Fast decreasing and orthogonal polynomials

Vilmos Totik

Abstract. This paper reviews some aspects of fast decreasing polynomials and some of their recent use in the theory of orthogonal polynomials.

1. Fast decreasing polynomials

Fast decreasing, or pin polynomials have been used in various situations. They imitate the "Dirac delta" best among polynomials of a given degree. They are an indispensable tool to localize results and to create well localized "partitions of unity" consisting of polynomials of a given degree.

We use the setup for them as was done in [**5**], from where the results of this section are taken. Let Φ be an even function on $[-1,1]$, increasing on $[0,1]$, and suppose that $\Phi(0) \le 0$. Consider $e^{-\Phi(x)}$, and our aim is to find polynomials P_n of a given degree $\le n$ such that

$$P_n(0) = 1, \qquad |P_n(x)| \le e^{-\Phi(x)}, \quad x \in [-1,1]. \tag{1.1}$$

Let $n_\Phi = n$ be the minimal degree for which this is possible. The following theorem gives an explicitly computable bound for this minimal degree.

Theorem 1.1. **(Ivanov–Totik [5])**

$$\frac{1}{6} N_\Phi \le n_\Phi \le 12 N_\Phi,$$

where

$$\begin{aligned} N_\Phi &= 2 \sup_{\Phi^{-1}(0)\le x<\Phi^{-1}(1)} \sqrt{\frac{\Phi(x)}{x^2}} \\ &+ \int_{\Phi^{-1}(1)}^{1/2} \frac{\Phi(x)}{x^2} dx + \sup_{1/2\le x<1} \frac{\Phi(x)}{-\log(1-x)} + 1. \end{aligned}$$

Here

$$\Phi^{-1}(t) = \sup\{u \mid \Phi(u) \le t\}$$

is the generalized inverse.

It can be shown that each term can be dominant in N_Φ, but, in the most important cases, the second term gives the order of n_Φ.

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Theorem 1.1 is universal in the sense that the function Φ may have parameters, in particular it may include the degree n of the polynomial. In applications mostly the following two types of decrease is used. Let φ, $\varphi(0) \le 0$, be an even function on $[-1,1]$. Assume that φ is increasing on $[0,1]$ and there is a constant M_0 such that $\varphi(2x) \le M_0\varphi(x)$ for all $0 < x \le 1/2$.

Corollary 1.2. *There are P_n of degree at most $n = 1, 2, \ldots$ with*

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-cn\varphi(x)}, \qquad x \in [-1,1],$$

(*where $c, C > 0$ are independent of n*), *if and only if*

$$\int_0^1 \frac{\varphi(u)}{u^2} du < \infty.$$

On the other hand, if ψ, $\psi(0) \le 0$, is an even function on $\mathbf{R}$ for which ψ is increasing on $[0,\infty)$ and there is a constant M_0 such that $\psi(2x) \le M_0\psi(x)$ for all $x > 0$, then we have

Corollary 1.3. *There are P_n of degree at most $n = 1, 2, \ldots$ with*

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-c\psi(nx)}, \qquad x \in [-1,1],$$

(*where $c, C > 0$ are independent of n*), *if and only if*

$$\int_{-\infty}^{\infty} \frac{\psi(u)}{1+u^2} du < \infty.$$

In Corollary 1.2 the decrease of $\{P_n(x)\}_{n=1}^{\infty}$ is exponential at every $x \ne 0$. In Corollary 1.3 this decrease is somewhat worse, but the polynomials P_n start to get small very close to 0 ($e^{-c\psi(nx)}$ start having effect from $|x| \sim 1/n$).

As concrete examples consider

Example 1.4.

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-cn|x|^{\alpha}}, \qquad x \in [-1,1],$$

with some P_n of degree at most $n = 1, 2, \ldots$ (and with some $c, C > 0$) is possible precisely for $\alpha > 1$.

Example 1.5.

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-c(n|x|)^{\beta}}, \qquad x \in [-1,1], \tag{1.2}$$

with some P_n of degree at most $n = 1, 2, \ldots$ (and with some $c, C > 0$) is possible precisely for $\beta < 1$.

In particular,

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-cn|x|}, \qquad x \in [-1,1],$$

is NOT possible for polynomials of degree at most n. It easily follows from Theorem 1.1 that to have this decrease one needs $\deg(P_n) \ge cn\log n$.

2. Quasi-uniform zero spacing of orthogonal polynomials

Let μ be a Borel-measure on $[-1,1]$ with infinite support, let $p_n(x) = \gamma_n x^n + \cdots$ denote the orthonormal polynomials with respect to μ, and let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ be the zeros of p_n. There is a vast literature on the spacing of these zeros. For example, in the case of Jacobi polynomials classical results show (see e.g. [**16**, Ch 6]) that if $x_{n,k} = \cos\theta_{n,k}$, then $\theta_{n,k} - \theta_{n,k+1} \sim 1/n$ (here, and in what follows, $A \sim B$ means that the ration A/B is bounded away from 0 and ∞). With

$$\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$$

this is the same as

$$x_{n,k+1} - x_{n,k} \sim \Delta_n(x_{n,k}),$$

and we call this behavior quasi-uniform spacing (B. Simon would probably use a terminology of some kind of “clock behavior”). One can visualize quasi-uniform spacing in the following way: project the zeros $x_{n,k}$ onto the unit circle (up and down) to get $2n$ points. These points divide the unit circle into $2n$ arcs. Now quasi-uniform behavior means that the length of these arcs is $\sim 1/n$, i.e. the ratio of the length of any two of these arcs is bounded by a constant independent of the arcs and of n. Note that this is also true for the arcs around ± 1 which are the projections of the segments $[-1, x_{1,n}]$ and $[x_{n,n}, 1]$.

In the paper [**11**] this quasi-uniform behavior was shown to be the case for a large class of measures, namely for the so called doubling measures. A measure μ with $\mathrm{supp}(\mu) = [-1,1]$ is called doubling if

$$\mu(2I) \le L\mu(I), \qquad \text{for all intervals } I \subset [-1,1]. \tag{2.1}$$

Here $2I$ is the interval I enlarged twice from its center. This is a fairly weak condition, for example, all generalized Jacobi weights

$$d\mu(x) = h(x) \prod |x - x_j|^{\gamma_j} dx, \qquad \gamma_j > -1, \quad h > 0 \text{ continuous},$$

are doubling. On the other hand, if $d\mu(x) = |x|^\gamma$ for $-1 \le x < 0$, and $d\mu(x) = |x|^\delta$ for $0 < x \le 1$, then this μ is doubling only if $\gamma = \delta$. Note also that by a result of Feffermann and Muckenhoupt [**3**], a doubling measure can vanish on a set of positive measure, so a doubling measure is not necessarily in the Szegő class. With this notion the aforementioned result states as

Theorem 2.1. **(Mastroianni-Totik [11])** *If μ is doubling, then*

$$A^{-1} \le \frac{x_{n,k+1} - x_{n,k}}{\Delta_n(x_{n,k})} \le A \tag{2.2}$$

with some constant A independent of n and k, i.e. the zeros are quasi-uniformly distributed.

Note that there is a “rule of thumb”: zeros accumulate where μ is large. The reason for this is that the monic orthogonal polynomials p_n/γ_n minimize the $L^2(\mu)$-norm:

$$\int \left(\frac{p_n(\mu,\cdot)}{\gamma_n}\right)^2 d\mu = \min\left\{ \int P_n^2 d\mu \,\middle|\, P_n(x) = x^n + \cdots \right\}. \tag{2.3}$$

But this is only a very crude rule, since for a weight like

$$d\mu(x) = |x - 1/2|^{200} |x + 1/2|^{-1/2} dx$$

the zero spacing is quasi-uniform regardless that the weight is much stronger around $-1/2$ than around $1/2$. Of course, finer spacing will distinguish such differences in the weight (see e.g. [**23**]).

Y. Last and B. Simon [**8**] had the first results on local zero spacing if only local information is used on the weight. They proved that
1. if $d\mu(x) = w(x)dx$ and for some $q > 0$

$$A|x - Z|^q \le w(x) \le B|x - Z|^q$$

in a neighborhood of a point Z, then (with a C independently of n)

$$|x_n^{(1)}(Z) - x_n^{(-1)}(Z)| \le \frac{C}{n}$$

where $x_n^{(-1)}(Z) \le Z \le x_n^{(1)}(Z)$ are the zeros enclosing Z, and
2. if w is bounded away from 0 and ∞ on I, then (with a c independently of n)

$$|x_n^{(1)}(y) - x_n^{(-1)}(y)| \ge \frac{c}{n}$$

inside I.

This was extended to locally doubling measures by T. Varga:

THEOREM 2.2. **(Varga [24])** *If μ is doubling on an interval I, then*

$$x_{n,k+1} - x_{n,k} \sim \frac{1}{n}$$

locally uniformly inside I.

The endpoint version of this is:

THEOREM 2.3. **(Totik-Varga [19])** *If μ is doubling on $I = [a, b]$ and $\mu((a - \varepsilon, a)) = 0$, then*

$$x_{n,k+1} - x_{n,k} \sim \frac{\sqrt{x_{n,k+1} - a}}{n} + \frac{1}{n^2}$$

locally uniformly for $x_{n,k} \in [a, b - \varepsilon]$.

So this holds around local endpoints of the support (i.e. at which, for some $\varepsilon > 0$, $\mu((a - \varepsilon, a)) = 0$ but $\mu((a, a + \varepsilon)) \ne 0$).

Zero spacing is connected to the measure via Christoffel functions and the Markov inequalities. So to see how fast decreasing polynomials enter the picture in connection with zero spacing we have to discuss Christoffel functions.

3. Christoffel functions

Recall the definition of the n-th Christoffel function associated with a measure μ:

$$\lambda_n(x) = \inf_{P_n(x)=1} \int |P_n|^2 d\mu,$$

where the infimum is taken for all polynomials of degree at most n taking the value 1 at the point x. It is well known (and easily comes from the minimality property (2.3)) that

$$\lambda_n(x) = \left(\sum_{k=0}^{n} |p_k(\mu, x)|^2 \right)^{-1}.$$

Their importance lies in the fact that Christoffel functions, unlike the orthogonal polynomials, are monotone in the measure μ (as well as in their index n). Hence, they are much easier to handle than the orthogonal polynomials themselves.

For them the following rough asymptotics was proved.

THEOREM 3.1. **(Mastroianni-Totik [11])** *If μ is doubling, then for $x \in [-1,1]$*

$$\lambda_n(x) \sim \mu([x - \Delta_n(x), x + \Delta_n(x)])$$

uniformly in n and $x \in [-1,1]$.

Recall also the Cotes numbers:

$$\lambda_{n,k} = \lambda_n(x_{n,k}),$$

which appear in Gaussian quadrature

$$\int_{-1}^{1} f d\mu \sim \sum_{k=1}^{n} \lambda_{n,k} f(x_{n,k}).$$

For them Theorems 2.1 and 3.1 easily give

THEOREM 3.2. **(Mastroianni-Totik [11])** *If μ is doubling, then for all n and $1 \le k < n$ we have*

$$B^{-1} \le \frac{\lambda_{n,k}}{\lambda_{n,k+1}} \le B, \tag{3.1}$$

with some constant B independent of n and k.

Now Theorems 2.1 and 3.2 have a converse:

THEOREM 3.3. **(Mastroianni-Totik [11])** *If μ is supported on $[-1,1]$ and* (2.2) *and* (3.1) *are true, then μ is doubling.*

We mention that it is an open problem if (2.2) (i.e. quasi-uniform zero spacing) alone is equivalent to μ being doubling.

Next, we show how fast decreasing polynomials are used in connection with zero spacing. Zero spacing of orthogonal polynomials is controlled by the Christoffel function via the Markov inequalities:

$$\sum_{j=1}^{k-1} \lambda_{n,j} \le \mu((-\infty, x_{n,k})) \le \mu((-\infty, x_{n,k}]) \le \sum_{j=1}^{k} \lambda_{n,j}. \tag{3.2}$$

If we apply this with the index k and the index i, then it follows that

$$\sum_{j=i+1}^{k-1} \lambda_{n,j} \le \int_{x_{n,i}}^{x_{n,k}} d\mu \le \sum_{j=i}^{k} \lambda_{n,j}. \tag{3.3}$$

Suppose we want to prove the upper estimate in Theorem 2.2. Thus, suppose that μ is a doubling weight on, say, $[-1,1]$, and we want to prove $x_{n,k+1} - x_{n,k} \le C/n$ for all zeros lying in, say, $[-1/2,1/2]$. We claim, that to this all we need is the bound

$$\lambda_n(x) \le C\mu([x - 1/n, x + 1/n]), \qquad x \in [-3/4, 3/4]. \tag{3.4}$$

Indeed, then from the Markov inequality (3.3) and from (3.4), we have

$$\begin{aligned} \mu([x_{n,k}, x_{n,k+1}]) &\le \lambda_{n,k} + \lambda_{n,k+1} \\ &\le C\Big(\mu([x_{n,k} - 1/n, x_{n,k} + 1/n]) + \mu([x_{n,k+1} - 1/n, x_{n,k+1} + 1/n])\Big). \end{aligned} \tag{3.5}$$

We may assume $x_{n,k+1} - x_{n,k} > 4/n$, since otherwise there is nothing to prove. Then we set $I = [x_{n,k} - 1/n, x_{n,k+1} + 1/n]$, $E_1 = [x_{n,k} - 1/n, x_{n,k} + 1/n]$ and $E_2 = [x_{n,k+1} - 1/n, x_{n,k+1} + 1/n]$. Using the doubling property (2.1) and the bound (3.5), we get

$$\mu(I) \le L\mu([x_{n,k}, x_{n,k+1}]) \le CL\left(\mu(E_1) + \mu(E_2)\right).$$

Now it can be shown that the doubling property implies that with some K and $r > 0$

$$\mu(E_1) \le K\left(\frac{|E_1|}{|I|}\right)^r \mu(I),$$

$$\mu(E_2) \le K\left(\frac{|E_2|}{|I|}\right)^r \mu(I).$$

Consequently, the preceding inequalities yield

$$1 \le 2CL\frac{K}{|I|^r}(|E_1| + |E_2|)^r$$

i.e.

$$x_{n,k+1} - x_{n,k} < |I| \le (2CLK)^{1/r}\frac{2}{n}.$$

Thus, it is enough to prove (3.4), and this is where fast decreasing polynomials enter the picture. Since we want to prove a local result like (3.4) from the local assumption that μ is doubling in a neighborhood I of x, we may assume that $x = 0 \in I = [-a, a]$ and $\mathrm{supp}(\mu) \subset [-1, 1]$. Take fast decreasing polynomials P_n of degree at most n such that

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-c(n|x|)^{1/2}}, \quad x \in [-1, 1]$$

(see (1.2)). On $[2^k/n, 2^{k+1}/n] \subset [-a, a]$ we have

$$|P_n(x)| \le C\exp(-c2^{k/2}),$$

and at the same time, by the doubling property of μ on $[-a, a]$, we have

$$\begin{aligned} \mu([2^k/n, 2^{k+1}/n]) &\le \mu([2^k/n, (2^k + 2^{k+1})/2n]) \le L^2\mu([2^{k-1}/n, 2^k/n]) \\ &\le \cdots \le L^{2k}\mu([0, 1/n]). \end{aligned}$$

Hence,

$$\int_0^1 |P_n|^2 d\mu \le C\sum_{2^k/n \le a} \exp(-c2^{k/2})L^{2k}\mu([0, 1/n]) + e^{-c(na/2)^{1/2}}\mu([-1, 1]).$$

Here the sum is convergent, and it is easy to see that the doubling property implies $\mu([0, 1/n]) \ge (c/n^s)$ with some s, so the preceding inequality gives

$$\int_0^1 |P_n|^2 d\mu \le C\mu([0, 1/n]).$$

A similar estimate holds for the integral over $[-1, 0]$, and this verifies (3.4).

∎

4. Nonsymmetric fast decreasing polynomials

Symmetric fast decreasing polynomials that we have considered up to now, are not enough to prove this way the endpoint case, namely Theorem 2.3. The problem is not in the requirement that the bound $e^{-\Phi(x)}$ is a symmetric function; indeed, if Φ is not even, then one can consider instead of it the symmetric $\Phi(x)+\Phi(-x)$ which is at least as large as $\Phi(x)$ (minus an irrelevant constant). However, so far we have requested that (1.1) should hold on $[-1,1]$, i.e. there is a control on P_n on a relatively long interval to the right and to the left from the peaking point 0. If the left-interval where one needs to control P_n is considerably shorter (like in the endpoint case), then one can get faster decrease.

THEOREM 4.1. **(Totik-Varga [19])** *For $\beta<1$ there are $C,c>0$ such that for all $x_0\in[0,1/2]$ there are polynomials $Q_n(t)$ of degree at most $n=1,2,\dots$ such that $Q_n(x_0)=1$,*

$$|Q_n(t)|\le C\exp\left(-\left(\frac{cn|t-x_0|}{\sqrt{|t-x_0|}+\sqrt{x}_0}\right)^{\beta}\right),\qquad t\in[0,1].$$

Note that here the denominator is $\sim\sqrt{x_0}$ on $[0,2x_0]$, which, for such x, results in a large positive factor in the exponent when compared to what we have in the symmetric case. For example, in the extreme case when $x_0=0$ we get: there are P_n of degree at most n such that $P_n(0)=1$ and

$$|P_n(x)|\le Ce^{-cnx^{\gamma}},\qquad x\in[0,1],$$

precisely if $\gamma>1/2$. Compare this with Example 1.4 according to which in the symmetric case

$$|P_n(x)|\le Ce^{-nx^{\beta}},\qquad x\in[-1,1],$$

is possible precisely if $\beta>1$.

Now the upper estimate in Theorem 4.1 goes through the Markov inequalities and the estimate of the Christoffel function:

$$\lambda_n(x)\le C\mu([x-\delta_n(x),x+\delta_n(x)]) \tag{4.1}$$

where

$$\delta_n(x)=\frac{\sqrt{x_{n,k+1}-a}}{n}+\frac{1}{n^2}$$

exactly as in the proof in the preceding section; and (4.1) follows from Theorem 4.1 as the analogous result (3.4) followed from (1.2).

5. Fast decreasing polynomials on the complex plane

For a long time fast decreasing polynomials and their applications were restricted to the real line. Recently it has turned out that they also exist on more general sets on the complex plane and they play a vital role in some questions related to orthogonal polynomials.

Let $K\subset\mathbf{C}$ be a compact subset of the complex plane and $Z\in K$. Of course, if Z lies in the interior of K (or in the interior of one of the connected components of its complement) then, by the maximum modulus principle, there are no fast decreasing polynomials on K that peak at Z. The situation is different if Z lies on the so called outer boundary of K, defined as the boundary $\partial\Omega$ of the unbounded component of the complement $\mathbf{C}\setminus K$.

Theorem 5.1. **(Totik [17], [20])** *Let $Z \in \partial\Omega$ be a point on the outer boundary of K. Assume there is a disk in Ω that contains Z on its boundary. Then, for $\gamma < 1$, there are a $c > 0$ and polynomials Q_n of degree at most $n = 1, 2, \ldots$ such that $Q_n(Z) = 1$, $|Q_n(z)| \le 1$ for $z \in K$ and*

(i): *type I:*

$$|Q_n(z)| \le Ce^{-c(n|z-Z|)^\gamma}, \quad z \in K,$$

(ii): *type II:*

$$|Q_n(z)| \le Ce^{-cn|z-Z|^{1/\gamma}}, \quad z \in K.$$

These two types of decrease are the analogues of Examples 1.4 and 1.5. Here, exactly as on the real line, $\gamma = 1$ is not possible.

We also mention, that the assumption that there is a disk in Ω containing Z on its boundary is very natural; in fact, it cannot be replaced e.g. by the assumption that there is a cone/wedge in the complement of opening $< \pi$ with vertex at Z.

In the next sections we shall give applications of these complex fast decreasing polynomials.

6. Christoffel functions on a system of Jordan curves

Recall that a Jordan curve is the homeomorphic image of the unit circle C_1, while a Jordan arc is the homeomorphic image of the interval $[0, 1]$.

Let E be a finite system of smooth (C^2) Jordan curves and let μ be a Borel-measure on E. We assume that there are infinitely many points in the support of μ. The definition of the Christoffel functions is the same:

$$\lambda_n(\mu, z) = \inf_{P_n(z)=1} \int |P_n|^2 d\mu,$$

and we have again that if $p_n(\mu, z)$ are the orthonormal polynomials, then

$$1/\lambda_n(\mu, z) = \sum_0^n |p_k(\mu, z)|^2.$$

To describe the asymptotic behavior of λ_n on E, we need the concept of equilibrium measures. The equilibrium measure μ_E of E minimizes the logarithmic energy

$$\int\int \log \frac{1}{|z-t|} d\nu(z) d\nu(t)$$

among all Borel-measures ν supported on E having total mass 1. We shall also define the equilibrium density ω_E as the density (Radon-Nikodym derivative) of the equilibrium measure with respect to arc length measure s on E: $d\mu_E = \omega_E ds$. The same concepts can be defined for arcs, and even for more general sets.

For example,

$$\omega_{[-1,1]}(x) = \frac{1}{\pi\sqrt{1-x^2}},$$

while for a circle/disk of radius r we have $\omega_E \equiv 1/2\pi r$, i.e. in this case the equilibrium measure lies on the bounding circle and it has constant density there (the constant coming from the normalization to have total mass 1). With these notions we can state

THEOREM 6.1. **(Totik [17], [20])** *Let E be a finite family of C^2 Jordan curves, and assume that μ is a Borel-measure on E for which $\log\mu' \in L^1(s)$, where μ' is the Radon-Nikodym derivative of μ with respect to arc measure s. Then at every Lebesgue-point z_0 of μ and $\log\mu'$*

$$\lim_{n\to\infty} n\lambda_n(\mu, z_0) = \frac{\mu'(z_0)}{\omega_E(z_0)}.$$

Recall, we say that $z_0 \in \gamma$ is a Lebesgue-point (with respect to s) for the integrable function w if

$$\lim_{s(J)\to 0} \frac{1}{s(J)} \int_J |w(\zeta) - w(z_0)| ds(\zeta) = 0,$$

where the limit is taken for subarcs J of E that contain z_0, the arc length $s(J)$ of which tends to 0. Also, if $d\mu = wds + d\mu_s$ is the decomposition of μ into its absolutely continuous and singular part with respect to s, then z_0 is a Lebesgue-point for μ if it is Lebesgue-point for w and

$$\lim_{s(J)\to 0} \frac{\mu_s(J)}{s(J)} = 0.$$

There is a local version of Theorem 6.1, where the smoothness of E and the Szegő condition $\mu' \in L^1(s)$ is assumed only in a neighborhood of z_0 (see [**17**], [**20**]).

The theorem is also true when some of the curves are replaced by arcs, but the proof for the arc case is completely different (the polynomial inverse image approach to be discussed below cannot be used; an arc has no interior, it cannot be exhausted by lemniscates), and is, again, based heavily on complex fast decreasing polynomials.

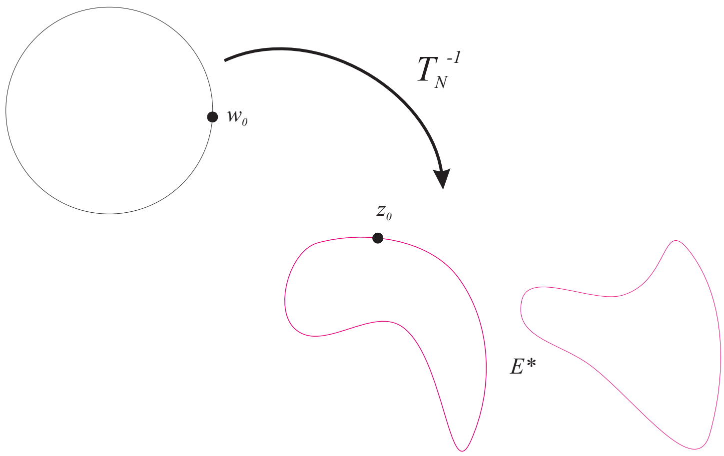

FIGURE 1. Creating lemniscates

Sketch of the proof of Theorem 6.1

There are two distinctively different parts: the continuous case has been dealt with in [**17**], while the case of general Lebesgue-points in [**20**].

Part I. Continuous case: *μ is absolutely continuous and $\mu' = w$ is continuous and positive at z_0.*

In this case we use a polynomial inverse mapping (see [**21**]).

a): The result is known for the unit circle C_1 (Szegő).

b): Go over to a lemniscate $E^* = T_N^{-1}(C_1)$ where T_N is an appropriate fixed polynomial (see Figure 1).

c): Approximate E by a leminscate $E^* = T_N^{-1}(C_1)$ containing z_0 (see Figure 2).

Here, in part c), fast decreasing polynomials of type II (see Theorem 5.1) are used in a very essential way. For the approximation in part c) one also needs an extension of Hilbert's lemniscate theorem: Suppose that Γ is another system of C^2 Jordan curves consisting of the same number of components as E such that each component of Γ lies in the corresponding component of E with the exception of the point z_0, where the two (system of) curves touch each other and have different curvatures. Then there is a lemniscate $E^* = T_N^{-1}(C_1)$ consisting of the same number of component and which separates E and Γ (and of course touch both at z_0).

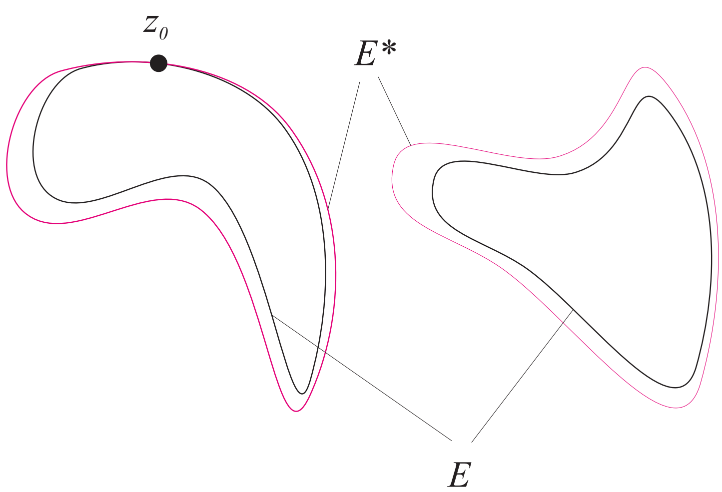

FIGURE 2. Approximating E by a lemniscate

Part II. Reduction to the continuous case.

1): Set $d\nu = \frac{\mu'(z_0)}{\omega_E(z_0)} ds$ on the component of E that contains z_0, and let $\nu = \mu$ on other components.

The density of this ν is just constant on the component of E which contains z_0, so for this ν Part I applies at z_0.

2): Show that $\lambda_n(\nu, z_0) = (1 + o(1))\lambda_n(\mu, z_0)$.

Here, and in many similar questions, the main problem is how to control the size of the optimal polynomials in

$$\lambda_n(\nu, z) = \inf_{P_n(z)=1} \int |P_n|^2 d\nu.$$

This problem is handled by the following inequality.

THEOREM 6.2. **(Totik [20])** *Let γ be a C^2 Jordan curve and $w \geq 0$ a measurable function on γ with $w, \log w \in L^1(s)$. If $z_0 \in \gamma$ is a Lebesgue-point for $\log w$, then there is a constant M such that for any polynomials P_n of degree at most $n = 1, 2, \ldots$ and for any $z \in \gamma$ (or for z lying inside γ)*

$$|P_n(z)|^2 \leq M e^{M\sqrt{n|z-z_0|}}\, n \int_\gamma |P_n|^2 w\, ds. \tag{6.1}$$

This is a fairly non-trivial estimate, for example nothing like this is true outside γ:

EXAMPLE 6.3. *Let γ be the unit circle, $w \equiv 1$, $P_n(z) = z^n$, $z_0 = 1$. Then, for $z > 1$,*

$$|P_n(z)|^2 = z^{2n} = (1 + (z-1))^{2n} \geq e^{n(z-1)},$$

and here the right hand side is far from being $\leq M e^{M\sqrt{n|z-1|}}\, n$.

The crucial idea is to combine Theorem 6.2 with fast decreasing polynomials of type I (see Theorem 5.1): $Q_{\varepsilon n}(z_0) = 1$,

$$|Q_{\varepsilon n}(z)| \leq C e^{-(\varepsilon n|z-z_0|)^{2/3}}, \quad z \in E.$$

Now no matter how small $\varepsilon > 0$ is, the factor $e^{-(\varepsilon n|z-z_0|)^{2/3}}$ kills the factor $e^{M\sqrt{n|z-\zeta_0|}}$ in (6.2), so the product $P_n Q_{\varepsilon n}$ is bounded on γ and is very small away from z_0. At the same time, it has almost the same degree at P_n, and we can use these as test polynomials to estimate the Christoffel functions for the measure ν (or μ) in Part II.1) above. Using the Lebesgue-point property and these test polynomials, it is relatively easy to verify Part II.2).

7. Universality

Let w be an integrable weight function on some compact set $\Sigma \subset \mathbf{R}$, and let p_k be the orthonormal polynomials associated with w. Form the so called reproducing kernel

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x)\overline{p_k(y)}.$$

A form of universality of random matrix theory/statistical physics at a point x claims

$$\frac{K_n\left(x + \frac{a}{w(x)K_n(x,x)}, x + \frac{b}{w(x)K_n(x,x)}\right)}{K_n(x, x)} \to \frac{\sin \pi(a-b)}{\pi(a-b)}$$

as $n \to \infty$. This was proved under analyticity of w in various settings by different authors (see e.g. Pastur [**12**], Deift, Kriecherbauer, McLaughlin, Venakides and Zhou, [**2**] or Kuijlaars and Vanlessen [**6**], [**7**]). D. S. Lubinsky [**10**] proved it under mere continuity: if $\Sigma = [-1, 1]$ and $w > 0$ is continuous in $(-1, 1)$, then universality is true at every $x \in (-1, 1)$. Actually, he proved universality at an $x \in (-1, 1)$ if $w(x)dx \in \mathbf{Reg}$ and $w > 0$ is continuous at x. Here $\mathbf{Reg}$ is the class of measures μ for which

$$\liminf \lambda_n(\mu, x)^{1/n} \geq 1$$

at every point x of the support with the exception of a set of zero logarithmic capacity. This is a weak global condition on the measure, and it says that for most points x in the support the value $|P_n(x)|$ of polynomials is not exponentially larger than their $L^2(\mu)$-norm $\|P_n\|_{L^2(\mu)}$. See [**15**] for various reformulations of regularity and for different regularity criteria.

Extension of Lubinsky's universality to general support and to almost everywhere convergence (under Szegő condition) was done by Simon [**13**], Findley [**4**] and Totik [**18**].

Lubinsky had a second, complex analytic approach to universality, which was abstracted by Avila, Last and Simon [**1**]: universality is true at a point $x_0 \in S$ if

(i):
$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu; x_0 + a/n, x_0 + a/n) = \frac{\omega_\Sigma(x_0)}{w(x_0)}$$
uniformly in $a \in [-A, A]$ for any fixed A,

(ii): there is a $C > 0$ such that for any $A > 0$, $|z| \le A$ and for sufficiently large $n \ge n_A$
$$\frac{1}{n} K_n(x_0 + z/n, x_0 + z/n) \le Ce^{C|z|}, \qquad z \in \mathbf{C}.$$

Since $1/\lambda_n(\mu, x) = K_n(x, x)$, property (i) is basically the asymptotics for Christoffel functions discussed before (with the small change $x_0 \to x_0 + a/n$, called by Simon the "Lubinsky wiggle", see Remark 3 on p. 225 of [**14**]). On the other hand, (ii) is not that easy to verify at a given non-continuity point. Now (ii) follows from the inequality (6.1) with the use of fast decreasing polynomials of type II (see Theorem 5.1) at every point which is a Lebesgue-point for w and $\log w$. This way one gets

Theorem 7.1. **(Totik [22])** *Let $\mu \in$ **Reg** and $d\mu(x) = w(x)dx$ on an interval I with $\log w \in L^1(I)$. Then universality is true at every $x_0 \in I$ which is a Lebesgue-point for both w and $\log w$. In particular, it is true a.e.*

That universality is true almost everywhere under a local Szegő condition was proved in [**18**] by a totally different method (using polynomial inverse images). It should be noticed that these two absolutely different approaches (namely in [**18**] and Theorem 7.1) need the same assumption, namely local Szegő condition $w \in L^1(I)$. It is an open problem if this Szegő condition can be replaced by something weaker (like $w > 0$ a.e. in I).

8. The Levin-Lubinsky fine zero spacing theorem

Let again w be an integrable weight, but now assume that its support is $[-1, 1]$, and let $x_{n,k}$ be the zeros of the associated orthogonal polynomials $p_n(\mu, x)$. The following remarkable result was proved as a consequence of Lubinsky's universality theorem.

Theorem 8.1. **(Levin-Lubinsky [9])** *If $w > 0$ is continuous on $(-1, 1)$, then*
$$x_{n,k+1} - x_{n,k} = (1 + o(1)) \frac{\sqrt{1 - x_{n,k}^2}}{n}$$
uniformly for $x_{n,k} \in [-1 + \varepsilon, 1 - \varepsilon]$.

Actually, Levin and Lubinsky proved more, namely that the same is true if it is only assumed that $w(x)dx \in$ **Reg**, w is continuous and positive at a point X, and $|x_{n,k} - X| = O(1/n)$.

The extension to more general support was given independently by Simon and Totik:

THEOREM 8.2. **(Simon [13], Totik [18])** *Let μ be a measure on the real line with compact support S in the* **Reg** *class. Assume also that $d\mu(x) = w(x)dx$, $\log w \in L^1(I)$ on some interval I. Then at every $X \in I$ which is a Lebesgue-point for w and $\log w$, we have*

$$x_{n,k+1} - x_{n,k} = \frac{1 + o(1)}{n\pi\omega_S(X)}, \qquad |X - x_{n,k}| = O(1/n), \tag{8.1}$$

where ω_S denotes the equilibrium density of S with respect to linear Lebesgue-measure.

In [**13**] the continuity and positivity of w was used, and in [**18**] a somewhat less precise result (as regards where (8.1) holds) was verified. The stated more precise form comes from Theorem 7.1, in the proof of which complex fast decreasing polynomials have played a crucial role.

References

1. A. Avila, Y. Last and B. Simon, Bulk universality and clock spacing of zeros for ergodic Jacobi matrices with absolutely continuous spectrum, *Anal. PDE*, **3**(2010), 81-108.
2. P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality limits in random matrix theory, *Comm. Pure Appl. Math.*, **52**(1999), 1335–1425.
3. C. Fefferman and B. Muckenhoupt, Two nonequivalent conditions for weight functions, *Proc. Amer. Math. Soc.*, **45**(1974), 99–104.
4. M. Findley, Universality for locally Szegő measures, *J. Approx. Theory.*, **155**(2008), 136–154.
5. K. G. Ivanov and V. Totik, Fast decreasing polynomials, *Constructive Approx.*, **6**(1990), 1–20.
6. A. B. J. Kuijlaars and M. Vanlessen, Universality for eigenvalue correlations at the origin of the spectrum, *Comm. Math. Phys.*, **243**(2003), 163-191.
7. A. B. J. Kuijlaars and M. Vanlessen, Universality for eigenvalue correlations from the modified Jacobi unitary ensemble, *Int. Math. Res. Not.* **30**(2002), 1575-1600.
8. Y. Last and B. Simon, Fine structure of the zeros of orthogonal polynomials, IV: A priori bounds and clock behavior, *Comm. Pure Appl. Math.*, **61**(2008), 486–538.
9. A. L. Levin, and D. S. Lubinsky, Applications of universality limits to zeros and reproducing kernels of orthogonal polynomials, *J. Approx. Theory.*, **150**(2008), 69-95.
10. D. S. Lubinsky, A new approach to universality limits involving orthogonal polynomials, *Annals Math.*, **170**(2009), 915-939.
11. G. Mastroianni and V. Totik, Uniform spacing of zeros of orthogonal polynomials, *Constructive Approx.*, **32**(2010), 181–192.
12. L. A. Pastur, Spectral and probabilistic aspects of matrix models. *Algebraic and geometric methods in mathematical physics* (Kaciveli, 1993), 207–242, *Math. Phys. Stud.*, **19**, Kluwer Acad. Publ., Dordrecht, 1996.
13. B. Simon, Two extensions of Lubinsky's universality theorem, *J. D´Analyse Math.*, **105**(2008), 345–362.
14. B. Simon, *Szegős theorem and its descendants. Spectral theory for L^2 perturbations of orthogonal polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.
15. H. Stahl and V. Totik, *General Orthogonal Polynomials*, Encyclopedia of Mathematics and its Applications, **43**, Cambridge University Press, Cambridge, 1992.
16. G. Szegő, *Orthogonal Polynomials*, Coll. Publ. , **XXIII**, Amer. Math. Soc., Providence, 1975.
17. V. Totik, Christoffel functions on curves and domains, *Trans. Amer. Math. Soc.*, **362**(2010), 2053–2087.
18. V. Totik, Universality and fine zero spacing on general sets, *Arkiv för Math.*, **47**(2009), 361–391.

19. V. Totik and T. Varga, Non-symmetric fast decreasing polynomials and applications (manuscript)
20. V. Totik, Szegő's problem on curves, (manuscript).
21. V. Totik, The polynomial inverse image method, Approximation Theory XIII: San Antonio 2010, Springer Proceedings in Mathematics **13**, M. Neamtu and L. Schumaker (eds.), DOI 10.1007/978-1-4614-0772-0
22. V. Totik, Local universality at Lebesgue-points, (manuscript)
23. M. Vanlessen, Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight, *J. Approx. Theory* **125**(2003), 198–237.
24. T. Varga, Uniform spacing of zeros of orthogonal polynomials for locally doubling measures, (manuscript).

Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave, PHY 114, Tampa, FL 33620-5700, USA, and
Bolyai Institute, Analysis Research Group of the Hungarian Academy os Sciences, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary

E-mail address: totik@mail.usf.edu