and at infinity it takes the value (see [19, (I.4.8)] or [17, p. 107]) $\log(\mathrm{cap}(\Gamma)/\mathrm{cap}(\Gamma_\tau))$, which tends to 0 as $\tau \to 0$ by what we have just established. Hence, by Harnack's principle, this function tends 0 (as $\tau \to 0$) locally uniformly in Ω. From this it follows via the maximum principle that $g_{\Gamma_\tau}(z) - g_\Gamma(z)$ tends to 0 locally uniformly inside any connected bounded component of $\mathbf{C} \setminus \Gamma$, as well (these are the interiors of those components of Γ that are Jordan curves). This and the fact that for sufficiently small $\tau > 0$ we have $g_{\Gamma_\tau}(z) - g_\Gamma(z) = 0$ on any small fixed arc $J \subset \Gamma$ about z_0 (so small that on J the function w is strictly positive) implies, by [12, Lemma 7.1], that if $\mathbf{n}$ is either of the normals to Γ at z_0, then

$$\frac{\partial g_{\Gamma_\tau}(z_0)}{\partial \mathbf{n}} \to \frac{\partial g_\Gamma(z_0)}{\partial \mathbf{n}} \qquad \text{as} \quad \tau \to 0.$$

Now the claim $\omega_{\Gamma_\tau}(z_0) \to \omega_\Gamma(z_0)$ as $\tau \to 0$ follows from here and from formula (5.3) below.

■

Finally, we give the

Proof of Theorem 1.2. The upper estimate (2.1) was given in Theorem 2.1, and that theorem holds under the assumptions of Theorem 1.2, so (2.1) is true.

In the proof of the lower estimate (3.1) the only place where we used the strict positivity of w was (3.10) (proved in Lemma 3.2), and it is clear from the proof that (3.10) can be replaced by

$$\|Q_n\|_\Gamma = e^{o(n)}. \tag{4.2}$$

But this is true in our case, since $\nu \in \mathbf{Reg}$ and

$$\int |Q_n|^2 d\nu \leq \frac{C}{n}$$

imply (4.2) (see (1.3)). Thus, (3.1) is also true under the conditions of Theorem 1.2, and hence Theorem 1.2 follows.

■

5 Appendix

Proof of Proposition 2.2. In short, the proof is that ω_Γ is given by the normal derivative of the Green's function (see formula (5.3) below) $g_{\mathbf{C}\setminus\Gamma}$, and away from the endpoints of the arc components of Γ, this Green's function is $C^{1+\alpha}$ smooth on Γ due to the $C^{1+\alpha}$ smoothness of Γ. We shall use a standard localization technique. The details are as follows.

As has already been said, the $\alpha > 0$ in the $C^{1+\alpha}$ smoothness assumption is assumed to be less than 1. First of all, note that the Green's function $g_{\mathbf{C}\setminus\Gamma}$ is continuous on $\mathbf{C}$ by Wiener's criterion [17, Theorem 5.4.1].

First, let J be a closed arc on Γ not containing an endpoint of an arc-component of Γ. Let G be a simply connected domain with $C^{1+\alpha}$ boundary that lies in the unbounded component Ω of $\mathbf{C}\setminus\Gamma$ such that J lies on the boundary of G, and let Φ be a conformal map from the unit disk Δ onto G. By the Kellogg-Warschawski theorem (see [16, Theorem 3.6]) this Φ is $C^{1+\alpha}$ on the closed unit disk and it has a nonzero derivative there. The function $h(z) = g_{\mathbf{C}\setminus\Gamma}(\Phi(z))$ is harmonic in Δ and continuous on the closed unit disk, so we have Poisson's formula for it:

$$h(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} h(e^{it})dt. \tag{5.1}$$

If J' is the arc of the unit circle that is mapped by Φ into J, then $h(e^{it}) = 0$ on J', so it follows from (5.1) that h (considered as a function on the closed unit disk) is C^∞ on any closed subarc of the interior of J'. Hence $g_{\mathbf{C}\setminus\Gamma}(z) = h(\Phi^{-1}(z))$ is $C^{1+\alpha}$-smooth on any closed subarc of the interior of J. Furthermore, (5.1) gives also that

$$h(re^{it}) \geq \frac{1-r}{1+r}\frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{it})dt = \frac{1-r}{1+r}h(0) > 0,$$

which gives via the mapping Φ

$$g_{\mathbf{C}\setminus\Gamma}(z + t\mathbf{n}) \geq ct$$

for any $z \in J$ with a positive constant $c > 0$ depending only on G, where $\mathbf{n}$ is the normal to Γ at z in the direction of G. As a consequence,

$$\frac{g_{\mathbf{C}\setminus\Gamma}}{\partial\mathbf{n}}(z) \geq c, \qquad z \in J. \tag{5.2}$$

Now all we need to do is to cite that in the interior of J we have (see e.g. [14, II.(4.1)] or [19, Theorem IV.2.3] and [19, (I.4.8)])

$$\omega_\Gamma(z) = \frac{1}{2\pi}\left(\frac{g_{\mathbf{C}\setminus\Gamma}}{\partial\mathbf{n}_+}(z) + \frac{g_{\mathbf{C}\setminus\Gamma}}{\partial\mathbf{n}_-}(z)\right), \tag{5.3}$$

where $\mathbf{n}_\pm$ are the two normals to Γ at z. The C^α smoothness of ω_Γ on J follows from the $C^{1+\alpha}$-smoothness of $g_{\mathbf{C}\setminus\Gamma}$ there, while the positivity is a consequence of (5.2) (where $\mathbf{n}$ is one of $\mathbf{n}_\pm$ pointing to Ω and note also that both normal derivatives in (5.3) are nonnegative).

Next, let the arc J contain an endpoint of an arc-component of Γ. We may assume that this endpoint is 0, and the positive semi-axis is a tangent to J at 0. Then in some small neighborhood of 0 the arc J has parametrization $t + i\gamma(t)$, $0 \le t \le t_0$ where $\gamma(0) = \gamma'(0) = 0$ and γ' is Lip α continuous on $[0, t_0]$. Hence $|\gamma'(t)| \le Ct^\alpha$ and $|\gamma(t)| \le Ct^{1+\alpha}$. Consider a small disk D_ρ with center at 0 and of radius ρ, and in $D_\rho \setminus J$ take the branch of $\sqrt{z}$ for which $\sqrt{it} = \sqrt{t}(1+i)/\sqrt{2}$ for $t > 0$. Then $w = \sqrt{z}$ maps $D_\rho \setminus J$ into a set D^* which is a subset of $D_{\sqrt{\rho}}$ the boundary of which consists of two parts: a half-circle of $D_{\sqrt{\rho}}$ and an arc J^*, which is the union of the two images $\pm J^*$ of J under this map (the two images are symmetric with respect to the origin). One of these images, say J^* has representation $\theta + i\sigma(\theta)$ (and the symmetric part has then the representation $-\theta - i\sigma(\theta)$) with $0 \le \theta \le \theta_0$ where $(\theta + i\sigma(\theta))^2 = t + i\gamma(t)$. Straightforward calculation gives that then $\sigma(0) = \sigma'(0) = 0$ and σ' is in the class Lip α. As a consequence, $J^* \cup (-J^*)$ is again $C^{1+\alpha}$ smooth. Now the argument that we used above gives that then $g^*(z) := g_{\mathbf{C}\setminus\Gamma}(z^2)$ defined on D^* is of class $C^{1+\alpha}$ on the (one dimensional) interior of $J^* \cup (-J^*)$ with positive and C^α-smooth normal derivatives there. Now if $z_0 \in D_\rho \cap J$ is any point, then the normal vector $\mathbf{n}^*$ at $\sqrt{z_0} \in J^*$ to J^* in the direction of D^* and (one of the) normal vector $\mathbf{n}$ at z_0 to J is related by $\mathbf{n} = (2\sqrt{z_0}/2\sqrt{|z_0|})\mathbf{n}^*$ since around $\sqrt{z_0}$ the mapping $z \to z^2$ is like multiplication by $2\sqrt{z_0}$. Hence

$$\begin{aligned}
\frac{\partial g^*}{\partial \mathbf{n}^*}(\sqrt{z_0}) &= \lim_{t\to 0+0} \frac{g((\sqrt{z_0} + t\mathbf{n}^*)^2) - g((\sqrt{z_0})^2)}{t} \\
&= \lim_{t\to 0+0} \frac{g(z_0 + 2\sqrt{z_0}t\mathbf{n}^* + O(t^2)) - g(z_0)}{2\sqrt{|z_0|}t} 2\sqrt{|z_0|} \\
&= 2\sqrt{|z_0|}\frac{\partial g_{\mathbf{C}\setminus\Gamma}}{\partial \mathbf{n}}(z_0).
\end{aligned}$$

This implies, in view of the fact that $\partial g^*/\partial \mathbf{n}^*$ is positive and Lip α around 0, that

$$\frac{\partial g_{\mathbf{C}\setminus\Gamma}}{\partial \mathbf{n}}(z_0) \sim 1/\sqrt{|z_0|}. \tag{5.4}$$

A similar formula is true for the normal derivative with respect to the other normal to J at z_0. Now $\omega_\Gamma(z_0) \sim 1/\sqrt{|z_0|}$ follows from these and from formula (5.3).

An alternative proof of (5.4) is to use [16, Theorem 3.9], which implies (5.4) for $g_{\mathbf{C}\setminus\Gamma_k}$ (the Green's function of the complement of the arc component Γ_k in question), and use the comparison $g_{\mathbf{C}\setminus\Gamma} \le g_{\mathbf{C}\setminus\Gamma_k} \le Cg_{\mathbf{C}\setminus\Gamma}$ valid in some neighborhood of Γ_k (apply the maximum principle in that neighborhood to the difference $g_{\mathbf{C}\setminus\Gamma_k} - g_{\mathbf{C}\setminus\Gamma}$).

$\blacksquare$

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