Bernstein-type inequalities*

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Abstract

It is shown that a Bernstein-type inequality always implies its Szegő-variant, and several corollaries are derived. Then, it is proven that the original Bernstein inequality on derivatives of trigonometric polynomials implies both Videnskii's inequality (which estimates the derivative of trigonometric polynomials on a subinterval of the period), as well as its half-integer variant. The method of these two results are then combined to derive the general sharp form of Videnskii's inequality on symmetric $E \subset [-\pi, \pi]$ sets. The sharp Bernstein factor turns out to be 2π times the equilibrium density of the set $\Gamma_E = \{e^{it} \,|\, t \in E\}$ on the unit circle C_1 that corresponds to E when we identify C_1 by $\mathbf{R}/(\mathrm{mod} 2\pi)$.

1 Introduction

Polynomial inequalities are very basic in several disciplines. There are hundreds, perhaps thousands of papers devoted to them, see e.g the two relatively recent books [4], [8].

Arguably the most important of them (which was also historically one of the first) is Bernstein's inequality: if T_n is a trigonometric polynomial of degree at most n, then

$$\|T_n'\| \le n\|T_n\|, \tag{1.1}$$

where $\|\cdot\|$ denotes the supremum norm. This is sharp, as is shown by $T_n(x) = \cos nx$. If P_n is an algebraic polynomial of degree at most n, then $T_n(t) = P_n(\cos t)$ is a trigonometric polynomial of degree at most n, and (1.1) yields

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}}\|P_n\|_{[-1,1]}, \qquad x \in (-1, 1), \tag{1.2}$$

which is also known as Bernstein inequality and which is also sharp.

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In (1.1) the norms are taken on $[-\pi,\pi]$, i.e. on the whole period of T_n. The analogous inequality on a subinterval of the complete period is due to Videnskii [15], who, in 1960, proved that if $\beta\in(0,\pi)$, then for $\theta\in(-\beta,\beta)$ we have

$$|T_n'(\theta)|\le n\frac{\cos\theta/2}{\sqrt{\sin^2\beta/2-\sin^2\theta/2}}\|T_n\|_{[-\beta,\beta]}, \tag{1.3}$$

(here, and in what follows, $\cos\theta/2=\cos(\theta/2)$), and this is sharp again. Videnskii had a variant for half-integer trigonometric polynomials (see [16]): let

$$Q_{n+1/2}(t)=\sum_{j=0}^{n}a_j\cos\left(\left(j+\frac{1}{2}\right)t\right)+b_j\sin\left(\left(j+\frac{1}{2}\right)t\right),\quad a_j,b_j\in\mathbf{R}. \tag{1.4}$$

Then for any $\theta\in(-\beta,\beta)$, we have

$$|Q_{n+1/2}'(\theta)|\le\left(n+\frac{1}{2}\right)\frac{\cos\theta/2}{\sqrt{\sin^2\beta/2-\sin^2\theta/2}}||Q_{n+1/2}||_{[-\beta,\beta]}. \tag{1.5}$$

These inequalities of Videnskii have always been considered as somewhat peculiar for the reason that $[-\beta,\beta]$ is not the natural domain of trigonometric polynomials. Their form on two or more intervals is not known. In section 3 we prove that both Videnskii inequalities (and even a sharper form of them) are simple consequences of Bernstein's inequality (1.1)–(1.2). In the last part of the paper we give their form on any set that is symmetric with respect to the origin. But before these, in the next section, we show that such Bernstein-type inequalities always have a sharper, so called Szegő form. The methods of sections 2 and 3 are used to derive the general form in section 4.

2 A general Szegő inequality

Bernstein's original paper [3] did not have the correct factor n in (1.1), it rather had 2n. The sharp form (1.1) was proved by M. Riesz [10]. G. Szegő [13] gave a result which implies the somewhat surprising extension: if T_n is a real trigonometric polynomial of degree at most n, then for all θ

$$|T_n'(\theta)|^2+n^2|T_n(\theta)|^2\le n^2\|T_n\|^2. \tag{2.1}$$

Note that the norm of the second term on the left is already what stands on the right-hand side. This inequality was also proven in by [12] by Schaake and van der Corput, so it is often referred to as the Schaake-van der Corput inequality, see e.g. [11].

The analogue of (2.1) for (1.2) reads as

$$(\sqrt{1-x^2}P_n'(x))^2+n^2P_n^2(x)\le n^2\|P_n\|_{[-1,1]}^2,\qquad x\in(-1,1). \tag{2.2}$$

Recently Erdélyi [5] proved the Szegő-version of Videnskii's inequality: with

$$V_\beta(\theta) = \frac{\cos\theta/2}{\sqrt{\sin^2\beta/2 - \sin^2\theta/2}} \tag{2.3}$$

$$\left|\frac{T_n'(\theta)}{V_\beta(\theta)}\right|^2 + n^2|T_n(\theta)|^2 \le n^2\|T_n\|_{[-\beta,\beta]}^2 \tag{2.4}$$

holds for real trigonometric polynomials. Our first result is that a Bernstein-type inequality implies its Szegő-version under very general circumstances.

Theorem 2.1 *Suppose that at a point x_0 a weak Bernstein inequality*

$$|Q_n'(x_0)| \le (1+o(1))nH(x_0)\|Q_n\|_E \tag{2.5}$$

holds for real trigonometric/algebraic polynomials of degree at most $n = 1, 2, \ldots$ with some $H(x_0)$, where $o(1)$ tends to 0 as $n \to \infty$ uniformly in Q_n. Then the strong Bernstein-Szegő inequality

$$\left(\frac{P_n'(x_0)}{H(x_0)}\right)^2 + n^2 P_n(x_0)^2 \le n^2\|P_n\|_E^2 \tag{2.6}$$

is true for all P_n for which P_n^2 is a real trigonometric/algebraic polynomial of degree at most $2n = 1, 2, \ldots$, provided P_n is differentiable at x_0.

We emphasize that only P_n^2 needs to be an algebraic/trigonometric polynomial of degree at most $2n = 1, 2, \ldots$, e.g. the result applies to $P_n(x) = \sqrt{1 + x^m}$, $m = 1, 2, \ldots$, in which case $n = m/2$, or to $P_n = Q_{m+\frac{1}{2}}$, $m = 1, 2, \ldots$ with the $Q_{m+\frac{1}{2}}$ from (1.4), in which case $n = m + \frac{1}{2}$.

Remarks. 1. It may happen that P_n is not differentiable at certain points (like $P_n(x) = |x - x_0|$).

2. The result is true only for real trigonometric/algebraic polynomials. Indeed, for example the original Szegő inequality (2.1) is clearly false for $T_n(x) = \cos nx + i \sin nx$. Nevertheless, if (2.5) is assumed for real trigonometric/algebraic polynomials, then we get that the sharper inequality

$$|P_n'(x_0)| \le nH(x_0)\|P_n\|_E \tag{2.7}$$

also holds for all real or complex trigonometric/algebraic polynomials of degree at most n. Indeed, we get from the theorem (2.7) for real polynomials. Now if P_n is a complex trigonometric/algebraic polynomial, then there is a complex number τ of modulus 1 such that $\tau P_n'(x_0) = |P_n'(x_0)|$. Then applying (2.7) to $P_n^* = \Re\tau P_n$ rather than to P_n gives us

$$|P_n'(x_0)| = \tau P_n'(x_0) = (P_n^*)'(x_0) \le nH(x_0)\|P_n^*\|_E \le nH(x_0)\|P_n\|_E. \tag{2.8}$$

3. The set E is not specified, it can be any subset of the real line. Actually, the proof trivially works in any dimension, in which case E can be a set in higher dimension (and then Q_n, P_n of several variables). For example, if for a convex body $E \subset \mathbf{R}^d$ and for some $x_0 \in E$ we have

$$|\nabla P_n| \le (1 + o(1))nH(x_0)\|P_n\|_E$$

for all real multivariate polynomials P_n of total degree at most n (where

$$|\nabla P_n| = \left(\sum_{j=1}^{d} \left(\frac{\partial P_n}{\partial x_j} \right)^2 \right)^{1/2}$$

is the Euclidean norm of the gradient), then

$$\left(\frac{|\nabla P_n(x_0)|}{H(x_0)} \right)^2 + n^2 P_n(x_0)^2 \le n^2 \|P_n\|_E^2 \tag{2.9}$$

automatically follows. Indeed, the proof of Theorem 2.1 works without any change for directional derivatives in $\mathbf{R}^d$, and the gradient is the largest of them.

4. Instead of polynomials we can have rational functions in both (2.5) and (2.6). Recall e.g. the following result (see [4], p. 324, Theorem 7.1.7): let C_1 be the unit circle, and for $a_k \in \mathbf{C} \setminus C_1$, $k = 1, \ldots, n$, set

$$B_n^+(z) := \sum_{k:|a_k|>1} \frac{|a_k|^2 - 1}{|a_k - z|^2}, \qquad B_n^-(z) := \sum_{k:|a_k|<1} \frac{1 - |a_k|^2}{|a_k - z|^2},$$

and let

$$B_n(z) := \max\left(B_n^+(z), B_n^-(z)\right).$$

Then, for every rational function $r(z)$ of the form $r(z) = Q(z)/\prod_{k=1}^{n}(z - a_k)$ where Q is a polynomial of degree at most n, we have

$$|r'(z)| \le B_n(z)||r||_{C_1} \qquad z \in C_1. \tag{2.10}$$

Now the proof that we give for Theorem 2.1 gives the following Szegő variant: if r is as above and it is real on C_1, then

$$\left(\frac{|r'(z)|}{B_n(z)} \right)^2 + |r(z)|^2 \le ||r||_{C_1}^2 \qquad z \in C_1. \tag{2.11}$$

In connection with this we mention the paper [7] by A. Lukashov that contains several Szegő-type inequalities for rational functions.

Before giving the (very simple) proof for Theorem 2.1 we mention a few consequences, in which we consider only real trigonometric/algebraic polynomials.

Corollary 2.2 *The Bernstein inequality* (1.1) *implies its Szegő version* (2.1)*. In a similar manner,* (1.2) *implies* (2.2)*.*

Corollary 2.3 *Videnskii's inequality* (1.3) *implies its half-integer variant* (1.5)*, and even its Szegő form:*

$$\left|\frac{Q'_{n+1/2}(\theta)}{V_\beta(\theta)}\right|^2 + \left(n+\frac{1}{2}\right)^2 |Q_{n+1/2}(\theta)|^2 \le \left(n+\frac{1}{2}\right)^2 \|Q_{n+1/2}\|^2_{[-\beta,\beta]}$$

for all $\theta \in (-\beta, \beta)$*.*

Indeed, all we have to mention is that if $Q_{n+1/2}$ is a half-integer trigonometric polynomial as in (1.4) of degree at most $n+1/2$, then $Q^2_{n+1/2}$ is a trigonometric polynomial of degree at most $2n+1$.

Corollary 2.4 *Videnskii's inequality* (1.3) *implies its Szegő form* (2.4)*.*

Let us also give a similar corollary for algebraic polynomials: if R_n is an algebraic polynomial of degree at most n which is nonnegative on $[-1,1]$, then for $x \in (-1,1)$

$$\left|\left(\sqrt{R_n(x)}\right)'\right| \le \frac{n}{2\sqrt{1-x^2}} \|R_n\|^{1/2}_{[-1,1]} . \tag{2.12}$$

This follows from Theorem 2.1 (with (1.2) as the reference inequality) if we apply it to $P_{n/2} = \sqrt{R_n}$.

As a consequence, we get

$$|R'_n(x)| \le \frac{n}{\sqrt{1-x^2}} R_n(x)^{1/2} \|R_n\|^{1/2}_{[-1,1]} , \tag{2.13}$$

and even (from the Szegő form of (2.12))

$$\left(\sqrt{1-x^2} R'_n(x)\right)^2 + n^2 R_n(x)^2 \le n^2 R_n(x) \|R_n\|_{[-1,1]}. \tag{2.14}$$

Note that for $R_2(x) = 1 - x^2$ we have equality in (2.14) for all $x \in [-1,1]$. These should be compared to

$$|R'_n(x)| \le \frac{n}{2\sqrt{1-x^2}} \|R_n\|_{[-1,1]} , \tag{2.15}$$

which follows from Berntein's inequality (1.2) for (on $[-1,1]$) nonnegative polynomials (apply (1.2) to $P_n = R_n - \|R_n\|_{[-1,1]}/2$). In particular, (2.13) gives that if $S_1, \dots S_j$ are algebraic polynomials of degree at most n, then for $x \in (-1,1)$

$$\left|S_1(x)S'_1(x) + \cdots S_j(x)S'_j(x)\right| \le \frac{n}{\sqrt{1-x^2}} \left(\sum_k S_k^2(x)\right)^{1/2} \left\|\sum_k S_k^2\right\|^{1/2}_{[-1,1]} . \tag{2.16}$$

Proof of Theorem 2.1. We use a similar argument that has been applied in [14].

Without loss of generality let $\|P_n\|_E = 1$. Assume first that $P_n(x_0) \neq 0$, and for a large integer m choose $0 \leq \alpha_m \leq 1$ in such a way that $\alpha_m P_n(x_0)$ is a zero $\cos((2k+1)\pi/4m)$ of the $2m$-th Chebyshev polynomial $\mathcal{T}_{2m}(x) = \cos(2m \arccos x)$. This α_m can be chosen so that $\alpha_m \to 1$ as $m \to \infty$. Now since P_n^2 is a a trigonometric/algebraic polynomial of degree $\leq 2n$ (n may be half-integer), $\mathcal{T}_{2m}(\alpha_m P_n(x))$ is a trigonometric/algebraic polynomial of degree $\leq 2mn$ with norm at most 1 on E, and so (2.5) gives for it

$$\begin{aligned} |\mathcal{T}_{2m}'(\alpha_m P_n(x_0)) P_n'(x_0)\alpha_m| &= \left| \Big(\mathcal{T}_{2m}(\alpha_m P_n(x))\Big)' \Big|_{x = x_0} \right| \\ &\leq (1 + o_m(1)) 2mnH(x_0). \end{aligned} \tag{2.17}$$

Since $\alpha_m P_n(x_0)$ is a zero of $\mathcal{T}_{2m}$, we have on the left

$$|\mathcal{T}_{2m}'(\alpha_m P_n(x_0))| = \frac{2m}{\sqrt{1 - (\alpha_m P_n(x_0))^2}}.$$

If we plug this into (2.17), divide by $2m$ and let m tend to ∞, we get

$$|P_n'(x_0)| \leq nH(x_0)\sqrt{1 - P_n(x_0)^2},$$

and this is (2.6).

When $P_n(x_0) = 0$ then apply what we have just proven to $P_n(x - \varepsilon)$ with some small $\varepsilon > 0$, and let $\varepsilon \to 0$.

■

3 Bernstein vs. Videnskii's inequality

In Corollaries 2.3 and 2.4 we saw that Videnskii's inequality easily gives its half-integer variant (1.5) and its Szegő form (2.4). In this section we show that to get these one does not even need Videnskii's inequality, actually all these follow in a very simple manner from Bernstein's inequality (1.1)–(1.2), i.e. in this section we give a simple argument to deduct Videnskii's inequality (1.3) from Bernstein's inequality (1.1)–(1.2). The same argument will be used in the next section to derive the general form of Videnskii's inequality.

First of all, by Remark 2 after Theorem 2.1 it is enough to consider real trigonometric polynomials.

In view of Theorem 2.1, we only need to derive the weak Videnskii inequality

$$|T_n'(t)| \leq (1 + o(1))nV_\beta(t)\|T_n\|_{[-\beta,\beta]}, \qquad t \in (-\beta, \beta), \tag{3.1}$$

(with $o(1)$ uniform in T_n) from Bernstein's inequality (1.2).

First of all, we remark that Bernstein's inequality (1.2) on the interval $[\cos\beta, 1]$ takes the form (apply linear transformation)

$$|P_n'(x)| \le \frac{n}{\sqrt{|x-\cos\beta||1-x|}}\|P_n\|_{[\cos\beta,1]}.$$

Setting here $x = \cos t$ and $T_n(t) = P_n(\cos t)$ we get

$$|T_n'(t)| \le nV_\beta(t)\|T_n\|_{[-\beta,\beta]}, \qquad t \in (-\beta, \beta), \tag{3.2}$$

for all even trigonometric polynomials T_n. This is precisely Videnskii's inequality (1.3) for even trigonometric polynomials, so all that remains is to get rid of the evenness of T_n.

Lemma 3.1 *If $\delta > 0$, then there is a C_δ such that for arbitrary trigonometric polynomials T_n of degree at most n*

$$|T_n'(u)| \le C_\delta n\|T_n\|_{[u-\delta,u+\delta]}, \qquad u \in \mathbf{R}. \tag{3.3}$$

Proof. We may assume $u = 0$ and that T_n is odd (the even part has zero derivative at 0, while for the odd part

$$T_{n,o}(x) := \frac{1}{2}\left(T_n(x) + T_n(-x)\right)$$

of T_n we have

$$\|T_{n,o}\|_{[-\delta,\delta]} \le \|T_n\|_{[-\delta,\delta]}).$$

Then $T_n(x) = \sin x R_n(\cos x)$ with some polynomial R_n of degree at most $n-1$, and then we have to show that

$$|T_n'(0)| = |R_n(1)| \le C_\delta n\|T_n\|_{[-\delta,\delta]}.$$

Since the norm on the right-hand side is

$$\|\sqrt{1-y^2}R_n(y)\|_{[\gamma,1]}, \qquad \gamma = \cos\delta,$$

we need to show for $S_{2n}(v) = R_n(1-v^2)$ that for $\sigma > 0$

$$|S_{2n}(0)| \le C_\sigma n\|vS_{2n}(v)\|_{[-\sigma,\sigma]},$$

which follows from (1.2) if we apply the latter to the polynomial $\sigma x S_{2n}(\sigma x)$ (of degree at most $2n$) at $x = 0$.

∎

Now let T_n be an arbitrary trigonometric polynomial of degree at most n, and we first prove (3.1) at a $t \in (-\beta, \beta)$, $t \ne 0$. With some $\varepsilon > 0$ consider

$$T_n^*(x) = T_n(x)\left(\frac{1+\cos(x-t)}{2}\right)^{[\varepsilon n]} + T_n(-x)\left(\frac{1+\cos(-x-t)}{2}\right)^{[\varepsilon n]}.$$

This is of degree $\leq n + \varepsilon n$, even, and, since $|1 + \cos(x - t)|/2 < q < 1$ if x lies outside any neighborhood of t (q depends on the neighborhood), we have

$$\|T_n^*\|_{[-\beta,\beta]} \leq (1 + o(1))\|T_n\|_{[-\beta,\beta]}.$$

Also,

$$\begin{aligned}(T_n^*)'(t) &= T_n'(t) - T_n'(-t)\left(\frac{1+\cos(-2t)}{2}\right)^{[\varepsilon n]} \\ &+ T_n(-t)\frac{[\varepsilon n]}{2}\left(\frac{1+\cos(-2t)}{2}\right)^{[\varepsilon n]-1}\sin(-2t),\end{aligned}$$

and view of Lemma 3.1 (apply it with $u = -t$) this gives

$$(T_n^*)'(t) = T_n'(t) + o(1)\|T_n\|_{[-\beta,\beta]}.$$

Thus, (3.2) for T_n^* yields

$$|T_n'(t)| \leq (1 + \varepsilon) n V_\beta(t)(1 + o(1))\|T_n\|_{[-\beta,\beta]},$$

and since here $\varepsilon > 0$ is arbitrary, (3.1) follows.

If $t = 0$, then apply the just proven (3.1) to $\tilde{T}_n(x) = T_n(x - \varepsilon)$ and to $[-\beta + \varepsilon, \beta - \varepsilon]$ with some small $\varepsilon > 0$ instead of $[-\beta, \beta]$. We get

$$|T_n'(0)| = |(\tilde{T}_n)'(\varepsilon)| \leq (1 + o(1)) n V_{\beta-\varepsilon}(\varepsilon)\|\tilde{T}_n\|_{[-\beta+\varepsilon,\beta-\varepsilon]}. \tag{3.4}$$

Since here

$$\|\tilde{T}_n\|_{[-\beta+\varepsilon,\beta-\varepsilon]} \leq \|T_n\|_{[-\beta,\beta]},$$

and $V_{\beta-\varepsilon}(\varepsilon)$ is as close to $V_\beta(0)$ as we wish if $\varepsilon > 0$ is sufficiently small, (3.1) follows also for $t = 0$.

■

4 The general form of the Videnskii inequality for symmetric sets

In this section we prove an extension of Videnskii's inequality to arbitrary compact sets symmetric with respect to the origin. As we have mentioned before, its form has not been known for any set consisting of more than one intervals (which is the case given in (1.3)).

We shall need the concept of the equilibrium measure μ_Γ of a compact subset Γ of the complex plane of positive logarithmic capacity. It is the unique measure minimizing the energy integral

$$\int\int \log\frac{1}{|z-t|}d\mu(z)d\mu(t)$$

among all Borel-measures μ that are supported on Γ and that have total mass 1. See [6], [9] for this concept and for the results that we use from potential theory.

If Γ is part of the unit circle, then on any subarc of Γ the measure μ_Γ is absolutely continuous with respect to arc measure s, and on such subarcs we denote its density $d\mu/ds$ with respect to the arc measure s by ω. Thus, on any subarc of Γ the equilibrium measure is of the form $\omega(e^{it})dt$.

Let C_1 be the unit circle, and for $E \subset [-\pi, \pi]$ let

$$\Gamma_E = \{e^{it} \,|\, t \in E\},$$

be the set that corresponds to E when we identify $(-\pi, \pi]$ with C_1.

Theorem 4.1 *Let $E \subset [-\pi, \pi]$ be compact and symmetric with respect to the origin. If $\theta \in E$ is an inner point of E then for any trigonometric polynomial T_n of degree at most $n = 1, 2, \dots$ we have*

$$|T_n'(\theta)| \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E. \tag{4.1}$$

The result is best possible:

Theorem 4.2 *Under the conditions of Theorem* 4.1 *for any θ lying in the interior of E there are nonzero trigonometric polynomials T_n of degree at most $n = 1, 2, \dots$ for which*

$$|T_n'(\theta)| \ge (1 - o(1)) n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E. \tag{4.2}$$

Via Theorem 2.1 the inequality (4.1) implies its Szegő form, as well as its half-integer variant, e.g.

$$\left| \frac{Q_{n+1/2}'(\theta)}{2\pi \omega_{\Gamma_E}(e^{i\theta})} \right|^2 + \left(n + \frac{1}{2}\right)^2 |Q_{n+1/2}(\theta)|^2 \le \left(n + \frac{1}{2}\right)^2 \|Q_{n+1/2}\|_E, \qquad \theta \in \mathrm{Int}(E).$$

for all half-integer real trigonometric polynomials as in (1.4).

The general statement in Theorem 4.1 easily follows (by taking limit) from its special case when E consists of a finite number of intervals, and in this case we can make the bound in (4.1) more concrete. In fact, let $E \subseteq [-\pi, \pi]$ be a set consisting of finitely many intervals such that E is symmetric with respect to the origin. In this case

$$E \cap [0, \pi] = \cup_{j=1}^m [\alpha_j, \beta_j],$$

where $0 \le \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m \le \pi$.

Lemma 4.3 *There are unique points* $\xi_j \in (\cos\alpha_{j+1}, \cos\beta_j)$, $j = 1, \ldots, m-1$ *satisfying the system of equations*

$$\int_{\cos\alpha_{j+1}}^{\cos\beta_j} \frac{\prod_{j=1}^{m-1}(u-\xi_j)}{\sqrt{\prod_{j=1}^{m}|u-\cos\alpha_j||u-\cos\beta_j|}}du = 0, \quad j = 1, \ldots, m-1. \tag{4.3}$$

With these points the density ω_{Γ_E} *in Theorem* 4.1 *has the form*

$$\omega_{\Gamma_E}(e^{i\theta}) = \frac{1}{2\pi} \frac{|\sin\theta| \prod_{j=1}^{m-1}|\cos\theta - \xi_j|}{\sqrt{\prod_{j=1}^{m}|\cos\theta - \cos\alpha_j||\cos\theta - \cos\beta_j|}}. \tag{4.4}$$

Note that the system (4.3) is a linear system for the coefficients of the polynomial

$$\prod_{j=1}^{m-1}(u-\xi_j) = u^{m-1} + c_2 u^{m-2} + \cdots + c_m.$$

It can be easily shown (cf. [14, Lemma 2.3]) that the system (4.3) is uniquely solvable for $c_2, \ldots, c_m$. Since the integrals in (4.3) over the $m-1$ intervals $[\cos\alpha_{j+1}, \cos\beta_j]$ are zero, it follows that $u^{m-1} + c_2 u^{m-2} + \cdots + c_m$ must have a zero on each of these intervals, so it has a unique zero on every $[\cos\alpha_{j+1}, \cos\beta_j]$, $j = 1, \ldots, m-1$, and this shows the existence and unicity of the ξ_j's.

Example 4.4 As an example, consider $E = [-\beta, -\alpha] \cup [\alpha, \beta]$ with some $0 \le \alpha < \beta \le \pi$. In this case $m = 1$, so the system (4.3) is empty, and we have

$$\omega_{\Gamma_E}(e^{i\theta}) = \frac{1}{2\pi} \frac{|\sin\theta|}{\sqrt{|\cos\theta - \cos\alpha||\cos\theta - \cos\beta|}}. \tag{4.5}$$

So for $\theta \in E$ we get from Theorem 4.1 the sharp inequality

$$|T_n'(\theta)| \le n \frac{|\sin\theta|}{\sqrt{|\cos\theta - \cos\alpha||\cos\theta - \cos\beta|}} \|T_n\|_{[-\beta,-\alpha]\cup[\alpha,\beta]}. \tag{4.6}$$

If $\alpha = 0$, then

$$\frac{|\sin\theta|}{\sqrt{|\cos\theta - 1||\cos\theta - \cos\beta|}} = \frac{\cos\theta/2}{\sqrt{\sin^2\beta/2 - \sin^2\theta/2}},$$

so (4.6) takes the form of the Videnskii inequality (1.3). Therefore, Videnskii's inequality is the $E = [-\beta, \beta]$ special case of Theorem 4.1.

Proof of Theorem 4.1. As has already been mentioned, the theorem follows from its special case when E consists of finitely many intervals. Indeed, if E is arbitrary, then there is a decreasing sequence $\{E_k\}$ of symmetric sets consisting of finitely many intervals such that $E = \cap_k E_k$. This clearly implies $\Gamma_E = \cap_k \Gamma_{E_k}$, and it is standard to verify that then $\mu_{\Gamma_{E_k}} \to \mu_{\Gamma_E}$ in the weak* topology. This then implies that $\omega_{\Gamma_{E_k}} \to \omega_{\Gamma_E}$ uniformly on compact subsets of any open arc J of E: this follows from the fact that if $I \subset J$ is any closed arc, then all ω_F, $J \subset F \subset C_1$ are uniformly equicontinuous on I. We leave the standard proofs of these to the reader (cf. [2, Lemma 3.1]).

Now if (4.1) is true for all E_k:

$$|T_n'(\theta)| \le n2\pi\omega_{\Gamma_{E_k}}(e^{i\theta})\|T_n\|_{E_k},$$

then by taking limit here for $k \to \infty$ and by making use of the fact that $\omega_{\Gamma_{E_k}}(e^{i\theta}) \to \omega_{\Gamma_E}(e^{i\theta})$, we get (4.1) in full generality.

With the argument of (2.8) we may also restrict our attention to real trigonometric polynomials.

Thus, in what follows we assume that E consists of finitely many intervals, say

$$E = \bigcup_{j=1}^{m} \left([\alpha_j, \beta_j] \cup [-\beta_j, -\alpha_j]\right),$$

where $0 \le \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m \le \pi$. Let

$$K = \bigcup_{j=1}^{m} [\cos\beta_j, \cos\alpha_j]$$

be the projection of Γ_E onto the real line. It is known (see e.g. [14, Lemma 2.3]) that the equilibrium measure is absolutely continuous with respect to Lebesgue-measure, and if $\omega_K(x)$ is its density, then

$$\omega_K(x) = \frac{1}{\pi}\frac{\prod_{j=1}^{m-1}|x-\xi_j|}{\sqrt{\prod_{j=1}^{m}|x-\cos\alpha_j||x-\cos\beta_j|}}, \tag{4.7}$$

where the $\xi_j \in (\cos\alpha_{j+1}, \cos\beta_j)$, $j = 1, \ldots, m-1$ are the unique points that satisfy the system of equations

$$\int_{\cos\alpha_{j+1}}^{\cos\beta_j} \frac{\prod_{j=1}^{m-1}(u-\xi_j)}{\sqrt{\prod_{j=1}^{m}|u-\cos\alpha_j||u-\cos\beta_j|}}\,du = 0, \quad j = 1, \ldots, m-1. \tag{4.8}$$

The following extension of Benstein's inequality (1.2) is also known (see [1], [14]):

$$|P_n'(x)| \le n\pi\omega_K(x)\|P_n\|_K, \qquad x \in K. \tag{4.9}$$

Now apply this with $T_n(t) = P_n(\cos t)$ to get for even (real) trigonometric polynomials T_n of degree at most n the inequality

$$|T_n'(\theta)| \leq n\pi\omega_K(\cos\theta)|\sin\theta|\|T_n\|_E, \qquad \theta \in E. \tag{4.10}$$

From here

$$|T_n'(\theta)| \leq (1+o(1))n\pi\omega_K(\cos\theta)|\sin\theta|\|T_n\|_E, \qquad \theta \in E, \tag{4.11}$$

follows for all trigonometric polynomials T_n of degree at most n with the method of section 3. An application of Theorem 2.1 then gives (4.10) for all (real) trigonometric polynomials T_n of degree at most n.

Thus, to complete the proof of Theorem 4.1, all we need to prove is that

$$\omega_{\Gamma_E}(e^{i\theta}) = \frac{1}{2}\omega_K(\cos\theta)|\sin\theta|, \qquad \theta \in E. \tag{4.12}$$

Note also that, in view of (4.7), formula (4.12) verifies (4.4), i.e. Lemma 4.3, as well.

Let $T(e^{it}) = \cos t$, and let $\nu(H) = \mu_{\Gamma_E}(T^{-1}(H))$, $H \subset [-1,1]$, be the pull-back of the measure μ_{Γ_E} under the map T. Then ν is a probability Borel-measure on K. We calculate its logarithmic potential

$$U^\nu(z) = \int \log\frac{1}{|z-t|}d\nu(t)$$

for $z \in K$. Note first of all, that, by the definition of ν, we have for $\cos\theta \in K$

$$\begin{aligned}
\int_K \log\frac{1}{|\cos\theta-\tau|}d\nu(\tau) &= \int_{\Gamma_E}\log\frac{1}{|\cos\theta-\cos t|}d\mu_{\Gamma_E}(e^{it}) \\
&= \int_{\Gamma_E}\log\frac{1}{\left|2\sin\frac{\theta-t}{2}\sin\frac{\theta+t}{2}\right|}d\mu_{\Gamma_E}(e^{it}) \\
&= \log 2 + \int_{\Gamma_E}\log\frac{1}{\left|2\sin\frac{\theta-t}{2}\right|}d\mu_{\Gamma_E}(e^{it}) \\
&\qquad + \int_{\Gamma_E}\log\frac{1}{\left|2\sin\frac{\theta+t}{2}\right|}d\mu_{\Gamma_E}(e^{it}).
\end{aligned}$$

Using the symmetry of Γ_E with respect to the real axis (which is equivalent to the symmetry of E with respect to the origin), we can see that the last two terms are equal to one another, and we can continue the preceding chain of equalities as

$$= \log 2 + 2\int_{\Gamma_E}\log\frac{1}{|e^{i\theta}-e^{it}|}d\mu_{\Gamma_E}(e^{it}).$$

The last term is $2U^{\mu_{\Gamma_E}}(e^{i\theta})$, and since the equilibrium potential $U^{\mu_{\Gamma_E}}$ is equal to $\log 1/\mathrm{cap}(\Gamma_E)$ on Γ_E, we can finally conclude

$$\int_K \log \frac{1}{|\cos\theta - \tau|} d\nu(\tau) = \mathrm{const}, \qquad \cos\theta \in K. \tag{4.13}$$

Since the equilibrium measure μ_K is characterized by the fact that its logarithmic potential is constant on K, we can conclude that $\nu = \mu_K$. Now under the map $T : \Gamma_E \to K$ every point in K has two inverse images (one on the upper and one on the lower part of the unit circle), so we get that the densities of $\mu_K = \nu = \mu_{\Gamma_E}(T^{-1})$ and μ_{Γ_E} are related as in (4.12).

■

The proof of Theorem 4.2 follows from the fact that (4.9) is sharp (see [14, Theorem 3.3]), and if for an x_0 lying in the interior of K the P_n are nonzero polynomials for which

$$|P_n'(x_0)| \geq (1 - o(1))n\pi\omega_K(x_0)\|P_n\|_K,$$

then $T_n(t) = P_n(\cos t)$ proves Theorem 4.2 at the point $\theta \in E$ for which $x_0 = \cos\theta$; see the argument in the preceding proof.

■

In conclusion we mention that Theorem 4.1 holds for non-symmetric sets, as well, but in that case one needs completely different arguments, and the form of the equilibrium density is not as simple as in the symmetric case treated in this paper.

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