Polynomials with zeros and small norm on curves*

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Abstract

This note considers the problem how zeros lying on the boundary of a domain influence the norm of polynomials (under the normalization that their value is fixed at a point). It is shown that k zeros raise the norm by a factor $(1+ck/n)$ (where n is the degree of the polynomial), while k excessive zeros on an arc compared to n times the equilibrium measure raise the norm by a factor $\exp(ck^2/n)$. These bounds are sharp, and they generalize earlier results for the unit circle which are connected to some constructions in number theory. Some related theorems of Andrievskii and Blatt will also be strengthened.

1 Results

Let $C_1 = \{z \mid |z| = 1\}$ be the unit circle. The paper [12] discussed monic polynomials with prescribed zeros on C_1 having as small norm as possible. The problem goes back to Turán's power sum method in number theory, in connection with which G. Halász [6] showed that there is a polynomial $Q_n(z) = z^n + \cdots$ with a zero at 1 and of norm $\|Q_n\|_{C_1} \le \exp(2/n)$, where $\|\cdot\|_K$ denotes supremum norm on the compact set K. See [7] for the smallest possible norm for such a polynomial. Halász' result implies that if $Z_1, Z_2, \ldots, Z_{k_n}$ are arbitrary $k_n < n/2$ points on the unit circle, then there is a $P_n = z^n + \cdots$ which has a zero at each Z_j and has norm

$$\|P_n\|_{C_1} \le \exp(4k_n^2/n) \tag{1}$$

It was shown in [12, Theorem 1] that, in general, one cannot have smaller norm, namely there is a constant $c > 0$ with the following property: for any monic polynomials $P_n(z) = z^n + \cdots$

(i) if P_n has k zeros (counting multiplicity) on C_1, then $\|P_n\|_{C_1} \ge 1 + c(k/n)$,

(ii) if P_n has $n|J|/2\pi + k$ zeros (counting multiplicity) on a subarc $J = J_n$ of the unit circle, then $\|P_n\|_{C_1} \ge \exp(ck^2/n)$.

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The polynomial $Q_{[n/k]}(z^k)$ shows that (i) is sharp, and $Q_{[n/k]}(z)^k$ (and, say, J a tiny interval around the point 1) shows that (ii) is sharp modulo constants.

For an alternative proof of part (ii) see the paper [4] by T. Erdélyi, where an improved version of a classical estimate of Erdős and Turán is used.

Another result of [12] showed that if the zeros are sufficiently well separated, then Halász' estimate can be improved. More precisely the following holds. Let $\alpha > 1$, and for each n let there be given a set X_n of k_n points on the unit circle such that the distance between different points of X_n is at least $\alpha 2\pi/n$. Then there are polynomials $P_n(z) = z^n + \cdots$ such that P_n vanishes at each point of X_n and

$$\|P_n\|_{C_1} \leq 1 + D_\alpha \sqrt{k_n/n}, \tag{2}$$

where the constant D_α depends only on α. In particular, if $k_n = o(n)$, then $\|P_n\|_{C_1} = 1 + o(1)$. We also mention that the conclusion is not valid for any $\alpha < 1$; this follows from (ii) above. The estimate (2) was improved by Andrievskii and Blatt [3] to

$$\|P_n\|_{C_1} \leq 1 + D_\alpha k_n/n, \tag{3}$$

which is a remarkable counterpart to (i) above.

By considering $z^n P_n(1/z)$, all these have a formulation for polynomials P_n with normalization $P_n(0) = 1$, and this is the form the problem was generalized in [3] to analytic Jordan curves Γ (multiple zeros) and in [2] to quasicirles (single zero). Note that if Γ is a Jordan curve and z_0 is a fixed point inside Γ, then, by the maximum modulus theorem, we must have $\|P\|_\Gamma \geq 1$ for all polynomials P with $P(z_0) = 1$. We are interested in the problem, how zeros lying on Γ influence this trivial lower estimate. For a single zero the analogue of Halász' result was settled even for quasicirles in the paper [2] where the zero can also occur, say, at a corner. For multiple zeros Andrievskii and Blatt [3] proved that if z_0 is a fixed point inside the analytic curve Γ and P_n is a polynomial of degree n with k_n separated zeros on Γ, then

$$\|P_n\|_\Gamma \geq 1 + c k_n/n. \tag{4}$$

On the other hand, if there are points $w_1, \ldots, w_{k_n}$ on Γ which are well separated (in terms of a conformal mapping of the outer domain onto the exterior of the unit disk), then there is a P_n of degree n such that $P_n(z_0) = 1$ and (3) holds (with $\|\cdot\|_\Gamma$ replacing $\|\cdot\|_{C_1}$).

The present paper was motivated by the aforementioned results of Andrievskii and Blatt, and in particular, we will drop the analyticity assumption on Γ, as well as the separation assumption in (4). Actually, we shall prove the complete analogue of the results mentioned above for the unit circle for all $C^{1+\alpha}$, $\alpha > 0$ Jordan curves. We emphasize that although the results match those for the unit circle, the proofs need ideas that do not use the special symmetry of the circle; in particular we cannot use trigonometric polynomials here. In fact, a Jordan curve can be pretty complicated from the point of view of polynomials.

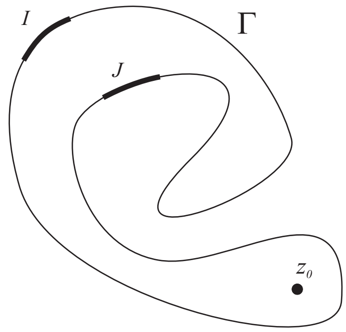

Figure 1: The arc I has bigger equilibrium measure than J, so there can be more zeros there without essentially raising the norm

For example, it follows from the result below that the arc I on Γ depicted in Figure 1 tolerates many more zeros without raising the norm of a polynomial than the arc J (more precisely, even though I and J have equal lengths, there is a c such that I can contain cn zeros of a P_n with $P_n(z_0) = 1$, $\|P_n\|_\Gamma = 1+o(1)$, while if a P_n has the same number of zeros of J, then necessarily $\|P_n\|_\Gamma \geq \exp(dn)$, with some $d > 0$, which is a very dramatic change).

Recall that Γ is a Jordan curve if it is the homeomorphic image of the unit circle. It is called of class $C^{1+\alpha}$ if in its arc-length parametrization the parameter function is differentiable and its derivative lies in the Lip α class.

Theorem 1 *Let Γ be a $C^{1+\alpha}$ Jordan curve, and let z_0 be a fixed point in the interior of Γ. If a polynomial P_n of degree at most n takes the value 1 at z_0 and has k_n zeros on Γ, then $\|P_n\|_\Gamma \geq 1 + ck_n/n$ with a $c > 0$ that depends only on Γ and z_0.*

This is sharp (at least for analytic Γ) because of the aforementioned result of Andrievskii and Blatt.

To formulate our next theorem let μ_Γ be the equilibrium measure of Γ (see e.g. [5], [10]). It is the unique unit Borel-measure on Γ for which the logarithmic potential

$$U^{\mu_\Gamma}(z) = \int \log \frac{1}{|z-t|} d\mu(t)$$

is constant on Γ. One should think of μ_Γ as the distribution of a unit charge placed on the conductor Γ when it is in equilibrium. Of course, if Γ is the unit circle, then μ_Γ is just the normalized arc measure. Therefore, the following result is an extension of (ii) to smooth Jordan curves.

Theorem 2 *Let Γ be a $C^{1+\alpha}$ Jordan curve and z_0 be a fixed point inside Γ. If P_n is a polynomial of degree at most n such that $P_n(z_0) = 1$ and P_n has at least $k_n + n\mu_\Gamma(J)$ zeros on a subarc J of Γ, then $\|P_n\|_\Gamma \geq \exp(ck_n^2/n)$ with a $c > 0$ that depends only on Γ and z_0.*

Corollary 3 *If P_n are polynomials with $P_n(z_0) = 1$, $\|P_n\|_\Gamma = 1 + o(1)$, then*

(a) *P_n has $o(n)$ zeros on Γ.*

(b) *P_n has at most $n\mu_\Gamma(J) + o(\sqrt{n})$ zeros on any subarc $J = J_n$ of the unit circle. In particular, if $w \in \Gamma$ is a zero of P_n, then its multiplicity is $o(\sqrt{n})$.*

Next we show that Theorem 2 is sharp for all C^2 curves.

Theorem 4 *Let Γ be a C^2 Jordan curve and z_0 a point inside Γ. Then there are a constant C and for every $w \in \Gamma$ and for every $n = 1, 2, \ldots$ a polynomial $P_{n,w}$ of degree at most n such that $P_{n,w}(z_0) = 1$, $P_{n,w}$ has a zero at w and $\|P_{n,w}\|_\Gamma \leq 1 + C/n$.*

Corollary 5 *Let Γ be a C^2 Jordan curve and z_0 a point inside Γ. Then there is a constant C with the following property: if $w_1, \ldots, w_{k_n} \in \Gamma$ are arbitrary $k_n \leq n$ points on Γ, then there is a polynomial P_n of degree n such that $P_n(z_0) = 1$, P_n has a zero at every w_j and $\|P_n\|_\Gamma \leq \exp(Ck_n^2/n)$.*

In particular, if $\{\delta_n\}$ is any positive sequence tending to 0 and if $w \in \Gamma$ is given, then there are polynomials P_n with $P_n(z_0) = 1$, $\|P_n\|_\Gamma = 1 + o(1)$ such that w is a zero of P_n of multiplicity $\geq \delta_n\sqrt{n}$. This shows that nothing more can be said about the multiplicities of zeros than what was stated in Corollary 3.

It should be mentioned that the results are true for Dini smooth curves instead of $C^{1+\alpha}$-curves (for Dini smoothness see [9]; it lies in between C^1 and $C^{1+\alpha}$, $\alpha > 0$, smoothness). Indeed, using [9, Theorem 3.5] one can derive Proposition 6 below for Dini smooth curves, and the rest of the argument remains the same. One should also mention that even though the simple proof we give for Theorem 4 is valid only for C^2 curves, the result itself follows also from formula (3) in [2] actually for Dini smooth curves. The author is thankful for these remarks to the referee.

2 Proof of Theorem 1

We shall need the following facts from potential theory. For the necessary concepts (like equilibrium measure, Green's function etc.) from logarithmic potential theory see e.g. [5] or [10]. Let Γ be a Jordan curve and Ω the unbounded component of $\overline{\mathbf{C}} \setminus \Gamma$. As before, we denote by μ_Γ the equilibrium measure of Γ

and by $g_{\overline{\mathbf{C}}\setminus\Gamma}(z,\infty)$ the Green's function of Ω with pole at infinity. For a domain G and a set $K\subset\partial G$ let $\omega(K,G,z)$ be the harmonic measure of K in G with respect to z (i.e. $\omega(K,G,z)$ is the value at z of the solution of the Dirichlet problem in G with boundary function equal to 1 on K and equal to 0 on the rest of the boundary). It is a unit Borel-measure on ∂G, and it is the unique measure on ∂G for which the Poisson-formula

$$u(z)=\int u d\omega(\cdot,G,z)$$

is valid for all u which is harmonic in G and continuous on $\overline{G}$. Harmonic measures are conformal invariant. For example, $\mu_\Gamma\equiv\omega(\cdot,\Omega,\infty)$ (see e.g. [10, Theorem 4.3.14]).

Proposition 6 *Let Γ be $C^{1+\alpha}$ Jordan-curve with some $0<\alpha<1$. The equilibrium measure μ_Γ has continuous (actually Lip α) and positive density with respect to the arc measure on Γ. The same is true of all harmonic measures $\omega(\cdot,G,\zeta_0)$, $\zeta_0\in G$, where G is either the bounded or the unbounded complement of Γ. Furthermore, the Green's function $g_{\overline{\mathbf{C}}\setminus\Gamma}(\cdot,\infty)$ of the unbounded component Ω with pole at infinity is uniformly* Lip 1 *(actually $C^{1+\alpha}$) on Γ.*

Proof. These are well known facts. For a reference to the statements concerning the equilibrium measure see [14, Proposition 2.2]. Now using the fact that the equilibrium measure is the harmonic measure at infinity, i.e. $\mu_\Gamma(K)=\omega(K,\Omega,\infty)$ where Ω is the unbounded component of $\overline{\mathbf{C}}\setminus\Gamma$ (see e.g. [10, Theorem 4.3.14]), the claim concerning the harmonic measures also follows for $\omega(K,\Omega,\infty)$. But harmonic measures are conformal invariant, so the claim follows in general by using Möbius inversion: if $T\Gamma$ is the curve under the conformal map $Tw=1/(w-z)$, then $\omega(K,\Gamma,z)=\omega(TK,T\Omega,\infty)=\mu_{T\Gamma}(TK)$, and clearly this conformal map preserves $C^{1+\alpha}$-smoothness.

The statement concerning the Green's function follows from the Kellogg-Warschawski theorem (see [9, Theorems 3.5, 3.6]) stating that the conformal map $\varphi(z)=cz+d++e/z+\cdots$, $c>0$, from Ω onto the exterior of the unit disk is of class $C^{1+\alpha}$ in $\overline{\Omega}$, since $g_{\overline{\mathbf{C}}\setminus\Gamma}(z)=\log|\varphi(z)|$.

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Next, we need

Lemma 7 *There are $\delta,\theta>0$ depending only on Γ such that if $J=\widehat{ab}$ is a subarc of Γ of length at most δ and if P_n has at least $\theta n|J|$ zeros on J, then $|P_n(b)|\le 1/3\|P_n\|_\Gamma$.*

Proof. According to the preceding proposition there is a C_1 such that for t close to Γ

$$g_{\overline{\mathbf{C}}\setminus\Gamma}(t,\infty)\le C_1\mathrm{dist}(t,\Gamma),$$

and for other t this is automatically true. Hence, by the Bernstein-Walsh lemma [15, p. 77] for $\mathrm{dist}(t,\Gamma)<\rho$ we have

$$|Q_n(t)|\le e^{ng_{\overline{C}\setminus\Gamma}(t,\infty)}\|Q_n\|_\Gamma\le e^{C_1n\rho}\|Q_n\|_\Gamma.$$

for any polynomial Q_n of degree at most $n=1,2,\ldots$. Therefore, by Cauchy's formula,

$$Q_n^{(m)}(z)=\frac{m!}{2\pi i}\int_{|t-z|=\rho}\frac{Q_n(t)}{(t-z)^{m+1}}dt$$

with integration on a circle with center at $z\in\Gamma$ and of radius ρ, we obtain for $z\in\Gamma$

$$|Q_n^{(m)}(z)|\le e^{C_1n\rho}m!\frac{1}{\rho^m}\|Q_n\|_\Gamma, \tag{5}$$

and here $\rho>0$ is arbitrary.

Let δ be selected so that on any arc of Γ of length at most 3δ the direction of the tangent line does not change more than $\pi/8$. Let $J=\widehat{ab}$ be a subarc of Γ of length at most δ, and let $z_1,\ldots,z_m$ be the zeros of P_n lying on J. Define the polynomial Q_n as

$$Q_n(z)=P_n(z)\prod_{j=1}^m\frac{z-a}{z-z_j},$$

i.e. we move all the zeros of P_n lying on J into a and leave all other zeros in place. Let $J'=\widehat{a'b'}$ be the arc of Γ that contains J and for which $|\widehat{a'a}|=|\widehat{bb'}|=|J|$, where $|J|$ denotes the arc length of J (i.e., J' is obtained by enlarging J three times with respect to arc length). By considering the individual factors $|z-a|/|z-z_j|$, it is easy to see that $|P_n(b)|\le|Q_n(b)|$, and if $z\notin\widehat{a'b'}$, $z\in\Gamma$, then $|z-a|/|z-z_j|\le C_2$ with some C_2 depending only on Γ, and hence for such z we have $|Q_n(z)|\le C_2^m|P_m(z)|$. Therefore, if the norm $\|Q_n\|_\Gamma$ is not attained on $J'=\widehat{a'b'}$, which we are going to show under the assumption that there are sufficiently many zeros on $\widehat{ab}$, then

$$\|Q_n\|_\Gamma\le C_2^m\|P_m\|_\Gamma. \tag{6}$$

Since $Q_n(z)$ has a zero at a of order m, we have

$$Q_n(z)=\int_a^z\int_a^{w_1}\cdots\int_a^{w_{m-1}}Q_n^{(m)}(w)dwdw_{m-1}\cdots dw_1.$$

If $z=\gamma(s)$, $s\in[0,|J|]$ is the arc length parametrization of J with $\gamma(0)=a$, then this takes the form

$$Q_n(z)=\int_0^s\int_0^{\tau_1}\cdots\int_0^{\tau_{m-1}}Q_n^{(m)}(\tau)\gamma'(\tau)\gamma'(\tau_{m-1})\cdots\gamma'(\tau_1)\,d\tau d\tau_{m-1}\cdots d\tau_1.$$

Clearly, this formula also holds for $z \in J'$ (with s in the extended range $[-|J|, 2|J|]$ then). Hence, $|\gamma'(\tau)| = 1$ and (5) with $\rho = \theta|J|$ gives for $z \in J'$ (note that during m-fold integration the factor $1/m!$ emerges)

$$|Q_n(z)| \le e^{C_1 n\theta|J|} m! \frac{1}{(\theta|J|)^m} \frac{|\widehat{az}|^m}{m!} \|Q_n\|_\Gamma \le e^{C_1 n\theta|J|} \left(\frac{2}{\theta}\right)^m \|Q_n\|_\Gamma$$

since for $z \in J'$ we have $|\widehat{az}| \le 2|J|$. Now if we assume that the number of zeros in J is $m \ge \theta n|J|$, then we obtain for $z \in J'$

$$|Q_n(z)| \le e^{C_1 m} \left(\frac{2}{\theta}\right)^m \|Q_n\|_\Gamma = \left(\frac{2e^{C_1}}{\theta}\right)^m \|Q_n\|_\Gamma,$$

and for $\theta > 2e^{C_1}$ this means that the norm $\|Q_n\|_\Gamma$ is not attained in J', and so (6) is true. Therefore, we get from the preceding inequality and (6)

$$|P_n(b)| \le |Q_n(b)| \le \left(\frac{2e^{C_1}}{\theta}\right)^m \|Q_n\|_\Gamma \le \left(\frac{2e^{C_1}C_2}{\theta}\right)^m \|P_n\|_\Gamma,$$

from which the claim immediately follows if $\theta > 6e^{C_1}C_2$ (we may assume $m \ge 1$ for otherwise there is nothing to prove).

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Proof of Theorem 1. In this proof $\omega(\cdot, z_0)$ denotes the harmonic measure in the interior of Γ. It follows from Proposition 6 that there is a constant C_0 such that for all arcs I on Γ we have

$$|I| \le C_0 \omega(I, z_0). \tag{7}$$

If $\|P_n\|_\Gamma \ge 3/2$, then we are ready. Otherwise consider the set H of those $z \in \Gamma$ for which $|P_n(z)| \le \|P_n\|_\Gamma/2$. This set consists of a finite number of arcs, say $J_1, \ldots, J_k$, on which $|P_n(z)| \le 3/4$. Since $\log|P_n(z)|$ is subharmonic, we have

$$0 = \log|P_n(z_0)| \le \int \log|P_n| d\omega(\cdot, z_0) = \int_H + \int_{\Gamma\setminus H} = I_1 + I_2. \tag{8}$$

Now for any j

$$I_1 \le \omega(H, z_0)\log(3/4) \le \omega(J_j, z_0)\log(3/4), \qquad I_2 \le \log\|P_n\|_\Gamma, \tag{9}$$

hence the preceding inequalities give the theorem if one of the J_j's is of length bigger than δ (with the δ from Lemma 7). Indeed, for then its harmonic measure is at least δ_1 with some $\delta_1 > 0$ depending only on Γ and z_0, and (8)–(9) give $\log \|P_n\|_\Gamma \geq c_1 > 0$.

If, on the other hand, all J_j has length at most δ, then, by Lemma 7, the number of zeros of P_n on J_j is at most $\theta n |J_j|$, since the value of P_n at the endpoints of J_j is $\|P_n\|_\Gamma/2$. Hence, using also (7), we have with some C_0

$$k_n \leq \theta n \sum_j |J_j| \leq \theta n C_0 \sum_j \omega(J_j, z_0) = \theta n C_0 \omega(H, z_0),$$

and so from (8) and (9) we obtain

$$\log \|P_n\| \geq I_2 \geq -I_1 \geq -\omega(H, z_0) \log(3/4) \geq (-\log(3/4)/\theta C_0) k_n/n,$$

and this completes the proof.

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3 Proof of Theorems 2 and 4

Proof of Theorem 2. Assume, without loss of generality, that $z_0 = 0$. Apply the transformation $w = 1/z$, and let Γ', J' be the image of Γ, J under this transformation, furthermore let $Q_n(z) = z^n P_n(1/z)$. If $\omega'(\cdot, 0)$ is the harmonic measure inside Γ' corresponding to the point 0, then the logarithmic potential of $\omega'(\cdot, 0)$ equals $\log 1/|z|$ on and outside Γ' (consider e.g. that if z is outside Γ' then $\log 1/|z - t|$ is a harmonic function of t inside Γ'), hence on Γ'

$$U^{n\omega'(\cdot,0)}(z) + \log |Q_n(z)| \leq \sup_{z \in \Gamma'} \log |P_n(1/z)| = \log \|P_n\|_\Gamma.$$

Let ν_n be the normalized zero counting measure on the zeros of Q_n. Then $-\log |Q_n(z)| = U^{\nu_n}(z)$. Let $\tilde{\nu}_n$ be the balayage (see [11, Theorems II.4.1, II.4.4]) of ν_n out of the two components of $\overline{\mathbf{C}} \setminus \Gamma'$ onto Γ'; in other words, $\tilde{\nu}_n$ is the unique measure on Γ that has total mass n for which $U^{\tilde{\nu}_n}(z) = \text{const} - \log |Q_n(z)|$ for all $z \in \Gamma$. Since taking the balayage out of a bounded region does not change the logarithmic potential on the boundary, while taking balayage out of an unbounded region increases it by a positive constant on the boundary (see [11, Theorems II.4.1, II.4.4]), it follows that

$$U^{\omega'(\cdot,0)}(z) - U^{\tilde{\nu}_n}(z) \leq \log \|P_n\|_\Gamma^{1/n}, \quad z \in \Gamma'.$$

Since the left-hand side is harmonic outside Γ' (including ∞), this inequality holds outside Γ', as well. Therefore, we can apply the one-sided discrepancy theorem [1, Theorem 4.1.1] with $\beta = 1$ to the curve Γ' and to the measure $\sigma = \omega'(\cdot, 0) - \tilde{\nu}_n$ to conclude that for some $C > 0$ and for any $\delta > 0$

$$|\sigma(J')| \leq C \left(\delta^{-1/2} \|P_n\|_\Gamma^{1/n} + \delta^{1/2} \right),$$

from which, with

$$\delta = \log \|P_n\|_\Gamma^{1/n},$$

we obtain

$$|\sigma(J')| \leq 2C\sqrt{\log \|P_n\|_\Gamma^{1/n}}. \tag{10}$$

Since the harmonic measure is conformal invariant, we have $\omega'(J', 0) = \omega(J, \infty) = \mu_\Gamma(J)$ (here the first harmonic measure is taken inside Γ', while the second one is taken in the unbounded component Ω of $\overline{\mathbf{C}} \setminus \Gamma$). Hence, by assumption, Q_n has at least $k_n + \omega'(J', 0)$ zeros on J', which implies that $\tilde{\nu}_n(J') \geq k_n + \omega'(J', 0)$, and therefore $|\sigma(J')| \geq k_n/n$. Now the claim follows from (10).

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Proof of Theorem 4. There is a polynomial $T_N = T_{N,w}$ of some degree N such that the lemniscate set $L_w = \{z \,|\, |T_N(z)| = 1\}$ consists of a single Jordan curve such that L_w contains Γ in its interior except for the point w, at which point L_w and Γ touch each other, see [8]. Furthermore, this is also true in the sense that a translated-rotated copy of L_w can serve as $L_{w'}$ for $w' \in \gamma$ lying sufficiently close to w (see [13, Theorem 2.3]). Then simple compactness tells us that there is a uniform bound on N, and the $T_{N,w}$'s can be chosen in such a way that they are obtained by a linear transformation of the argument in a fixed finite family of polynomials. Let us call this fact by saying that the $T_{n,w}$'s form a compact family. We may also assume that $T_{N,w}(w) = 1$ (just multiply $T_{N,w}$ by a constant of modulus 1 if this is not the case).

Now let Q_m be polynomials of degree $m = 1, 2, \ldots$ such that $Q_m(0) = 1$, $Q_m(1) = 0$ and $|Q_m(z)| \leq 1 + 4/m$ (see [6]), and set

$$P_{n,w}(z) = Q_{[n/N]}(T_{N,w}(z))/Q_{[n/N]}(T_{N,w}(z_0)).$$

For this $P_{n,w}(w) = 0$. Simple calculation shows that by replacing a factor $z - a$ with $|a| < 1$ in $Q_m(z)$ by $|a|^2(z - 1/\overline{a})$, we decrease the norm of Q_m on the unit circle (keeping the normalization $Q_m(0) = 1$), so we may assume that Q_m has no zeros inside the unit circle. But then $\log |Q_m(z)|$ is harmonic in the unit disk, it takes the value 0 at the origin and has the bound $\leq \log(1 + 4/m)$ throughout the disk. Hence we can derive from Harnack's inequality [10, Theorem 1.3.1] that for any compact subset K of the open unit disk there is a constant C_K such that $|Q_m(z)| \geq 1 - C_K/m$ for $z \in K$. From the fact that the polynomials $T_{N,w}$ form a compact family it follows that the set $\{T_{N,w}(z_0) \,|\, w \in \Gamma\}$ lies in a fixed compact subset K of the unit disk. Therefore, $Q_{[n/N]}(T_{N,w}(z_0)) \geq 1 - 2C_K N/n$, and $|Q_{[n/N]}(T_{N,w}(z))| \leq 1 + 8N/n$ for $z \in \Gamma$, which show that $\|P_{n,w}\|_\Gamma \leq 1 + C/n$ with some C.

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