Bernstein inequality in L^α norms

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Abstract

The classical Bernstein inequality estimates the derivative of a polynomial at a fixed point with the supremum norm and a factor depending on the point only. Recently, this classical inequality was generalized to arbitrary compact subsets on the real line. That generalization is sharp and naturally introduces potential theoretical quantities. It also gives a hint how a sharp L^α Bernstein inequality should look like.

In this paper we prove this conjectured L^α Bernstein type inequality and we also prove its sharpness. [1] [2]

1 Introduction

The classical Bernstein inequality states the following (see also [6] or [2])

$$\left|P_n'(t)\right| \le n\frac{1}{\sqrt{1-t^2}}||P_n||_{I,\infty}, \tag{1}$$

where P_n is an arbitrary real polynomial with degree n, $t \in (-1,1)$ and $||P_n||_{I,\infty}$ is the supremum norm over $I := [-1,+1]$.

There is a recent generalization.

Theorem 1. *Let $K \subset \mathbf{R}$ be a compact set and assume that its equilibrium measure ν_K is absolutely continuous w.r.t. the Lebesgue measure, t is in the interior of K so that the density $d\nu_K(t)/dt = \omega_K(t)$ exists and is finite, and* $\deg(P_n) = n$. *Then*

$$|P_n'(t)| \le n\pi\omega_K(t)||P_n||_{K,\infty} \tag{2}$$

where $||P_n||_{K,\infty}$ denotes the supremum norm over K.

For the equilibrium measure ν_K and its density $\omega_K(.)$, we refer to [6]. This theorem was proved independently in [1] and [7].

We work with compact sets on the real line, namely,

$$K \subset \mathbf{R}, \text{ is a compact set consisting of finitely many, disjoint, closed intervals and none of them is a single point.} \tag{3}$$

We assume these throughout this paper. Denote the components of K by $K_{c,1}, \ldots, K_{c,\ell_1}$. That is, $K_{c,j}$'s are closed, disjoint intervals. By reindexing them, we can assume that if $i < j$, $x \in K_{c,i}, y \in K_{c,j}$, then $x < y$. We also need a special class of these sets which is defined as follows.

Consider those (algebraic, real) polynomials r which have $\deg r$ distinct zeros on the real line and if $r'(t) = 0$, then $|r(t)| \ge 1$. These polynomials are called *admissible polynomials* in [7].

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Definition 2. *We call a compact set $K \subset \mathbf{R}$ a* real lemniscate*, if $K = r^{-1}[-1,+1]$ for some admissible polynomial r.*

Since at the extreme places of r the modulus is greater than or equal to 1, $r^{-1}[(-1,1)] = \{t : -1 < r(t) < 1\}$ consists of $\deg(r)$ open intervals. We call the closures of the intervals branches of K, and denote them by $K_{b,1}, \ldots, K_{b,\deg r}$. Two different branches are either disjoint or have one endpoint t in common for which $r'(t) = 0$ and $|r(t)| = 1$. As above, by reindexing them, we can assume that if $i < j$, $x \in K_{b,i}, y \in K_{b,j}$, then $x \le y$ with equality only when $j = i+1$ and $x = y$ is the only one common point of $K_{b,i}$ and $K_{b,j}$ provided $K_{b,i} \cap K_{b,i+1} \ne \emptyset$.

We denote by ν_K the equilibrium measure of K. If K is as in (3), the equilibrium measure is absolutely continuous. Furthermore, if K is a real lemniscate, then its equilibrium measure is known explicitly, see (18) later.

The following two theorems state the main results of this paper.

Theorem 3. *Let $K \subset \mathbf{R}$ be a compact set as in (3), let ν_K be its equilibrium measure and $\omega_K(t)$ its density, $\omega_K(t) = d\nu_K(t)/dt$. Furthermore, let $1 \le \alpha < \infty$. Then*

$$\int_K \left|\frac{P_n'(t)}{n\pi\omega_K(t)}\right|^\alpha d\nu_K(t) \le \big(1 + o(1)\big) \int_K \big|P_n(t)\big|^\alpha d\nu_K(t) \ , \tag{4}$$

where P_n is an arbitrary polynomial of degree n and $o(1)$ means an error term that tends to 0 as $n \to \infty$ and is independent of P_n.

Inequality (2) corresponds to the $\alpha = \infty$ case. Note that inequality (4) does not include inequality (2) and we have no information on the error term as $\alpha \to \infty$.

Theorem 4. *The constant 1 on the right hand side of (4) is asymptotically sharp.*

The proof of (4) we found is quite technical and consists of several steps. The technique used is the polynomial inverse image method, for a nice survey, we refer to [9].

First we prove it in Section 2 for the special case when K is the interval $[-1,1]$. Then we prove it for real lemniscates. Now, the case when P_n is a polynomial of the lemniscate-defining polynomial r, is easier, this is handled in Section 3. The general case, when K is still a real lemniscate is treated in Section 5, after some technical preparations in Section 4. Then Section 6 completes the proof for general K consisting of finitely many intervals. Finally, Section 7 proves the sharpness and Section 8 contains the proofs of the lemmas from Sections 4, 5 and 6.

2 The proof of (4) when $K = [-1,+1]$

We use Zygmund's inequality, see [2], p. 390 or [3], p. 584, Theorem 1.7.1.

Theorem 5. *Let $1 \le \alpha < \infty$. If Q_n is a trigonometric polynomial of degree n, then*

$$\int_{-\pi}^{\pi} |Q_n'(t)|^\alpha dt \le n^\alpha \int_{-\pi}^{\pi} |Q_n(t)|^\alpha dt.$$

This is sharp, if $Q_n(t) = \cos(nt)$, then the two sides are equal.

If $K=[-1,1]$, then it is known that $d\nu_{[-1,+1]}(t)=\frac{1}{\pi\sqrt{1-t^2}}dt$. So (4) simplifies to

$$\int_{-1}^{1}\Big|\frac{P_n'(t)\cdot\sqrt{1-t^2}}{n}\Big|^\alpha d\nu_{[-1,1]}(t)\le\big(1+o(1)\big)\cdot\int_{-1}^{1}\big|P_n(t)\big|^\alpha d\nu_{[-1,1]}(t)\ ,$$

which is easy to prove even without the factor $1+o(1)$. Let P_n be an arbitrary polynomial with real coefficients ($P_n\in\mathbf{R}[t]$), $n=\deg P_n$. Define $q(t):=P_n(\cos t)$. So $q(t)$ is an even function, $q(-t)=q(t)$ and $q'(t)=P_n'(\cos t)(-\sin t)$ so

$$\int_{-\pi}^{0}|q(t)|^\alpha\frac{dt}{2\pi}=\int_{0}^{\pi}|q(t)|^\alpha\frac{dt}{2\pi}$$

and

$$\int_{-\pi}^{0}\Big|\frac{q'(t)}{n}\Big|^\alpha\frac{dt}{2\pi}=\int_{0}^{\pi}\Big|\frac{q'(t)}{n}\Big|^\alpha\frac{dt}{2\pi}\ .$$

Furthermore q is actually a trigonometric polynomial with real coefficients of (trigonometric) degree n. So Zygmund's inequality can be applied to obtain

$$\int_{-\pi}^{+\pi}\Big|\frac{q'(t)}{n}\Big|^\alpha dt\le\int_{-\pi}^{+\pi}\big|q(t)\big|^\alpha dt.$$

That is,

$$\int_{0}^{\pi}\Big|\frac{q'(t)}{n}\Big|^\alpha dt\le\int_{0}^{\pi}\big|q(t)\big|^\alpha dt.$$

Now substitute $t=\arccos u$ ($t\in[0,\pi]$ and $u\in[-1,1]$) with $\frac{dt}{du}=\frac{-1}{\sqrt{1-u^2}}$ to obtain

$$\int_{-1}^{+1}\Big|\frac{P_n'(u)\cdot\sqrt{1-u^2}}{n}\Big|^\alpha\frac{1}{\pi\sqrt{1-u^2}}du\le\int_{-1}^{+1}\big|P_n(u)\big|^\alpha\frac{1}{\pi\sqrt{1-u^2}}du\ .$$

Since $d\nu_{[-1,+1]}(t)=\frac{1}{\pi\sqrt{1-t^2}}dt$, this inequality is nothing else than (4) without the error term $1+o(1)$ on $I=[-1,+1]$, that is

$$\int_I\left|\frac{P_n'(t)}{n\pi\omega_I(t)}\right|^\alpha d\nu_I(t)\le\int_I\big|P_n(t)\big|^\alpha d\nu_I(t). \tag{5}$$

As for its sharpness, consider the trigonometric polynomials $Q_n(t)=\cos(nt)$. Using the sharpness of Zygmund's inequality with the $t=\arccos u$ substitution, we arrive at the Chebyshev polynomials T_n of $[-1,+1]$ with degree n. Then, the inequality (5) is sharp with these polynomials, that is, the left hand side is equal to the right hand side.

3 The proof of (4) when K is a real lemniscate and P_n is a polynomial of r

This case is very similar to the previous one, but we have to inspect carefully the substitution $r(u)=t$, since r is $\deg r$-to-1 mapping.

In this section we assume that P_n is a polynomial of r, that is, there exists a polynomial p such that $P_n(t)=p\big(r(t)\big)$.

It is known (see e.g. [7] (3.7)) that the equilibrium measure of $K = r^{-1}[I]$ in this case can be expressed as follows

$$\nu_K(A) = \frac{1}{\deg r} \nu_I(r(A)) \ , \tag{6}$$

where $A \subset K$ is an arbitrary subset with the property that r is 1-to-1 from A to $r(A)$. Then for the density function, it easily follows that

$$\omega_{r^{-1}[I]}(u) = \frac{1}{\deg r} \left| r'(u) \right| \omega_I\big(r(u)\big) \ . \tag{7}$$

We will use the substitution $t = r(u)$. Starting from the left hand side of (4) for $P_n = p(r)$

$$\int\limits_{r^{-1}[I]} \left| \frac{p'\big(r(u)\big) \cdot r'(u)}{\big(\deg p \cdot \deg r\big) \cdot \pi \cdot \omega_{r^{-1}[I]}(u)} \right|^{\alpha} d\nu_{r^{-1}[I]}(u) =$$

replacing the measure and the density function with the help of (6) and (7)

$$= \frac{1}{\deg r} \int\limits_{r^{-1}[I]} \left| \frac{p'\big(r(u)\big)}{\deg p \cdot \pi \cdot \omega_I\big(r(u)\big)} \right|^{\alpha} |r'(u)| \omega_I\big(r(u)\big) du$$

$$= \int\limits_{[-1,+1]} \left| \frac{p'(t)}{\deg p \cdot \pi \cdot \omega_I(t)} \right|^{\alpha} \omega_I(t) dt$$

which we continue later. This substitution is valid since $r(u)$ runs through $[-1, +1]$ $\deg(r)$ times as u runs through $r^{-1}[I]$ and each time we get

$$\frac{1}{\deg r} \int_{[-1,+1]} \left| p'(t) / \big(\deg p \, \pi \, \omega_I(t) \big) \right|^{\alpha} \omega_I(t) dt.$$

Continuing with the already proved inequality on $I = [-1, +1]$

$$\int_{[-1,+1]} \left| \frac{p'(t)}{\deg p \cdot \pi \cdot \omega_I(t)} \right|^{\alpha} \omega_I(t) dt \leq \int_I \left| p(t) \right|^{\alpha} \omega_I(t) dt$$

$$= \int_{r^{-1}[I]} \left| p\big(r(u)\big) \right|^{\alpha} \frac{1}{\deg r} |r'(u)| \omega_I\big(r(u)\big) du$$

$$= \int_{r^{-1}[I]} \left| p\big(r(u)\big) \right|^{\alpha} \omega_{r^{-1}[I]}(u) du \ ,$$

which is the right hand side of (4) for $P_n = p(r)$. So we have derived

$$\int\limits_K \left| \frac{P_n'(u)}{\deg(P_n) \, \pi \omega_K(u)} \right|^{\alpha} d\nu_K(u) \leq \int_K \left| P_n(u) \right|^{\alpha} d\nu_K(u) \tag{8}$$

that is, (4) without the error term when P_n is a polynomial of r and $K = r^{-1}[-1, 1]$.

4 Splitting the set

Let K be an arbitrary set consisting of finitely many intervals as in (3). Suppose that K is given in the following form $K = \cup_{i=1}^{k_1} K_i$, $K_i = [u_{2i-1}, u_{2i}]$ where $u_{2i-1} < u_{2i} \le u_{2i+1} < u_{2i+2}$, $i = 1, \ldots, k_1 - 1$. In other words, these intervals can touch each other, but none of them can be a single point. For example, K can be a real lemniscate, $K = r^{-1}[-1, 1]$, and u_i's are all those places where $|r| = 1$, $k_1 = \deg r$.

Split K into small closed intervals whose length is at most

$$\lambda_n := c_1/n^\kappa \tag{9}$$

where $0 < c_1 < 1/4$, $0 < \kappa < 1$ and every two of these small intervals have at most one common point. More precisely, we form a family of closed subintervals of K such that their union is K, any two of them can have at most one common point, none of the u_i's of K are in the interior of any small intervals and the length of the intervals is $\lambda_n/2$ except for those when any of the u_i's is in the interval, then in this case, its length is in between $\lambda_n/2$ and λ_n.

If n is large enough depending on K, more precisely,

$$\lambda_n < \min\{u_i - u_{i-1} : u_i \ne u_{i-1}, i = 2, 3, \ldots, 2k_1\}, \tag{10}$$

then λ_n is smaller than the shortest interval of K and smaller than the shortest gap between the u_i's of K, and so such a family of subintervals exists.

This way we have $O\big(1/\lambda_n\big) = O\big(n^\kappa\big)$ small closed intervals, denote them by I_j where j runs through J_n, $J_n := \big[1, O(n^\kappa)\big] \cap \mathbf{N}$. We assume that if $i, j \in J_n$ and $i < j$, then $I_i \le I_j$, that is, $x \le y$ for all $x \in I_i, y \in I_j$ and equality holds only if $j = i+1$ and x and y are the only one common point of I_i and I_j provided $I_i \cap I_{i+1} \ne \emptyset$.

Consider the following seven properties of a $J \subset J_n$:

$$H = H(J) := \cup_{j \in J} I_j \text{ is an interval} \tag{I}$$

or the weaker

$$H = H(J) \text{ is the union of at most } k_1 \text{ intervals} \tag{I'}$$

where k_1 is defined above. Frequently we need that $H(J)$ is in a branch of K, that is,

$$H(J) \subset K_i \text{ for some } i. \tag{II}$$

Let $H = H(J)$ be given for some $J \subset J_n$ where H is not necessarily an interval. For each $j \in J_n$ we consider the following small intervals:

- if $j - 1 \in J$, $j \notin J$ and I_{j-1}, I_j are in the same K_i, then I_j,
- if $j + 1 \in J$, $j \notin J$ and I_{j+1}, I_j are in the same K_i, then I_j,
- if $k \in J$ and the right endpoint of I_k coincides with the right endpoint of K_i which is u_{2i}, and $u_{2i} < u_{2i+1}$, then $[u_{2i}, u_{2i} + \lambda_n]$,
- if $k \in J$ and the left endpoint of I_k coincides with the left endpoint of K_i which is u_{2i-1}, and $u_{2i-1} < u_{2i}$, then $[u_{2i-1} - \lambda_n, u_{2i-1}]$.

Denote the union of these intervals by $H_b = H_b(J)$. We think of $H_b(J)$ as the "boundary" of $H(J)$. Since K consists of finitely many intervals, if n is large enough so that (10) is satisfied, then the intervals given in the latter two cases do not overlap with K.

Sometimes we need that H is well inside that K_i, that is,

$$\text{if } H \subset K_i, \text{ then } H_b \cap K \subset K_i. \tag{III}$$

For the polynomial P and $X \subset \mathbf{R}$ we define $A(X) = A_P(X) = A(P, X)$, $B(X) = B_P(X) = B(P, X)$ and $a(X), b(X)$ as follows

$$\begin{aligned} A_P(X) &:= \int_{X\cap K} \Big| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t), \\ B_P(X) &:= \int_{X\cap K} \big|P(t)\big|^\alpha d\nu_K(t), \\ a(X) &:= A_P(X)/A_P(K), \\ b(X) &:= B_P(X)/B_P(K), \end{aligned}$$

and if $\deg(P) = 0$, that is, $P \equiv const$, then set $A(P, X) = 0$ for all X.

If we want to emphasize the polynomial P, then we write $a_P(X) = a(P, X)$, $b_P(X) = b(P, X)$. If $X \cap K = \emptyset$, we set $A_P(X) = B_P(X) = 0$.

If $X \cap Y$ consists of finitely many points (or empty), then $a(X \cup Y) = a(X) + a(Y)$ and $b(X \cup Y) = b(X) + b(Y)$ and the same holds for A and B too. In other words, A, B, a, b are additive, and this is why we do not raise them to the power $1/\alpha$.

Note that $\sum_{j\in J_n} a(I_j) = 1$ and $\sum_{j\in J_n} b(I_j) = 1$.

We also need that most of the $a(I_j)$'s and $b(I_j)$'s tend to 0 simultaneously. Consider the bound $n^{-\gamma}$ where $0 < \gamma < 1$. The next two properties for J and $H_b = H_b(J)$ are

$$a(H_b) < 2n^{-\gamma}, \tag{IV-a}$$

$$b(H_b) < 2n^{-\gamma}. \tag{IV-b}$$

There are at most $\lceil n^\gamma \rceil + \lceil n^\gamma \rceil$ indices j with $a(I_j) > n^{-\gamma}$ or $b(I_j) > n^{-\gamma}$. If

$$\kappa > \gamma, \tag{11}$$

this is few, $O(n^\kappa) - 2\lceil n^\gamma \rceil = O(n^\kappa)\big(1 - o(1)\big)$ where $o(1)$ is obviously independent of P_n. So on most of the intervals, P_n and P_n' are relatively small. In other words, let

$$J_n' := \{j \in J_n : a(I_j), b(I_j) < n^{-\gamma}\} \tag{12}$$

and then

$$|J_n'| = O(n^\kappa) - 2\lceil n^\gamma \rceil = O(n^\kappa)\big(1 - o(1)\big) \tag{13}$$

where $|J_n'|$ denotes the number of indices in $J_n' \subset J_n$. It implies that if n is large, then for each K_i there is a $j \in J_n'$ such that $I_j \subset K_i$. And it also implies that if n is large and

$$\text{if } J \subset J_n,\ J \cap J_n' = \emptyset \text{ and } H(J) \text{ is an interval,}$$
$$\text{then } |H(J)| \le 2\lceil n^\gamma \rceil \lambda_n = O(n^{\gamma-\kappa}) = o(1) \tag{V}$$

where $|H(J)|$ denotes the Lebesgue measure of $H(J) = \cup_{j\in J} I_j$.

We approximate characteristic functions of intervals.

Lemma 6. *Assume that K, H and the "boundary" H_b of H are as above and H is an interval. Denote the characteristic function of H by $\chi_H(t)$. Fix θ, $0 < 2\theta < 1$ with the property that*

$$\theta > 2\kappa. \tag{14}$$

Then there exists $C_2 > 0$ which depends on $\inf K$, $\sup K$ and there exist polynomials $q(t) = q(H, n; t)$ of small degree, $\deg q(H, n; t) \le O(n^{2\theta})$ which satisfy $0 \le q(t) \le 1$ on $[\inf K - 1, \sup K + 1]$, and

$$\big|q(t) - \chi_H(t)\big| \le O\big(\exp(-C_2 n^\theta)\big) \tag{15}$$

$$\big|q'(t)\big| \le O\big(\exp(-C_2 n^\theta)\big) \tag{16}$$

for all n and all $t \in [\inf K - 1, \sup K + 1] \setminus H_b$.

Note that the degree of q and the error estimates depend on n only, and they are independent of $H = H(J)$. The proof of this Lemma is in Section 8. Further, property (14) implies that for large n,

$$n^{-\theta/2} < \lambda_n = c_1 n^{-\kappa}$$

which implies that, roughly speaking, q can increase from 0 to 1 on any of the small intervals I_j.

There will be no further assumptions on κ, γ and θ. For example, $\theta = 1/4$, $\kappa = 1/16$ and $\gamma = 1/32$ is a good choice.

5 The proof of (4) when K is still a real lemniscate but P_n is an arbitrary polynomial

Let $K := r^{-1}[-1, +1]$ be a real lemniscate where r is a (real) admissible polynomial. Denote the interior of K in $\mathbf{R}$ by $\operatorname{Int} K$. Note that there may exist places, where $t \in \operatorname{Int} K$ and $|r(t)| = 1$. Then necessarily $r'(t) = 0$. Notations from the previous Section are as follows: $k_1 = \deg r$, $K = K_1 \cup \ldots \cup K_{\deg r}$ where $K_i = K_{b,i} = [u_{2i-1}, u_{2i}]$, $i = 1, 2, \ldots, \deg r$ and $u_{2i-1} < u_{2i} \le u_{2i+1} < u_{2i+2}$. Recall that $K_{b,i}$ denotes the i-th branch of K. By the admissibility of r, $1 = |r(u_{2i+1})| = |r(u_{2i})|$, $r(u_{2i}) = r(u_{2i+1})$ and $r(u_{2i-1}) = -r(u_{2i})$ and r is strictly monotone on each $K_i = [u_{2i-1}, u_{2i}]$.

Denote the inverse of r restricted to $K_{b,i}$ by r_i^{-1}. That is, if $t \in K_{b,i}$, then $r_i^{-1}\big(r(t)\big) = t$. For the sake of simplicity, we also use the following notation $t_i := r_i^{-1}\big(r(t)\big)$. Note that t_i is a function of t. By elementary calculations, we have

$$\frac{d}{dt} t_i = \big(r_i^{-1}\big(r(t)\big)\big)' = \frac{r'(t)}{r'(t_i)} \ . \tag{17}$$

We use the following form of (7)

$$\omega_K(t) = \frac{1}{\pi \deg r} \frac{\big|r'(t)\big|}{\sqrt{1 - r^2(t)}}, \tag{18}$$

which is well known, see e.g. [7], p. 151, (3.8). It immediately follows that $\omega_K(t) = O(|t - t_0|^{-1/2})$ if $t \to t_0, t \in K$ where $t_0 \in K \setminus \operatorname{Int} K$.

Let $z_1 < z_2 < \ldots < z_{\deg r}$ denote the zeros of r and let $\zeta_1 < \zeta_2 < \ldots < \zeta_{\deg(r)-1}$ denote the zeros of r'.

The ideas of the proof are as follows. We try to find intervals from which we can extend the polynomial periodically (see (19)) so that we can apply the previous case and do something else on the remaining part.

In the first case (see Subsection 5.3), these intervals, which we denote by H, have to be in a "branch" of K (see (II) and (III)). To extract this part, we use special polynomials (the q's) which approximate the characteristic function of H. Near the endpoints of H, where a particular q decays to zero, we have to guarantee that P_n and P_n' are small (see (IV-a), (IV-b)).

In the second case (see Subsection 5.4), if H contains an inner extremal point of r, that is, there is a k_2 such that $|r(\zeta_{k_2})| = 1$ and $r'(\zeta_{k_2}) = 0$, we slightly modify the set and use the first case on this modified set.

In the third case, if H contains a non-inner extremal point of r, that is, there is a k_3 such that $|r(\zeta_{k_3})| = 1$ and $r'(\zeta_{k_3}) \neq 0$, we use the argument as in the first case.

5.1 Symmetrization

Let P be an arbitrary polynomial and assume $H = H(J) \subset K$ satisfies (I), (II), that is, H is an interval ($H \subset K_{b,i_0}$ for some i_0) and $H_b \cap K \subset K_{b,i_0}$. Then using $q(H, \deg(P); t)$ from Lemma 6, we can define

$$P^*(t) = \sum_{i=1}^{\deg r} P\big(t_i\big) \cdot q\big(H, \deg P; t_i\big) \ . \tag{19}$$

This P^* is a polynomial of r, that is, there exists a polynomial p such that $P^*(t) = p\big(r(t)\big)$. And $\deg P^* \leq \big(1 + o(1)\big) \deg P$, where $o(1)$ is independent of P (cf. [4], p. 454). Roughly speaking, P^* is a periodic extension of $P|_H$ to $r^{-1}[-1, 1]$. Note that $\deg(P^*)$ can be much smaller than $\deg(P)$.

The following two lemmas compare the left and the right hand side of (4). Their proofs are in Section 8.

Lemma 7. *Using the setting described above, assume that we have an admissible polynomial r and the set $K = r^{-1}[-1, 1]$ and an arbitrary polynomial P. We also have a set $H = H(J) \subset K$ and its "boundary" H_b satisfying* (I), (II) *and* (IV-a). *We allow that $H_b \not\subset K$, but we assume* (III). *Then, for P^* from* (19) *defined for P, we have*

$$\begin{aligned}
\Bigg| \left(\int_K \left| \frac{\big(P^*\big)'(t)}{\deg(P)\,\pi\,\omega_K(t)} \right|^\alpha d\nu_K(t) \right)^{1/\alpha} & \\
- (\deg r)^{1/\alpha} \left(\int_H \left| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \right|^\alpha d\nu_K(t) \right)^{1/\alpha} \Bigg| & \\
\leq o(1) \left(\int_K \left| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \right|^\alpha d\nu_K(t) \right)^{1/\alpha} & \\
+ o(1) \left(\int_K \big|P(t)\big|^\alpha d\nu_K(t) \right)^{1/\alpha} &
\end{aligned} \tag{20}$$

where $o(1)$ tends to 0 as $\deg(P) \to \infty$ and depends on α and $\deg(r)$ but is independent of P. Or, for short

$$\Big|\frac{\deg(P^*)}{\deg(P)} A_{P^*}^{1/\alpha}(K) - (\deg r)^{1/\alpha} A_P^{1/\alpha}(H)\Big| \le o(1) A_P^{1/\alpha}(K) + o(1) B_P^{1/\alpha}(K). \quad (21)$$

We also need its power-free version

$$\Big|\Big(\frac{\deg(P^*)}{\deg(P)}\Big)^\alpha A(P^*,K) - \deg(r) A(P,H)\Big| \\ \le o(1)A(P,K) + o(1)B(P,K). \quad (22)$$

It is important to note that the $o(1)$'s on the right hand side do not depend on H directly, only through $a(P, H_b)$, see (68). Also note that (20), (21) and (22) hold even if $\deg(P^*) = 0$. In this case, the Lemma simply states that $A(P,H)$ is small.

Lemma 8. *With the same assumptions as in the previous Lemma, except that we need* (IV-b) *instead of* (IV-a), *we have*

$$\Bigg|\bigg(\int_K \big|P^*(t)\big|^\alpha d\nu_K(t)\bigg)^{1/\alpha} - (\deg r)^{1/\alpha} \bigg(\int_H \big|P(t)\big|^\alpha d\nu_K(t)\bigg)^{1/\alpha}\Bigg| \\ \le o(1) \bigg(\int_K \big|P(t)\big|^\alpha d\nu_K(t)\bigg)^{1/\alpha}.$$

Or, with the short notation

$$\big|B_{P^*}^{1/\alpha}(K) - (\deg r)^{1/\alpha} B_P^{1/\alpha}(H)\big| \le o(1) B_P^{1/\alpha}(K) \quad (23)$$

and its power-free version is

$$\big|B(P^*,K) - \deg(r) B(P,H)\big| \le o(1) B(P,K). \quad (24)$$

It is important to note that the $o(1)$'s on the right hand side do not depend on H directly, only through $b(P, H_b)$, see (77).

5.2 Splitting into three cases

Let $K = r^{-1}[-1,1]$ be a real lemniscate and $K_{b,1}, \ldots, K_{b,\deg r}$ be its branches as above and P_n be an arbitrary polynomial with degree n. We use the intervals I_j's introduced in Section 4 as well as J_n, J'_n.

For each $K_{b,i}$, $i = 1, 2, \ldots, \deg r$, let $k_{l,i} := \min\{j \in J'_n : I_j \subset K_{b,i}\}$ and $k_{r,i} := \max\{j \in J'_n : I_j \subset K_{b,i}\}$. With these particular indices $k_{l,i}$, $k_{r,i}$, l and r in the subscripts mean left and right hand sides.

First, let $J_{1,i} := [k_{l,i} + 1, k_{r,i} - 1] \cap \mathbf{N}$ and let $P_{n,1,i}(t) := P_n(t)$. This is called the first case and is discussed in Subsection 5.3 with the notations: $P(t) = P_{n,1,i}(t)$, $J = J_{1,i}$, $H = H(J)$. If n is large, then $J_{1,i} \ne \emptyset$, see (10).

The second case is the following. If $1 \le i < \deg r$ is such that $K_{b,i} \cap K_{b,i+1} \ne \emptyset$, then let $J_{2,i} := [k_{r,i} + 1, k_{l,i+1} - 1] \cap \mathbf{N}$. If $k_{r,i} + 1 > k_{l,i+1} - 1$, then $J_{2,i} = \emptyset$, and there is nothing to be done.

If $J_{2,i} \neq \emptyset$, then let k_2 and ζ_{k_2} be such that $\{\zeta_{k_2}\} = K_{b,i} \cap K_{b,i+1}$ and let $P_{n,2,i}(t) := P_n(t)$. This case is discussed in Subsection 5.4 with $P(t) = P_{n,2,i}(t)$, $J = J_{2,i}$, $H = H(J)$.

Third, if i is such that $K_{b,i}$ and $K_{b,i+1}$ are disjoint, $1 \leq i < \deg(r)$, then let $J_{3,i,r} := \{j \in J_n : I_j \subset K_{b,i}, j > k_{r,i}\}$ and $J_{3,i+1,l} := \{j \in J_n : I_j \subset K_{b,i+1}, j < k_{l,i+1}\}$. And let $J_{3,1,l} := \{j \in J_n : j < k_{l,1}\}$, $J_{3,\deg(r),r} := \{j \in J_n : j > k_{r,\deg(r)}\}$. With these particular sets, the third subscripts l and r refer to left and right end of the branch.

If $J_{3,i,r} \neq \emptyset$, then let $P_{n,3,i,r}(t) := P_n(t)$ and this case is discussed in Subsection 5.3 with $P(t) = P_{n,3,i,r}(t)$, $J = J_{3,i,r}$, $H = H(J)$. Similarly, if $J_{3,i,l} \neq \emptyset$, then let $P_{n,3,i,l}(t) := P_n(t)$ and this case is discussed in Subsection 5.3 with $P(t) = P_{n,3,i,l}(t)$, $J = J_{3,i,l}$, $H = H(J)$.

For completeness, let $J_{2,i} := \emptyset$ if $K_{b,i} \cap K_{b,i+1} = \emptyset$, and let $J_{3,i,r} = J_{3,i+1,l} := \emptyset$ if $K_{b,i} \cap K_{b,i+1} \neq \emptyset$ and let $J_{2,\deg(r)} := \emptyset$.

Note that, in all these cases, $H_b \cap K$ is one interval or union of two nondegenerate (consisting of infinitely many numbers) closed intervals.

Let

$$\mathcal{J} := \{J_{1,i} : 1 \leq i \leq \deg(r), J_{1,i} \neq \emptyset\} \cup \{J_{2,i} : 1 \leq i \leq \deg(r), J_{2,i} \neq \emptyset\} \cup \{J_{3,i,l} : 1 \leq i \leq \deg(r), J_{3,i,l} \neq \emptyset\} \cup \{J_{3,i,r} : 1 \leq i \leq \deg(r), J_{3,i,r} \neq \emptyset\}.$$

Obviously,

$$\text{if } J^1, J^2 \in \mathcal{J}, \text{ then } J^1 = J^2, \text{ or } J^1 \cap J^2 = \emptyset, H(J^1) \cap H(J^2) = \emptyset \tag{25}$$

and

$$|\mathcal{J}| \leq 4\deg(r). \tag{26}$$

Note that there are at most $2\deg(r)$ small intervals I_j with $j \in J_n'$ which are not covered by $H(\cup\mathcal{J})$, that is,

$$|J_n \setminus \cup\mathcal{J}| \leq 2\deg(r) \tag{27}$$

and, by construction, if $j \in J_n \setminus \cup\mathcal{J}$, then $j \in J_n'$, so with (12),

$$a(I_j), b(I_j) < n^{-\gamma} = o(1) \quad \text{for all } j \in J_n \setminus \cup\mathcal{J}. \tag{28}$$

Obviously, if $J \in \mathcal{J}$, then $\deg(P) = \deg(P_n(.)q(H(J), n; .)) = n + O(n^{2\theta}) = (1 + o(1))n$ where $o(1)$ here is independent of K and P_n.

5.3 The first and the third cases

In these two cases, we have a polynomial $P = P_n$ and a set $J \subset J_n$ such that $H = H(J)$ satisfies (I), (II), (III) and (IV-a)-(IV-b), that is, H is an interval, $H \subset K_{b,i_0}$ for some i_0 and P and P' are small on $H_b \cap K$, that is, $a_P(H_b), b_P(H_b) < 2n^{-\gamma}$. We use the polynomial P^* defined in Subsection 5.1 for P. For now, we assume $\deg(P^*) > 0$. We discuss the situation $\deg(P^*) = 0$ at the end of this subsection.

From Lemmas 7, 8, we know that the error terms are ”small”, so, instead, let us write just ”error terms” for now.

$$
\begin{aligned}
(\deg r) \int_H \Big| \frac{P'(t)}{\deg(P)\, \pi\omega_K(t)} \Big|^\alpha d\nu_K(t) + \text{ error terms} \\
\le (\frac{\deg(P^*)}{\deg(P)})^\alpha \int_K \Big| \frac{\big(P^*\big)'(t)}{\deg(P^*)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \\
\le (\frac{\deg(P^*)}{\deg(P)})^\alpha \int_K \big|P^*(t)\big|^\alpha d\nu_K(t) \\
\le (\deg r)(\frac{\deg(P^*)}{\deg(P)})^\alpha \int_H \big|P(t)\big|^\alpha d\nu_K(t) + \text{ error terms},
\end{aligned}
$$

where at the first inequality Lemma 7 is used, at the second inequality the asymptotic Bernstein inequality in the case when the polynomial (here P^*) is polynomial of r (which is the case now), at the third inequality, Lemma 8. This way we obtain

$$
\begin{aligned}
A(P, H) = \int_H \Big| \frac{P'(t)}{\deg(P)\, \pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \\
\le \left(\frac{\deg(P^*)}{\deg(P)}\right)^\alpha \int_H \big|P(t)\big|^\alpha d\nu_K(t) \quad + \text{ error terms } \cdot \frac{1}{\deg r}\left(\frac{\deg(P^*)}{\deg(P)}\right)^\alpha \\
= \left(\frac{\deg(P^*)}{\deg(P)}\right)^\alpha B(P, H) + \text{ error terms } \cdot \frac{1}{\deg r}\left(\frac{\deg(P^*)}{\deg(P)}\right)^\alpha .
\end{aligned}
$$

As for the error terms, $\deg r$ is fixed and $\deg(P^*) \le \big(1 + o(1)\big) \deg(P)$, so

$$
\frac{1}{\deg r}\left(\frac{\deg(P^*)}{\deg(P)}\right)^\alpha = O(1), \tag{29}
$$

therefore

$$
\begin{aligned}
\text{error terms } \cdot \frac{1}{\deg r}\left(\frac{\deg(P^*)}{\deg(P)}\right)^\alpha \\
= o(1) \int_K \Big| \frac{P'(t)}{\deg(P)\, \pi\omega_K(t)} \Big|^\alpha d\nu_K(t) + o(1) \int_K \big|P(t)\big|^\alpha d\nu_K(t).
\end{aligned}
$$

So we obtain with $P = P_n$ in this case

$$
A(P_n, H) \le (1 + o(1))B(P_n, H) \ + o(1)A(P_n, K) + o(1)B(P_n, K) \tag{30}
$$

where $o(1)$ tends to 0 as $\deg(P_n) \to \infty$ and depends on α (and on K and $\deg r$, of course) but is independent of P_n.

It is worth noting that the $o(1)$ error term of $A(P_n, K)$ in (30) depends on the set H through $a(P_n, H_b)$ (see (75)), and similarly, the $o(1)$ error term of $B(P_n, K)$ depends on the set H through $b(P_n, H_b)$ (see (80)).

If $\deg(P^*) = 0$, then, by definition, $A(P^*, H) = 0$ and $\int_H \big| \frac{(P^*)'(t)}{\deg(P)\pi\omega_K(t)} \big|^\alpha d\nu_K(t) = 0$ and with (22),

$$
A(P_n, H) \le o(1)A(P_n, K) + o(1)B(P_n, K).
$$

Increasing the right hand side with $B(P_n, H)$, we immediately have (30) in this situation too.

5.4 The second case

In this case, we investigate the polynomial P near an inner extremal point of K which we denote by ζ_{k_2}, $\zeta_{k_2} \in \operatorname{Int} K$, $|r(\zeta_{k_2})| = 1$. In other words, $H = H(J)$ is such that H is an interval (see (I)), at the endpoints, P and P' are small (see (IV-a)-(IV-b)), H is minimal, that is, there is no smaller interval with these properties, and H intersects K_{b,i_2} and K_{b,i_2+1} for some i_2. Since H is minimal, the Lebesgue measure of H is smaller than $O(n^{\gamma-\kappa}) = o(1)$, see (V).

Let ℓ_2 and K_{c,ℓ_2} be fixed such that $\zeta_{k_2} \in K_{c,\ell_2}$. Recall that K_{c,ℓ_2} denotes the ℓ_2-th component (interval) of K.

5.4.1 Deforming the set

Let us write K in the form

$$K = \cup_{\ell=1}^{\ell_1} [v_{2\ell-1}, v_{2\ell}], \quad v_1 < v_2 < v_3 < v_4 < \ldots < v_{2\ell_1}, \tag{31}$$

where $K_{c,\ell} = [v_{2\ell-1}, v_{2\ell}]$ and $\zeta_{k_2} \in K_{c,\ell_2}$.

Proposition 9. *For every small $\delta > 0$ there exists an admissible polynomial $\tilde{r}$ with $\tilde{K}(\delta) = \tilde{r}^{-1}[-1,1]$ such that*

$$\deg \tilde{r} = \deg r \tag{32}$$

and

$$\tilde{K}(\delta) = [v_1, \tilde{v}_2(\delta)] \cup \ldots \cup [v_{2\ell_1-1}, \tilde{v}_{2\ell_1}(\delta)], \\ \text{where } v_1 < \tilde{v}_2(\delta) < \ldots < v_{2\ell_1-1} < \tilde{v}_{2\ell_1}(\delta) \tag{33}$$

so that $\tilde{v}_{2\ell} = \tilde{v}_{2\ell}(\delta)$ are continuous functions of δ for all ℓ and $\tilde{v}_{2\ell}(\delta)$'s are strictly increasing as δ decreases to 0 and $\tilde{v}_{2\ell}(\delta) \to v_{2\ell}$ for all ℓ as $\delta \to 0$.

Furthermore,

$$\nu_{\tilde{K}(\delta)}([v_{2\ell-1}, \tilde{v}_{2\ell}(\delta)]) = \nu_K([v_{2\ell-1}, v_{2\ell}]), \qquad \ell = 1, \ldots, \ell_1. \tag{34}$$

This is essentially Corollary 11 and using Lemma 12 in [8].

With the previous proposition, we can assume that

$$\tilde{v}_{2\ell_2}(\delta) = v_{2\ell_2} - \delta. \tag{35}$$

For notational simplicity, we use both sides of (35) in the following.

Denote the zeros of $\tilde{r}'(\delta; .)$ by $\tilde{\zeta}_1(\delta) < \ldots < \tilde{\zeta}_{\deg(r)-1}(\delta)$ and the density of the equilibrium measure $\nu_{\tilde{K}(\delta)}(.)$ by $\omega_{\tilde{K}(\delta)}(t)$.

Proposition 10. *Using the notations introduced so far,*

$$\tilde{r}(\delta; t) \to r(t), \\ \tilde{r}'(\delta; t) \to r'(t),$$

as $\delta \to 0$ for every $t \in \mathbf{R}$, and

$$\tilde{\zeta}_\ell(\delta) \to \zeta_\ell \text{ for all } \ell = 1, \ldots, \deg(r) - 1.$$

Furthermore, for all closed set $X \subset \operatorname{Int} K$, as $\delta \to 0$,

$$\frac{\omega_{\tilde{K}(\delta)}(t)}{\omega_K(t)} \to 1 \text{ uniformly in } t \in X, \tag{36}$$

and there exists $C_3 > 0$ depending on K only such that for all small $\delta > 0$ and all ℓ such that $\zeta_\ell \in K_{c,\ell_2}$, we have

$$|\tilde{\zeta}_\ell(\delta) - \zeta_\ell| > C_3\delta \tag{37}$$

where $K_{c,\ell}$'s are the connected components of K.

Proof. First, we prove that $\tilde{r}(\delta; t) \to r(t)$. The previous proposition implies that $\nu_{\tilde{K}(\delta)} \to \nu_K$ in weak-star sense. Obviously,

$$U(\delta; z) := \int \log(z - t) d\nu_{\tilde{K}(\delta)}(t) \to U(z) := \int \log(z - t) d\nu_K(t) \tag{38}$$

pointwise for all $z \in \mathbf{C} \setminus K$. Since $\log(z - t)$ is equicontinuous away from K, this convergence is locally uniform away from K. Furthermore, $r(z)$ and $\tilde{r}(\delta; z)$ can be written as

$$r(z) = \cosh\left((\deg r)(U(z) - \log \operatorname{cap} K)\right)$$
$$\tilde{r}(\delta; z) = \cosh\left((\deg r)(U(\delta; z) - \log \operatorname{cap} \tilde{K}(\delta))\right)$$

for all $z \in \mathbf{C}$, see [7], p. 142. Therefore $\tilde{r}(\delta; z) \to r(z)$ locally uniformly away from K, and, since $\tilde{r}(\delta; z)$ and $r(z)$ are polynomials with the same degree, $\tilde{r}(\delta; z) \to r(z)$ pointwise everywhere on $\mathbf{C}$.

Again, using that $\tilde{r}(\delta; z)$ and $r(z)$ are polynomials with the same degree, we immediately have $\tilde{r}'(\delta; z) \to r'(z)$ and $\tilde{\zeta}_\ell(\delta) \to \zeta_\ell$ for all ℓ. Since near the inner extremal point ζ_ℓ of r, we have $0/0$ limit, so we rather use the following equation which follows from (18).

$$\frac{\omega_{\tilde{K}(\delta)}(t)}{\omega_K(t)} = \frac{\left|\frac{\tilde{r}'_\delta(t)}{t - \tilde{\zeta}_\ell(\delta)}\right|}{\left|\frac{r'(t)}{t - \zeta_\ell}\right|} \cdot \frac{\sqrt{\frac{1 - r^2(t)}{(t - \zeta_\ell)^2}}}{\sqrt{\frac{1 - \tilde{r}_\delta^2(t)}{(t - \tilde{\zeta}_\ell(\delta))^2}}}$$

Obviously, all terms have nonzero limit, so we obtain (36).

In order to prove (37), we use the density of the balayage from [7], p. 144. With our notations, the density at $s \in \tilde{K}(\delta)$ of the balayage of the Dirac delta at $t \in (\tilde{v}_{2\ell_2}(\delta), v_{2\ell_2+1})$ onto $\tilde{K}(\delta)$ can be expressed as

$$\operatorname{Bal}(s, \tilde{K}(\delta); t) = \frac{1}{\pi} \frac{\prod_{\ell=1}^{\ell_1} |(t - v_{2\ell-1})(t - \tilde{v}_{2\ell}(\delta))|^{1/2}}{\prod_{\ell=1}^{\ell_1} |(s - v_{2\ell-1})(s - \tilde{v}_{2\ell}(\delta))|^{1/2}} \left|\frac{R(\delta, t; s)}{R(\delta, t; t)}\right| \frac{1}{|t - s|} \tag{39}$$

where $R(\delta, t; s)$ is a certain polynomial. For further properties of $R(\delta, t; s)$, we refer to [7], p. 144, but all we need is that it is a monic polynomial with degree $\ell_1 - 1$ and it has exactly one zero in each $(\tilde{v}_{2\ell}(\delta), v_{2\ell+1})$, $\ell = 1, \ldots, \ell_1 - 1$, $\ell \neq \ell_2$ and one in $(-\infty, v_1) \cup (\tilde{v}_{2\ell_1}(\delta), \infty)$, and $R(\delta, t; s) \to R(0, t; s)$ as $\delta \to 0$ and the zeros of $R(\delta, t; .)$ depend continuously on δ and t. It implies that there exist

$C_4(K), C_5(K) > 0$ such that for all small $\delta > 0$ and all possible polynomials $R(.;.)$ as above, we have for all $t \in [v_{2\ell_2} - \delta/2, v_{2\ell_2}]$ and $s \in K_{c,\ell_2} = [v_{2\ell_2-1}, v_{2\ell_2}]$

$$C_4 < \Big| \frac{R(\delta,t;s)}{R(\delta,t;t)} \Big| < C_5$$

because we use the above mentioned properties and these functions are polynomials with fixed degree and their zeros stay at a positive distance from K_{c,ℓ_2} and $R(\delta,t;s)$ converges as $\delta \to 0$.

We are going to prove that for all y_1, y_2, $v_{2\ell_2-1} < y_1 < y_2 < v_{2\ell_2}$, there exists $C_6(K, y_1, y_2) > 0$ such that for all $x \in [y_1, y_2]$ and all small $\delta > 0$

$$\frac{\nu_{\tilde{K}(\delta)}([v_{2\ell_2-1}, x]) - \nu_K([v_{2\ell_2-1}, x])}{\delta} \geq C_6. \tag{40}$$

Actually, we will specify y_1 and y_2 later. Since $\nu_{\tilde{K}(\delta)}$ is the balayage of ν_K onto $\tilde{K}(\delta)$, and using the properties of balayage,

$$\begin{aligned}
&\frac{\nu_{\tilde{K}(\delta)}([v_{2\ell_2-1}, x]) - \nu_K([v_{2\ell_2-1}, x])}{\delta} \\
&\qquad \geq \frac{1}{\delta} \int_{v_{2\ell_2-1}}^{x} \int_{v_{2\ell_2}-\delta}^{v_{2\ell_2}} \mathrm{Bal}(s, \tilde{K}(\delta), t)\omega_K(t) dt ds \\
&\qquad\qquad \geq \frac{1}{\delta} \int_{v_{2\ell_2-1}}^{x} \int_{v_{2\ell_2}-\delta/2}^{v_{2\ell_2}} \mathrm{Bal}(s, \tilde{K}(\delta), t)\omega_K(t) ds dt.
\end{aligned}$$

Using (18) and the formula (39) for the balayage, we can write

$$\mathrm{Bal}(s, \tilde{K}(\delta), t)\omega_K(t) = \frac{1}{\sqrt{|(s - v_{2\ell_2-1})(s - v_{2\ell_2} + \delta)|}} \sqrt{\Big|\frac{t - v_{2\ell_2} + \delta}{t - v_{2\ell_2}}\Big|} \frac{1}{|t-s|} F(t,s)$$

where $F(t,s)$ is a suitable positive, continuous function if $v_{2\ell_2-1} < y_1 \leq x \leq y_2 < v_{2\ell_2}$, $v_{2\ell_2-1} \leq s \leq x$, $y_2 < v_{2\ell_2} - \delta/2 \leq t \leq v_{2\ell_2}$ and is bounded from above and below by some positive constants depending on K and y_1, y_2.

Now we integrate with respect to t, and the last two terms are bounded from below (and above) and the first one does not depend on t, all we have to use is

$$\int_{v_{2\ell_2}-\delta/2}^{v_{2\ell_2}} \sqrt{\Big|\frac{t - v_{2\ell_2} + \delta}{t - v_{2\ell_2}}\Big|} dt = \delta \frac{2 + \pi}{4}.$$

And now we integrate with respect to s, and use that there exists $C_7 = C_7(K, y_1, y_2) > 0$ such that for all $v_{2\ell_2-1} < y_1 \leq x \leq y_2$

$$\int_{v_{2\ell_2-1}}^{x} \frac{1}{\sqrt{|(s - v_{2\ell_2-1})(s - v_{2\ell_2} + \delta)|}} ds > C_7.$$

This way we obtain that (40) holds.

There is another representation of r and $\tilde{r}$: for all $x \in \tilde{K}(\delta) \subset K$

$$\begin{aligned}
r(x) &= \cos\big(\deg(r)\pi\nu_K([x,\infty))\big), \\
\tilde{r}(\delta; x) &= \cos\big(\deg(r)\pi\nu_{\tilde{K}(\delta)}([x,\infty))\big)
\end{aligned}$$

see [7], p.142, second to last displayed formula. So, using property (34), $\tilde{r}(\delta;x) = \cos\big(-\deg(r)\pi\nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},x)) - C_8\big) = \cos\big(\deg(r)\pi\nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},x)) + C_8\big)$ for some $C_8 \in \mathbf{R}$, and $\tilde{r}'(\delta;x) = 0$ if and only if $\nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},x)) = (k - C_8/\pi)/\deg(r)$ for some integer k. (Note that $d/dx\, \nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},x)) = \omega_{\tilde{K}(\delta)}(x)$ which is strictly positive on $\tilde{K}(\delta)$.)

Now let $y_1 := \frac{1}{2}(v_{2\ell_2-1} + \min\{\zeta_\ell : \zeta_\ell \in K_{c,\ell_2}\})$ and $y_2 := \frac{1}{2}(v_{2\ell_2} + \max\{\zeta_\ell : \zeta_\ell \in K_{c,\ell_2}\})$. By Proposition 9, if $\delta > 0$ is small, then for all ℓ if $\tilde{\zeta}_\ell(\delta) \in K_{c,\ell_2}$, then $\tilde{\zeta}_\ell(\delta) \in [y_1, y_2]$.

$$\begin{aligned}\int_{\tilde{\zeta}_\ell(\delta)}^{\zeta_\ell} \omega_{\tilde{K}(\delta)}(x)dx = \int_{\tilde{\zeta}_\ell(\delta)}^{\zeta_\ell} \frac{d}{dx}\nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},x))dx \\ = \nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},\zeta_\ell)) - \nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},\tilde{\zeta}_\ell(\delta)) \\ = \nu_{\tilde{K}(\delta)}([v_{2\ell_2-1},\zeta_\ell)) - \nu_K([v_{2\ell_2-1},\zeta_\ell)), \quad (41)\end{aligned}$$

where in the last equality we used that $\nu_{\tilde{K}(\delta)}([v_{2l_2-1},\tilde{\zeta}_l(\delta)))$ remains constant if δ is small (see the above remark). Now $\omega_{\tilde{K}(\delta)}(x)$ is bounded from above on $[\tilde{\zeta}_l(\delta),\zeta_l]$ by a constant independent of small δ, say $\omega_{\tilde{K}(\delta)}(x) \le C_9$ on $x \in [\tilde{\zeta}_l(\delta),\zeta_l]$. Thus, the leftmost term in (41) is less then or equal to $C_9(\zeta_l - \tilde{\zeta}_l(\delta))$, while the rightmost term can be estimated from below by $C_6\delta$ according to (40). This gives (37) and completes the proof of the proposition.

□

5.4.2 Proving the inequality on the deformed set

On the fixed set

$$\Big[\frac{v_{2\ell_2-1}+\zeta_{k_2}}{2}, \frac{v_{2\ell_2}+\zeta_{k_2}}{2}\Big]$$

denote the supremum of

$$\Big|\frac{\omega_{\tilde{K}(\delta)}(t)}{\omega_K(t)} - 1\Big|$$

by $\delta_1 = \delta_1(\delta)$. From (36), if $\delta \to 0$, then $\delta_1 \to 0$.

For given $\delta > 0$, taking account of (IV-a)-(IV-b), (V) and (37), if the degree n of the original polynomial P_n satisfies

$$\big(2\lceil n^\gamma\rceil + 2\big)\frac{c_1}{n^\kappa} < C_3\delta \quad (42)$$

then $H(J) \cup H_b(J)$ does not contain any extremal point of $\tilde{r}(\delta;.)$.

Recall the notations

$$A(P,H) = \int_{H\cap K}\Big|\frac{P'(t)}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t),$$
$$\tilde{A}_\delta(P,H) := \int_{H\cap\tilde{K}(\delta)}\Big|\frac{P'(t)}{\deg(P)\pi\omega_{\tilde{K}(\delta)}(t)}\Big|^\alpha d\nu_{\tilde{K}(\delta)}(t)$$

and, $B(P,H)$, $\tilde{B}_\delta(P,H)$ are defined similarly.

We also have to introduce a bigger interval $\tilde{H}$ as follows. Let $\tilde{J} = \tilde{J}(n,J) \subset J_n$ be the smallest set such that $\tilde{H} := H(\tilde{J})$ is an interval, $H(J) \cup H_b(J) \subset \tilde{H}$

and $\tilde{H}_b \subset H(J'_n)$. Such set $\tilde{J}$ and interval $\tilde{H}$ exist, and the length of $\tilde{H}$ tends to 0 as n tends to infinity. This follows from (V) and the remarks after that. Furthermore, let $\tilde{q} := q(\tilde{H}, n; t)$.

We need the following Lemma whose proof is in Section 8.

Lemma 11. *Using the notations introduced above, if $X \subset \tilde{H}$ is an interval, then*

$$\big|A(P_n\tilde{q}, X) - A(P_n, X)\big| \le o(1)A(P_n, K) + o(1)B(P_n, K), \tag{43}$$

$$\big|B(P_n\tilde{q}, X) - B(P_n, X)\big| \le o(1)B(P_n, K), \tag{44}$$

$$\big|\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) - \tilde{A}_\delta(P_n\tilde{q}, \tilde{H})\big| \le o(1)A(P_n, K) + o(1)B(P_n, K), \tag{45}$$

$$\big|\tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) - \tilde{B}_\delta(P_n\tilde{q}, \tilde{H})\big| \le o(1)B(P_n, K), \tag{46}$$

where the $o(1)$ error terms do not depend on X.

We start our estimate. First, we use (43) with $X = H$, then use the definition of δ_1,

$$\begin{aligned} A(P_n, H) &\le A(P_n\tilde{q}, H) + o(1)A(P_n, K) + o(1)B(P_n, K) \\ &\le (1+\delta_1)^{\alpha-1}\tilde{A}_\delta(P_n\tilde{q}, H) \ + o(1)A(P_n, K) + o(1)B(P_n, K) = \end{aligned} \tag{47}$$

Now we want to use case one for the polynomial $P_n\tilde{q}$ for H on the set $\tilde{K}(\delta)$. We know that the $o(1)$ error terms depend on the set H through (68) and (77).

Properties (II) and (III) are satisfied because of (42). So showing this dependence

$$\begin{aligned} \tilde{A}_\delta(P_n\tilde{q}, H) \le &\big(1 + o(1)\big)\tilde{B}_\delta(P_n\tilde{q}, H) \\ &+ o(1)\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) + o(1)\tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) \\ &+ C_{10}\tilde{a}_\delta(P_n\tilde{q}, H_b)\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) + C_{11}\tilde{b}_\delta(P_n\tilde{q}, H_b)\tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) \end{aligned} \tag{48}$$

with some constants $C_{10}, C_{11} > 0$, see (29), (75) and (80).

We apply (43) twice, with the two intervals of which H_b consists, we can write

$$\begin{aligned} \tilde{a}_\delta(P_n\tilde{q}, H_b)\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) &= \tilde{A}_\delta(P_n\tilde{q}, H_b) \le (1-\delta_1)^{1-\alpha}A(P_n\tilde{q}, H_b) \\ &\le (1-\delta_1)^{1-\alpha}A(P_n, H_b) \ + o(1)A(P_n, K) + o(1)B(P_n, K) \\ &= (1-\delta_1)^{1-\alpha}a(P_n, H_b)A(P_n, K) \ + o(1)A(P_n, K) + o(1)B(P_n, K) \end{aligned}$$

and

$$C_{10}\tilde{a}_\delta(P_n\tilde{q}, H_b)\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) = o(1)A(P_n, K) + o(1)B(P_n, K).$$

Roughly speaking, if P_n is small on H_b with respect to K, then $P_n\tilde{q}$ is small on the same H_b with respect to $\tilde{K}(\delta)$.

Similarly for B, applying (44) twice, with $X = H_b$, we can write

$$\begin{aligned} \tilde{b}_\delta(P_n\tilde{q}, H_b)\tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) &= \tilde{B}_\delta(P_n\tilde{q}, H_b) \le (1+\delta_1)B(P_n\tilde{q}, H_b) \\ &= (1+\delta_1)B(P_n, H_b) \ + o(1)B(P_n, K) \\ &= (1+\delta_1)b(P_n, H_b)B(P_n, K) \ + o(1)B(P_n, K) \end{aligned}$$

and

$$C_{11}\tilde{b}_\delta(P_n\tilde{q},H_b)\tilde{B}_\delta(P_n\tilde{q},\tilde{K}(\delta))=o(1)B(P_n,K).$$

These imply that the error term is uniform in $\delta>0$. We can also use the following two estimates: first (45), definition of δ_1 and (43) with $X=\tilde{H}$,

$$\begin{aligned}\tilde{A}_\delta(P_n\tilde{q},\tilde{K}(\delta)) &\le \tilde{A}_\delta(P_n\tilde{q},\tilde{H})+o(1)A(P_n,K)+o(1)B(P_n,K)\\ &\le (1-\delta_1)^{1-\alpha}A(P_n\tilde{q},\tilde{H})+o(1)A(P_n,K)+o(1)B(P_n,K)\\ &\le (1-\delta_1)^{1-\alpha}A(P_n,\tilde{H})+o(1)A(P_n,K)+o(1)B(P_n,K)\\ &\le (1-\delta_1)^{1-\alpha}A(P_n,K)+o(1)A(P_n,K)+o(1)B(P_n,K). \quad (49)\end{aligned}$$

Similarly for $\tilde{B}_\delta(P_n\tilde{q},\tilde{K}(\delta))$: we use first (46), the definition of δ_1, $|\tilde{q}|\le 1$, and the monotonicity of B in both variables:

$$\begin{aligned}\tilde{B}_\delta(P_n\tilde{q},\tilde{K}(\delta)) &\le \tilde{B}_\delta(P_n\tilde{q},\tilde{H})+o(1)B(P_n,K)\\ &\le (1+\delta_1)B(P_n\tilde{q},\tilde{H})+o(1)B(P_n,K)\\ &\le (1+\delta_1)B(P_n,\tilde{H})+o(1)B(P_n,K)\\ &\le (1+\delta_1)B(P_n,K)+o(1)B(P_n,K). \quad (50)\end{aligned}$$

Now we use (48) and then (44) with $X=H$, for the first error term on the right of (48) we use (49), for the second error term we use (50), and for the third and fourth error terms we use the four unnumbered displayed formulas. So we continue (47) as

$$\begin{aligned}&=(1+\delta_1)^{\alpha-1}\tilde{A}_\delta(P_n\tilde{q},H)\ +o(1)A(P_n,K)+o(1)B(P_n,K)\\ &\le (1+\delta_1)^{\alpha-1}(1+o(1))\tilde{B}_\delta(P_n\tilde{q},H)\ +o(1)A(P_n,K)+o(1)B(P_n,K)\\ &\le (1+\delta_1)^{\alpha-1}(1+\delta_1)(1+o(1))B(P_n\tilde{q},H)\ +o(1)A(P_n,K)+o(1)B(P_n,K)\\ &\le (1+\delta_1)^{\alpha}(1+o(1))B(P_n,H)\ +o(1)A(P_n,K)+o(1)B(P_n,K).\end{aligned}$$

So we obtained in this case that

$$A(P_n,H)\le(1+\delta_1)^\alpha(1+o(1))B(P_n,H)+o(1)A(P_n,K)+o(1)B(P_n,K),$$

that is,

$$A(P_n,H)\le\big(1+o(1)\big)B(P_n,H)+\ o(1)A(P_n,K)+o(1)B(P_n,K). \quad (51)$$

5.5 Putting these cases together

We use the notations introduced in Subsection 5.2. For all $J^1,J^2\in\mathcal{J}$ we know (25), the additivity of $A(.)$ and $B(.)$, and for all $J\in\mathcal{J}$ case one, case two or case three holds, so we have (30) or (51).

Therefore, with (26), we can write

$$\begin{aligned}A(P_n,H(\cup\mathcal{J}))\le&\big(1+o(1)\big)B(P_n,H(\cup\mathcal{J}))\\ &+4\deg(r)o(1)B(P_n,K)+4\deg(r)o(1)A(P_n,K).\end{aligned}$$

And, if $j\in J_n\setminus\cup\mathcal{J}$, then by (27) and (28),

$$\begin{aligned}A\big(P_n,H(J_n\setminus\cup\mathcal{J})\big)&\le\deg(r)2n^{-\gamma}A(P_n,K)=o(1)A(P_n,K),\\ B\big(P_n,H(J_n\setminus\cup\mathcal{J})\big)&\le\deg(r)2n^{-\gamma}B(P_n,K)=o(1)B(P_n,K),\end{aligned}$$

so adding up these, we obtain (4) on real lemniscates for arbitrary polynomials.

6 Finitely many intervals

Now let K consist of arbitrary finitely many intervals, denote them by $K_{c,1},\dots,K_{c,\ell_1}$. That is, $K = K_{c,1}\cup\ldots\cup K_{c,\ell_1}$ and $K_{c,m_1}\cap K_{c,m_2}=\emptyset$ if $m_1\neq m_2$, where $K_{c,i}=[v_{2i-1},v_{2i}]$.

We decompose K into smaller intervals as constructed in Section 5.2. Further, we use the introduced J_n, $a(P_n,H)$, $b(P_n,H)$ notations for K and the arbitrarily fixed P_n.

Let $J_n' = \{j\in J_n : a(I_j), b(I_j) < n^{-\gamma}\}$ as introduced in Subsection 5.5. If n is large enough, then J_n' has lots of elements, and for all $i=1,\dots,\ell_1$

$$\text{there exists } k(i)\in J_n' \quad \text{such that } I_{k(i)}\subset[(2/3)v_{2i-1}+(1/3)v_{2i},(1/3)v_{2i-1}+(2/3)v_{2i}]. \tag{52}$$

This is true, because (V) holds even if $H(J)$ is not an interval, therefore for any fixed interval, there will be an I_j, $j\in J_n'$ lying in that interval if n is large enough.

For each $i=1,\dots,\ell_1$ let

$$\begin{aligned} H_{i,l} &:= \cup\{I_j : j<k(i), I_j\subset K_{c,i}\},\\ H_{i,r} &:= \cup\{I_j : k(i)<j, I_j\subset K_{c,i}\}. \end{aligned}$$

Let $\delta_2>0$ be arbitrary. Then by Theorem 2.1 and the remarks after that of [7], there exist $K_l, K_r\subset K$ real lemniscates such that K_l, K_r consist of ℓ_1 disjoint intervals like K, for all $i=1,\dots,\ell_1$, $H_{i,l}\subset K_l$, $H_{i,r}\subset K_r$ and

$$\left|\Big(\frac{\omega_{K_l}(t)}{\omega_K(t)}\Big)^{1-\alpha}-1\right|,\left|\frac{\omega_{K_l}(t)}{\omega_K(t)}-1\right|<\delta_2 \quad (t\in H_{i,l})$$

and

$$\left|\Big(\frac{\omega_{K_r}(t)}{\omega_K(t)}\Big)^{1-\alpha}-1\right|,\left|\frac{\omega_{K_r}(t)}{\omega_K(t)}-1\right|<\delta_2 \quad (t\in H_{i,r})$$

Then K_l and K_r depend only on K and δ_2. And in other words, K_l covers each of the components from the left endpoints up to $I_{k(i)}$, while K_r does the opposite way, it covers from the right endpoints toward the left endpoints on each component. Obviously, $K=K_l\cup K_r\bigcup\cup_{i=1}^{\ell_1}I_{k(i)}$. If $\delta_2>0$ is small enough, then K_l and K_r must cover almost the entire K, so $K=K_l\cup K_r$.

We also use with $Y=K_l$, $X=H_{i,l}$ or $Y=K_r$, $X=H_{i,r}$ the following notations

$$\begin{aligned} A(P,X,Y) &:= \int_{Y\cap X}\left|\frac{P'(t)}{\deg(P)\pi\omega_Y(t)}\right|^{\alpha}d\nu_Y(t),\\ B(P,X,Y) &:= \int_{Y\cap X}\left|P(t)\right|^{\alpha}d\nu_Y(t). \end{aligned}$$

We need the following Lemma whose proof is in Section 8.

Lemma 12. *Using the notations above, if $Y=K_l$, and with $H:=H_{i,l}$, $q(t):=q(H_{i,l},\deg(P_n);t)$ or if $Y=K_r$, and with $H:=H_{i,r}$, $q(t):=q(H_{i,r},\deg(P_n);t)$, then*

$$\left|A(P_nq,Y,Y)-A(P_n,H,Y)\right|\le o(1)A(P_n,K,K)+o(1)B(P_n,K,K) \tag{53}$$

$$\left|B(P_nq,Y,Y)-B(P_n,H,Y)\right|\le o(1)B(P_n,K,K) \tag{54}$$

where the error terms are independent of P_n and of Y and δ_2, but tend to 0 *as* $\deg(P_n) \to \infty$.

With P_n given, let us consider $P_{n,l,i}(t) := P_n(t)q(H_{i,l}, n; t)$ on the real lemniscate K_l and $P_{n,r,i}(t) := P_n(t)q(H_{i,r}, n; t)$ on the real lemniscate K_r. Based on the previous section

$$A(P_n q, K_l, K_l) \le \big(1 + o_{K_l}(1)\big) B(P_n q, K_l, K_l).$$

Now using (53) and (54) with $H = H_{i,l}$,

$$A(P_n, H, K_l) \le \big(1 + o_{K_l}(1)\big) B(P_n, H, K_l) \\ + o(1)A(P_n, K, K) + o(1)B(P_n, K, K).$$

Obviously, $A(P_n, K, K) = A(P_n, K)$, $B(P_n, K, K) = B(P_n, K)$ and using δ_2, we can write

$$A(P_n, H, K) \le \big(1 + o_{K_l}(1)\big) \frac{1+\delta_2}{1-\delta_2} B(P_n, H, K) \\ + o(1)A(P_n, K, K) + o(1)B(P_n, K, K).$$

As $o_{K_l}(1)$ tends to 0 for each $\delta_2 > 0$, we can have

$$\frac{1+\delta_2}{1-\delta_2}\big(1 + o_{K_l}(1)\big) = 1 + o(1)$$

that is,

$$A(P_n, H) \le \big(1 + o(1)\big) B(P_n, H) + o(1)A(P_n, K) + o(1)B(P_n, K).$$

Similarly, if $H = H_{i,r}$, and we consider $P_{n,r,i}(t) = P_n(t)q(H_{i,r}, n; t)$ on the real lemniscate K_r, then we get

$$A(P_n, H) \le \big(1 + o(1)\big) B(P_n, H) + o(1)A(P_n, K) + o(1)B(P_n, K).$$

Summing up these last two estimates for $i = 1, \ldots, \ell_1$ as in Subsection 5.5, we obtain (4) on finitely many intervals. Note that we do not add up $A(P_n, I_{k(i)})$ and $B(P_n, I_{k(i)})$, because they are small (since $k(i) \in J'_n$), see (52).

7 Sharpness

Let us recall that for a polynomial P and a set $X \subset \mathbf{R}$

$$A(P, X) = \int_{X\cap K} \Big| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t),$$
$$B(P, X) = \int_{X\cap K} \big|P(t)\big|^\alpha d\nu_K(t),$$

and if $\deg(P) = 0$, then, by definition, $A(P, K) = 0$.

In this Section we prove Theorem 4.

If K is a real lemniscate, $K = r^{-1}[-1, 1]$, then there are polynomials P with arbitrarily large degree such that $A(P, K) = B(P, K)$.

Indeed, let $P(x) := T_n(r(x))$ where T_n is the Chebyshev polynomial with degree n. Then, with the same calculations as in Section 3,

$$
\begin{aligned}
A(P,K) &= \int_K \left|\frac{T_n'(r(x))\cdot r'(x)}{\deg(T_n)\deg(r)\pi\omega_K(x)}\right|^\alpha d\nu_K(x) \\
&= \int_K \left|\frac{T_n'(r(x))}{\deg(T_n)(1-r^2(x))^{-1/2}}\frac{\frac{r'(x)}{\deg(r)\pi\sqrt{1-r^2(x)}}}{\omega_K(x)}\right|^\alpha \omega_K(x)dx \\
&= \int_K \left|\frac{T_n'(r(x))}{\deg(T_n)(1-r^2(x))^{-1/2}}\right|^\alpha \frac{|r'(x)|}{\deg(r)\pi\sqrt{1-r^2(x)}}dx \\
&= \int_{[-1,1]} \left|\frac{T_n'(u)}{\deg(T_n)(1-u^2)^{-1/2}}\right|^\alpha \frac{1}{\pi\sqrt{1-u^2}}dx = A(T_n, I)
\end{aligned}
$$

and using the $t = \arccos(u)$ substitution and the definition of Chebyshev polynomials, we can continue

$$
\begin{aligned}
\int_{-1}^{1} \left|\frac{T_n'(u)\sqrt{1-u^2}}{\deg(T_n)}\right|^\alpha \frac{1}{\pi\sqrt{1-u^2}}du &= \frac{1}{\pi}\int_0^\pi |\sin(nt)|^\alpha dt = \frac{1}{\pi}\int_0^\pi |\sin(t)|^\alpha dt \\
&= \frac{1}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)} > 0
\end{aligned}
$$

where we also used the π-periodicity of $|\sin(.)|^\alpha$.

So if $K = r^{-1}[-1,1]$ and $P = T_n \circ r$, then

$$
A(P,K) = \frac{1}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)}. \tag{55}
$$

For the right hand side, we follow the same steps

$$
\begin{aligned}
B(P,K) = \int_K |T_n(r(x))|^\alpha \, d\nu_K(x) &= \int_K |T_n(r(x))|^\alpha \frac{|r'(x)|}{\deg(r)\pi\sqrt{1-r^2(x)}}dx \\
&= \int_{[-1,1]} |T_n(u)|^\alpha \frac{1}{\pi\sqrt{1-u^2}}du = B(T_n, I)
\end{aligned}
$$

and using the $t = \arccos(u)$ substitution and the definition of Chebyshev polynomials, we can continue

$$
\begin{aligned}
\int_{-1}^{1} |T_n(u)|^\alpha \frac{1}{\pi\sqrt{1-u^2}}du &= \frac{1}{\pi}\int_0^\pi |\cos(nt)|^\alpha dt = \frac{1}{\pi}\int_0^\pi |\cos(t)|^\alpha dt \\
&= \frac{1}{\pi}\int_0^\pi |\sin(t)|^\alpha dt = \frac{1}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)} > 0
\end{aligned}
$$

where we also used the π-periodicity of $|\cos(.)|^\alpha$.

In short, if $K = r^{-1}[-1,1]$ and $P = T_n \circ r$, then

$$
B(P,K) = \frac{1}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)}. \tag{56}
$$

These show that if $K = r^{-1}[-1,1]$ and $P = T_n \circ r$, then actually $A(P,K) = B(P,K)$.

Now assume that K is a finite union of intervals as in (3). We use the components of K, $K=\cup_{\ell=1}^{\ell_1}[v_{2\ell-1},v_{2\ell}]$. Fix $\delta_3>0$ and define

$$\underline{K}(\delta_3):=\cup_{\ell=1}^{\ell_1}[v_{2\ell-1}+\delta_3,v_{2\ell}-\delta_3].$$

We use [7], Section 2 now. So there exists an admissible polynomial $\overline{r}(\delta_3;x)=\overline{r}(x)$, with

$$\overline{K}(\delta_3):=\{x\in\mathbf{R}:|\overline{r}(\delta_3;x)|\le 1\}$$

so that $\overline{K}(\delta_3)\supset K$, $\overline{K}(\delta_3)\subset\cup_{\ell=1}^{\ell_1}[v_{2\ell-1}-\delta_3,v_{2\ell}+\delta_3]$, and $\omega_{\overline{K}(\delta_3)}(x)\ge(1-\delta_3)\omega_K(x)$ on $x\in\underline{K}(\delta_3)$. By $K\subset\overline{K}(\delta_3)$, $\omega_{\overline{K}(\delta_3)}(x)\le\omega_K(x)$ on $x\in K$.

Consider $P(x):=T_n(\overline{r}(\delta_3;x))$. Then

$$\begin{aligned}A(P,K)&=\int_K\Big|\frac{T_n'(\overline{r}(x))\cdot\overline{r}'(x)}{\deg(T_n)\deg(\overline{r})\pi\omega_K(x)}\Big|^\alpha d\nu_K(x)\\&=\int_K\Big|\frac{T_n'(\overline{r}(x))}{\deg(T_n)(1-\overline{r}^2(x))^{-1/2}}\frac{\frac{\overline{r}'(x)}{\deg(\overline{r})\pi\sqrt{1-\overline{r}^2(x)}}}{\omega_K(x)}\Big|^\alpha\omega_K(x)dx\\&=\int_K\Big|\frac{T_n'(\overline{r}(x))}{\deg(T_n)(1-\overline{r}^2(x))^{-1/2}}\Big|^\alpha\Big|\frac{\omega_{\overline{K}(\delta_3)}(x)}{\omega_K(x)}\Big|^\alpha\omega_K(x)dx\\&=\int_{\underline{K}(\delta_3)}+\int_{K\setminus\underline{K}(\delta_3)}.\end{aligned}$$

The second integral is nonnegative. We estimate the first integral from below as follows

$$\begin{aligned}&\int_{\underline{K}(\delta_3)}\left|\frac{T_n'(\overline{r}(x))}{\deg(T_n)(1-\overline{r}^2(x))^{-1/2}}\right|^\alpha\left|\frac{\omega_{\overline{K}(\delta_3)}(x)}{\omega_K(x)}\right|^\alpha\omega_K(x)dx\\&\quad\ge(1-\delta_3)^\alpha\int_{\underline{K}(\delta_3)}\left|\frac{T_n'(\overline{r}(x))}{\deg(T_n)(1-\overline{r}^2(x))^{-1/2}}\right|^\alpha\omega_K(x)dx\\&\quad\ge(1-\delta_3)^\alpha\int_{\underline{K}(\delta_3)}\left|\frac{T_n'(\overline{r}(x))}{\deg(T_n)(1-\overline{r}^2(x))^{-1/2}}\right|^\alpha\omega_{\overline{K}(\delta_3)}(x)dx.\end{aligned}$$

Now we want to replace $\underline{K}(\delta_3)$ with the larger set $\overline{K}(\delta_3)$. We need the followings. If $x\in\overline{K}(\delta_3)$, then $|T_n'(\overline{r}(x))|\le(1-\overline{r}^2(x))^{-1/2}\deg(T_n)$. Since $\overline{K}(\delta_3)\subset\cup_{\ell=1}^{\ell_1}[v_{2\ell-1}-\delta_3,v_{2\ell}+\delta_3]$, the Lebesgue measure of $\overline{K}(\delta_3)\setminus\underline{K}(\delta_3)$ is at most $2\ell_1\delta_3$, and there is a constant $C_{12}>0$ independent of δ_3 such that for all measure $\nu_{\overline{K}(\delta_3)}$, we know

$$\nu_{\overline{K}(\delta_3)}(\overline{K}(\delta_3)\setminus\underline{K}(\delta_3))\le C_{12}\sqrt{\delta_3}. \tag{57}$$

This follows from the representation (2.4) from [7].

Furthermore, as in (55),

$$\int_{\overline{K}(\delta_3)}\Big|\frac{T_n'(\overline{r}(x))}{\deg(T_n)(1-\overline{r}^2(x))^{-1/2}}\Big|^\alpha\omega_{\overline{K}(\delta_3)}(x)dx=\frac{1}{\sqrt{\pi}}\frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)}>0$$

so we can continue the estimate

$$(1-\delta_3)^\alpha \int_{\underline{K}(\delta_3)} = (1-\delta_3)^\alpha \left(\int_{\overline{K}(\delta_3)} - \int_{\overline{K}(\delta_3)\setminus \underline{K}(\delta_3)} \right)$$
$$\geq (1-\delta_3)^\alpha \frac{1}{\sqrt{\pi}} \frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)} - (1-\delta_3)^\alpha C_{12}\sqrt{\delta_3}.$$

Summarizing what we have

$$A(P,K) \geq (1-\delta_3)^\alpha \frac{1}{\sqrt{\pi}} \frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)} - (1-\delta_3)^\alpha C_{12}\sqrt{\delta_3}. \tag{58}$$

We estimate $B(P,K)$ from above following similar steps.

$$B(P,K) = \int_K |T_n(\overline{r}(x))|^\alpha \omega_K(x)dx = \int_{\underline{K}(\delta_3)} + \int_{K\setminus \underline{K}(\delta_3)} .$$

We estimate the second integral using $K \setminus \underline{K}(\delta_3) \subset \overline{K}(\delta_3) \setminus \underline{K}(\delta_3)$ and (57), so

$$\int_{K\setminus \underline{K}(\delta_3)} |T_n(\overline{r}(x))|^\alpha d\nu_K(x) \leq ||T_n \circ \overline{r}||_K^\alpha \cdot \nu_K(K \setminus \underline{K}(\delta_3)) \leq 1 \cdot C_{12}\sqrt{\delta_3}.$$

The first integral can be estimated as follows

$$\int_{\underline{K}(\delta_3)} |T_n(\overline{r}(x))|^\alpha \omega_K(x)dx \leq (1-\delta_3)^{-1} \int_{\underline{K}(\delta_3)} |T_n(\overline{r}(x))|^\alpha \omega_{\overline{K}(\delta_3)}(x)dx$$
$$\leq (1-\delta_3)^{-1} \int_{\overline{K}(\delta_3)} |T_n(\overline{r}(x))|^\alpha \omega_{\overline{K}(\delta_3)}(x)dx$$

This last integral can be calculated as in (56), so the estimate for $B(P,K)$ is

$$B(P,K) \leq (1-\delta_3)^{-1} \frac{1}{\sqrt{\pi}} \frac{\Gamma((\alpha+1)/2)}{\Gamma(\alpha/2+1)} + C_{12}\sqrt{\delta_3}.$$

The lower estimate of $A(P,K)$ and the upper estimate of $B(P,K)$ can be arbitrarily close to one another if $\delta_3 > 0$ is small enough. This shows asymptotical sharpness on every set K, for example along the polynomial sequence $\{T_n(\overline{r}(1/n;x))\}_n$.

8 Proof of the lemmas

In this section, for simplicity, we use the notations

$$E_n := O\big(\exp(-C_2 n^\theta)\big) \tag{59}$$

for the error term appearing in Lemma 6.

Proof of Lemma 6. Lemma 6 is a fairly simple result in simultaneous approximation, but for the sake of completeness, we present a proof. Let $K^+ = \big[\inf K - 1, \sup K + 1\big]$.

We construct the polynomial q in three steps. There exists $C_{13} > 0$ depending on $\inf K, \sup K$ only such that

$$0 \le 1 - C_{13}(x - x_0)^2 \le 1$$

for all $x \in K^+$ and $x_0 \in K^+$.

Let

$$Q_1(x_0, n^{2\theta}; x) := \left(1 - C_{13}(x - x_0)^2\right)^{[n^{2\theta}]}.$$

If $|x - x_0| \ge n^{-\theta/2}$ and $x \in K^+$, then $C_{13}(x - x_0)^2 \ge C_{13} n^{-\theta}$, so

$$Q_1(x_0, n^{2\theta}; x) \le \left(1 - C_{13} n^{-\theta}\right)^{[n^{2\theta}]} = \left(\left(1 - C_{13} n^{-\theta}\right)^{n^\theta}\right)^{\frac{[n^{2\theta}]}{n^\theta}} \approx O\big(\exp(-C_{13} n^\theta)\big).$$

So, for some $C_{14} = C_{14}(C_{13}) > 0$, if $|x - x_0| \ge n^{-\theta/2}$ and $x \in K^+$,

$$Q_1(x_0, n^{2\theta}; x) \le C_{14} \exp(-C_{13} n^\theta)$$

and $\deg Q_1 = 2[n^{2\theta}]$

The supremum norm of Q_1 over K^+ is 1, $||Q_1||_\infty = 1$, so using a Nikolskii-type inequality (see [3] p. 498. Theorem 3.1.4.), we obtain with some $C_{15} = C_{15}(\inf K, \sup K) > 0$

$$\int_{K^+} Q_1(x_0, n^{2\theta}; t) dt \ge C_{15} n^{-4\theta}.$$

Let

$$Q_2(x_0, n^{2\theta}; x) := \frac{\int_{\inf K - 1}^{x} Q_1(x_0, n^{2\theta}; t) dt}{\int_{K^+} Q_1(x_0, n^{2\theta}; t) dt}.$$

It is easy to see that if $x \in K^+$, $x \le x_0 - n^{-\theta/2}$, then for some $C_{16} > 0$, $0 < C_2 < C_{13}$,

$$0 \le Q_2(x_0, n^{2\theta}; x) \le \frac{|K^+| C_{14} \exp(-C_{13} n^\theta)}{C_{15} n^{-4\theta}} \le C_{16} \exp(-C_2 n^\theta)$$

and if $x \in K^+$, $x_0 + n^{-\theta/2} \le x$, then

$$1 \ge Q_2(x_0, n^{2\theta}; x) \ge 1 - C_{16} \exp(-C_2 n^\theta).$$

For simplicity, let $F_n := C_{16} \exp(-C_2 n^\theta)$ As for the derivative,

$$Q_2'(x_0, n^{2\theta}; x) = \frac{Q_1(x_0, n^{2\theta}; x)}{\int_{K^+} Q_1(x_0, n^{2\theta}; t) dt}$$

and we can also assume (by choosing smaller C_{13} and larger C_{16}) that

$$|Q_2'(x_0, n^{2\theta}; x)| \le C_{16} \exp(-C_2 n^\theta) = F_n.$$

Since H is an interval, H_b consists of two intervals and denote by u' and u'' the midpoints of these two intervals, $u'' > u'$. By construction (see (9) and the definition of H_b), the length of these intervals are in between $\lambda_n/2$ and $\lambda_n = c_1/n^\kappa$. Let $Q_3(H,n;x) := Q_2(u', n^{2\theta}; x) - Q_2(u'', n^{2\theta}; x)$. Summarizing what we have

$$
\begin{gathered}
-F_n \le Q_3(H,n;x) \le F_n, \text{ if } x \in K^+ \setminus (H \cup H_b), \\
-F_n \le Q_3(H,n;x) \le 1, \text{ if } x \in H_b, \\
1 - 2F_n \le Q_3(H,n;x) \le 1, \text{ if } x \in H, \\
|Q_3'(H,n;x)| \le 2F_n, \text{ if } x \in K^+ \setminus H_b.
\end{gathered}
$$

Finally, let $q(H,n;x) := \frac{Q_3(H,n;x)+F_n}{1+F_n}$. Using the estimates for Q_3, we can write

$$
\begin{gathered}
0 \le q \le \frac{2F_n}{1+F_n}, \text{ if } x \in K^+ \setminus (H \cup H_b), \\
0 \le q \le 1, \text{ if } x \in H_b, \\
\frac{1-F_n}{1+F_n} \le q \le 1, \text{ if } x \in H, \\
|q'| \le \frac{2F_n}{1+F_n}, \text{ if } x \in K^+ \setminus H_b.
\end{gathered}
$$

Finally, if $E_n \ge 2F_n$, then we are done.

Remark.

The $O(.)$ in this proof depend only on $\inf K$ and $\sup K$ not on the whole K (in other words, it does not depend on the "structure" of K). □

We use the following calculations frequently (see (17)).

$$
\begin{aligned}
\big(P^*(t)\big)' &= \Big(\sum_{j=1}^{\deg r} P_n\big(r_j^{-1}\big(r(t)\big)\big) q\big(r_j^{-1}\big(r(t)\big)\big)\Big)' \\
&= \sum_{j=1}^{\deg r} P_n'\big(r_j^{-1}\big(r(t)\big)\big) \frac{r'(t)}{r'(t_j)} \cdot q\big(r_j^{-1}\big(r(t)\big)\big) \\
&+ \sum_{j=1}^{\deg r} P_n\big(r_j^{-1}\big(r(t)\big)\big) \cdot q'\big(r_j^{-1}\big(r(t)\big)\big) \frac{r'(t)}{r'(t_j)} \\
&= \sum_{j=1}^{\deg r} P_n'\big(t_j\big) \frac{r'(t)}{r'(t_j)} \cdot q\big(t_j\big) + \sum_{j=1}^{\deg r} P_n\big(t_j\big) \cdot q'\big(t_j\big) \frac{r'(t)}{r'(t_j)}.
\end{aligned}
$$

We also use that

$$
\frac{\omega_K(t_i)|r'(t)|}{\omega_K(t)|r'(t_i)|} = 1 \tag{60}
$$

which easily follows from (17) and (18). With this, we can derive the following two calculations.

If f is any continuous function and $X \subset K_{b,j}$ for some j, then

$$
\int_X f(t_i) d\nu_K(t) = \int_X f(t_i) \frac{1}{\pi \deg r} \frac{\big|r'(t)\big|}{\sqrt{1-r^2(t)}} dt =
$$

where we use the $s=t_i=r_i^{-1}\big(r(t)\big)$ substitution with $r(s)=r(t_i)$ and $r'(s)ds=r'(t)dt$ (see (17)), so we can continue

$$=\int_X f(s)\frac{1}{\pi\deg r}\frac{\big|r'(t)\big|}{\sqrt{1-r^2(s)}}dt=\int_{X_i} f(s)\frac{1}{\pi\deg r}\frac{\big|r'(s)\big|}{\sqrt{1-r^2(s)}}ds$$
$$=\int_{X_i} f(s)d\nu_K(s)$$

where $X_i=r_i^{-1}\big(r(X)\big)$. That is,

$$\int_X f(t_i)d\nu_K(t)=\int_{X_i} f(s)d\nu_K(s) \tag{61}$$

Similarly, if f is any continuously differentiable function and $X\subset K_{b,j}$ for some j, then, with the help of (17),

$$\int_X \Big|\frac{(f(t_i))'}{\omega_K(t)}\Big|^\alpha d\nu_K(t)=\int_X \Big|\frac{f'(t_i)\frac{r'(t)}{r'(t_i)}}{\omega_K(t)}\Big|^\alpha d\nu_K(t)$$
$$=\int_X \big|f'(t_i)\big|^\alpha\Big|\frac{r'(t)}{r'(t_i)}\pi\deg(r)\frac{\sqrt{1-r^2(t)}}{r'(t)}\Big|^\alpha\frac{1}{\deg(r)\pi}\frac{|r'(t)|}{\sqrt{1-r^2(t)}}dt$$
$$=\int_X \big|f'(s)\big|^\alpha\Big|\pi\deg(r)\frac{\sqrt{1-r^2(s)}}{r'(s)}\Big|^\alpha\frac{1}{\deg(r)\pi}\frac{|r'(t)|}{\sqrt{1-r^2(s)}}dt$$
$$=\int_{X_i} \big|f'(s)\big|^\alpha\Big|\pi\deg r\frac{\sqrt{1-r^2(s)}}{r'(s)}\Big|^\alpha\frac{1}{\deg(r)\pi}\frac{|r'(s)|}{\sqrt{1-r^2(s)}}ds$$
$$=\int_{X_i}\Big|\frac{f'(s)}{\omega_K(s)}\Big|^\alpha d\nu_K(s)$$

where $X_i=r_i^{-1}\big(r(X)\big)$. That is, all the following integrals are equal:

$$\int_X \Big|\frac{(f(t_i))'}{\omega_K(t)}\Big|^\alpha d\nu_K(t)=\int_X \Big|\frac{f'(t_i)\frac{r'(t)}{r'(t_i)}}{\omega_K(t)}\Big|^\alpha d\nu_K(t)=\int_{X_i}\Big|\frac{f'(s)}{\omega_K(s)}\Big|^\alpha d\nu_K(s). \tag{62}$$

Proof of Lemma 7. The next assumptions, tools will be frequently used in this proof: $|q(t)|\le 1$ ($t\in K$), inequality (2) for q and Lemma 6.

We split the integrals into three terms depending on the set: $K\setminus r^{-1}\big[r(H\cup H_b)\big]$, $r^{-1}\big[r(H)\big]$ and $r^{-1}\big[r(H_b)\big]\cap K$. Because of (I), (II) and (III), these three sets are "almost" disjoint, that is, they have only finitely many common points and H is a subset of some branch of K, say $H\subset K_{b,i_0}$.

On $K\setminus r^{-1}\big[r(H\cup H_b)\big]$:

$$\left(\int_{K\setminus r^{-1}\left[r(H\cup H_b)\right]}\Big|\frac{\big(P^*(t)\big)'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$\le\sum_{i=1}^{\deg r}\left(\int\Big|\frac{\big(P(t_i)q(t_i)\big)'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$\le\sum\left(\int\Big|\frac{\big|\big(P(t_i)\big)'q(t_i)\big|\;+\;\big|P(t_i)\big(q(t_i)\big)'\big|}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}\le$$

where $t \in K \setminus r^{-1}\big[r(H \cup H_b)\big]$, so $t_i \notin H \cup H_b$ for every $i = 1, \dots, \deg r$. We continue this estimate later. By Lemma 6, this implies that $|q(t_i)|, |q'(t_i)| \le O\big(\exp(-C_2 n^\theta)\big) = E_n$. We use inequality (2) for q, and with the notation (59) from now on, we can continue

$$\le \sum_{i=1}^{\deg r} \left(E_n \left(\int_{K\setminus r^{-1}\left[r(H\cup H_b)\right]} \Big| \frac{\big(P(t_i)\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} + \frac{\deg(q)}{\deg(P)} \left(\int_{K\setminus r^{-1}\left[r(H\cup H_b)\right]} \big|P(t_i)\big|^\alpha d\nu_K(t) \right)^{1/\alpha} \right) \le$$

which we continue later again. Now, for each fixed i, we estimate the first term in $(\dots)^{1/\alpha}$ by increasing the set from $K \setminus r^{-1}\big[r(H \cup H_b)\big]$ to K and using (62) on each branch $K_{b,j}$ of K. And again, increasing the integral by increasing the set from $K_{b,j}$ to K. These steps bring in a factor $(\deg r)^{1+1/\alpha}$.

For the second term in $(\dots)^{1/\alpha}$ we do it similarly: increasing the set from $K \setminus r^{-1}\big[r(H \cup H_b)\big]$ to K and using (61) on each branch $K_{b,j}$ of K. And again, increasing the integral by increasing the set from $K_{b,j}$ to K. Again, these steps bring in a factor $(\deg r)^{1+1/\alpha}$.

So we can continue as follows

$$\le \big(\deg r\big)^{1+\frac{1}{\alpha}} E_n \left(\int_K \Big| \frac{P'(t)}{\deg(P)\,\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} + \big(\deg r\big)^{1+\frac{1}{\alpha}} \frac{\deg(q)}{\deg(P)} \left(\int_K \big|P(t)\big|^\alpha d\nu_K(t) \right)^{1/\alpha}.$$

So on $K \setminus r^{-1}\big[r(H \cup H_b)\big]$ the upper estimate is

$$\left(\int_{K\setminus r^{-1}\left[r(H\cup H_b)\right]} \Big| \frac{\big(P^*(t)\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} \le \big(\deg r\big)^{1+\frac{1}{\alpha}} E_n \left(\int_K \Big| \frac{P'(t)}{\deg(P)\,\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} + \big(\deg r\big)^{1+\frac{1}{\alpha}} \frac{\deg(q)}{\deg(P)} \left(\int_K \big|P(t)\big|^\alpha d\nu_K(t) \right)^{1/\alpha}. \quad (63)$$

Raising to power α and using generalized mean inequality $(X+Y)^\alpha \le 2^{\alpha-1}(X^\alpha + Y^\alpha)$, we can write

$$\int_{K\setminus r^{-1}[r(H\cup H_b)]} \left| \frac{(P^*(t))'}{\deg(P)\pi\omega_K(t)} \right|^\alpha d\nu_K(t) \le 2^{\alpha-1}(\deg r)^{\alpha+1} E_n^\alpha A(P,K) + 2^{\alpha-1}(\deg r)^{\alpha+1} \left(\frac{\deg(q)}{\deg(P)} \right)^\alpha B(P,K). \quad (64)$$

On $r^{-1}\big[r(H_b)\big]\cap K$:

$$\left(\int_{r^{-1}\left[r(H_b)\right]\cap K}\Big|\frac{\big(P^*(t)\big)'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$\le \sum_{i=1}^{\deg r}\left(\int\Big|\frac{\big(P(t_i)q(t_i)\big)'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$\le \sum\left(\int\Big|\frac{\big|\big(P(t_i)\big)'q(t_i)\big|\ +\ \big|P(t_i)\big(q(t_i)\big)'\big|}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}\le$$

where we split this sum as follows. By (III), $H_b\cap K$ lays in the same branch as H which is denoted by K_{b,i_0}. We continue as follows

$$\le\left(\int_{r^{-1}[r(H_b)]\cap K}\Big|\frac{\big|\big(P(t_{i_0})\big)'q(t_{i_0})\big|\ +\ \big|P(t_{i_0})\big(q(t_{i_0})\big)'\big|}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$+\sum_{i\neq i_0}\left(\int_{r^{-1}[r(H_b)]\cap K}\Big|\frac{\big|\big(P(t_i)\big)'q(t_i)\big|\ +\ \big|P(t_i)\big(q(t_i)\big)'\big|}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}. \quad (65)$$

The first term in (65) can be rewritten as

$$\left(\int_{r^{-1}\left[r(H_b)\right]\cap K}\Big|\frac{\big|\big(P(t_{i_0})\big)'q(t_{i_0})\big|\ +\ \big|P(t_{i_0})\big(q(t_{i_0})\big)'\big|}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$\le\sum_{j=1}^{\deg r}\left(\int_{r_j^{-1}\left[r(H_b)\right]\cap K}\dots\right)^{1/\alpha}$$
$$=\sum_{j=1}^{\deg r}\left(\int_{r_j^{-1}\left[r(H_b)\right]\cap K}\Big|\frac{\big(P(t_{i_0})\big)'q(t_{i_0})}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$+\sum_{j=1}^{\deg r}\left(\int_{r_j^{-1}\left[r(H_b)\right]\cap K}\Big|\frac{P(t_{i_0})\big(q(t_{i_0})\big)'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}.$$

For the terms in the first sum, we can use (17), $|q|\le 1$, (62), and the definition of $a(P,H_b)$, so we can write

$$\sum_{j=1}^{\deg r}\left(\int_{r_j^{-1}\left[r(H_b)\right]\cap K}\Big|\frac{\big(P(t_{i_0})\big)'q(t_{i_0})}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}$$
$$\le \deg(r)\big(a(P,H_b)A(P,K)\big)^{1/\alpha}. \quad (66)$$

For the terms in the second sum, we can use (17), (2) for q, (61), and the definition of $b(P,H_b)$, so we can write

$$\sum_{j=1}^{\deg r}\left(\int_{r_j^{-1}\left[r(H_b)\right]\cap K}\Big|\frac{P(t_{i_0})\big(q(t_{i_0})\big)'}{\deg(P)\pi\omega_K(t)}\Big|^{\alpha}d\nu_K(t)\right)^{1/\alpha} \leq \deg(r)\frac{\deg(q)}{\deg(P)}\big(b(P,H_b)B(P,K)\big)^{1/\alpha}. \quad (67)$$

The rest of the terms in (65) are small because $i\neq i_0$ so $t_i\notin H_b$. More precisely, we increase the term in the first line below by splitting into two groups in $(...)^{1/\alpha}$. Then we split the terms in $(...)^{1/\alpha}$ for each $r_j^{-1}[r(H_b)]$, $j=1,\dots,\deg(r)$, and for the terms with $\big(P(t_i)\big)'q(t_i)$ inside $(...)^{1/\alpha}$ for each j we use $|q|\leq E_n$ and (62) and increase the set from $r_j^{-1}[r(H_b)]$ to $r^{-1}[r(H_b)]$. For the terms with $P(t_i)\big(q(t_i)\big)'$ inside $(...)^{1/\alpha}$ for each j we use (62) and then (2) for q, and increase the set from $r_j^{-1}[r(H_b)]$ to $r^{-1}[r(H_b)]$. This way we obtain

$$\begin{aligned}
&\sum_{i\neq i_0}\left(\int_{r^{-1}[r(H_b)]\cap K}\Big|\frac{\big|\big(P(t_i)\big)'q(t_i)\big|\ +\ \big|P(t_i)\big(q(t_i)\big)'\big|}{\deg(P)\pi\omega_K(t)}\Big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}\\
&\qquad\leq\sum_{i\neq i_0}\left(\int_{r^{-1}[r(H_b)]\cap K}\Big|\frac{\big(P(t_i)\big)'q(t_i)}{\deg(P)\pi\omega_K(t)}\Big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}\\
&\qquad\quad+\sum_{i\neq i_0}\left(\int_{r^{-1}[r(H_b)]\cap K}\Big|\frac{P(t_i)\big(q(t_i)\big)'}{\deg(P)\pi\omega_K(t)}\Big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}\\
&\leq\big(\deg(r)-1\big)(\deg r)^{1/\alpha}E_n\left(\int_{r^{-1}[r(H_b)]\cap K}\Big|\frac{P'(t)}{\deg(P)\pi\omega_K(t)}\Big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}\\
&\qquad+\big(\deg r\ -1\big)(\deg r)^{1/\alpha}\frac{\deg(q)}{\deg(P)}\left(\int_{r^{-1}[r(H_b)]}\big|P(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}.
\end{aligned}$$

So on $r^{-1}\left[r(H_b)\right]\cap K$, the upper estimate of

$$(\int_{r^{-1}\left[r(H_b)\right]\cap K}|\frac{\big(P^*(t)\big)'}{\deg(P)\pi\omega_K(t)}|^{\alpha}d\nu_K(t))^{1/\alpha}$$

is

$$
\begin{aligned}
&\left(\int_K \Big|\frac{P'(t)}{\deg(P)\,\pi\,\omega(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}\\
&\qquad\cdot\left(\deg(r)\big(a(P,H_b)\big)^{1/\alpha}+(\deg(r)-1)(\deg r)^{1/\alpha}E_n\right)\\
&\qquad\qquad+\left(\int_K \big|P(t)\big|^\alpha d\nu_K(t)\right)^{1/\alpha}\\
&\quad\cdot\left(\deg(r)\frac{\deg(q)}{\deg(P)}\big(b(P,H_b)\big)^{1/\alpha}+(\deg r-1)(\deg r)^{1/\alpha}\frac{\deg(q)}{\deg(P)}\right)\\
&\quad\le A(P,K)^{1/\alpha}\Big(\deg(r)\big(a(P,H_b)\big)^{1/\alpha}+(\deg(r)-1)(\deg r)^{1/\alpha}E_n\Big)\\
&\qquad\quad+B(P,K)^{1/\alpha}\frac{\deg(q)}{\deg(P)}\Big(\deg(r)+(\deg(r)-1)(\deg r)^{1/\alpha}\Big). \qquad (68)
\end{aligned}
$$

Note that $a(P,H_b)$ is small, by (IV-a) and $b(P,H_b)$ is small too, by (IV-b).

Again, we can remove the power $1/\alpha$ as above: first raising to power α, then using the generalized mean inequality for three terms (terms with $B(P,K)$, the term with $A(P,K)$ and $a(P,H_b)$, and the term with $A(P,K)$), so we can write

$$
\begin{aligned}
&\int_{r^{-1}\left[r(H_b)\right]\cap K}\Big|\frac{\big(P^*(t)\big)'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\\
&\qquad\le B(P,K)3^{\alpha-1}\left(\frac{\deg(q)}{\deg(P)}\right)^\alpha(\deg(r)+(\deg(r)-1)(\deg r)^{1/\alpha})^\alpha\\
&+A(P,K)3^{\alpha-1}(\deg r)^\alpha a(P,H_b)+A(P,K)3^{\alpha-1}(\deg(r)-1)^\alpha(\deg r)E_n^\alpha. \qquad (69)
\end{aligned}
$$

On $r^{-1}\big[r(H)\big]$:
We need the estimates of the following three terms. First, if $i\neq i_0$, then, using properties of q and (62), we have

$$
\begin{aligned}
&\left(\int_H \Big|\frac{\big(P(t_i)\big)'q(t_i)}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}\\
&\qquad\le E_n\left(\int_{H_i}\Big|\frac{P'(t)}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}\\
&\qquad\qquad\le E_n\left(\int_K\Big|\frac{P'(t)}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\right)^{1/\alpha}. \qquad (70)
\end{aligned}
$$

Second, we estimate the following two terms very similarly: using (2) for q,

$t_{i_0} = t$ $(t \in H)$, we have

$$\left(\int_H \Big| \frac{P(t_{i_0})\big(q(t_{i_0})\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$= \left(\int_H \Big| \frac{P(t)q'(t)}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$\leq \left(\int_H \Big| P(t) \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} \frac{\deg q}{\deg P} = \frac{\deg q}{\deg P} \big(b(P,H)B(P,K) \big)^{1/\alpha}$$

and for $i \neq i_0$, we have for each j using (61) and inequality (2) for q,

$$\left(\int_{H_j} \Big| \frac{P(t_i)\big(q(t_i)\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$\leq \left(\int_{r_i^{-1}\left[r(H)\right]} \Big| P(t) \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} \frac{\deg(q)}{\deg(P)}$$

$$\leq \frac{\deg(q)}{\deg(P)} \left(\int_K \big| P(t) \big|^\alpha d\nu_K(t) \right)^{1/\alpha}. \quad (71)$$

Furthermore, we need the following short calculation, which comes from (62),

$$\deg(r) \int_H \Big| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t)$$

$$= \sum_{i=1}^{\deg r} \int_{r_i^{-1}[r(H)]} \Big| \frac{(P(t_{i_0}))'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t)$$

$$= \int_{r^{-1}[r(H)]} \Big| \frac{(P(t_{i_0}))'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t).$$

We obtain

$$\Bigg| \left(\int_{r^{-1}\left[r(H)\right]} \Big| \frac{\big(P^*(t)\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$- (\deg r)^{1/\alpha} \left(\int_H \Big| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} \Bigg| \leq$$

where, for the second term, we use the previous short calculation and the triangle inequality for the norm $||.|| = \big(\int_{r^{-1}[r(H)]} \big| \frac{\cdot}{\deg(P)\pi\omega_K(t)} \big|^\alpha d\nu_K(t) \big)^{1/\alpha}$, we can continue

$$\leq \left(\int_{r^{-1}\left[r(H)\right]} \Big| \frac{\big(P^*(t)\big)' - \big(P(t_{i_0})\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$\leq \left(\int_{r^{-1}\left[r(H)\right]} \Big| \frac{\big(P(t_{i_0})\big)' q(t_{i_0}) - \big(P(t_{i_0})\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$+ \left(\int_{r^{-1}\left[r(H)\right]} \Big| \frac{P(t_{i_0})\big(q(t_{i_0})\big)'}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$+ \left(\int_{r^{-1}\left[r(H)\right]} \Big| \frac{\sum_{i\neq i_0} \big|\big(P(t_i)\big)' q(t_i)\big|}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$+ \left(\int_{r^{-1}\left[r(H)\right]} \Big| \frac{\sum_{i\neq i_0} \big|P(t_i)\big(q(t_i)\big)'\big|}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha} \leq$$

which we continue later.

For the first term, we use $1 - E_n \leq q \leq 1$ on H, and on each $r_j^{-1}[r(H)]$, increase $r_j^{-1}[r(H)]$ to K. For the second, inequality (2) for q and again the same set-increasing step, for the third, we use (70) on each $r_j^{-1}[r(H)]$, and the same set-increasing step. For the fourth term, we use (71) on each $r_j^{-1}[r(H)]$, and the same set-increasing step, so we can continue

$$\leq E_n (\deg r)^{1/\alpha} \left(\int_K \Big| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$+ \frac{\deg(q)}{\deg(P)} (\deg r)^{1/\alpha} \left(\int_K \big|P(t)\big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$+ E_n \big(\deg r\big)^{1+1/\alpha} \left(\int_K \Big| \frac{P'(t)}{\deg(P)\pi\omega_K(t)} \Big|^\alpha d\nu_K(t) \right)^{1/\alpha}$$

$$+ \frac{\deg(q)}{\deg(P)} \big(\deg r\big)^{1+1/\alpha} \left(\int_K \big|P(t)\big|^\alpha d\nu_K(t) \right)^{1/\alpha}.$$

So on $r^{-1}\big[r(H)\big]$ we have

$$\Bigg|\Bigg(\int_{r^{-1}\big[r(H)\big]}\Big|\frac{\big(P^*(t)\big)'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Bigg)^{1/\alpha} - (\deg r)^{1/\alpha}\Bigg(\int_H\Big|\frac{P'(t)}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Bigg)^{1/\alpha}\Bigg|$$
$$\le E_n\deg(r)^{1/\alpha}\Big(\deg(r)+1\Big)\Bigg(\int_K\Big|\frac{P'(t)}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Bigg)^{1/\alpha}$$
$$+\frac{\deg(q)}{\deg(P)}\deg(r)^{1/\alpha}\Big(\deg(r)+1\Big)\Bigg(\int_K\big|P(t)\big|^\alpha d\nu_K(t)\Bigg)^{1/\alpha}. \quad (72)$$

Summing up the estimates (63), (68) and (72) we obtain the first assertion of the lemma, (21).

To prove the power free assertion (22), we remove the power $1/\alpha$ in (72) and introduce some notations:

$$X := \Big(\int_{r^{-1}[r(H)]}\Big|\frac{(P^*(t))'}{\deg(P)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Big)^{1/\alpha}, \qquad Y := (\deg r)^{1/\alpha}A^{1/\alpha}(P,H).$$

Our goal is that we deduce

$$|X^\alpha - Y^\alpha| \le o(1)A(P,K) + o(1)B(P,K) \quad (73)$$

from

$$|X-Y| \le o(1)A^{1/\alpha}(P,K) + o(1)B^{1/\alpha}(P,K).$$

If we are done with this last displayed inequality, then summing it up with (64) and (69) we obtain (22).

First, assume that $X > Y$, so we know

$$X - Y \le o(1)A^{1/\alpha}(P,K) + o(1)B^{1/\alpha}(P,K).$$

We start with

$$X^\alpha \le \Big(Y + o(1)A^{1/\alpha}(P,K) + o(1)B^{1/\alpha}(P,K)\Big)^\alpha$$
$$= \big(1+o(1)+o(1)\big)^\alpha\Big(\frac{Y + o(1)A^{1/\alpha}(P,K) + o(1)B^{1/\alpha}(P,K)}{1+o(1)+o(1)}\Big)^\alpha \le$$

where we apply the generalized weighted mean inequality for Y, $A^{1/\alpha}(P,K)$ and $B^{1/\alpha}(P,K)$ with the weights 1, $o(1)$ and $o(1)$ respectively in the big bracket between power 1 and power α, so we can continue

$$\le \big(1+o(1)+o(1)\big)^{\alpha-1}\Big(Y^\alpha + o(1)A(P,K) + o(1)B(P,K)\Big).$$

Rearranging and noting that $Y^\alpha = \big(\deg r\big)A(P,H)$, $A(P,H) \le A(P,K)$ and $o(1)Y^\alpha = o(1)A(P,K)$, we obtain the power free assertion (73) in this case. Note that $(1+o(1)+o(1))^{\alpha-1}$, the error terms $o(1)$ in front of $A(P,K)$ and $B(P,K)$ are independent of P, H and H_b but depend on $\deg(P)$.

If $Y > X$, then we follow the same steps and the following two estimates. Actually, these estimates are similar to those in this proof above, but now we can be more "generous". First, we use the generalized mean inequality, and for each term, split K as $K = \cup K_{b,j}$, then on each branch, we use (62), that is,

$$\int_K \left|\frac{(P^*(t))'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t) = \int_K \left|\frac{\left(\sum_k P(t_k)q(H;t_k)\right)'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t)$$
$$\leq (\deg r)^{\alpha-1}\sum_k \int_K \left|\frac{\left(P(t_k)q(H;t_k)\right)'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t)$$
$$= (\deg r)^{\alpha-1}\sum_k\sum_j \int_{K_{b,j}} \left|\frac{\left(P(t_k)q(H;t_k)\right)'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t)$$
$$= (\deg r)^{\alpha-1}\sum_k\sum_j \int_{K_{b,k}} \left|\frac{\left(P(t)q(H;t)\right)'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t)$$
$$= (\deg r)^{\alpha} \int_K \left|\frac{\left(P(t)q(H;t)\right)'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t) = \left(\deg(r)\frac{\deg(Pq)}{\deg(P)}\right)^\alpha A(Pq,K).$$

Then, using the generalized mean inequality, $|q| \leq 1$, and inequality (2) for q, we have

$$A(Pq,K) \leq 2^{\alpha-1}\left(\frac{\deg(P)}{\deg(Pq)}\right)^\alpha A(P,K) + 2^{\alpha-1}\left(\frac{\deg(q)}{\deg(Pq)}\right)^\alpha B(P,K). \tag{74}$$

Thus it follows that

$$\int_K \left|\frac{(P^*(t))'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t) \leq 2^{\alpha-1}(\deg r)^\alpha\left(A(P,K) + \left(\frac{\deg q}{\deg P}\right)^\alpha B(P,K)\right).$$

Therefore, we have $o(1)X^\alpha = o(1)A(P,K) + o(1)B(P,K)$. So we obtain the power free assertion in this case, too.

Tracing back the error terms in (64), (69) and the remark before $Y > X$, we can write

$$\left|\int_K \left|\frac{(P^*(t))'}{\deg(P)\pi\omega_K(t)}\right|^\alpha d\nu_K(t) - \deg(r)A(P,H)\right|$$
$$\leq \left(o(1) + 3^{\alpha-1}(\deg r)^\alpha a(P,H_b)\right)A(P,K) + o(1)B(P,K) \tag{75}$$

where the $o(1)$ error terms here do not depend on P, H and H_b. □

Proof of Lemma 8. The proof runs very similarly to the previous proof. We split the left hand side of (23) into three terms and estimate them as follows.

On $K \setminus r^{-1}\big[r(H \cup H_b)\big]$:

$$\left(\int_{K\setminus r^{-1}[r(H\cup H_b)]} \big|P^*(t)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}$$
$$\leq \sum_{i=1}^{\deg r}\left(\int_{K\setminus r^{-1}[r(H\cup H_b)]} \big|P(t_i)q(t_i)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}$$
$$\leq E_n(\deg r)^{1+1/\alpha}\left(\int_{K\setminus r^{-1}[r(H\cup H_b)]} \big|P(t)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}$$
$$\leq E_n(\deg r)^{1+1/\alpha}\left(\int_{K} \big|P(t)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}$$

where we used Lemma 6. Raising to power α, we can write

$$B(P^*, K \setminus r^{-1}[r(H \cup H_b)]) \leq E_n^{\alpha}(\deg r)^{\alpha+1} B(P, K). \tag{76}$$

On $r^{-1}\big[r(H_b)\big] \cap K$, $H_b \cap K_i \neq \emptyset$ if and only if $i = i_0$.

$$\left(\int_{r^{-1}[r(H_b)]} \big|P^*(t)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha} \leq \sum_{i=1}^{\deg r}\left(\int \big|P(t_i)q(t_i)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}$$

For all $i \neq i_0$, $t_i \notin H_b$, so $|q(t_i)| \leq E_n$ and

$$\left(\int_{r^{-1}[r(H_b)]} \big|P(t_i)q(t_i)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}$$
$$\leq (\deg r)^{1/\alpha}\, E_n\left(\int_{r_i^{-1}[r(H_b)]} \big|P(t)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}.$$

If $i = i_0$, on each $r_j^{-1}[r(H_b)] \cap K$ we use the substitution $s = t_{i_0}$,

$$\left(\int_{r^{-1}[r(H_b)]} \big|P(t_{i_0})q(t_{i_0})\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha} = (\deg r)^{1/\alpha}\left(\int_{H_b} \dots\right)^{1/\alpha} \leq$$

then the definition of the $b(P, H_b)$

$$\leq (\deg r)^{1/\alpha}\left(\int_{H_b} \big|P(s)q(s)\big|^{\alpha} d\nu_K(s)\right)^{1/\alpha}$$
$$\leq (\deg r)^{1/\alpha} b(P, H_b)^{1/\alpha}\left(\int_{K} \big|P(t)\big|^{\alpha} d\nu_K(t)\right)^{1/\alpha}.$$

So on $r^{-1}\big[r(H_b)\big]\cap K$ the upper estimate is

$$\left(\int_{r^{-1}[r(H_b)]}\big|P^*(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}$$
$$\leq \Big(\big(\deg r\big)^{1+1/\alpha}E_n+(\deg r)^{1/\alpha}b(P,H_b)^{1/\alpha}\Big)\left(\int_K\big|P(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}. \quad (77)$$

Raising to power α and using the generalized mean inequality $(X+Y)^{\alpha}\leq 2^{\alpha-1}(X^{\alpha}+Y^{\alpha})$ for the error terms, we can write

$$B(P^*,r^{-1}[r(H_b)])$$
$$\leq\big(2^{\alpha-1}(\deg r)^{\alpha+1}E_n^{\alpha}+2^{\alpha-1}(\deg r)b(P,H_b)\big)\,B(P,K). \quad (78)$$

On $r^{-1}\big[r(H)\big]$, we use $H\subset K_{i_0}$, the following calculation with the help of (61)

$$\deg(r)\int_H|P(t)|^{\alpha}d\nu_K(t)$$
$$=\sum_{i=1}^{\deg r}\int_{r_i^{-1}[r(H)]}|P(t_{i_0})|^{\alpha}d\nu_K(t)$$
$$=\int_{r^{-1}[r(H)]}|P(t_{i_0})|^{\alpha}d\nu_K(t)$$

the triangle inequality for the norm $||.||=(\int_{r^{-1}[r(H)]}|.(t)|^{\alpha}d\nu_K(t))^{1/\alpha}$ and we can write

$$\Bigg|\left(\int_{r^{-1}[r(H)]}\big|P^*(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}-(\deg r)^{1/\alpha}\left(\int_H\big|P(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}\Bigg|$$
$$\leq\Bigg|\left(\int_{r^{-1}[r(H)]}\big|P(t_{i_0})q(t_{i_0})\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}-(\deg r)^{1/\alpha}\left(\int_H\big|P(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}\Bigg|$$
$$+\sum_{i\neq i_0}\left(\int_{r^{-1}[r(H)]}\big|P(t_i)q(t_i)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}$$
$$\leq\left(\deg(r)\int_H\big|P(t)\big|^{\alpha}\cdot\big|q(t)-1\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}$$
$$+(\deg r-1)E_n(\deg r)^{1/\alpha}\left(\int_K\big|P(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}$$
$$\leq\Big((\deg r)^{1/\alpha}E_n+(\deg r-1)E_n(\deg r)^{1/\alpha}\Big)\left(\int_K\big|P(t)\big|^{\alpha}d\nu_K(t)\right)^{1/\alpha}$$
$$=E_n(\deg r)^{1/\alpha+1}B(P,K)^{1/\alpha}.$$

Summing up all these together we obtain assertion (23).

Now we prove (24) from (23) similarly as in the proof of Lemma 7. So let $X := B^{1/\alpha}(P^*, r^{-1}[r(H)])$ and $Y := (\deg r)^{1/\alpha} B^{1/\alpha}(P, H)$ and assume that $X > Y$.

$$X^\alpha \le \big(Y + o(1)B^{1/\alpha}(P,K)\big)^\alpha = \big(1+o(1)\big)^\alpha \left(\frac{Y + o(1)B^{1/\alpha}(P,K)}{1+o(1)}\right)^\alpha \le$$

where we apply the generalized weighted mean inequality for Y and $B^{1/\alpha}(P,K)$ with the weights 1 and $o(1)$ respectively in the big bracket between power 1 and power α, so we can continue

$$\le \big(1+o(1)\big)\Big(Y^\alpha + o(1)B(P,K)\Big).$$

Rearranging and using that by assumption $Y^\alpha \le \deg(r)B(P,K)$ we obtain (24) if $X > Y$.

On the other hand, if $Y > X$, then we can do the same steps and use the following estimates. First, we use the generalized mean inequality, and for each term, split K as $K = \cup K_{b,j}$, then on each branch, we use (61), that is,

$$\begin{aligned} B(P^*, K) &= \int_K \big|\sum_k P(t_k) q(H; t_k)\big|^\alpha d\nu_K(t) \\ &\le (\deg r)^{\alpha-1} \sum_k \int_K \big|P(t_k)q(H;t_k)\big|^\alpha d\nu_K(t) \\ &= (\deg r)^{\alpha-1} \sum_k \sum_j \int_{K_{b,j}} \big|P(t_k)q(H;t_k)\big|^\alpha d\nu_K(t) \\ &= (\deg r)^{\alpha-1} \sum_k \sum_j \int_{K_{b,k}} \big|P(t)q(H;t)\big|^\alpha d\nu_K(t) \\ &= (\deg r)^\alpha \int_K \big|P(t)q(H;t)\big|^\alpha d\nu_K(t) = (\deg r)^\alpha B(Pq, K). \end{aligned}$$

Then, using $|q| \le 1$, we have $o(1)B(P^*, K) = o(1)B(P,K)$. Therefore, we obtained

$$|B(P^*, r^{-1}[r(H)]) - (\deg r)B(P,H)| \le o(1)B(P,K) \tag{79}$$

where $o(1)$ is independent of P_n, H and H_b.

Now, summing up (76), (78) and (79), we obtain (24) in the form

$$|B(P^*, K) - (\deg r)B(P,H)| \le \big(2^{\alpha-1}(\deg r)b(P, H_b) + o(1)\big)\, B(P,K) \tag{80}$$

where $o(1)$ is independent of P_n, H and H_b. □

Proof of Lemma 11. The proofs in this Lemma are similar to the proofs above.

First we prove (43).

$$\begin{aligned}
&\big|A(P_n\tilde{q},X)^{1/\alpha}-A(P_n,X)^{1/\alpha}\big|\\
&\leq \Big|\Big(\int_X\Big|\frac{P_n'(t)\tilde{q}(t)}{\deg(P_n\tilde{q})\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Big)^{1/\alpha}-\Big(\int_X\Big|\frac{P_n'(t)}{\deg(P_n)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Big)^{1/\alpha}\Big|\\
&\qquad+\Big(\int_X|P_n(t)|^\alpha\Big|\frac{\tilde{q}'(t)}{\deg(P_n\tilde{q})\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Big)^{1/\alpha}\\
&\leq \Big(\int_X\Big|\frac{P_n'(t)}{\deg(P_n)\pi\omega_K(t)}\Big|^\alpha\Big|\tilde{q}(t)\frac{\deg(P_n)}{\deg(P_n\tilde{q})}-1\Big|^\alpha d\nu_K(t)\Big)^{1/\alpha}\\
&\qquad+\Big(\int_X|P_n(t)|^\alpha\Big|\frac{\tilde{q}'(t)}{\deg(P_n\tilde{q})\pi\omega_K(t)}\Big|^\alpha d\nu_K(t)\Big)^{1/\alpha}\leq
\end{aligned}$$

which we continue later. To estimate the last two terms, we use that $q(\tilde{H},\deg(P_n);t)\rightrightarrows 1$ as $\deg(P_n)\to\infty$ on $t\in X$, where $X\subset\tilde{H}$ by assumption, $\deg(P_n)/\deg(P_n\tilde{q})\to 1$ and inequality (2) for $\tilde{q}$, where $|\tilde{q}(t)|\leq 1$ $(t\in K)$, so we can continue

$$\begin{aligned}
&\leq o(1)A(P_n,X)^{1/\alpha}+\frac{\deg(\tilde{q})}{\deg(P_n\tilde{q})}B(P_n,X)^{1/\alpha}\\
&\qquad\qquad\leq o(1)A(P_n,K)^{1/\alpha}+o(1)B(P_n,K)^{1/\alpha}.
\end{aligned}$$

Now, we get rid of the powers $1/\alpha$ as follows. If $A(P_n,X)\geq A(P_n\tilde{q},X)$, then using the weighted generalized mean inequality, we have

$$\begin{aligned}
A(P_n,X)&\leq\big(A(P_n\tilde{q},X)^{1/\alpha}+o(1)A(P_n,K)^{1/\alpha}+o(1)B(P_n,K)^{1/\alpha}\big)^\alpha\\
&\leq\big(1+o(1)\big)\frac{A(P_n\tilde{q},X)+o(1)^\alpha A(P_n,K)+o(1)^\alpha B(P_n,K)}{1+o(1)+o(1)}\\
&\leq\big(1+o(1)\big)A(P_n\tilde{q},X)+o(1)A(P_n,K)+o(1)B(P_n,K).
\end{aligned}$$

Now we have to prove that $o(1)A(P_n\tilde{q},X)\leq o(1)A(P_n,K)+o(1)B(P_n,K)$. As in (74), we use the generalized mean inequality, $|\tilde{q}|\leq 1$, and inequality (2) for $\tilde{q}$, so

$$\begin{aligned}
A(P_n\tilde{q},X)&\leq 2^{\alpha-1}\Big(\frac{\deg(P_n)}{\deg(P_n\tilde{q})}\Big)^\alpha A(P_n,X)+2^{\alpha-1}\Big(\frac{\deg(\tilde{q})}{\deg(P_n\tilde{q})}\Big)^\alpha B(P_n,X)\\
&\leq 2^{\alpha-1}A(P_n,K)+2^{\alpha-1}B(P_n,K).
\end{aligned}$$

Using this, and condition $A(P_n,X)\geq A(P_n\tilde{q},X)$, we can write

$$0\leq A(P_n,X)-A(P_n\tilde{q},X)\leq o(1)A(P_n,K)+o(1)B(P_n,K).$$

Similarly, if $A(P_n\tilde{q},X)\geq A(P_n,X)$:

$$\begin{aligned}
A(P_n\tilde{q},X)&\leq\big(A(P_n,X)^{1/\alpha}+o(1)A(P_n,K)^{1/\alpha}+o(1)B(P_n,K)^{1/\alpha}\big)^\alpha\\
&\leq\big(1+o(1)\big)\frac{A(P_n,X)+o(1)^\alpha A(P_n,K)+o(1)^\alpha B(P_n,K)}{1+o(1)+o(1)}\\
&\leq\big(1+o(1)\big)A(P_n,X)+o(1)A(P_n,K)+o(1)B(P_n,K).
\end{aligned}$$

Rearranging, and using $A(P_n\tilde{q}, X) \geq A(P_n, X)$, we are done with (43).

For (44) all we have to use is that $|q(\tilde{H}, \deg(P_n); t)|^\alpha \rightrightarrows 1$ as $\deg(P_n) \to \infty$ on $t \in X$ because $X \subset \tilde{H}$.

Now we prove (45). Without loss of generality, we may assume that $||P_n||_{K,\infty} = 1$ (when P_n is not identically zero, this can be achieved by multiplying through by a suitable constant).

$$|\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) - \tilde{A}_\delta(P_n\tilde{q}, \tilde{H})| \leq \tilde{A}_\delta(P_n\tilde{q}, \tilde{H}_b) + \tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)).$$

We estimate the first term on the right hand side using the definition of δ_1 from Section 5.4.2, and the following two relations between ω_K and $\omega_{\tilde{K}(\delta)}$

$$\begin{gathered} \omega_{\tilde{K}(\delta)}(t) \geq \omega_K(t) \quad (t \in \tilde{K}(\delta)), \\ \left(\omega_{\tilde{K}(\delta)}(t)\right)^{1-\alpha} \leq (\omega_K(t))^{1-\alpha} \quad (t \in \tilde{K}(\delta)) \end{gathered} \tag{81}$$

since $\tilde{K}(\delta) \subset K$. So

$$\begin{aligned} \tilde{A}_\delta(P_n\tilde{q}, \tilde{H}_b) &= \int_{\tilde{H}_b} \Big|\frac{P_n'(t)\tilde{q}(t) + P_n(t)\tilde{q}'(t)}{\deg(P_n\tilde{q})\pi\omega_{\tilde{K}(\delta)}}\Big|^\alpha d\nu_{\tilde{K}(\delta)}(t) \\ &\leq 2^{\alpha-1} \int_{\tilde{H}_b} \Big|\frac{P_n'(t)\tilde{q}(t)}{\deg(P_n\tilde{q})\pi\omega_{\tilde{K}(\delta)}(t)}\Big|^\alpha d\nu_{\tilde{K}(\delta)}(t) \\ &\quad + 2^{\alpha-1} \int_{\tilde{H}_b} \Big|\frac{P_n(t)\tilde{q}'(t)}{\deg(P_n\tilde{q})\pi\omega_{\tilde{K}(\delta)}(t)}\Big|^\alpha d\nu_{\tilde{K}(\delta)}(t) \\ \leq 2^{\alpha-1}(1-\delta_1)^{1-\alpha}&\Big(\frac{\deg(P_n)}{\deg(P_n\tilde{q})}\Big)^\alpha \int_{\tilde{H}_b} \Big|\frac{P_n'(t)}{\deg(P_n)\pi\omega_K(t)}\Big|^\alpha d\nu_K(t) \\ + 2^{\alpha-1}(1-\delta_1)^{1-\alpha}&\Big(\frac{\deg(\tilde{q})}{\deg(P_n\tilde{q})}\Big)^\alpha \int_{\tilde{H}_b} |P_n(t)|^\alpha \Big|\frac{\tilde{q}'(t)}{\deg(\tilde{q})\pi\omega_K(t)}\Big|^\alpha d\nu_K(t) \\ &\leq 2^{\alpha-1}(1-\delta_1)^{1-\alpha}\Big(\frac{\deg(P_n)}{\deg(P_n\tilde{q})}\Big)^\alpha a(P_n, \tilde{H}_b)A(P_n, K) \\ &\quad + 2^{\alpha-1}(1-\delta_1)^{1-\alpha}\Big(\frac{\deg(\tilde{q})}{\deg(P_n\tilde{q})}\Big)^\alpha b(P_n, \tilde{H}_b)B(P_n, K) \\ &= o(1)A(P_n, K) + o(1)B(P_n, K) \end{aligned} \tag{82}$$

where we used inequality (2) for $\tilde{q}$ and that $\tilde{H}_b \subset H(J_n')$.

As for the second term, using inequality (2) again

$$\begin{aligned} &\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)) \\ &\quad \leq 2^{\alpha-1}\Big(\frac{\deg(P_n)}{\deg(P_n\tilde{q})}\Big)^\alpha \int_{\tilde{K}(\delta)\setminus(\tilde{H}\cup\tilde{H}_b)} |\tilde{q}(t)|^\alpha \Big|\frac{P_n'(t)}{\deg(P_n)\pi\omega_{\tilde{K}(\delta)}}\Big|^\alpha d\nu_{\tilde{K}(\delta)}(t) \\ &\quad + 2^{\alpha-1}\Big(\frac{1}{\pi\deg(P_n\tilde{q})}\Big)^\alpha \int_{\tilde{K}(\delta)\setminus(\tilde{H}\cup\tilde{H}_b)} |P_n(t)\tilde{q}'(t)|^\alpha \big(\omega_{\tilde{K}(\delta)}(t)\big)^{1-\alpha} dt \\ &\qquad \leq 2^{\alpha-1}\Big(\frac{\deg(P_n)}{\deg(P_n\tilde{q})}\Big)^\alpha E_n^\alpha \int_{\tilde{K}(\delta)\setminus(\tilde{H}\cup\tilde{H}_b)} 1 d\nu_{\tilde{K}(\delta)}(t) \\ &\qquad\qquad + 2^{\alpha-1}\Big(\frac{1}{\pi\deg(P_n\tilde{q})}\Big)^\alpha \int_{\tilde{K}(\delta)} 1 E_n^\alpha \big(\omega_{\tilde{K}(\delta)}(t)\big)^{1-\alpha} dt. \end{aligned}$$

We use (81), therefore

$$\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)) = O(E_n^\alpha).$$

On the other hand, $||P_n||_{K,\infty} = 1$, so there exists ℓ_4 such that $1 = ||P_n||_{K,\infty} = ||P_n||_{K_{c,\ell_4},\infty}$ and with $C_{17} := \min_{t\in K} \omega_K(t) > 0$,

$$\begin{aligned} B(P_n, K) \geq B(P_n, K_{c,\ell_4}) &\geq \int_{K_{c,\ell_4}} |P_n(t)|^\alpha C_{17} dt \\ &\geq C_{17} \frac{1}{(1+\alpha)\big(\deg P_n\big)^2} \frac{|K_{c,\ell_4}|}{2} ||P_n||^\alpha_{K_{c,\ell_4},\infty} \geq C_{18} \frac{1}{\big(\deg P_n\big)^2} = \frac{C_{18}}{n^2} \end{aligned}$$

where $C_{18} = C_{17}/(1+\alpha)\frac{|K_{c,\ell_4}|}{2}$ and we used the Nikolskii inequality for P_n on the interval K_{c,ℓ_4}. For the Nikolskii inequality on $[-1,1]$, see [3] p. 498. Theorem 3.1.4. That is, $B(P_n, K)$ cannot be small,

$$B(P_n, K) \geq \frac{C_{18}}{n^2}, \tag{83}$$

where C_{18} depends on α, K, but is independent of P_n. This implies

$$\tilde{A}_\delta(P_n\tilde{q}, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)) \leq o(1)B(P_n, K). \tag{84}$$

Summing up (82) and (84), we obtain (45).

Finally, we prove (46),

$$\tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta)) - \tilde{B}_\delta(P_n\tilde{q}, \tilde{H}) = \tilde{B}_\delta(P_n\tilde{q}, \tilde{H}_b) + \tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)).$$

Now,

$$\begin{aligned} \tilde{B}_\delta(P_n\tilde{q}, \tilde{H}_b) &\leq \tilde{B}_\delta(P_n, \tilde{H}_b) \leq (1+\delta_1)B(P_n, \tilde{H}_b) \\ &= (1+\delta_1)b(P_n, \tilde{H}_b)B(P_n, K) = o(1)B(P_n, K). \end{aligned} \tag{85}$$

On the other hand,

$$\begin{aligned} \tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)) &\leq E_n^\alpha \tilde{B}_\delta(P_n, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)) \\ &\leq E_n^\alpha \tilde{B}_\delta(P_n, \tilde{K}(\delta)) \leq E_n^\alpha \tilde{B}_\delta(1, \tilde{K}(\delta)) = E_n^\alpha \end{aligned}$$

and using (83) again, we have

$$\tilde{B}_\delta(P_n\tilde{q}, \tilde{K}(\delta) \setminus (\tilde{H} \cup \tilde{H}_b)) = o(1)B(P_n, K). \tag{86}$$

So summing up (85) and (86) gives (46).

Therefore the Lemma is proved.

□

Proof of Lemma 12. The proofs of (53) and (54) are similar to the proofs of (45) and (46) replacing $\tilde{K}(\delta)$ by Y and $\tilde{H}$ by H. □

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