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On the variance of the mean width of random polytopes circumscribed around a convex body

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Abstract

Let K be a convex body in $\mathbb{R}^d$ in which a ball rolls freely and which slides freely in a ball. Let $K^{(n)}$ be the intersection of n i.i.d. random half-spaces containing K chosen according to a certain prescribed probability distribution. We prove an asymptotic upper bound on the variance of the mean width of $K^{(n)}$ as $n \to \infty$. We achieve this result by first proving an asymptotic upper bound on the variance of the weighted volume of random polytopes generated by n i.i.d. random points selected according to certain probability distributions, then, using polarity, we transfer this to the circumscribed model.

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1 | INTRODUCTION AND RESULTS

In this paper, we study both random polytopes contained in a convex body and random polyhedral sets that contain a convex body. In the literature, the overwhelming majority of results are about the former types of models. Our results are asymptotic upper bounds on variances and laws of large numbers. The first-order asymptotic properties of random polytopes have been investigated extensively since the ground-breaking works of Rényi and Sulanke [17–19] in the 1960s, and their literature has grown enormous since then. Results on variances, higher moments, and limit theorems are, however, much more scarce in the literature. For an overview of these extensive topics, we refer to the surveys by Bárány [2], Hug [12], Reitzner [15], Schneider [21, 22], and Weil and Wieacker [24], and the references therein. In this paper, we only mention those results that most directly related to our investigations.

We work in d-dimensional Euclidean space $\mathbb{R}^d$ with scalar product $\langle\cdot,\cdot\rangle$, and induced norm $\|\cdot\|$. Let B^d be the unit ball centred at the origin o, and S^{d-1} the unit sphere, the boundary ∂B^d of B^d. We denote by $V(X)$ the volume (Lebesgue measure) of a measurable set $X \subset \mathbb{R}^d$, and by $\sigma(Y)$ the spherical Lebesgue measure of a measurable set $Y \subset S^{d-1}$. We set $\kappa_d = V(B^d)$ and $\omega_d = \sigma(S^{d-1})$. A convex body $K \subset \mathbb{R}^d$ is a compact convex set with non-empty interior. We use $\kappa(x)$ to denote the generalised Gauss–Kronecker curvature at $x \in \partial K$; for precise definition and properties see, for example, [20, sections 1.5, 2.5, 2.6].

If the functions f and g are defined on a space I and there exists a constant $\gamma > 0$ such that $|f| < \gamma g$ on I, then we denote this fact by $f \ll g$, or $f = O(g)$.

In the first part of the paper, we study the following probability model. Let $\varrho : K \to \mathbb{R}$ be a probability density function with respect to the Lebesgue measure which is positive on ∂K and continuous in a neighbourhood of ∂K (relative to K). Then for any measurable set $A \subset K$, $\mathbb{P}_{\varrho,K}(A) := \int_A \varrho(x)\,\mathrm{d}x$. Let $p_1, \dots, p_n$ be i.i.d. random points from K distributed according $\mathbb{P}_{\varrho,K}$. The convex hull $K_{(n)} = [p_1, \dots, p_n]$ is a random polytope in K. Expectation and variance with respect to $\mathbb{P}_{\varrho,K}$ will be denoted by $\mathbb{E}_{\varrho,K}$ and $\mathrm{Var}_{\varrho,K}$, respectively. If K is clear from the context, we may also use the simpler notations $\mathbb{P}_\varrho$, $\mathbb{E}_\varrho$ and Var_ϱ. If $\varrho \equiv 1/V(K)$, then one obtains the uniform model (in that case, we use the even simpler notations K_n, $\mathbb{E}$ and Var).

Let $\lambda : K \to \mathbb{R}$ be a non-negative integrable function which is positive on ∂K and continuous in a neighbourhood of ∂K. For a measurable set $A \subset K$, let $V_\lambda(A) = \int_A \lambda(x)\,\mathrm{d}x$.

Throughout the paper, the probability density ϱ and weight function λ are always assumed to be as defined above without further mention.

It was proved in [6] that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_\varrho \int_{K\setminus K_{(n)}} \lambda(x)\,\mathrm{d}x = c_d \int_{\partial K} \varrho(x)^{\frac{-2}{d+1}} \lambda(x)\kappa(x)^{\frac{1}{d+1}} \mathcal{H}^{d-1}(\mathrm{d}x), \tag{1}$$

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}} \mathbb{E}_\varrho(f_0(K_{(n)})) = c_d \int_{\partial K} \varrho(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \mathcal{H}^{d-1}(\mathrm{d}x),$$

where $f_0(K_{(n)})$ is the number of vertices of $K_{(n)}$ and integration is with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$ on ∂K.

The exact value of the constant c_d was determined by Wieacker [25]. The special case of (1) when $\varrho \equiv 1/V(K)$ and $\lambda \equiv 1$ was proved by Rényi and Sulanke [17] for $d = 2$ when ∂K is C^3_+, by Wieacker [25] for $K = B^d$ and general d, and by Bárány [1] for general d when ∂K is C^3_+. Schütt [23] removed the smoothness condition on K, and Böröczky, Fodor and Hug [6] introduced the density ϱ and weight function λ.

In the uniform case (for $\varrho \equiv 1/\kappa_d$ and $\lambda \equiv 1$), Küfer [14] proved that $\mathrm{Var}(V(B^d_n)) \ll n^{-(d+3)/(d+1)}$. Reitzner [16], using the Efron–Stein jackknife inequality [9], extended this upper bound $\mathrm{Var}(V(K_n)) \ll n^{-(d+3)/(d+1)}$ for C^2_+ bodies for general d and proved the strong law of large numbers for the volume. Reitzner's results were extended to all intrinsic volumes by Bárány, Fodor and Vígh [3] in the case when ∂K is C^2_+. We note that only in the planar case ($d = 2$) is an asymptotic upper bound known for $\mathrm{Var}(V(K_n))$ for general convex bodies without smoothness condition, see Bárány and Steiger [5]. Upper bounds were also proved for $\mathrm{Var}(V(K_n))$ by Bárány and Reitzner [4] for polytopes. For further results on second-order results and limit laws, we refer to the surveys mentioned above.

We say that a ball of radius $r > 0$ rolls freely in K if for any $x \in \partial K$ there exists a $v \in \mathbb{R}^d$ such that $x \in rB^d + v \subset K$. The body K slides freely in a ball of radius $R > 0$ if for each $x \in RS^{d-1}$ there exists $p \in \mathbb{R}^d$ with $x \in K + p \subset RB^d$. If K has a rolling ball and slides freely in a ball at the same time, then ∂K is C^1 and strictly convex. However, ∂K need not be C^2.

Let $\sigma(K, \cdot) : \partial K \to S^{d-1}$ denote the spherical image map, and let $\tau(K, \cdot) : S^{d-1} \to \partial K$ be the reverse spherical image map of K, cf. [20, section 2.2]. If K has a rolling ball and slides freely in a ball, both $\sigma(K, \cdot)$ and $\tau(K, \cdot)$ are well defined and inverses to each other.

Our first main result is the following upper bound on the variance of $V_\lambda(K_{(n)})$.

Theorem 1.1. *For a convex body $K \subset \mathbb{R}^d$ that has a rolling ball and which slides freely in a ball, it holds that*

$$\mathrm{Var}_\varrho(V_\lambda(K_{(n)})) \ll n^{-\frac{d+3}{d+1}},$$

where the implied constant depends only on K, ϱ, λ and the dimension d.

Theorem 1.1 is a generalisation of Theorem 1 of Reitzner [16, p. 2138]. The need for this level of generality in ϱ and λ will be explained by its applicability in the circumscribed model in Theorem 1.4.

From Theorem 1.1, one can derive the law of large numbers for $V_\lambda(K \setminus K_{(n)})$.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1,*

$$\lim_{n\to\infty} V_\lambda(K \setminus K_{(n)})\, n^{\frac{2}{d+1}} = c_d \int_{\partial K} \varrho(x)^{-\frac{2}{d+1}} \lambda(x) \kappa(x)^{\frac{1}{d+1}}\, \mathrm{d}x \text{ with probability 1.}$$

The proof of Theorem 1.2 is very similar to that of Theorem 2 in Reitzner [16, pp. 2150–2151].

Reitzner [16] proved that if ∂K is C^2_+, then $\mathrm{Var}(f_0(K_n)) \ll n^{\frac{d-1}{d+1}}$. With a minor modification of the proof of Theorem 1.1, we can obtain the following.

Theorem 1.3. *Under the same assumptions as in Theorem 1.1,*

$$\mathrm{Var}_\varrho\,(f_0(K_{(n)})) \ll n^{\frac{d-1}{d+1}},$$

where the implied constant depends only on K, ϱ and the dimension d.

The proof of Theorem 1.3 is essentially the same as that of Theorem 1.1 with only minor adjustments, so we refer to the relevant part of Reitzner's paper for the details. We note that Reitzner [16] also proved the strong law of large numbers for the number of vertices in the case when $d \geqslant 4$.

Next we turn to the circumscribed model, which was recently studied, for example, in Böröczky and Schneider [8], Böröczky, Fodor and Hug [6] and Fodor, Hug and Ziebarth [10]. We also mention the more recent paper by Hug and Schneider [13] which deals with the closely related question of circumscribed Poisson polyhedra.

Let $W(K)$ denote the mean width of K, which is the average distance between parallel supporting hyperplanes of K over all directions, for precise definition and properties, see, for example, [20, section 1.7].

Let $K_1 = K + B^d$ be the radius 1 parallel domain of K. By $A(d, d-1)$ we denote the space of hyperplanes in $\mathbb{R}^d$ with its usual topology, and by $\mathcal{H}_K$ the subspace of $A(d, d-1)$ with the property that for any $H \in \mathcal{H}_K$, $H \cap K_1 \neq \emptyset$ and $H \cap \operatorname{int} K = \emptyset$. For $H \in \mathcal{H}_K$, let H^- denote the closed half-space bounded by H that contains K. Let the motion invariant Borel measure μ_d on $A(d, d-1)$ be normalised in such a way that $\mu_d(\{H \in A(d, d-1) : H \cap M \neq \emptyset\}) = W(M)$ for every convex body $M \subset \mathbb{R}^d$. Let $2\mu_K$ be the restriction of μ_d to $\mathcal{H}_K$. Thus, μ_K is a probability measure on $\mathcal{H}_K$. Let $H_1, \ldots, H_n$ be i.i.d. random hyperplanes in $\mathbb{R}^d$, distributed according to μ_K. Then, $K^{(n)} = \bigcap_{i=1}^n H_i^-$ is a (possibly unbounded) random polyhedron containing K. As already noted in [6], the choice of K_1 does not affect the asymptotic behaviour of $W(K^{(n)} \cap K_1)$, only some normalisation constants. Since $K^{(n)}$ is unbounded with positive probability, we consider $K^{(n)} \cap K_1$ instead (which is no longer a polyhedron).

It was proved in [6] that

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mu_K}(W(K^{(n)} \cap K_1) - W(K)) = 2c_d \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} \mathcal{H}^{d-1}(\mathrm{d}x).$$

Our main statement regarding this circumscribed model is the following theorem.

Theorem 1.4. *For a convex body $K \subset \mathbb{R}^d$ that has a rolling ball and which slides freely in a ball, it holds that*

$$\operatorname{Var}_{\mu_K}\left(W\left(K^{(n)} \cap K_1\right)\right) \ll n^{-\frac{d+3}{d+1}},$$

where the implied constant depends only on K and d.

We note that we prove a more general statement in Theorem 3.1. From Theorem 1.4, we can also obtain the strong law of large numbers by standard methods.

Theorem 1.5. *Under the same hypotheses as in Theorem 1.4,*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \left(W\left(K^{(n)} \cap K_1\right) - W(K)\right) = 2\, c_d\, \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} \mathcal{H}^{d-1}(\mathrm{d}x)$$

with probability 1.

Using Theorem 1.3, we also prove an upper bound for the number of facets $f_{d-1}(K^{(n)})$ of $K^{(n)}$, see Theorem 3.3.

Finally, we note that analogous statements were proved in Fodor, Hug and Ziebarth [10] for the weighted mean width for inscribed polytopes and the volume difference of circumscribed polytopes involving probability densities and weight functions.

2 | PROOF OF THEOREM 1.1

Our proof follows the argument of Reitzner [16, section 4] with the addition that it also takes into account the properties of ϱ and λ. For this reason, we only describe the most important steps of the proof.

The main idea is to use the Efron–Stein jackknife inequality [9] to bound the variance from above by the second moment of the increment of the weighted volume of $K_{(n)}$ when adding a new random point. Then, one obtains a geometric integral that involves cap volumes, which can be estimated based on the geometric assumptions on K. This is where the existence of the rolling ball and sliding ball are important.

For $u \in S^{d-1}$ and $t \geqslant 0$, let $H(t,u)$ be the hyperplane $H(t,u) = \{x \in \mathbb{R}^d : \langle x, u\rangle = t\}$. Let $H^+(t,u)$ and $H^-(t,u)$ be the closed half-spaces bounded by $H(t,u)$ such that $H^-(t,u)$ contains the origin.

The intersection of K with a closed half-space is a cap. In particular, let $C(t,u) = K \cap H^+(t,u)$ and let $V(t,u) = V(C(t,u))$. The (unique) boundary point $\tau(K,u)$ is the vertex, and $h = h(K,u) - t$ is the height of $C(t,u)$. We also use the notation $\overline{C}(h,u)$ ($\overline{V}(h,u) = V(\overline{C}(h,u))$) when we describe the cap $C(t,u)$ using its height.

Assume that the radius of the rolling ball is r and K slides freely in a ball of radius R. Then for all $h \leqslant r$ and $u \in S^{d-1}$, it holds that

$$\gamma_1 h^{\frac{d+1}{2}} = \frac{2\kappa_{d-1} r^{\frac{d-1}{2}} h^{\frac{d+1}{2}}}{d+1} \leqslant V(\overline{C}(h,u)) \leqslant \gamma_2 h^{\frac{d+1}{2}} \tag{2}$$

for some positive constant γ_2 that depends on R.

For $\varepsilon > 0$, let $K(\varepsilon) = K \cap (\partial K + \varepsilon B^d)$. Assume that ε is sufficiently small that both λ and ϱ are positive and continuous on $K(\varepsilon)$, and let $\varrho_m(\varepsilon)$ be the minimum and $\varrho_M(\varepsilon)$ and $\lambda_M(\varepsilon)$ be the maximum of $\varrho(x)$ and $\lambda(\varepsilon)$, respectively, for $x \in K(\varepsilon)$. Then for any measurable set $A \subset K(\varepsilon)$,

$$\varrho_m(\varepsilon)V(A) \leqslant \mathbb{P}_\varrho(A) \leqslant \varrho_M(\varepsilon)V(A), \text{ and } V_\lambda(A) \leqslant \lambda_M(\varepsilon)V(A). \tag{3}$$

In order to prove the upper bound in Theorem 1.1, we use the Efron–Stein jackknife inequality [9], which, when applied to $V_\lambda(K_{(n)})$, yields

$$\operatorname{Var}_\varrho V_\lambda(K_{(n)}) \leqslant (n+1)\mathbb{E}_\varrho V_\lambda^2(K_{(n+1)} \setminus K_{(n)}). \tag{4}$$

Let $c_1 = 18R/r$, and let $\varepsilon_0 > 0$ be sufficiently small that the following conditions are all satisfied:

(i) $c_1\varepsilon_0 < r/2$.
(ii) Both λ and ϱ are positive and continuous on $K(c_1\varepsilon_0)$.
(iii) $\varrho_0\gamma_1\varepsilon_0^{\frac{d+1}{2}} < 1$, where $\varrho_0 = \varrho_m(c_1\varepsilon_0)$.

Let $\delta(\cdot,\cdot)$ denote the Hausdorff distance of compact sets in $\mathbb{R}^d$. Let D denote the event $\delta(K_{(n)}, K) < \varepsilon_0$ and let D^c be its complement. Using the bounds in (3), a similar argument as in [16, pp. 2146–2147] shows

$$\mathbb{P}_\varrho(D^c) \leqslant O\big(n^d(1-c_0)^n\big)$$

for some suitable constant c_0 depending on ε_0. Therefore,

$$\operatorname{Var}_\varrho V_\lambda\big(K_{(n)}\big) \leqslant (n+1)\int_K \dots \int_K \mathbb{1}(D) V_\lambda^2\big(K_{(n+1)} \setminus K_{(n)}\big)\,\mathrm{d}p_1 \dots \mathrm{d}p_{n+1}$$
$$+ O\big(n^{d+1}(1-c_0)^n\big). \tag{5}$$

Let $x_1, \dots, x_n, x_{n+1}$ be arbitrary points in K. For an integer $0 \leqslant k \leqslant d$, let $F_1 = [x_1, \dots, x_d]$ and $F_2 = [x_{d-k+1}, \dots, x_{2d-k}]$. Denote by H_1 and H_2 the affine hulls of F_1 and F_2, respectively. Then, the F_i, $i = 1, 2$ are almost always $(d-1)$-dimensional simplices and H_i, $i = 1, 2$ are hyperplanes. If H_i is a supporting hyperplane of the polytope $[x_1, \dots, x_n]$, then we denote the half-space of H_i containing $[x_1, \dots, x_n]$ by H_i^-, and the other half-space by H_i^+. Let $F_i^+ = K \cap H_i^+$ be the cap corresponding to F_i, and $V_i^+ = V(F_i^+)$ for $i = 1, 2$.

Here we refer to the argument of Reitzner in [16, Section 4] as it is essentially the same as ours with the only difference that we also use the properties of ϱ and λ. Thus, one obtains for the integral in (5) that it is less than

$$n^{2d-k+1}\sum_{k=0}^{d}\int_K \dots \int_K \mathbb{1}(D)(1-\mathbb{P}_\varrho(F_1^+))^{n-2d+k}\mathbb{P}_\varrho(F_1^+)V_1^+$$
$$\times\, \mathbb{1}\big(F_1^+ \cap F_2^+ \neq \emptyset\big)\mathbb{1}(A)V_2^+ \prod_{i=1}^{2d-k}\varrho(x_i)\,\mathrm{d}x_1 \dots \mathrm{d}x_{2d-k}, \tag{6}$$

where A denotes the event that the diameter of F_2^+ is less than the diameter of F_1^+.

Now, for a fixed $0 \leqslant k \leqslant d-1$ and $x_1, \dots, x_d$, we evaluate the following integral:

$$\int_K \dots \int_K \mathbb{1}(D)\mathbb{1}\big(F_1^+ \cap F_2^+ \neq \emptyset\big)\mathbb{1}(A)V_2^+ \prod_{i=d+1}^{2d-k}\varrho(x_i)\,\mathrm{d}x_{d+1} \dots \mathrm{d}x_{2d-k}. \tag{7}$$

In order to do this, we need the following statement. Let y_i be the vertex and h_i the height of the cap F_i^+, $i = 1, 2$. We show that if $h_1 < \varepsilon_0$, then

$$F_2^+ = \overline{C}(y_2, h_2) \subset \overline{C}(y_1, c_1 h_1). \tag{8}$$

We note that a careful analysis of the argument in Reitzner [16] shows that, under the assumptions on K, this statement holds in each case when ∂K is twice differentiable in the generalised sense at both y_1 and y_2, from which it follows that it is true for almost all pairs y_1, y_2 with the prescribed conditions on F_1^+ and F_2^+. However, here we give a short and direct proof that verifies (8) for all possible combinations of y_1 and y_2.

Let H be the supporting hyperplane of K at y_1. Let B be the radius R ball (in which K slides freely) that supports K at y_1, that is, $y_1 \in \partial B$, $K \subset B$, and let B' be the radius r rolling ball containing y_1.

Then the intersection $H_1 \cap B$ is a $(d-1)$-dimensional ball of radius $\sqrt{2Rh_1 - h_1^2} < \sqrt{2Rh_1}$. From $F_1^+ \subset H_1^+ \cap B$, it follows that $\operatorname{diam}(F_1^+) < 2\sqrt{2Rh_1}$. Since $\operatorname{diam}(F_2^+) < \operatorname{diam}(F_1^+)$ and $F_1^+ \cap F_2^+ \neq \emptyset$, the orthogonal projection of F_2^+ to H is contained in the $(d-1)$-dimensional ball B'' of radius $3\sqrt{2Rh_1}$ centred at o. Let h' be chosen such that $\sqrt{rh'} = 3\sqrt{2Rh_1}$, that is, $h' = 18(R/r)h_1 = c_1h_1 < r/2$ by the choice of ε_0. The hyperplane H' parallel to H at height c_1h_1 intersects the rolling ball B' in a $(d-1)$-dimensional ball of radius at least $\sqrt{rc_1h_1} = 3\sqrt{2Rh_1}$, so the orthogonal projection of $H' \cap B'$ to H contains B'', therefore, $F_2^+ \subset \overline{C}(y_1, c_1h_1)$.

Using (8), (2) and (3), we obtain that for fixed $x_1, \dots, x_d$,

$$
\begin{aligned}
(7) \ll \int_K \dots \int_K \mathbb{1}\Big(x_{d+1}, \dots, x_{2d-k} \in \overline{C}(y_1, c_1h_1)\Big) \\
\times V(\overline{C}(y_1, c_1h_1)) \prod_{i=d+1}^{2d-k} \varrho(x_i)\,\mathrm{d}x_{d+1} \dots \mathrm{d}x_{2d-k} \\
\leqslant (\varrho_M(c_1\varepsilon_0))^{d-k}(V(\overline{C}(y_1, c_1h_1)))^{d-k+1} \ll (V_1^+)^{d-k+1},
\end{aligned}
$$

which yields

$$
\begin{aligned}
(6) \ll n^{2d-k+1} \sum_{k=0}^{d} \int_K \dots \int_K \mathbb{1}(D)(1 - \mathbb{P}_\varrho(F_1^+))^{n-2d+k}(V_1^+)^{d-k+3} \\
\times \prod_{i=1}^{d} \varrho(x_i)\mathrm{d}x_1 \dots \mathrm{d}x_d.
\end{aligned} \tag{9}
$$

We will show that the order of magnitude of (9) is less than $n^{-\frac{d+3}{d+1}}$.

We use the following special case of the affine Blaschke–Petkantschin formula (see, for example, [22, Theorem 7.2.7]). Let $\Delta_{d-1} = \Delta_{d-1}(x_1, \dots, x_d)$ be the $(d-1)$-dimensional volume of the simplex spanned by $x_1, \dots, x_d$.

Theorem 2.1. *Let $f : (\mathbb{R}^d)^d \to \mathbb{R}$ be a non-negative measurable function. Then*

$$
\int_{(\mathbb{R}^d)^d} f\,\mathrm{d}x_1 \dots \mathrm{d}x_d = \frac{\omega_d}{\omega_1}(d-1)! \int_{A(d,d-1)} \int_{H^d} f\Delta_{d-1}\,\mathrm{d}x_1 \dots \mathrm{d}x_d\mathrm{d}\mu_d(H). \tag{10}
$$

The measure $\mathrm{d}\mu_d = \mathrm{d}u\mathrm{d}t$ assuming that $\mathrm{d}u$ is the surface area element of the unique rotation invariant probability measure on S^{d-1} and $\mathrm{d}t$ is the volume element of the one-dimensional Lebesgue measure. Let $0 \leqslant k \leqslant d$ be fixed. Using (10), the condition $\mathbb{1}(D)$ and the boundedness of ϱ on $K(\varepsilon_0)$, we obtain

$$
\begin{aligned}
\int_K \dots \int_K (1 - \mathbb{P}_\varrho(F_1^+))^{n-2d+k}(V_1^+)^{d-k+3} \prod_{i=1}^{d} \varrho(x_i)\,\mathrm{d}x_1 \dots \mathrm{d}x_d \\
\ll \int_{S^{d-1}} \int_{h(K,u)-\varepsilon_0}^{h(K,u)} (1 - \mathbb{P}_\varrho(C(t,u)))^{n-2d+k} V(t,u)^{d-k+3} \\
\times \left(\int_{H(t,u)\cap K} \dots \int_{H(t,u)\cap K} \Delta_{d-1}\,\mathrm{d}x_1 \dots \mathrm{d}x_d\right)\mathrm{d}t\mathrm{d}u.
\end{aligned} \tag{11}
$$

For a fixed $u \in S^{d-1}$, let B be the supporting ball (of radius R) of K at $\tau(K,u)$. Then $H \cap B$ is a $(d-1)$-dimensional ball with $H \cap K \subset H \cap B$. Its radius is $r(t) = \sqrt{2R(h(K,u)-t)-(h(K,u)-t)^2} \ll h^{1/2}$, where $h = h(K,u) - t$.

Since the innermost d-fold integral in (11) is monotone with respect to the integration domain, we obtain

$$(11) \ll \int_{S^{d-1}} \int_{h(K,u)-\varepsilon_0}^{h(K,u)} (1-\mathbb{P}_\varrho(C(t,u)))^{n-2d+k} V(t,u)^{d-k+3} \times \left(\int_{r(t)B^{d-1}} \dots \int_{r(t)B^{d-1}} \Delta_{d-1}\, \mathrm{d}x_1 \dots \mathrm{d}x_d \right) \mathrm{d}t\mathrm{d}u. \tag{12}$$

Let us substitute $h = h(K,u) - t$ in (12). By the choice of ε_0, if $h(K,u) - \varepsilon_0 \leqslant t \leqslant h(K,u)$, then $\mathbb{P}_\varrho(C(t,u)) = \mathbb{P}_\varrho(\overline{C}(h,u)) > \varrho_0\gamma_1 h^{(d+1)/2}$, and $\varrho_0\gamma_1\varepsilon_0^{(d+1)/2} < 1$. Using that the degree of homogeneity of the innermost d-fold integral in (12) is d^2-1, we obtain

$$(12) \ll \int_{S^{d-1}} \int_0^{\varepsilon_0} \left(1-\varrho_0\gamma_1 h^{\frac{d+1}{2}}\right)^{n-2d+k} h^{\frac{d+1}{2}(d-k+3)} h^{\frac{d^2-1}{2}}\, \mathrm{d}h\mathrm{d}u$$
$$\ll \int_0^{\varepsilon_0} \left(1-\varrho_0\gamma_1 h^{\frac{d+1}{2}}\right)^{n-2d+k} h^{\frac{d+1}{2}(d-k+3)} h^{\frac{d^2-1}{2}}\, \mathrm{d}h. \tag{13}$$

We evaluate (13) using the following asymptotic formula (see, for example, [7, formula (11)]). For any $\beta \geqslant 0$, $\omega > 0$ and $\alpha > 0$, it holds that

$$\int_0^{g(n)} h^\beta (1-\omega h^\alpha)^n\, \mathrm{d}h \sim \frac{1}{\alpha\omega^{\frac{\beta+1}{\alpha}}} \Gamma\left(\frac{\beta+1}{\alpha}\right) n^{-\frac{\beta+1}{\alpha}}$$

as $n \to \infty$, assuming $\left(\frac{(\beta+\alpha+1)\ln n}{\alpha\omega n}\right)^{\frac{1}{\alpha}} \leqslant g(n) \leqslant \omega^{-\frac{1}{\alpha}}$ for sufficiently large n. The symbol $\sim$ denotes the asymptotic equality of sequences.

By the choice of ε_0, it holds that $\varepsilon_0 < (\varrho_0\gamma_1)^{-2/(d+1)}$. Let $g(n) = \varepsilon_0$, and $\alpha = (d+1)/2$, $\beta = (d+1)(d-k+3)/2 + (d^2-1)/2$ and $\omega = \varrho_0\gamma_1$. Simple calculation yields $(\beta+1)/\alpha = (d+3)/(d+1) + 2d - k + 1$.

Since $0 \leqslant k \leqslant d$ was arbitrary, this finishes the proof of the theorem.

3 | THE VARIANCE OF THE MEAN WIDTH OF CIRCUMSCRIBED POLYHEDRAL SETS

We recall some of the notations and arguments from [6]. Assume that the convex body $K \subset \mathbb{R}^d$ contains the origin in its interior. Let $K^* = \{z \in \mathbb{R}^d : \langle x,z\rangle \leqslant 1 \ \ \forall x \in K\}$ be the polar body of K. It was proved by Hug [11] (see Propositions 1.40 and 1.45) that if K has a rolling ball and it slides freely in a ball, then K^* also has a rolling ball and slides freely in a ball.

The circumscribed model is based on random hyperplanes with the following distribution (see [6, (5.1) on p. 515]):

$$\mu_q = 2\int_{S^{d-1}} \int_0^\infty \mathbb{1}(H(t,u) \in \cdot)q(t,u)\, \mathrm{d}t\mathrm{d}u,$$

where $q : [0,\infty)\times S^{d-1} \to [0,\infty)$ is a measurable function with the following properties: It is

(1) concentrated on $D_K = \{(t,u) \in [0,\infty)\times S^{d-1} : h(K,u) \leqslant t \leqslant h(K_1,u)\}$,
(2) positive and continuous in the neighbourhood of $\{(t,u) \in [0,\infty)\times S^{d-1} : t = h(K,u)\}$ relative to D_K,
(3) μ_q is a probability measure, that is, $\mu_q(\mathcal{H}_K) = 1$.

Probabilities, expectations and variances with respect to μ_q are denoted by $\mathbb{P}_{\mu_q}$, $\mathbb{E}_{\mu_q}$ and Var_{μ_q}, respectively.

Let $H_1,\dots,H_n$ be i.i.d. random hyperplanes in $\mathbb{R}^d$ with distribution μ_q. For each H_i, let H_i^- be the closed half-space that contains the origin. Let $K^{(n)} = \cap_{i=1}^n H_i^-$, a random polyhedron containing K. We note that $K^{(n)}$ may be unbounded with positive probability, so we consider $K^{(n)} \cap K_1$ instead, or the conditional event that $K^{(n)} \subset K_1$, which has the same asymptotics as $n \to \infty$, see Böröczky and Schneider [8].

The polar body of $K^{(n)}$ is the convex hull of the points $x_i = t_i^{-1}u_i$, where t_i is the distance between o and H_i, and $u_i \in S^{d-1}$ is the (outer) unit normal vector of H_i, namely

$$(K^{(n)})^* = \left[t_1^{-1}u_1,\dots,t_n^{-1}u_n\right].$$

Let $\rho(K,x) = \sup\{\lambda \geqslant 0 : \lambda x \in K\}$, $x \in \mathbb{R}^d \setminus \{o\}$ be the radial function of K. We also introduce the following extension of q:

$$\tilde{q}(x) = q\left(\frac{1}{\|x\|}, \frac{x}{\|x\|}\right), \quad x \in K^* \setminus \{o\}.$$

It was proved in [6] (see p. 516) that the probability density function of the points $t_1^{-1}u_1,\dots,t_n^{-1}u_n$ in the polar model is

$$\varrho(x) = \begin{cases} \omega_d^{-1}\tilde{q}(x)\|x\|^{-(d+1)}, & x \in K^* \setminus K_1^*, \\ 0, & x \in K_1^*. \end{cases}$$

Note that $\varrho(x)$ is a probability density function on K^* that is positive and continuous in a neighbourhood of ∂K^* with respect to K^*, so it satisfies the conditions of Theorem 1.1. Following the notation conventions in [6], we denote $(K^*)_{(n)}$ by the simpler symbol $K^*_{(n)}$. We prove the following theorem.

Theorem 3.1. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \mathrm{int}\, K$ which has a rolling ball and which slides freely in a ball. If $q : [0,\infty)\times S^{d-1} \to [0,\infty)$ satisfies properties (1)–(3), then*

$$\mathrm{Var}_{\mu_K}\left(W(K^{(n)} \cap K_1)\right) \ll n^{-\frac{d+3}{d+1}},$$

where the implied constant depends only on K, q and d.

Proof. It was proved in [8] that $\mathbb{P}_{\mu_q}(K^{(n)} \not\subset K_1) \ll \alpha^n$ for a suitable $\alpha \in (0,1)$ depending on K and μ_q. Furthermore, it was proved in [6, Proposition 5.1] that $K^{(n)}$ and $(K^*_{(n)})^*$ are equal in distribution. Thus, we obtain from the Efron–Stein inequality that

$$\begin{aligned}
\operatorname{Var}_{\mu_q}\left(W\left(K^{(n)}\cap K_1\right)\right) &\ll n\mathbb{E}_{\mu_q}\left(W\left(K^{(n)}\cap K_1\right)-W\left(K^{(n+1)}\cap K_1\right)\right)^2\\
&\ll n\left(\mathbb{E}_{\mu_q}\left(\mathbb{1}\left(K^{(n)}\subset K_1\right)\left(W\left(K^{(n)}\right)-W\left(K^{(n+1)}\right)\right)^2\right)+O(\alpha^n)\right)\\
&= n\left(\mathbb{E}_{\varrho,K^*}\left(\mathbb{1}\left(\left(K^*_{(n)}\right)^*\subset K_1\right)\left(W\left(\left(K^*_{(n)}\right)^*\right)-W\left(\left(K^*_{(n+1)}\right)^*\right)\right)^2\right)+O(\alpha^n)\right).
\end{aligned}$$

It was proved in [6] that

$$\begin{aligned}
\mathbb{1}(K^{(n)}\subset K_1)(W(K^{(n)}\cap K_1)-W(K)) &= \mathbb{1}\left(\left(K^*_{(n)}\right)^*\subset K_1\right)\int_{K^*\setminus K^*_{(n)}}\lambda(x)\,\mathrm{d}x\\
&= \mathbb{1}\left(\left(K^*_{(n)}\right)^*\subset K_1\right)\left(V_\lambda(K^*)-V_\lambda\left(K^*_{(n)}\right)\right), \qquad (14)
\end{aligned}$$

where

$$\lambda(x)=\begin{cases}\omega_d^{-1}\|x\|^{-(d+1)}, & x\in K^*\setminus K_1^*,\\ 0, & x\in K_1^*.\end{cases}$$

Note that $\lambda(x)$ is integrable on K^* and it is positive and continuous on a neighbourhood of ∂K^* with respect to K^*, thus, it satisfies the conditions of Theorem 1.1. Therefore, it follows that

$$\begin{aligned}
&\operatorname{Var}_{\mu_q}\left(W\left(K^{(n)}\cap K_1\right)\right)\\
&\ll n\left(\mathbb{E}_{\varrho,K^*}\left(\mathbb{1}\left(\left(K^*_{(n)}\right)^*\subset K_1\right)\left(V_\lambda\left(K^*_{(n+1)}\right)-V_\lambda\left(K^*_{(n)}\right)\right)^2\right)+O(\alpha^n)\right)\\
&= n\left(\mathbb{E}_{\varrho,K^*}\left(V_\lambda\left(K^*_{(n+1)}\right)-V_\lambda\left(K^*_{(n)}\right)\right)^2+O(\alpha^n)\right)\ll n^{-\frac{d+3}{d+1}}.
\end{aligned}$$

□

The following asymptotic formula was also proved in [6]. Under the same assumptions as in Theorem 3.1, it holds that

$$\begin{aligned}
&\lim_{n\to\infty} n^{\frac{2}{d+1}}\mathbb{E}_{\mu_K}\left(W\left(K^{(n)}\cap K_1\right)-W(K)\right)\\
&\qquad= 2c_d\omega_d^{-\frac{d-1}{d+1}}\int_{\partial K} q(h(K,\sigma(K,x)),\sigma(K,x))^{-\frac{2}{d+1}}\kappa^{\frac{d}{d+1}}(x)\,\mathcal{H}^{d-1}(\mathrm{d}x).
\end{aligned}$$

Using the asymptotic upper bound of Theorem 1.1 and taking into account the monotone decreasing property of $W(K^{(n)}\cap K_1)$, essentially the same argument as in [7] yields the following statement.

Theorem 3.2. *Under the same hypotheses as in Theorem 3.1, it holds that*

$$\lim_{n\to\infty}\left(W\left(K^{(n)}\cap K_1\right)-W(K)\right)n^{\frac{2}{d+1}} = 2c_d\omega_d^{-\frac{d-1}{d+1}}\int_{\partial K} q(h(K,\sigma(K,x)),\sigma(K,x))^{-\frac{2}{d+1}}\varkappa^{\frac{d}{d+1}}(x)\,\mathcal{H}^{d-1}(\mathrm{d}x)$$

with probability 1.

Finally, we turn to the number of facets $f_{d-1}(K^{(n)})$ of $K^{(n)}$. It was proved in [6] that, under the same hypotheses as in Theorem 3.1,

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\mathbb{E}_{\mu_q}\left(f_{d-1}\left(K^{(n)}\right)\right) = c_d\omega_d^{-\frac{d-1}{d+1}}\int_{\partial K} q(h(K,\sigma(K,x)),\sigma(K,x))^{\frac{d-1}{d+1}}\varkappa^{\frac{d}{d+1}}(x)\,\mathcal{H}^{d-1}(\mathrm{d}x).$$

Since for any polyhedral set $P\subset\mathbb{R}^d$ with $o\in\operatorname{int}P$, $f_0(P)=f_{d-1}(P^*)$, and $K^{(n)}$ and $(K^*_{(n)})^*$ are equal in distribution (cf. [6, Proposition 5.1]), we obtain by the Efron–Stein inequality that

$$\begin{aligned}
\operatorname{Var}_{\mu_q}(f_{d-1}(K^{(n)})) &\ll n\mathbb{E}_{\mu_q}\left(f_{d-1}\left(K^{(n+1)}\right)-f_{d-1}\left(K^{(n)}\right)\right)^2\\
&\ll n\left(\mathbb{E}_{\mu_q}\left(\mathbb{1}\left(K^{(n)}\subset K_1\right)\left(f_{d-1}\left(K^{(n+1)}\right)-f_{d-1}\left(K^{(n)}\right)\right)^2\right)+O\left(n^2\cdot\alpha^n\right)\right)\\
&= n\mathbb{E}_{\varrho,K^*}\left(\mathbb{1}\left(\left(K^*_{(n)}\right)^*\subset K_1\right)\left(f_{d-1}\left(\left(K^*_{(n+1)}\right)^*\right)-f_{d-1}\left(\left(K^*_{(n)}\right)^*\right)\right)^2\right)+O\left(n^3\cdot\alpha^n\right)\\
&= n\mathbb{E}_{\varrho,K^*}\left(\mathbb{1}\left(\left(K^*_{(n)}\right)^*\subset K_1\right)\left(f_0\left(K^*_{(n+1)}\right)-f_0\left(K^*_{(n)}\right)\right)^2\right)+O\left(n^3\cdot\alpha^n\right)\\
&= n\mathbb{E}_{\varrho,K^*}\left(f_0\left(K^*_{(n+1)}\right)-f_0\left(K^*_{(n)}\right)\right)^2+O\left(n^3\cdot\alpha^n\right)\ll n^{\frac{d-1}{d+1}}
\end{aligned}$$

by Theorem 1.3. Thus, we have proved the following statement.

Theorem 3.3. *Under the same hypotheses as in Theorem 3.1, it holds that*

$$\operatorname{Var}_{\mu_q}\left(f_{d-1}\left(K^{(n)}\right)\right)\ll n^{\frac{d-1}{d+1}},$$

where the implied constant depends only on K, q and d.

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