Approximation by homogeneous polynomials*

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Abstract

A new, elementary proof is given for the fact that on a centrally symmetric convex curve on the plane every continuous even function can be uniformly approximated by homogeneous polynomials. The theorem has been proven before by Benko and Kroó, and independently by Varjú using the theory of weighted potentials. In higher dimension the new method recaptures a theorem of Kroó and Szabados, which is the strongest result for homogeneous polynomial approximation on smooth convex surfaces.

1 Introduction

Let $\mathcal{S} = \partial K$ be the boundary of a centrally symmetric convex body K in $\mathbf{R}^d$, more precisely, K is symmetric with respect to the origin: $z \in K \Rightarrow -z \in K$. A. Kroó conjectured (see [1]) that every (real) continuous function f on $\mathcal{S}$ can be uniformly approximated by sums $Q_m^1 + Q_{m+1}^2$, $m = 1, 2, \ldots$, where Q_m^1 and Q_{m+1}^2 are (real) homogeneous polynomials of degree m and $m+1$, respectively (note that a homogeneous polynomial is either even or odd, so in general one needs two terms for approximation). The conjecture is equivalent to the claim (see [7, Proposition 2.1]) that every even continuous function on $\mathcal{S}$ can be uniformly approximated by homogeneous polynomials

$$P_{2m}(x_1, x_2, \ldots, x_d) = \sum_{j_1+\cdots+j_d=2m} a_{m,j_1,\ldots,j_d} x_1^{j_1} \cdots x_d^{j_d}$$

of degree $2m = 2, 4, \ldots$ (a function f defined on $\mathcal{S}$ is even if $f(z) = f(-z)$ for all $z \in \mathcal{S}$). This is a beautiful conjecture, it is the Weierstrass theorem for approximation by homogeneous polynomials (it is easy to see that approximation in this sense is possible by homogeneous P_{2m} only on surfaces which are centrally symmetric). See [7] for the connection to approximation of general surfaces by level surfaces of homogeneous polynomials.

The conjecture has been proven in the following cases:

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(i) K is a polytope (P. Varjú [7]),

(ii) K has at every boundary point at most one supporting hyperplane (A. Kroó and J. Szabados [4]),

(iii) $d = 2$ (D. Benko and A. Kroó [1] and P. Varjú [7]).

Thus, the complete solution has only been found in dimension 2 in the papers [1] and [7], and both proofs are quite involved and are based on the theory of weighted polynomial approximation with varying weights and on the theory of weighted logarithmic potentials. In this note we give a new, more elementary proof for the $d = 2$ case that does not use potential theory. In higher dimension this approach yields the Kroó-Szabados result from (ii). Since (i) follows in a few lines from (iii) via a marvellous trick of P. Varjú (see the proof of [7, Theorem 1.4,(c)]), in a sense the method gives a new proof for all (i)–(iii), i.e. it is as strong as the methods applied so far. Actually, the proof is easy to modify so as to give the claim for some other bodies K in $\mathbf{R}^d$, but the exact geometric conditions are not clear, so we do not elaborate on it, and definitely the general case in $\mathbf{R}^d$, $d \geq 3$ remains open.

Thus, in this note we prove

Theorem 1 *Let K be a centrally symmetric convex set with non-empty interior in $\mathbf{R}^2$. Then every even continuous function on ∂K can be uniformly approximated by homogeneous polynomials P_{2n} of degree $2n = 2, 4, \ldots$.*

Let W be the set of functions f on $\mathcal{S} = \partial K$ for which there is a sequence P_{2m} of homogeneous polynomials of degree $2m = 2, 4, \ldots$ such that $P_{2m} \to f$ uniformly on $\mathcal{S}$. Suppose that the identically 1 function is in W, i.e. there is a sequence P_{2m} of homogeneous polynomials of degree $2m = 2, 4, \ldots$ such that $P_{2m} \to 1$ uniformly on $\mathcal{S}$. Then is easy to see (c.f. [7]) that W is a subalgebra of the set of continuous functions on $\mathcal{S}$ which separates every non-symmetric point pair on $\mathcal{S}$, and then the proof is completed by the Stone-Weierstrass theorem (see e.g. [6, Theorem 7.32]). Hence, all we need to do is to show that the identically 1 function is in W.

Let $L = L(K)$ be the smallest number such that K contains the disk about 0 of radius $1/L$ and it is contained in the disk about 0 of radius L.

Call K ε-regular, if at any point on the boundary the angle of any two supporting lines is at most ε. The theorem clearly follows from the following two propositions.

Proposition 2 *If K is as in Theorem 1, then for every $\varepsilon > 0$ there are centrally symmetric ε-regular sets K_1, K_2, K_3, K_4 such that $L(K_i) \leq 2L(K)$ and $K = K_1 \cap K_2 \cap K_3 \cap K_4$.*

Here the constant 2 could be replaced by any constant bigger than 1.

Proposition 3 *For every $\eta > 0$ and L there is an $\varepsilon > 0$ such that if $K = K_1 \cap K_2 \cap K_3 \cap K_4$ is the intersection of four centrally symmetric ε-regular sets K_1, K_2, K_3, K_4 with $L(K_i) \le L$, then for every m there are homogeneous polynomials P_{2m} of degree $2m$ such that for sufficiently large m*

$$1 - \eta \le P_{2m}(x, y) \le 1 + \eta, \qquad (x, y) \in \partial K. \tag{1}$$

We are going to prove these propositions in Section 3, but first we need to shows that for an ε-regular set K the constant 1 can be approximated by homogeneous polynomials with en error $C\varepsilon^{1/3}$, which is the content of the next section.

2 Approximating on ε-regular sets

In this section we prove

Proposition 4 *If K is ε-regular, then for every m there are homogeneous polynomials H_{2m} of degree $2m$ such that for sufficiently large m*

$$1 - A\varepsilon^{1/3} \le H_{2m}(x, y) \le 1 + A\varepsilon^{1/3}, \qquad (x, y) \in \partial K \tag{2}$$

where the constant A depends only on L.

This is the heart of the matter, and it is worth while to explain the main idea. Basically, the proof is based on fast decreasing polynomials of a single variable (see [3]), more precisely on their variant that approximate the signum function well on $[-1, 1]$ (with a transition interval around 0). By simple transformation we get then polynomials of a single variable of some large degree m that approximate well the characteristic function of an interval $[-M^2/m, M^2/m]$, with some fixed $M \ll m$, and from there we get for each point $T \in \partial K$ a positive homogeneous polynomial R_m^T of degree am with some fix a such that on the boundary of K this R_m^T is approximately 1 on an arc around T of central opening M^2/m, and is small outside that arc (with some transition intervals around the endpoints of that arc). Now the sum of these R_m^T, where T runs through the $2m$ points on the boundary of K for which the argument is $j2\pi/2m$, $j = 0, 1 \ldots, 2m-1$, will be approximately $2M^2$ on the boundary, so by dividing it trough by $2M^2$ we get a homogeneous polynomial that is approximately 1 on ∂K (depending how large M is).

Proof. It was proved in [3, Theorem 3] (see also example 2 on p. 5 of that paper) that for every $m = 1, 2, \ldots$ there is an odd polynomial U_m of degree at most m such that $-1 \le U_m(t) \le 1$ and

$$|U_m(t) - \operatorname{sign}(t)| \le C_0 e^{-c_0\sqrt{m|t|}}, \qquad t \in [-1, 1],$$

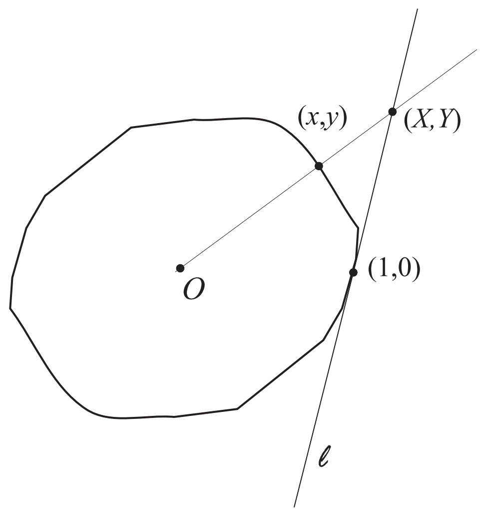

Figure 1: The supporting line ℓ and the points (x, y) and (X, Y)

with some absolute constants C_0, c_0. Choose and fix a large M and consider for $m > M^2$

$$S_{2m}(t) = \frac{1}{4}\left[1 + U_m\left(\frac{t + M^2/m}{2L+1}\right)\right]\left[1 - U_m\left(\frac{t - M^2/m}{2L+1}\right)\right].$$

Then S_{2m} is an even polynomial of degree at most $2m$, $0 \le S_{2m}(t) \le 1$ for $t \in [-2L, 2L]$, and with $I_m = [-M^2/m, M^2/m]$ the inequalities

$$0 \le S_{2m}(t) \le C_0 \exp\left(-c_1\sqrt{m \cdot \operatorname{dist}(t, I_m)}\right), \quad t \in [-2L, 2L], \tag{3}$$

and

$$0 \le 1 - S_{2m}(t) \le C_0 \exp\left(-c_1\sqrt{m \cdot \operatorname{dist}(t, \mathbf{R} \setminus I_m)}\right), \quad t \in I_m, \tag{4}$$

are satisfied, where c_1 depends only on L. In particular,

$$0 \le 1 - S_{2m}(t) \le C_0 \exp\left(-c_1\sqrt{M}\right), \quad t \in \left[\frac{-M^2 + M}{m}, \frac{M^2 - M}{m}\right], \tag{5}$$

and

$$0 \le S_{2m}(t) \le C_0 \exp\left(-c_1\sqrt{M}\right), \quad |t| \in \left[\frac{M^2 + M}{m}, \frac{2M^2}{m}\right]. \tag{6}$$

This latter inequality holds also for $|t| \in [2M^2/m, 2L]$, but in this range we shall need better estimates, see (11).

Assume first that the point $(1, 0)$ belongs to ∂K and the line ℓ defined by $x + by = 1$ is a supporting line to K at $(1, 0)$. Then, by the symmetry of K, we have $-1 \le x + by \le 1$ for all $(x, y) \in K$.

Consider, with some fixed positive even integer a, the polynomial

$$R_m(x, y) = (x + by)^{am} S_{2m}\left(\frac{y}{x + by}\right). \tag{7}$$

It is an even homogeneous polynomial of degree am. First we estimate R_m on ∂K at some point (x, y), and we may assume $y \geq 0$, $x + by \geq 0$ (the case $y \leq 0$, $x + by \geq 0$ is perfectly analogous, and the remaining cases follow by symmetry). For the time being assume that the half-line emanating from 0 and going through (x, y) intersects ℓ in some point (X, Y), see Figure 1. Then $X + bY = 1$ and hence $Y = Y/(X + bY) = y/(x + by)$. The choice of L gives $y \leq L$, hence for $x + by \geq 1/2$ we have $Y \leq 2L$, in which case we use

$$R_m(x, y) \leq S_{2m}\left(\frac{y}{x + by}\right) = S_{2m}(Y),$$

and for the right-hand side we can use (3)–(4). This can actually be said for all x, y for which $Y \leq 2L$. On the other hand, if $Y > 2L$ then necessarily $x + by \leq 1/2$, and then, with $\|\cdot\|_{[-1,1]}$ denoting the supremum norm on $[-1, 1]$, we use the well-known inequality (see [2, Proposition 4.2.3])

$$\begin{aligned} |P_n(Y)| &\leq \|P_n\|_{[-1,1]}\frac{1}{2}\left\{\left(|Y| + \sqrt{Y^2 - 1}\right)^n + \left(|Y| - \sqrt{Y^2 - 1}\right)^n\right\} \\ &\leq (2Y)^n \|P_n\|_{[-1,1]} \end{aligned}$$

for $n = 2m$, $P_n = S_{2m}$ combined with $\|S_{2m}\|_{[-1,1]} \leq 1$ to conclude

$$\begin{aligned} |R_m(x, y)| &\leq (x + by)^{am}(2Y)^{2m} = (x + by)^{am}\left(\frac{2y}{x + by}\right)^{2m} \\ &\leq (2L)^{2m}(x + by)^{(a-2)m} \leq (2L)^{2m}\left(\frac{1}{2}\right)^{(a-2)m} \leq 2^{-m} \qquad (8) \end{aligned}$$

if a is sufficiently large (depending only on L). Choose such an a.

It follows by continuity that if the the half-line emanating from 0 and going through (x, y) does not intersect ℓ, then (8) is still true (approach such a point with points for which (8) has been verified).

Next, we investigate more closely the behavior of R_m close to the point $(1, 0)$, and for that purpose now we drop the assumption $y \geq 0$. Set $z = x + iy \in \mathbf{C}$, $Z = X + iY \in \mathbf{C}$, and let φ be the common argument of z and Z. Note that $\frac{1}{2}|z|\sin|\varphi| = |y|/2$ is the area of the triangle $\{(0, 0), (1, 0), (x, y)\}$ while $\frac{1}{2}|Z|\sin|\varphi| = |Y|/2$ is the area of the triangle $\{(0, 0), (1, 0), (X, Y)\}$. Therefore, for small φ we have $Y \approx \varphi$, in fact $Y - \varphi = O(\varphi^2)$ (note that $|Z| = 1 + O(|\varphi|)$), and in general $|Y| \geq b_1|\varphi|$ with some $b_1 > 0$ (depending in this case on the angle in between ℓ and the positive x-axis, but since the point $(1, 0)$ should be replaced by any point on the boundary of K, b_1 depends eventually on the geometry of K); see Figure 1. It follows from (5), (6), (8) and (3) (for large m)

$$0 \leq 1 - S_{2m}(Y) \leq C_0 \exp\left(-c_1\sqrt{M}\right), \quad \varphi \in \left[\frac{-M^2 + 2M}{m}, \frac{M^2 - 2M}{m}\right], \qquad (9)$$

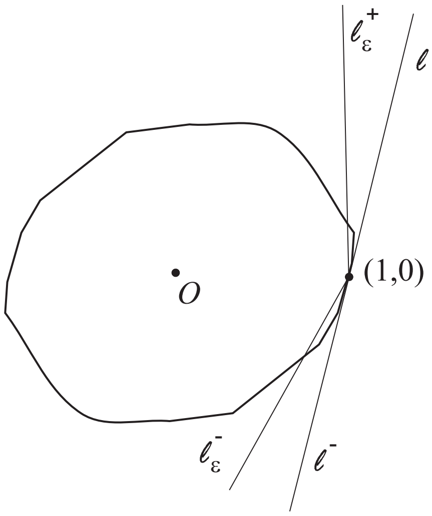

Figure 2: The supporting line ℓ and the rotated half-lines

and

$$0 \le R_m(x,y) \le S_{2m}(Y) \le C_0 \exp\left(-c_1\sqrt{M}\right), \quad |\varphi| \in \left[\frac{M^2+2M}{m}, \frac{2M^2}{m}\right]. \tag{10}$$

Similarly for $k = 1, 2, \ldots$

$$0 \le R_m(x,y) \le S_{2m}(Y) \le C_0 \exp\left(-c_1\sqrt{b_1 M^2 2^{k-1}}\right), \quad |\varphi| \in \left[\frac{2^k M^2}{m}, \frac{2^{k+1}M^2}{m}\right], \tag{11}$$

provided in the last inequality $|Y| \le 2L$, and

$$|R_m(x,y)| \le 2^{-m} \tag{12}$$

if this is not the case, see (8).

These are the upper estimates we need. They do not cover the case when $|\varphi| \in [(M^2-2M)/m, (M^2+2M)/m]$, in which case we just use

$$0 \le R_n(x,y) \le S_{2m}(Y) \le 1, \quad |\varphi| \in \left[\frac{M^2-2M}{m}, \frac{M^2+2M}{m}\right].$$

Our next aim is to get a lower estimate for R_m for points $(x,y) \in \partial K$ lying close to $(1,0)$. The point $(1,0)$ divides the line ℓ into the half-lines ℓ^+ and ℓ^-, ℓ^+ lying in the upper half-plane, see Figure 2. Rotate ℓ^+ by angle 2ε in counterclockwise direction to get ℓ_ε^+ and rotate ℓ^- in clockwise direction by angle 2ε to get ℓ_ε^-. The half-lines $\ell_\varepsilon^\pm$ form a cone of opening angle $\pi - 4\varepsilon$, and since K is ε-regular, there is a $\delta = \delta_{(1,0)}$ such that for $|\varphi| \le \delta$ the point $z \in \partial K$ $(z \ne (1,0))$ lies outside this cone (recall that φ is the argument of the points $z = x + iy$, $Z = X + iY$). This gives, by comparing again the areas of the triangles $\{(0,0),(1,0),(x,y)\}$ and $\{(0,0),(1,0),(X,Y)\}$, that for some

$b_2 > 0$ the inequality $|z|/|Z| \geq \exp(-b_2\varepsilon|\varphi|)$ is true, where b_2 depends only on L. Hence, if we compare the values of a homogeneous polynomial R_s^* of degree s at (x, y) and at (X, Y), we can infer from the homogeneity

$$|R_s^*(x, y)| = (|z|/|Z|)^s |R_s^*(X, Y)| \geq \exp(-b_2\varepsilon|\varphi|)^s.$$

Therefore, for large m and for $\varphi \in [(-M^2 + 2M)/m, (M^2 - 2M)/m]$ we get from (9)

$$R_m(x, y) \geq R_m(X, Y) \left(\exp(-b_2\varepsilon M^2/m)\right)^{am} \geq \exp(-b_2 a\varepsilon M^2) \left(1 - C_0 e^{-c_1\sqrt{M}}\right),$$

and for $M = \varepsilon^{-1/3}$ this yields

$$R_m(x, y) \geq e^{-c_2\varepsilon^{1/3}}, \qquad \varphi \in \left[\frac{-M^2 + 2M}{m}, \frac{M^2 - 2M}{m}\right] \tag{13}$$

with a c_2 depending only on L (recall that the constant a in (7) depended only on L).

All these were done for the point $T = (1, 0) = 1 + i0$, but it is clear that the same construction can be carried out for any point $T \in \partial K$. Let the corresponding R_m be denoted by R_m^T. Simple compactness shows that the $\delta = \delta_T > 0$ introduced above can be chosen independently of $T \in \partial K$. Choose now $T_1, \ldots, T_{2m} \in \partial K$ so that for the corresponding arguments we have $\varphi_j = 2\pi j/2m$, i.e. the points $T_1, \ldots, T_{2m}$ are equidistributed regarding their arguments. Set

$$H_{am}(x, y) = \frac{\pi}{2M^2} \sum_{j=1}^{2m} R_m^{T_j}(x, y). \tag{14}$$

(13) shows that for $(x, y) \in \partial K$

$$H_{am}(x, y) \geq \frac{\pi}{2M^2} \frac{2M^2 - 4M - 2\pi}{\pi} e^{-c_2\varepsilon^{1/3}},$$

while (9)–(12) give

$$H_{am}(x, y) \leq \frac{\pi}{2M^2} \left[\frac{2M^2 + 4M + 2\pi}{\pi} + 2C_0 M^2 e^{-c_1\sqrt{M}} + 2\sum_k C_0 \cdot 2^{k+1} M^2 e^{-c_1\sqrt{b_1 M^2 2^{k-1}}} + 2m2^{-m}\right].$$

Hence, for small ε, i.e. for large $M = \varepsilon^{-1/3}$, and for all large m we have

$$e^{-c_2\varepsilon^{1/3}} - \frac{3}{M} \leq H_{am}(x, y) \leq 1 + \frac{3}{M}.$$

Therefore,

$$1-(c_2+3)\varepsilon^{1/3} \leq H_{am}(x,y) \leq 1+3\varepsilon^{1/3},$$

which shows the claim for the degree am.

It is also clear that these reasonings give for $k=1,2,\ldots,(a/2)-1$ that if we define

$$H_{am}^{2k}(x,y)=\frac{\pi}{2M^2}\sum_{j=1}^{2m}\frac{1}{x_j^{2k}+y_j^{2k}}R_m^{T_j}(x,y); \qquad T_j=:(x_j,y_j) \tag{15}$$

then

$$\left|H_{am}^{2k}(x,y)-\frac{1}{x^{2k}+y^{2k}}\right| \leq C_2\varepsilon^{1/3},$$

and so

$$H_{am+2k}(x,y)=(x^{2k}+y^{2k})H_{2m}^{2k}(x,y), \qquad m=1,2,\ldots,\ k=1,2,\ldots,(a/2)m-1$$

together with the above H_{am} give a sequence of homogeneous polynomials with the desired property for the full sequence of even integers.

■

3 Proof of Propositions 2 and 3

Proof of Proposition 2. Let $P_1,\ldots,P_{2k}$ be the points on the boundary of K where there are two supporting lines with angle $>\varepsilon$. Their number is finite, since the total rotation of supporting lines is 2π as we move around ∂K once. Also, by symmetry, their number is even and the set $\{P_1,\ldots,P_{2k}\}$ is centrally symmetric. First assume that k is even. Then all we have to do is to replace the arcs $\overline{P_1,P_2},\overline{P_3,P_4},\ldots,\overline{P_{2j+1},P_{2j+2}},\ldots,$ $j<k/2$ on ∂K by some suitable smooth arcs lying outside K, and these arcs, the arcs $\overline{P_2,P_3},\ldots,\overline{P_{2j},P_{2j+2}},\ldots,$ $j<k/2$ together with their reflections on the origin form the boundary of K_1. If we exchange the roles of the arcs $\overline{P_{2j+1},P_{2j+2}}$ and $\overline{P_{2j},P_{2j+2}}$ then we obtain K_2, and the intersection of K_1 and K_2 is K. Indeed, Figure 3 explains everything.

Hence, when k is even we only need two sets K_1 and K_2 (and if we want to have formally the 4-intersection in the proposition then just use K_1,K_1,K_2,K_2). When k is odd, then in between P_1 and P_2 (and symmetrically in between P_{k+1} and P_{k+2}) we add a vertex $P_{3/2}$ (and $P_{k+3/2}$) that we count among the P_i's as is shown in Figure 4 to get the set K'. Similarly, in between P_2 and P_3 (and symmetrically in between P_{k+2} and P_{k+3}) we add a vertex $P_{5/2}$ (and $P_{k+5/2}$) that now we count among the P_i's to get the set K''. Now $K=K'\cap K''$, and the number k (which is now actually $k+1$ with the original k) for the sets K'

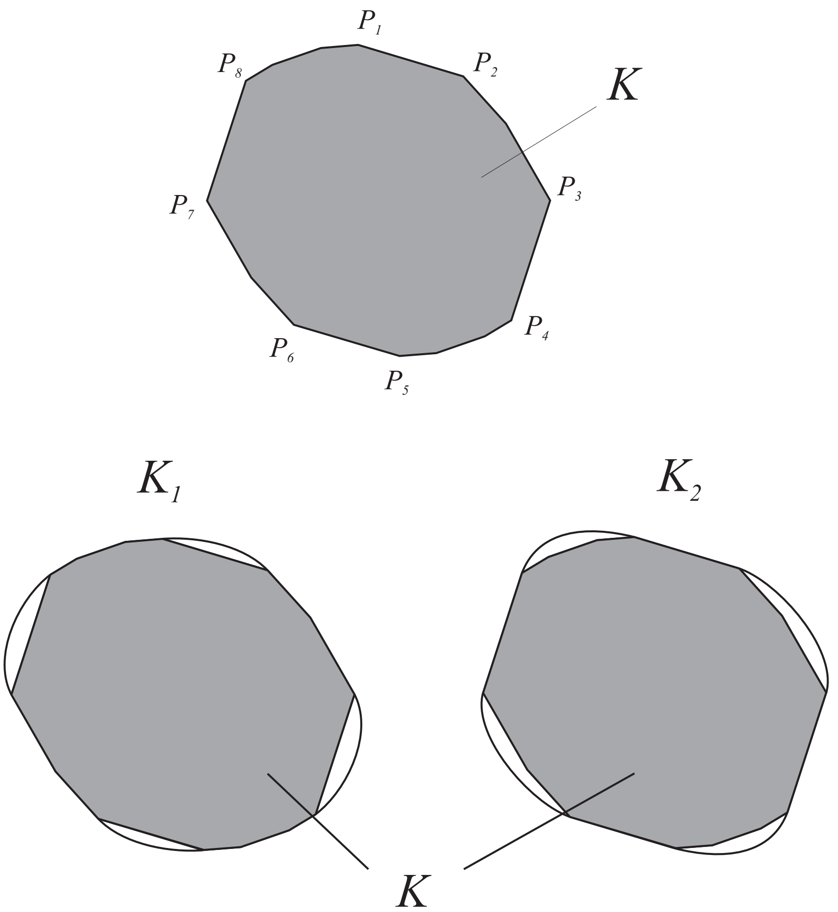

Figure 3: K and the associated K_1 and K_2

and K'' is even. Hence, according to what we have just seen, $K' = K_1' \cap K_2'$ and $K'' = K_1'' \cap K_2''$ with some ε-regular sets K_1', K_2', K_1'', K_2'', so these 4 sets are suitable in the proposition.

■

Proof of Proposition 3. We repeat an argument of [7, Theorem 1.4,(c)]. Select homogeneous polynomials $V_s(X_1, X_2, X_3, X_4)$, $V_{s+1}(X_1, X_2, X_3, X_4)$ of four variables and of some degrees s, $s+1$, respectively, such that

$$|V_s(X_1, X_2, X_3, X_4) - 1| + |V_{s+1}(X_1, X_2, X_3, X_4) - 1| < \eta/4 \tag{16}$$

provided $\max(X_1, X_2, X_3, X_4) = 1$, $X_j \in [0, 1]$ (see Proposition 5 below). Then there is a $\delta > 0$ (depending also on s) such that (16) is true also for all $|\max(X_1, X_2, X_3, X_4) - 1| \le \delta$, $X_j \in [0, 2]$. Now let $H_{2m}^{(j)}$, be the polynomials from Proposition 4 for the sets K_j, $j = 1, 2, 3, 4$, and for them we may assume that the A is the same in (2) (recall that A depended only on L). If $A\varepsilon^{1/3} < \delta$, then for

$$R_{2(m+k)s+2k}(x, y) \quad = \quad V_s\Big(H_{2m}^{(1)}(x, y), H_{2m}^{(2)}(x, y), H_{2m}^{(3)}(x, y), H_{2m}^{(4)}(x, y)\Big) \times$$

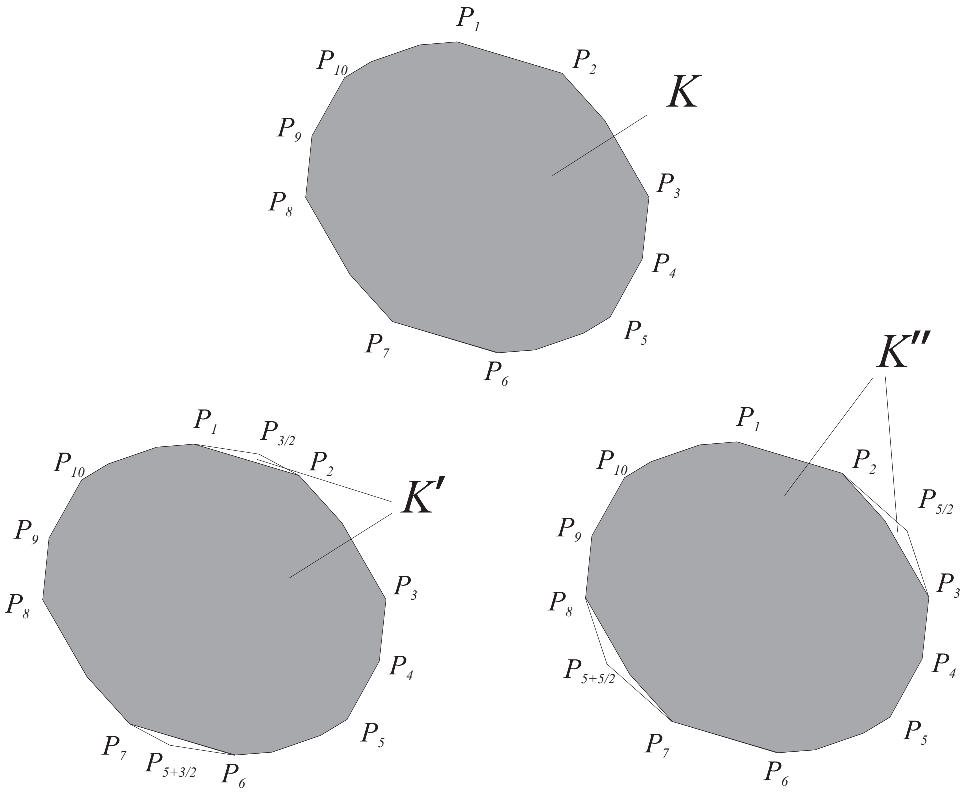

Figure 4: K and the construction of K' and K''

$$\times \quad V_{s+1}\Big(H^{(1)}_{2k}(x,y), H^{(2)}_{2k}(x,y), H^{(3)}_{2k}(x,y), H^{(4)}_{2k}(x,y)\Big)$$

we will have for $(x,y) \in \partial K$

$$(1-\eta/4)^2 \le R_{2(m+k)s+2k}(x,y) \le (1+\eta/4)^2,$$

because $(x,y) \in \partial K$ means that the point (x,y) lies on the boundary of some of the K_j's (at least for one of them), and it lies in the interior of the others. Since for any m_0 all large even integers are of the form $2(m+k)s+2k$ with $m \ge m_0$, $m_0 < k \le m_0 + s$, the proof is complete.

■

To have a complete proof we need to show

Proposition 5 *Let $r \ge 2$ be an integer. For every $m \ge 1$ there are homogeneous polynomials $V_{r,m}(X_1,\dots,X_r)$ of degree m in the variables $X_1,\dots,X_r$ such that $V_m(X_1,\dots,X_r)$ uniformly tends to 1 on the set*

$$E := \{(X_1,\dots,X_r) \,|\, \max\{X_1,\dots,X_r\} = 1,\ X_j \in [0,1]\}.$$

Proof. First we deal with the $r=2$ case. Once it is done, the claim is obtained for all r by an induction similar to the one from the preceding proof.

We note first of all that for every $\varepsilon>0$ there is a k such that for sufficiently large m

$$\frac{1}{(1+u)^m}\left|(1-u)^m-\sum_{j=0}^{k-1}\binom{m}{j}(-1)^j u^j\right|\le\varepsilon,\qquad u\in[0,1]. \tag{17}$$

Indeed, by the remainder formula for Taylor expansions the left-hand side equals (with some $\xi\in(0,u)$)

$$\frac{1}{(1+u)^m}\binom{m}{k}(1-\xi)^{m-k}u^k\le\frac{m^k u^k}{k!(1+u)^m},$$

and here the right-hand side takes its maximum on $[0,1]$ at $u=k/(m-k)$, the maximum being

$$\frac{m^k}{k!}\frac{k^k}{(m-k)^k}\frac{(m-k)^m}{m^m}\to\frac{k^k}{k!e^k}\sim\frac{1}{\sqrt{2\pi k}},$$

as $m\to\infty$, where we also used Stirling's formula.

Next we invoke the solution to Bernstein's approximation problem according to which all continuous functions g with $g(x)/e^{|x|/2}\to 0$ as $|x|\to\infty$ can be uniformly approximated on the whole real line by polynomials with the weight $e^{-|x|/2}$ (see e.g. [5, Theorem 1.3]). Thus, there is a polynomial H such that

$$e^{-|x|/2}\left|\sum_{j=0}^{k-1}(-1)^j\frac{|x|^j}{j!}-H(x)\right|\le\varepsilon,$$

which, with the substitution $x=mu$ gives

$$e^{-m|u|/2}\left|\sum_{j=0}^{k-1}(-1)^j\frac{m^j|u|^j}{j!}-H(mu)\right|\le\varepsilon. \tag{18}$$

Finally, for each fixed j the expression

$$e^{-m|u|/2}\left|\binom{m}{j}|u|^j-\frac{m^j}{j!}|u|^j\right|=e^{-m|u|/2}\frac{(m|u|)^j}{j!}\left|\frac{m(m-1)\cdots(m-j+1)}{m^j}-1\right|$$

tends uniformly to 0 as $m\to\infty$, so for sufficiently large m

$$e^{-m|u|/2}\left|\sum_{j=0}^{k-1}\binom{m}{j}(-1)^j|u|^j-\sum_{j=0}^{k-1}\frac{m^j}{j!}(-1)^j|u|^j\right|\le\varepsilon,\quad u\in\mathbf{R} \tag{19}$$

Now noting that $1+|u| \geq e^{|u|/2}$ for $|u| \leq 1$, formulae (17), (18) and (19) give

$$\frac{\Big|(1-|u|)^m - H(mu)\Big|}{(1+|u|)^m} \leq 3\varepsilon, \qquad |u| \leq 1. \tag{20}$$

Here we may assume H to be even (just replace it by $(H(x)+H(-x))/2$ if not), so $Q(u) = H(m\sqrt{u})$ is a polynomial, and with it

$$\frac{\Big|(1-\sqrt{u})^m - Q(u)\Big|}{(1+\sqrt{u})^m} \leq 3\varepsilon, \qquad u \in [0,1].$$

Setting here $u = 1 - 1/z^2$ and multiplying through both the numerator and denominator by z^m we obtain with $Q_m^*(z) = z^m Q(1-1/z^2)$, which is a polynomial of degree m for large m,

$$\frac{\Big|(z-\sqrt{z^2-1})^m - Q_m^*(u)\Big|}{(z+\sqrt{z^2-1})^m} \leq 3\varepsilon, \qquad z \geq 1.$$

This gives for $S_m(z) = -Q_m^*(z) + 2T_m(z)$, where

$$T_m(z) = \frac{1}{2}\left[(z+\sqrt{z^2-1})^m + (z-\sqrt{z^2-1})^m\right]$$

are the Chebyshev polynomials, the estimate

$$\left|\frac{S_m(z)}{(z+\sqrt{z^2-1})^m} - 1\right| \leq 3\varepsilon, \qquad z \geq 1. \tag{21}$$

Finally, apply the Zhoukovskii transformation

$$x = \frac{1}{z+\sqrt{z^2-1}}, \qquad z = \frac{1}{2}\left(x + \frac{1}{x}\right),$$

and write

$$x^m S_m\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right) := a_m x^m + \sum_{j=1}^{m} a_{m-j}(x^{m+j} + x^{m-j}).$$

If we set

$$V_{2,2m}(x,y) = a_m x^m y^m + \sum_{j=1}^{m} a_{m-j}(x^{m+j}y^{m-j} + x^{m-j}y^{m+j}),$$

then $V_{2,2m}$ is a homogeneous polynomial of two variables of degree $2m$, and we have by (21)

$$|V_{2,2m}(x,1) - 1| \leq 3\varepsilon, \qquad x \in [0,1].$$

and

$$|V_{2,2m}(1,y)-1|\le 3\varepsilon, \qquad y\in[0,1].$$

This is the claim for even degrees.

Standard Stone–Weierstrass-type argument gives that then every continuous function on the set E can be uniformly approximated by such $V_{2,2m}$'s (i.e. homogeneous polynomials of degree $2m=2,4,\ldots$). Hence for some $V^*_{2,2m}$ we have $V^*_{2,2m}(x,y)\to\frac{1}{x+y}$ uniformly on E, and then $(x+y)V^*_{2,2m}(x,y)$ are suitable for odd degrees. This proves Proposition 5 for $r=2$.

For $r>2$ we use induction. Suppose that the existence of $V_{r-1,m}$ has already been verified. We repeat the argument from the proof of Theorem 3, see [7, Theorem 1.4,(c)]. For some small $\eta>0$ select homogeneous polynomials $V_{2,s}(X,Y)$, $V_{2,s+1}(X,Y)$ of two variables and of some degrees s, $s+1$, respectively, such that

$$|V_{2,s}(X,Y)-1|+|V_{2,s+1}(X,Y)-1|<\eta/4 \tag{22}$$

provided $\max(X,Y)=1$, $X,Y\in[0,1]$ (this is the just verified $r=2$ case). Then there is a $\delta>0$ (depending also on s) such that (22) is true also for all $|\max(X,Y)-1|\le\delta$, $X,Y\in[0,2]$. Now consider

$$\begin{aligned} R_{r,2(m+k)s+2k}(X_1,\ldots,X_r) &= V_{2,s}\Big(V_{r-1,m}(X_1,\ldots,X_{r-1}),V_{r-1,m}(X_2,\ldots,X_r)\Big)\times\\ &\times\ V_{2,s+1}\Big(V_{r-1,k}(X_1,\ldots,X_{r-1}),V_{r-1,k}(X_2,\ldots,X_r)\Big).\end{aligned}$$

For large m we have uniformly in $\max_{1\le j\le r}X_j=1$, $X_j\in[0,1]$ the relations

$$0\le V_{r-1,m}(X_1,\ldots,X_{r-1}),\quad V_{r-1,m}(X_2,\ldots,X_r)<1+\delta.$$

Indeed, for example with $M:=\max\{X_1,\ldots,X_{r-1}\}\le 1$ we have for large m

$$|V_{r-1,m}(X_1/M,\ldots,X_{r-1}/M)-1|<\delta$$

and

$$V_{r-1,m}(X_1,\ldots,X_{r-1})=M^{r-1}V_{r-1,m}(X_1/M,\ldots,X_{r-1}/M).$$

Therefore, for an r-tuple $(X_1,\ldots,X_r)$ with $\max\{X_1,\ldots,X_r\}=1$, $X_j\in[0,1]$ the inequality

$$\Big|V_{2,s}\Big(V_{r-1,m}(X_1,\ldots,X_{r-1}),V_{r-1,m}(X_2,\ldots,X_r)\Big)-1\Big|\ge\eta/4 \tag{23}$$

can only happen if

$$\max\{V_{r-1,m}(X_1,\ldots,X_{r-1}),V_{r-1,m}(X_2,\ldots,X_r)\}\le 1-\delta.$$

This in turn, for large m, would mean that

$$\max\{X_1,\ldots,X_{r-1}\}<1,\quad \max\{X_2,\ldots,X_r\}<1,$$

i.e.

$$\max\{X_1,\dots,X_r\}<1,$$

which is not the case.

Hence, (23) cannot happen for large m, and we can deduce

$$1-\eta/4<V_{2,s}\Big(V_{r-1,m}(X_1,\dots,X_{r-1}),V_{r-1,m}(X_2,\dots,X_r)\Big)<1+\eta/4.$$

A similar bound can be given for the second factor in $R_{r,2(m+k)s+2k}(X_1,\dots,X_r)$ for large k, and we obtain

$$(1-\eta/4)^2\le R_{r,2(m+k)s+2k}(X_1,\dots,X_r)\le(1+\eta/4)^2. \tag{24}$$

Since for any m_0 all large even integers are of the form $2(m+k)s+2k$ with $m\ge m_0$, $m_0<k\le m_0+s$, we get that $R_{r,2(m+k)s+2k}(X_1,\dots,X_r)$ has the desired property if in its definition we let $s\to\infty$ very slowly.

To be absolutely clear, the selection of the indices $2(m+k)s+2k$ in $R_{2(m+k)s+2k}$ is as follows. We set $\eta=1/l$, $l=1,2,\dots$. Then (22) holds with some $s=s_l$. To this s_l choose $\delta=\delta_l$ so that (22) is true for $|\max(X,Y)-1|<\delta_l$, $X,Y\in[0,2]$. Then for $m\ge m_l$ (23) is impossible, so we get (24) for all $2(m+k)s+2k$ with $m\ge m_l$, $m_l<k\le m_l+s$. These integers cover all even integers $2n\ge 2n_l$ with some number n_l. Now we keep these $R_{r,2(m+k)s+2k}=:R_{2,2n}$ for all $2n=2(m+k)s+2k$ for which $n_l<n\le n_{l+1}$ (and then move to the same construction with l replaced by $l+1$ etc.). Hence, for $n_l<n\le n_{l+1}$ we have

$$\left(1-\frac{1}{4}\right)^2\le R_{r,2n}(X_1,\dots,X_r)\le\left(1+\frac{1}{4}\right)^2$$

provided $\max X_j=1$, $X_j\in[0,1]$.

■

4 The Kroó-Szabados theorem

We have already mentioned that Kroó's conjecture is true in any dimension for convex sets K which have only one supporting hyperplane at any boundary point. This was proved in [4]. Now this theorem actually follows from the proof in Proposition 4 in the $\varepsilon=0$ case. Indeed, if the point $(1,0,\dots,0)$ belongs to ∂K and the hyperplane ℓ defined by $x_1+b_2x_2+\cdots+b_nx_d=1$ is the supporting hyperplane to K at $(1,0,\dots,0)$, then the only change needed in the proof of Proposition 4 is to consider instead of the R_m in (7) the polynomial

$$R_m(x_1,\cdots,x_n)=(x_1+b_2x_2+\cdots+b_dx_d)^{am}S_{2m}\left(\frac{\sqrt{x_2^2+\cdots+x_d^2}}{x_1+b_2x_2+\cdots+b_dx_d}\right) \tag{25}$$

(note that S_{2m} is even, so this is a polynomial; here $\sqrt{x_2^2+\cdots+x_d^2}$ plays the role of $|Y|$ from the proof of Proposition 4), and then do the analogue of (14) for some fairly uniformly chosen rays.

Since this latter requirement is not as straightforward in $\mathbf{R}^d$ than in $\mathbf{R}^2$, we sketch it. A "ray" from the origin is given by a point P on the $(d-1)$-sphere S^{d-1}. One can get "fairly uniformly chosen rays" of "density $\sim 1/m$" (that could replace $e^{ij2\pi/2m}$, $j=1,\dots,2m$ that were used in (15)) as follows. Let us put as many points as possible on S^{d-1} such that the distance in between any two is at least $1/m$. Let $X_m \subset S^{d-1}$ be a point system with this (admissibility) property for which $|X_m|$, the cardinality of X_m, is maximal. It is easy to see that $|X_m| \sim m^{d-1}$ (see the proof of (A) below).

Let $P_0 = (1,0,\dots,0) \in S^{d-1}$ the point considered above, and define the spherical cap around P_0 of radius r as

$$S(P_0,r) := \{(x_1,\dots,x_d)\,|\,x_1>0,\ x_2^2+x_3^2+\cdots+x_d^2 \le r^2\}.$$

We define similarly $S(P,r)$ around any point $P \in S^{d-1}$. Set $N_m = |X_m \cap S(P_0, M^2/m)|$. What we need when we want to copy (14) is the following:

(A) $N_m \sim M^{2(d-1)}$

(B) For any $P \in S^{d-1}$

- **(a)** $|X_m \cap S(P, 2^k M^2/m)| \le C(2^k M^2)^{d-1}$
- **(b)** $|X_m \cap S(P,(M^2-2M)/m)| \ge (1-c/M)N_m$
- **(c)** $|X_m \cap S(P,(M^2+2M)/m)| \le (1+c/M)N_m$

(on the right $(1\pm c/M)$ is not really necessary—it could be replaced by $(1+o(1))$—, but that is what was used before).

Indeed, once this is established, we can proceed as in the proof of Proposition 4: let $R_m^T(x_1,\cdots,x_n)$ be the analogue of (25) for a point $T \in \partial K$, and set, in analogy with (14),

$$H_{am}(x_1,\dots,x_d) = \frac{1}{N_m}\sum_{T/\|T\|\in X_m} R_m^T(x_1,\dots,x_d), \tag{26}$$

where the summation is taken for the point set on ∂K that corresponds to the rays in X_m. In this case we have the analogues of (10)–(13), e.g.

$$R_m^T(x_1,\dots,x_d) \ge e^{-c_2\varepsilon^{1/3}}$$

if $(x_1,\dots,x_d)$ belongs to the region of ∂K determined by the spherical cap $S\big(T/\|T\|,(M^2-2M)/m\big)$ (this corresponds to $\varphi \in [(-M^2+2M)/m,(M^2-2M)/m]$ in (13)). Now using the properties (A) and (B) the proof given in Proposition 4 shows that

$$e^{-c_2\varepsilon^{1/3}} - \frac{C}{M} \le H_{am}(x,y) \le 1+\frac{C}{M}.$$

To prove (A) note first of all that one can put $cM^{2(d-1)}$ points into the cap $S(P_0, M^2/m)$ with mutual distances $\geq 1/m$: consider for example, with some small c_1, points around P_0 with their spherical coordinates forming a rectangular grid of size $c_1M^2 \times \cdots \times c_1M^2$ and of mesh size $4/m$. This combined with the replacement argument below shows that $N_m \geq cM^{2(d-1)}$. On the other hand the spherical cap of radius $1/4m$ about the points of X_m are all disjoint, so the total $(d-1)$-dimensional volume of those caps (of radii $1/4m$) the center of which lie in the set $X_m \cap S(P_0, M^2/m)$ is at least $N_m c(1/4m)^{d-1}$, and that must be smaller than the volume of $S(P_0, M^2/m)$, which is $\sim (M^2/m)^{d-1}$. This gives $N_m \leq CM^{2(d-1)}$.

The proof of (B)(a) is the same. By looking again at the volume of spherical caps of radius $1/4m$ with center in the set $X_m \cap \big(S(P_0, M^2/m) \setminus S(P_0, (M^2 - 4M)/m)\big)$ we get that

$$|X_m \cap S(P_0, M^2/m)| \leq |X_m \cap S(P_0, (M^2 - 4M)/m)| + CM^{2d-3}$$

(since the $(d-1)$-dimensional volume of the spherical ring $S(P_0, (M^2+1)/m) \setminus S(P_0, (M^2 - 4M - 1)/m)$ is at most CM^{2d-3}/m^{d-1} and this ring contains the disjoint spherical caps of radii $1/4m$ with center in the set $X_m \cap \big(S(P, M^2/m) \setminus S(P, (M^2 - 4M)/m)\big)$). Now if we remove all points from X_m that lie in $S(P, (M^2 - 2M)/m)$ and replace them by a rotated copy of $X_m \cap S(P_0, (M^2 - 4M)/m)$ (by a rotation that takes P_0 into P), then we get an admissible point system, hence

$$\begin{aligned} |X_m \cap S(P, (M^2 - 2M)/m)| &\geq |X_m \cap S(P_0, (M^2 - 4M)/m)| \\ &\geq |X_m \cap S(P_0, M^2/m)| - CM^{2d-3}, \end{aligned}$$

which proves (B)(b) in view of (A). The proof of (B)(c) is similar.

■

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