Chebyshev polynomials on compact sets*

Vilmos Totik†

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Abstract

In connection with a problem of H. Widom it is shown that if a compact set K on the complex plane contains a smooth Jordan arc on its outer boundary, then the minimal norm of monic polynomials of degree $n = 1, 2, \ldots$ is at least $(1 + \beta)\mathrm{cap}(K)^n$ with some $\beta > 0$, where $\mathrm{cap}(K)^n$ would be the theoretical lower bound. It is also shown that the rate $(1 + o(1))\mathrm{cap}(K)^n$ is possible only for compact for which the unbounded component of the complement is simply connected. A related result for sets lying on the real line is also proven.

1 Results

Let K be a compact subset on the complex plane consisting of infinitely many points, and let $T_n(z) = z^n + \cdots$ be the unique monic polynomial of degree $n = 1, 2, \ldots$ which minimizes the supremum norm $\|T_n\|_K$ on K among all monic polynomial of the same degree. This T_n is called the n-th Chebyshev polynomial on K. Chebyshev polynomials originated from a problem in classical mechanics, and due to their extremal properties they are connected with numerical analysis, potential theory, continued fractions, orthogonal polynomials, number theory, function theory, approximation theory, polynomial inequalities etc. For their importance and various uses and appearances we refer to [9].

In what follows we shall use potential theoretic concepts such as logarithmic capacity, Green's function, equilibrium measure etc., see [1], [2], [6] or [7] for these concepts and their properties.

It is a simple fact (see e.g. [6, Theorem 5.5.4]) that

$$\|T_n\|_K \geq \mathrm{cap}(K)^n, \tag{1}$$

where $\mathrm{cap}(K)$ is the logarithmic capacity of K, and it is a delicate problem how close the minimal norm $\|T_n\|_K$ can get to the theoretical lower bound $\mathrm{cap}(K)^n$.

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That $\|T_n\|_K$ is not exponentially larger than $\mathrm{cap}(K)^n$ is a theorem of Szegő (see e.g. [6, Corollary 5.5.5]):

$$\lim_{n\to\infty} \|T_n\|_K^{1/n} = \mathrm{cap}(K).$$

In a deep paper H. Widom [13] described (in terms of some extremal problems for analytic functions) the behavior of $\|T_n\|_K$ in the case when K consists of finitely many disjoint C^{2+} smooth Jordan curves. Recall that a Jordan curve is a homeomorphic image of the unit circle, while a Jordan arc is a homeomorphic image of the interval $[0,1]$. Widom's theory was less complete when K had arc components, and he conjectured in that case that necessarily

$$\liminf_{n\to\infty} \frac{\|T_n\|_K}{\mathrm{cap}(K)^n} \geq 2.$$

That this is true if $K \subset \mathbf{R}$ follows from [8], but the general case is still open.

If K consists of smooth Jordan curves then it follows from the results of [13] that

$$\liminf_{n\to\infty} \frac{\|T_n\|_K}{\mathrm{cap}(K)^n} = 1. \tag{2}$$

Our first theorem shows that this is not possible if K contains an arc on its outer boundary.

In what follows Ω denotes the unbounded connected component of $\overline{\mathbf{C}} \setminus K$.

Theorem 1 *Suppose that for some disk Δ the intersection $\Delta \cap K$ is a $C^{1+\alpha}$, $\alpha > 0$, Jordan arc and $\Delta \setminus K \subset \Omega$. Then there is a $\beta > 0$ such that for all $n = 1, 2, \ldots$ the inequality $\|T_n\|_K \geq (1+\beta)\mathrm{cap}(K)^n$ holds.*

Therefore, in this case for any monic polynomials P_n we have $\|P_n\|_K \geq (1+\beta)\mathrm{cap}(K)^n$. In a sense this result proves a weak form of Widom's conjecture in a general setting.

The claim in the theorem should be compared to Pommerenke's result in [4] on Fekete points for sets symmetric with respect to the real line which contain at least one line segment on $\mathbf{R}$.

There are sets for which $\|T_n\|_K$ can be very close to $\mathrm{cap}(K)^n$ for all n. The extreme case is a circle/disk when there is equality in (1) for all n, but also when K consist of a single analytic curve then $\|T_n\|_K \leq (1+Cq^n)\mathrm{cap}(K)^n$ with some $0 < q < 1$. In each of these cases the outer domain Ω is simply connected. Next, we show that such a relation is possible only if Ω is simply connected.

Theorem 2 *If Ω is not simply connected, then there is a $c > 0$ and a subsequence $\mathcal{N}$ of the natural numbers such that for $n \in \mathcal{N}$ we have $\|T_n\|_K \geq (1+c)\mathrm{cap}(K)^n$.*

Note that, on the other hand, if K consists of smooth Jordan curves, then along another subsequence $\mathcal{N}'$ we have

$$\lim_{n\to\infty,\ n\in\mathcal{N}'} \frac{\|T_n\|_K}{\mathrm{cap}(K)^n} = 1$$

by (2).

We have already mentioned the fact from [8] that for sets K on the real line

$$\|T_n\|_K \geq 2\text{cap}(K)^n, \tag{3}$$

and it is classical that for an interval we have equality. Our final result says that this is the only case when $\|T_n\|_K$ is close to $2\text{cap}(K)^n$ for all n.

Theorem 3 *If $K \subset \mathbf{R}$ is a compact set which is not an interval, then there is a $c > 0$ and a subsequence $\mathcal{N}$ of the natural numbers such that for $n \in \mathcal{N}$ we have $\|T_n\|_K \geq (2+c)\text{cap}(K)^n$.*

On the other hand, if K consists of finitely many intervals, then there is another subsequence $\mathcal{N}'$ such that

$$\lim_{n\to\infty,\ n\in\mathcal{N}'} \frac{\|T_n\|_K}{\text{cap}(K)^n} = 2.$$

see [11].

2 Preliminaries for the proofs

The proofs of the results in this paper rely on results from logarithmic potential theory, see e.g. [1], [2], [6] or [7] for the concepts appearing below.

For a compact subset K of the complex plane let $\text{cap}(K)$ denote its logarithmic capacity and μ_K its equilibrium measure. Then, by Frostman's theorem [6, Theorem 3.3.4], for the logarithmic potential

$$U^{\mu_K}(z) = \int \log \frac{1}{|z-t|} d\mu_K(t)$$

we have

$$U^{\mu_K}(z) \leq \log \frac{1}{\text{cap}(K)}, \qquad z \in \mathbf{C} \tag{4}$$

and

$$U^{\mu_K}(z) = \log \frac{1}{\text{cap}(K)}, \qquad \text{for quasi-every } z \in K, \tag{5}$$

i.e. with the exception of a set of zero capacity. If K consists of finitely many Jordan curves or arcs then (5) is true everywhere on K by Wiener's criterion [6, Theorem 5.4.1]. Let Ω be the unbounded connected component of $\overline{\mathbf{C}} \setminus K$ and let $g_\Omega(z,\infty) \equiv g_{\overline{\mathbf{C}}\setminus K}(z,\infty)$ be the Green's function in Ω with pole at infinity. Then (see e.g. [6, Sec. 4.4] or [7, (I.4.8)])

$$g_{\overline{\mathbf{C}}\setminus K}(z,\infty) = \log \frac{1}{\text{cap}(K)} - U^{\mu_K}(z). \tag{6}$$

If G is a domain for which the boundary is of positive capacity and $z_0 \in G$ is a fixed point, then let $\omega(\cdot, z_0, G)$ denote the harmonic measure for z_0 relative to G.

Let, as before, Ω be the unbounded component of $\overline{\mathbf{C}} \setminus K$, where K is a compact set of positive capacity. We shall need the notion of balayage out of Ω: if ρ is a finite Borel-measure with compact support in Ω, then (see [7, Theorem II.4.4]) there is a measure $\widehat{\rho}$ supported on $\partial\Omega$, called its balayage, such that $\widehat{\rho}$ has the same total mass as ρ has and

$$U^{\widehat{\rho}}(z) = U^{\rho}(z) + \text{const} \tag{7}$$

quasi-everywhere (i.e. with the exception of a set of zero capacity) on K. When we require that $\widehat{\rho}$ should vanish on sets of zero capacity, then $\widehat{\rho}$ is unique. The constant in (7) can be expressed via the Green's function, namely (see e.g. [7, Theorem 4.4])

$$U^{\widehat{\rho}}(z) = U^{\rho}(z) + \int_{\Omega} g_{\Omega}(a, \infty) d\rho(a). \tag{8}$$

There is a related concept: balayage out of a bounded region G. If ρ is a Borel-measure on G then (see e.g. [7, Theorem 4.1]) there is a measure $\widehat{\rho}$ on ∂G such that $\widehat{\rho}$ has the same total mass as ρ has and

$$U^{\widehat{\rho}}(z) = U^{\rho}(z) \tag{9}$$

for quasi-every $z \notin G$.

Note that $\omega(E, z_0, \Omega) = \widehat{\delta_{z_0}}(E)$ for any Borel-set $E \subseteq \partial\Omega$, and $\omega(\cdot, \infty, \Omega) = \widehat{\delta_{\infty}}$ is just the equilibrium measure of the set K.

The set $\text{Pc}(K) = \overline{\mathbf{C}} \setminus \Omega$ is called the polynomial convex hull of K (it is the union of K with all the bounded components of $\overline{\mathbf{C}} \setminus K$). Clearly, Ω is simply connected if and only if $\text{Pc}(K)$ is connected.

With these we prove first

Lemma 4 *Let* $\text{Pc}(K)_r = \{z \mid \text{dist}(z, \text{Pc}(K)) < r\}$ *be the* r*-neighborhood of* $\text{Pc}(K)$*. Then there is an* $\varepsilon_r > 0$ *such that if* P_n *is a monic polynomial of degree* n *satisfying* $\|P_n\|_K \le e^{\varepsilon_r} \text{cap}(K)^n$*, then all zeros of* P_n *lie inside* $\text{Pc}(K)_r$.

Proof. Let $z_{1,n}, \ldots, z_{n,n}$ be the zeros of P_n, and of these let $z_{1,n}, \ldots, z_{k_n,n}$ lie outside K_r. Consider $\widehat{\delta_{z_{j,n}}}$, the balayage of the Dirac delta $\delta_{z_{j,n}}$ at $z_{j,n}$ out of $\Omega = \overline{\mathbf{C}} \setminus \text{Pc}(K)$. Since

$$U^{\widehat{\delta_a}}(z) = U^{\delta_a}(z) + g_{\Omega}(a, \infty)$$

for quasi-every $z \in K$ and

$$U^{\mu_K}(z) \le \log \frac{1}{\text{cap}(K)}, \qquad z \in K,$$

with equality for quasi-every $z \in K$, it follows that for the measure

$$\widehat{\nu}_n = \sum_{j=1}^{k_n} \widehat{\delta}_{z_{j,n}} + \sum_{j=k_n+1}^{n} \delta_{z_{j,n}}$$

we have for quasi-every $z \in K$

$$-U^{\widehat{\nu_n}}(z) + nU^{\mu_K}(z) = \log|P_n(z)| + nU^{\mu_K}(z) - \sum_{j=1}^{k_n} g_\Omega(z_{j,n}, \infty).$$

Now if we assume $\|P_n\|_K \le e^\varepsilon \mathrm{cap}(K)^n$, then

$$\log|P_n(z)| + nU^{\mu_K}(z) \le \log|P_n(z)| - n\log\mathrm{cap}(K) \le \varepsilon \tag{10}$$

for all $z \in K$, and so

$$-U^{\widehat{\nu_n}}(z) + nU^{\mu_K}(z) \le \varepsilon - \sum_{j=1}^{k_n} g_\Omega(z_{j,n}, \infty) \tag{11}$$

follows quasi-everywhere on K. By the principle of domination (see e.g. [7, Theorem II3.2]) (note that μ_K has finite logarithmic energy), this inequality pertains for all $z \in \overline{\mathbf{C}}$. But at ∞ the left-hand side is zero, therefore we obtain

$$\sum_{j=1}^{k_n} g_\Omega(z_{j,n}, \infty) \le \varepsilon. \tag{12}$$

Now the lemma follows (i.e. there cannot be any $z_{j,n} \notin \mathrm{Pc}(K)_r$ for sufficiently small ε) since $g_\Omega(z,\infty)$ has a strict lower bound outside $\mathrm{Pc}(K)_r$, being a positive harmonic function there.

■

3 Proof of Theorem 1

Let $\gamma = \overline{\Delta} \cap K$ be the arc component in question in the theorem, and let s_γ denote the arc measure on γ.

Suppose to the contrary that for some sequence $\mathcal{N} \subset \mathbf{N}$ we have

$$\|T_n\|_K = (1 + o(1))\mathrm{cap}(K)^n \qquad \text{as } n \to \infty,\ n \in \mathcal{N}, \tag{13}$$

and let ν_n be the normalized counting measure on the zeros of T_n. Then

$$|T_n(z)|^{1/n} = \exp(-U^{\nu_n}(z)).$$

Choose a closed subarc γ_1 of γ that does not contain the endpoints of γ, and then a subarc γ_2 of γ_1 that does not contain the endpoints of γ_1.

First we mention that the equilibrium measure μ_K is absolutely continuous on γ with respect to arc measure s_γ, and its density is continuous and positive in the (one dimensional) interior of γ. This is very classical, it is basically a localized form of the Kellogg-Warschawski theorem (see [5, Theorem 3.6]). For a reference see e.g. [12, Proposition 2.2] in the special case if γ is a connected component of K. For arbitrary K just follow the proof of [12, Proposition 2.2]; we do not repeat the details. Let us also note that the argument of [12, Proposition 2.2] gives on γ the strict positivity of both $\partial g_{\overline{\mathbf{C}}\setminus K}(z,\infty)\Big/\partial \mathbf{n}_{\pm}$, where $\mathbf{n}_{\pm}$ are the two normals to γ, and hence

$$g_{\overline{\mathbf{C}}\setminus K}(z,\infty) \geq c \cdot \mathrm{dist}(z,\gamma) \tag{14}$$

in a neighborhood of γ_1 with a positive constant c.

In order to verify Theorem 1 we are going to prove several statements.

Claim I. If τ is a weak limit point of ν_n, $n \in \mathcal{N}$, then $\tau\big|_{\gamma_1} = \mu_K\big|_{\gamma_1}$.*

Indeed, note first of all that τ is supported on $\mathrm{Pc}(K)$ by Lemma 4. Next, (13) shows that on K we have $U^{\mu_K}(z) - U^{\nu_n}(z) \leq o(1)$ (recall (4)), and hence, by the principle of domination (see e.g. [7, Theorem II3.2]), this holds then throughout $\mathbf{C}$. On making limit along a subsequence for which $\nu_n \to \tau$ in the weak*-topology we can conclude the inequality $U^{\mu_K}(z) - U^{\tau}(z) \leq 0$ for all $z \in \Omega$. Now if $\widehat{\tau}$ is the balayage of τ onto $\partial\Omega$ (i.e. we sweep out τ from each bounded component of $\overline{\mathbf{C}} \setminus \Omega$), then $U^{\widehat{\tau}}(z) = U^{\tau}(z)$ on Ω (see (9)), and hence we have in Ω the inequality $U^{\mu_K}(z) - U^{\widehat{\tau}}(z) \leq 0$. Since the left-hand side is harmonic in Ω and vanishes at infinity, we can conclude by the maximum principle for harmonic functions that $U^{\mu_K}(z) - U^{\widehat{\tau}}(z) \equiv 0$ in Ω. But here both measures μ_K and $\widehat{\tau}$ are supported on $\partial\Omega$, hence Carleson's unicity theorem [7, Theorem II.4.13] gives $\mu_K = \widehat{\tau}$. Finally, the claim follows, since $\tau = \widehat{\tau}$ on γ.

As a consequence, it follows that in any neighborhood U of γ_2 there are more than $n\mu_K(\gamma_2)/2$ zeros of T_n for large $n \in \mathcal{N}$. Since inside γ the measures μ_γ and the arc measure s_γ on γ are comparable, this also gives that there is a $c_0 > 0$ such that in any neighborhood U of γ_2 there are more than $c_0 n s_\gamma(\gamma_2)$ zeros of T_n for large $n \in \mathcal{N}$.

Claim II. If $z_{1,n}, \ldots, z_{k_n,n}$ are the zeros of T_n lying in the unbounded component Ω of $\overline{\mathbf{C}} \setminus K$, then

$$\sum_{k=1}^{k_n} g_{\overline{\mathbf{C}}\setminus K}(z_{n,k},\infty) \to 0 \qquad \text{as } n \to \infty,\ n \in \mathcal{N}. \tag{15}$$

See the proof of (12) above.

Claim III. Let

$$E_n = \{z \in \gamma_1 \,|\, |T_n(z)| \leq \mathrm{cap}(K)^n/2\}. \tag{16}$$

Then $s_\gamma(E_n) \to 0$ as $n \to \infty$, $n \in \mathcal{N}$.

To prove this, let $\widehat{\nu_n}$ be the balayage of ν_n out of Ω. Since quasi-everywhere on $\partial\Omega$ we have $U^{\widehat{\nu_n}}(z) \geq U^{\nu_n}(z)$ (see (8)), we get for quasi-every $z \in E_n$ the inequality

$$n(U^{\mu_K}(z) - U^{\widehat{\nu_n}}(z)) \leq \log\frac{1}{\mathrm{cap}(K)^n} + \log|T_n(z)| \leq -\log 2. \tag{17}$$

At the same time, by (4) and by the assumption, we have quasi-everywhere on K (and hence on $\partial\Omega$)

$$n(U^{\mu_K}(z) - U^{\widehat{\nu_n}}(z)) \leq \log \frac{1}{\mathrm{cap}(K)^n} + \log |T_n(z)| \leq o(1) \tag{18}$$

as $n \to \infty$, $n \in \mathcal{N}$. Recall now that μ_K has positive and continuous derivative with respect to the arc measure s_γ, and μ_K is the same as the harmonic measure at ∞, furthermore sets of zero logarithmic capacity have zero μ_K-measures since μ_K has finite logarithmic energy. So if we had $s_\gamma(E_n) \geq c_1$ with some $c_1 > 0$ on some subsequence of $\mathcal{N}_1 \subset \mathcal{N}$, then $\omega(E_n, \infty, \overline{\mathbf{C}} \setminus K) = \mu_K(E_n) \geq c_2$ would be also true with some $c_2 > 0$ on the same subsequence. This and (17)–(18) would then imply

$$\begin{aligned} 0 &= n(U^{\mu_K}(z) - U^{\widehat{\nu_n}}(z))\Big|_{z=\infty} = \int_{\partial\Omega} \Big(n(U^{\mu_K}(\cdot) - U^{\widehat{\nu_n}}(\cdot)\Big) d\omega(\cdot, \infty, \Omega) \\ &= \int_{\partial\Omega} \Big(n(U^{\mu_K}(\cdot) - U^{\widehat{\nu_n}}(\cdot)\Big) d\mu_K(t) = \int_{\partial\Omega \setminus E_n} + \int_{E_n} = o(1) - c_2 \log 2, \end{aligned}$$

which is impossible. This shows that, indeed, $s_\gamma(E_n) \to 0$ along $\mathcal{N}$.

Claim IV. There is a C_0 such that for all polynomials P_n of degree at most $n = 1, 2, \ldots$ and for all $r \geq 1$ we have

$$|P_n^{(r)}(z)| \leq e^{C_0 n\delta} r! \delta^{-r} \|P_n\|_K, \qquad \textit{for } \mathrm{dist}(z, \gamma_1) \leq \delta, \tag{19}$$

where $\delta > 0$ is arbitrary.

Since the Green's function $g_{\overline{C}\setminus\gamma}(t, \infty)$ is Lip 1 continuous on γ_1 (see e.g. [10, Corollary 7.4] and note that the conformal map appearing in the proof is Lip 1 continuous even for $C^{1+\alpha}$ arcs by the Kellogg-Warschawski theorem ([5, Theorem 3.6]), it follows from

$$g_{\overline{C}\setminus K}(t, \infty) \leq g_{\overline{C}\setminus\gamma}(t, \infty)$$

that with some C the estimate $g_{\overline{C}\setminus K}(t, \infty) \leq C\mathrm{dist}(t, \gamma_1)$ holds. Therefore, by the Bernstein-Walsh lemma [14, p. 77], for all $\mathrm{dist}(t, \gamma_1) < 2\delta$ we have

$$|P_n(t)| \leq e^{n g_{\overline{C}\setminus K}(t,\infty)} \|P_n\|_K \leq e^{2Cn\delta} \|P_n\|_K.$$

Now if we use Cauchy's formula

$$P_n^{(r)}(z) = \frac{r!}{2\pi} \int_{|t-z|=\delta} \frac{P_n(t)}{(t-z)^{r+1}} dt$$

for z lying of distance $\leq \delta$ from γ_1, the claim follows.

Claim V. With the C_0 from Claim IV we have

$$|P_n'(z)| \leq e^{C_0} n \|P_n\|_K, \qquad \textit{for } \mathrm{dist}(z, \gamma_1) \leq 1/n$$

for all polynomials P_n of degree at most n.

This follows from (19) with $\delta = 1/n$.

Let $\overline{ab}$ be a subarc of γ, and denote by $\Delta_r(a)$ the disk of radius r with center at a.

Claim VI. There is a C_1 such that if $\overline{ab}$ is an arc-component of the set E_n from (16) that has non-empty intersection with γ_2, then there are at most $C_1 n s_\gamma(\overline{ab})$ zeros of T_n in the set $V_{\overline{ab}} := \Delta_{|b-a|}(a) \cup \Delta_{|b-a|}(b)$.

Indeed, for large n the arc $\overline{ab}$ lies strictly inside γ_1 by Claim III. Now let C_1 be some fixed number, and suppose there are $2\overline{M} > 2C_1 n s_\gamma(\overline{ab})$ zeros of T_n in the set $V_{\overline{ab}}$. Then there are at least $M \geq \overline{M} > C_1 n s_\gamma(\overline{ab}) > C_1 n|b-a|$ zeros either in $\Delta_{|b-a|}(a)$ or in $\Delta_{|b-a|}(b)$, say in $\Delta_{|b-a|}(a)$. Let Q_n be the polynomial that we obtain from T_n by moving all its zeros lying in $\Delta_{|b-a|}(a)$ into a. Outside the set $\Delta_{2|b-a|}(a)$ clearly $|Q_n(z)| \leq 2^M |T_n(z)|$, and we also have $|T_n(b)| \leq 2^M |Q_n(b)|$. Next we show that the polynomial Q_n cannot attain its absolute maximum on K in the set $\gamma \cap \Delta_{2|b-a|}(a)$, and then, from what we have just said,

$$\|Q_n\|_K \leq 2^M \|T_n\|_K < 2^{M+1} \mathrm{cap}(K)^n \tag{20}$$

will follow for large $n \in \mathcal{N}$. To prove this claim note that $Q_n(z)$ has a zero at a of order M, we can write

$$Q_n(z) = \int_a^z \int_a^{w_1} \cdots \int_a^{w_{M-1}} Q_n^{(M)}(w) dw dw_{M-1} \cdots dw_1.$$

If $z = \gamma(s)$, $s \in [0, s_\gamma(\overline{ab})]$ is the arc-length parametrization of $\overline{ab}$ with $\gamma(0) = a$, then this takes the form

$$Q_n(z) = \int_0^s \int_0^{\tau_1} \cdots \int_0^{\tau_{M-1}} Q_n^{(M)}(\gamma(\tau))\gamma'(\tau)\gamma'(\tau_{M-1}) \cdots \gamma'(\tau_1)\, d\tau d\tau_{M-1} \cdots d\tau_1. \tag{21}$$

Here $|\gamma'(\tau)| = 1$, and (19) with $\delta = C_1|b-a|$ gives for $\tau \in \gamma \cap \Delta_{2|b-a|}(a)$

$$|Q_n^{(M)}(\gamma(\tau))| \leq e^{C_0 C_1 |b-a| n} M! \frac{1}{(C_1|b-a|)^M} \|Q_n\|_K.$$

In repeated integration in (21) the $1/M!$ comes in, hence we obtain from (21)

$$|Q_n(z)| \leq e^{C_0 C_1 |b-a| n} \frac{1}{(C_1|b-a|)^M} s_\gamma(\overline{az})^M \|Q_n\|_K.$$

Here $s_\gamma(\overline{az}) \leq 4|b-a|$ for $z \in \gamma \cap \Delta_{2|b-a|}(a)$, and we increase the right-hand side if in the first exponent we write instead of $C_1 n|b-a|$ the larger value M, hence

$$|Q_n(z)| \leq e^{C_0 C_1 |b-a| n} \left(\frac{4}{C_1}\right)^M \|Q_n\|_K \leq \left(\frac{4e^{C_0}}{C_1}\right)^M \|Q_n\|_K.$$

Now if $C_1 > 4e^{C_0}$ then the factor in front of $\|Q_n\|_K$ on the right-hand side is smaller than 1 (we may assume $M \geq 1$ for otherwise there is nothing to prove).

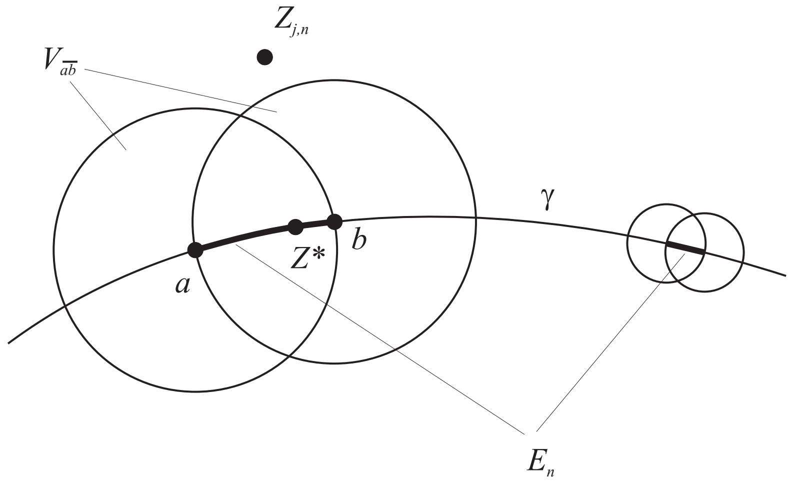

Figure 1: The sets $V_{\overline{ab}}$ and the points $Z_{j,n}$, Z^*

This means that the norm $\|Q_n\|_K$ is not attained in $\gamma \cap \Delta_{2|b-a|}(a)$, and so (20) is true.

Therefore, for large $n \in \mathcal{N}$ the preceding inequality and (20) give for $z = b$

$$|T_n(b)| \leq 2^M |Q_n(b)| \leq \left(\frac{2 \cdot 4e^{C_0}}{C_1}\right)^M \|Q_n\|_K < \left(\frac{2 \cdot 2 \cdot 4e^{C_0}}{C_1}\right)^M 2\mathrm{cap}(K)^n.$$

Now if $C_1 > 64e^{C_0}$ then the right-hand side is smaller than $\mathrm{cap}(K)^n/2$, which is not the case, since $|T_n(b)| = \mathrm{cap}(K)^n/2$ by the choice of b (it was an endpoint of a subarc of E_n). This contradiction proves claim VI.

After these preparations let us turn to the proof of Theorem 1. Claims I (see the consequence mentioned just before Case II), III and VI show that for large $n \in \mathcal{N}$ there are $r_n \geq c_0 n s_\gamma(\gamma_2)/2$ zeros of T_n close to γ_2 (in any fixed neighborhood) lying outside the set

$$\bigcup \{V_{\overline{ab}} \,\big|\, \overline{ab} \text{ is a subarc of } E_n\}.$$

Let these be $Z_{1,n}, \ldots, Z_{r_n,n}$. By Claim II for at least one of them we must have $g_{\overline{\mathbf{C}}\setminus K}(Z_{j,n}, \infty) \leq \varepsilon/n$ for large n, whatever $\varepsilon > 0$ is. This means, in view of (14), that $\mathrm{dist}(Z_{j,n}, \gamma_1) \leq C_2\varepsilon/n$ with some fixed C_2. Now it is easy to see that there must be a point $Z \in \gamma_1 \setminus E_n$ which is of distance $\leq 4\mathrm{dist}(Z_{j,n}, \gamma_1) \leq 4C_2\varepsilon/n$ from $Z_{j,n}$. Indeed, if the closest point Z^* to $Z_{j,n}$ on γ lies outside E_n then this is clear with $Z = Z^*$. On the other hand, if Z^* lies in a subarc $\overline{ab}$ of E_n (see Figure 1), then, by the choice of the set $V_{\overline{ab}}$, we have $\mathrm{dist}(Z_{j,n}, a) < 3\mathrm{dist}(Z_{j,n}, \gamma_1) \leq 3C_2\varepsilon/n$, and hence a point $Z \notin E_n$ lying close to a suffices. Now Claim V gives (via integration along the segment connecting $Z_{j,n}$ and Z) that then for sufficiently small $\varepsilon > 0$

$$|T_n(Z)| = \left|\int_{Z_{j,n}}^{Z} T_n'(\xi)d\xi\right| \leq e^{C_0} n \|T_n\|_K (4C_2\varepsilon/n) < \mathrm{cap}(K)^n/3,$$

which is impossible by the definition of E_n, since then Z would have to belong to E_n.

■

4 Proof of Theorems 2 and 3

Proof of Theorem 2. First we prove the following lemma. In it we use the notations from Lemma 4.

Lemma 5 *For every $r > 0$ there is a C_r such that if P_n is a monic polynomial of degree n for which $\|P_n\|_K \le e^{\varepsilon}\mathrm{cap}(K)^n$ with some $\varepsilon \le \varepsilon_{r/2}$, then*

$$|\log|P_n(z)| + nU^{\mu_K}(z)| \le C_r\varepsilon$$

for $z \notin \mathrm{Pc}(K)_r$.

Proof. Since for $\varepsilon \le \varepsilon_{r/2}$ the polynomial has no zero in $\overline{\mathbf{C}} \setminus \overline{\mathrm{Pc}(K)_{r/2}}$, the function

$$\varepsilon - (\log|P_n(z)| + nU^{\mu_K}(z))$$

is harmonic there. Furthermore, this is a nonnegative function in $\overline{\mathbf{C}} \setminus \overline{K_{r/2}}$ (actually on the whole complex plane) by the principle of domination (see e.g. [7, Theorem II3.2]), because it is nonnegative on K (see (4)). Since it also takes the value ε at infinity and since $\overline{\mathbf{C}} \setminus \mathrm{Pc}(K)_r$ is a closed subset of $\overline{\mathbf{C}} \setminus \overline{\mathrm{Pc}(K)_{r/2}}$, Harnack's inequality gives that there is a C such that

$$0 \le \varepsilon - (\log|P_n(z)| + nU^{\mu_K}(z)) \le C\varepsilon, \qquad z \notin \mathrm{Pc}(K)_r.$$

■

After these let us return to the proof of Theorem 2. Since Ω is not simply connected, we have that $\mathrm{Pc}(K)$ is not connected, and hence there is a C^2 Jordan curve γ in Ω that separates two points of K. Let $\delta > 0$ be so small that the set $V_\delta = \{z \,|\, \mathrm{dist}(z, \gamma) \le \delta\}$ is still part of Ω.

First suppose that both components of $\overline{\mathbf{C}} \setminus \gamma$ intersect K in a set of positive capacity. Let K^* be one of these intersections, say K^* is the intersection of K with the interior of γ. Then $0 < \mu_K(K^*) < 1$, so there are infinitely many n's (let these form the sequence $\mathcal{N}$ in the theorem) such that $N + 1/3 \le n\mu_K(K^*) \le N + 2/3$ with some integer N (which of course depends on n).

Now assume that $\|T_n\|_K \le e^{\varepsilon}\mathrm{cap}(K)^n$ for some $n \in \mathcal{N}$ and $\varepsilon > 0$. Let r be so small that $\mathrm{Pc}(K)_r \cap V_\delta = \emptyset$. By Lemma 5 if $\varepsilon < \varepsilon_{r/2}$ we have

$$|\log|T_n(z)| + nU^{\mu_K}(z)| \le C\varepsilon \tag{22}$$

for all $z \in V_\delta$. Then for the normal derivative with respect to the inner normal $\mathbf{n}$ to γ we have with some C_1 (that may depend on δ) the inequality

$$\left| \frac{\partial(\log |T_n(z)| + nU^{\mu_K}(z))}{\partial \mathbf{n}} \right| \le C_1 \varepsilon, \qquad z \in \gamma. \tag{23}$$

In fact, for $z \in \gamma$ the disk $D_\delta(z)$ of radius δ and with center at z lies in V_δ, hence for the harmonic function $\log |T_n(z)| + nU^{\mu_K}(z)$ the estimate (22) is true in $D_\delta(z)$. Now if we apply Poisson's formula in $D_\delta(z)$, then (23) follows with $C_1 = 2C/\delta$.

Recall now that

$$\log |T_n(z)| + nU^{\mu_K}(z) = U^{-\nu_n + n\mu_K}(z),$$

where ν_n is the counting measure on the zeros of P_n. By Gauss' theorem (see e.g. [7, Theorem II.1.1])

$$\frac{1}{2\pi} \int_\gamma \frac{\partial(\log |T_n(z)| + nU^{\mu_K}(z))}{\partial \mathbf{n}} ds_\gamma = -\nu_n(G) + n\mu_K(G), \tag{24}$$

where G is the domain enclosed by γ. Hence

$$|\nu_n(G) - n\mu_K(K^*)| \le C_1 \varepsilon \frac{s_\gamma(\gamma)}{2\pi}, \tag{25}$$

which is impossible for $\varepsilon < 1/C_1 s_\gamma(\gamma)$ by the choice of the numbers in $\mathcal{N}$ and by the fact that $\nu_n(G)$ is an integer (the number of zeros of P_n inside G). This contradiction shows that $\|T_n\|_K \le e^\varepsilon \mathrm{cap}(K)^n$ is impossible $n \in \mathcal{N}$ if $\varepsilon > 0$ is small.

It is left to consider the case when the intersection of K with one of the components of $\overline{\mathbf{C}} \setminus \gamma$ is of zero capacity, say in the exterior of γ the set K has only a zero capacity (but non-empty) portion K^{**}, and let $K^* = K \setminus K^{**}$. Then the capacity and Green's function of K^* is the same as those of K (the Chebyshev polynomials are NOT the same!). Let T_n denote the Chebyshev polynomials for K, and suppose again that for some n we have $\|T_n\|_K \le e^\varepsilon \mathrm{cap}(K)^n$ with some small $\varepsilon > 0$. Apply Lemma 5 with $P_n = T_n$ and with some small r, but with the set K^* replacing K. It follows that for $z \in K^{**}$ we have with some C_0

$$|\log |T_n(z)| + nU^{\mu_K}(z)| \le C_0 \varepsilon.$$

Now if Ω^* is the unbounded component of $\overline{\mathbf{C}} \setminus K^*$, then $K^{**} \subset \Omega^*$, and the preceding inequality takes the form

$$|\log |T_n(z)| - ng_{\Omega^*}(z, \infty) - n \log \mathrm{cap}(K)| \le C_0 \varepsilon, \qquad z \in K^{**}.$$

Therefore, at any $z \in K^{**}$

$$|T_n(z)| \ge \exp\bigl(ng_{\Omega^*}(z, \infty) - C_0 \varepsilon\bigr) \mathrm{cap}(K)^n \ge \exp\bigl(n\rho^* - C_0 \varepsilon\bigr) \mathrm{cap}(K)^n,$$

where ρ^* is the minimum of g_{Ω^*} on K^{**}. On the other hand, the left-hand side is at most $e^\varepsilon \mathrm{cap}(K)^n$ (note that $z \in K^{**} \subset K$), hence we must have $n\rho^* - (1 + C_0)\varepsilon \le 0$, which is not the case for large n. This shows that for $\varepsilon < \varepsilon_{r/2}$ with an r for which $(K^*)_{r/2} \cap K^{**} = \emptyset$ (to be able to apply Lemma 5), the bound $\|T_n\|_K \le e^\varepsilon \mathrm{cap}(K)^n$ is not possible for large $n \in \mathcal{N}$.

■

Proof of Theorem 3. We follow the ideas in the preceding proof, but we need to make substantial modifications for (10) now would take the form

$$\log |P_n(z)| - n \log \mathrm{cap}(K) \le \log 2 + \varepsilon$$

which only yields

$$\sum_{j=1}^{k_n} g_\Omega(z_{j,n}, \infty) \le \log 2 + \varepsilon$$

instead of (12), and due to that the preceding proof breaks down.

.

Since K is not an interval, there is a C^2 Jordan curve γ in Ω that separates two points of K. Let $r, \delta > 0$ be so small that the set

$$V_\delta = \{z \,|\, \mathrm{dist}(z, \gamma) \le \delta\} \tag{26}$$

is part of $\mathbf{C} \setminus K_r$, where $K_r = \{z \,|\, \mathrm{dist}(z, K) < r\}$.

Now suppose that $\|T_n\|_K \le 2e^\varepsilon \mathrm{cap}(K)^n$ for some $n \in \mathcal{N}$ and $\varepsilon > 0$. Then the set

$$H_n = \{z \,|\, T_n(z) \in [-\|T_n\|_K, \|T_n\|_K]\},$$

which is the inverse image of the interval $[-\|T_n\|_K, \|T_n\|_K]$ under the map $z \to T_n(z)$, contains K and lies on the real line since T_n is a real polynomial with all its zeros on the real axis (indeed, if T_n had a zero $\alpha + i\beta$ outside $\mathbf{R}$, then by replacing this zero by α we would get a monic polynomial with smaller norm on K than what T_n has). Hence

$$H_n = T_n^{-1}[-\|T_n\|_K, \|T_n\|_K].$$

Now use that the capacity of an interval equals one quarter of its length and the fact that by [6, Theorem 5.2.5] $\mathrm{cap}(T_n^{-1}(E)) = (\mathrm{cap}(E))^{1/n}$ for all E, to get

$$\begin{aligned} \mathrm{cap}(H_n) &= \Big(\mathrm{cap}([-\|T_n\|_K, \|T_n\|_K])\Big)^{1/n} \\ &\le \Big(\mathrm{cap}([-2e^\varepsilon \mathrm{cap}(K)^n, 2e^\varepsilon \mathrm{cap}(K)^n])\Big)^{1/n} = e^{\varepsilon/n}\mathrm{cap}(K). \end{aligned} \tag{27}$$

First we claim that for sufficiently small $\varepsilon = \varepsilon_r$ the set H_n lies inside K_r for all large n. Suppose this is not a case, and there is a point $z_n \in H_n \setminus K_r$. The set H_n consists of at most n intervals, and let I_n be that subinterval of H_n that contains z_n. We are going to prove

$$\mu_{H_n}(I_n \setminus K_{r/2}) \geq \frac{1}{n} \tag{28}$$

for sufficiently large n. Indeed, if $I_n \cap K_{r/2} = \emptyset$ then this is clear, since the equilibrium measure μ_{H_n} of H_n has mass of the form p/n, $p \in \mathbf{N}$ on each subinterval of H_n (see e.g. [3, Proposition 1.1]). On the other hand, if $I_n \cap K_{r/2} \neq \emptyset$, then I_n contains a subinterval J_n connecting z_n to a point of $K_{r/2}$, and hence the length of J_n is at least $r/2$. Now it is easy to see that if $J = [a, b]$ is the convex hull of K (i.e. the smallest interval containing K) then $H_n \subset J$, and hence

$$\mu_{H_n} \geq \mu_J\big|_{H_n}$$

(since the left-hand side is the balayage of μ_J onto H_n by [7, TheoremIV.1.6(e)]). Now

$$d\mu_J(x) = \frac{1}{\pi}\frac{1}{\sqrt{(x-a)(b-x)}}dx,$$

so it follows that

$$\mu_{H_n}(I_n) \geq \mu_J(J_n) \geq c|J_n| \geq cr/2$$

with some $c > 0$, and this is $> 1/n$ for large n. This completes the proof of (28).

Now μ_K is the balayage of μ_{H_n} onto K, and for this balayage measure we have the formula

$$U^{\mu_K}(z) = U^{\mu_{H_n}}(z) + \int_{H_n \setminus K} g_{\overline{\mathbf{C}}\setminus K}(a, \infty)d\mu_{H_n}(a)$$

for quasi-every $z \in K$. Since the left-hand side is $\log 1/\mathrm{cap}(K)$ for quasi-every $z \in K$, and the first term on the right-hand side is $\log 1/\mathrm{cap}(H_n)$ for all $z \in H_n \supset K$, we get that

$$\log\frac{1}{\mathrm{cap}(K)} = \log\frac{1}{\mathrm{cap}(H_n)} + \int_{H_n \setminus K} g_{\overline{\mathbf{C}}\setminus K}(a, \infty)d\mu_{H_n}(a),$$

which, in view of (27), implies

$$\int_{H_n \setminus K} g_{\overline{\mathbf{C}}\setminus K}(a, \infty)d\mu_{H_n}(a) \leq \frac{\varepsilon}{n}.$$

Since outside the set $K_{r/2}$ the Green's function has a positive lower bound ρ_r, we can infer

$$\rho_r\mu_{H_n}(I_n \setminus K_{r/2}) \leq \frac{\varepsilon}{n},$$

which is impossible for small ε in view of (28). This contradiction proves the claim that $H_n \subset K_r$ for sufficiently small ε.

What we have just proven implies that if $r > 0$ is fixed and ε is sufficiently small, then the function

$$U^{\mu_K}(z) - U^{H_n}(z)$$

is harmonic outside K_r, and takes the value $\log\Big(\mathrm{cap}(H_n)/\mathrm{cap}(K)\Big) \le \varepsilon/n$ quasi-everywhere on K (see (27)). Then, by the principle of domination, the inequality

$$U^{\mu_K}(z) - U^{H_n}(z) \le \frac{\varepsilon}{n}$$

holds for all z. On applying Harnack's inequality to the nonnegative function

$$\frac{\varepsilon}{n} - (U^{\mu_K}(z) - U^{H_n}(z)),$$

which takes the value ε/n at ∞, we can conclude that on the set V_δ (see (26)) we have

$$|U^{\mu_K}(z) - U^{H_n}(z)| \le C_0\frac{\varepsilon}{n}$$

with some C_0 independent of ε and n. Exactly as in (23) this gives

$$\left|\frac{\partial(U^{\mu_K}(z) - U^{H_n}(z))}{\partial\mathbf{n}}\right| \le C_1\frac{\varepsilon}{n}, \qquad z \in \gamma,$$

and then, as in (25), we obtain

$$|\mu_K(G) - \mu_{H_n}(G)| \le \frac{C_1 s_\gamma(\gamma)}{2\pi}\frac{\varepsilon}{n} =: C_2\frac{\varepsilon}{n}, \tag{29}$$

where G is the interior of γ.

Now we can easily complete the proof of Theorem 2. Let K^* be the intersection of K with the interior of γ: $K^* = K \cap G$. If $0 < \mu_K(K^*) < 1$, then, exactly as in the preceding proof, there are infinitely many n's (these form $\mathcal{N}$) for which $N + 1/3 \le n\mu_K(K^*) \le N + 2/3$ with some integer N. Now if for such an $n \in \mathcal{N}$ we had $\|T_n\|_K \le 2e^\varepsilon \mathrm{cap}(K)^n$ for some small $\varepsilon < 1/3C_2$, then (29) was also true, i.e. we would have

$$|n\mu_K(K^*) - n\mu_{H_n}(G)| \le C_2\varepsilon < 1/3, \tag{30}$$

which is impossible by the choice of $n \in \mathcal{N}$ since $G \cap H_n$ consists of some connected components of H_n, hence $n\mu_{H_n}(G)$ is an integer (see e.g. [3, Proposition 1.1]).

If $\mu_K(K^*) = 0$, then (30) means that $n\mu_{H_n}(G)$ is also zero (it must be an integer), which implies that $G \cap H_n = \emptyset$, and then the more so $K^* \cap H_n = \emptyset$, which is impossible since $K^* \subset K \subset H_n$.

In the same way if $\mu_K(K^*) = 1$, then (30) shows that $\mu_{H_n}(G)$ must be also 1, and this means that $H_n \subset G$, which is again impossible since H_n contains K and $K \not\subseteq G$ by the choice of γ.

Thus, in the last two cases $\|T_n\|_K \le 2e^\varepsilon \mathrm{cap}(K)^n$ for some small $\varepsilon < 1/3C_2$ is not possible for any n, and the proof is complete.

■

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Bolyai Institute
Analysis Research Group of the Hungarian Academy os Sciences
University of Szeged
Szeged
Aradi v. tere 1, 6720, Hungary

and

Department of Mathematics and Statistics
University of South Florida
4202 E. Fowler Ave, CMC342
Tampa, FL 33620-5700, USA

totik@mail.usf.edu