Variance Bounds for Disc-Polygons

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Abstract. We prove asymptotic lower bounds on the variance of the number of vertices and missed area of random disc-polygons in convex discs whose boundary is C_+^2 smooth. The established lower bounds are of the same order as the upper bounds proved previously in [10].

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1   Results

We work in the Euclidean plane $\mathbb{R}^2$ with origin centred closed unit ball $B = B^2$ whose boundary is the unit circle $S^1 = \partial B$. We denote the (origin centred) open unit ball by B°. We denote by $A(\cdot)$ the area of measurable sets in $\mathbb{R}^2$. For general information about convex sets we refer to the books by Gruber [11] and Schneider [21].

For asymptotic inequalities, we use the following common notation: for two real sequences f, g, we write $f \ll g$ if there is a positive constant γ such that $|f(n)| \leq \gamma g(n)$ for every $n \in \mathbb{N}$.

Let $K \subset \mathbb{R}^2$ be a convex disc with C_+^2 smooth boundary (twice continuously differentiable with positive curvature $\kappa(x) > 0$ at every point $x \in \partial K$). Let $r_M = 1/\kappa_m$, where $\kappa_m = \min_{x \in \partial K} \kappa(x) > 0$. It is known (see [21, Theorem 3.2.12 on p. 164]) that K slides freely in a circle of radius r_M, that is, for every $x \in \partial K$, there exists a $v \in \mathbb{R}^2$ with $x \in r_M S^1 + v$ and $K \subset r_M B + v$ (cf. [21, p. 156]). For $r \geq r_M$ and a set X, let $\mathrm{conv}_r(X)$ denote the intersection of all radius r closed circular discs that contain X, that is,

$$\mathrm{conv}_r(X) := \bigcap_{\substack{v \in \mathbb{R}^2, \\ X \subset rB + v}} (rB + v).$$

The set $\mathrm{conv}_r(X)$ is called the closed r-spindle convex hull or r-hull of X. It is known that for $X \subset K$, $\mathrm{conv}_r(X) \subset K$, see Bezdek et al [6]. For more information about the geometric properties of the r-spindle convex hull see, for example, [6] and [8] and the references therein.

We investigate the following probability model. Let $X_n = \{x_1, \ldots, x_n\} \subset K$ be a sample of n i.i.d. random points selected according to the uniform probability distribution. Let $K_n^r := \mathrm{conv}_r(X_n)$. The set K_n^r is a *(uniform) random convex r-disc-polygon in K* whose sides are arcs of radius r circles. The edges, vertices and angles of K_n^r are defined the usual way. Let $f_0(K_n^r)$ denote the number of vertices of K_n^r. We call $A(K \setminus K_n^r)$ the missed area of K_n^r.

If $\overline{K}_n$ denotes the (usual) convex hull of X_n, then $\overline{K}_n \subset K_n^r \subset K$ for all $r \geq r_M$. Since the containment $\overline{K}_n \subset K_n^r$ is strict, K_n^r approximates K better than K_n from the point of view of area and perimeter. It is also clear that for fixed X_n, the r-disc-polygons K_n^r tend to $\overline{K}_n$ in the Hausdorff-metric as $r \to \infty$. Furthermore, $f_0(K_n^r) \leq f_0(\overline{K}_n)$ for all $r \geq r_M$.

The geometric properties of the (classical) random polygon $\overline{K}_n$ have been investigated extensively. Starting with the seminal papers of Rényi and Sulanke [18–20] the asymptotic behaviour of the expected number of vertices, area and perimeter have been determined. For a detailed overview of known results about the classical model, see for example, the surveys [1, 2, 14] and [22].

Fodor, Kevei and Vígh [8, Theorem 1.1 on p. 901] proved the following asymptotic formulas: for $r > r_M$, it holds that

$$\lim_{n \to \infty} \mathbb{E} f_0(K_n^r) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A(K)}} \Gamma\left(\frac{5}{3}\right) c_1(K, r), \qquad (1)$$

$$\lim_{n \to \infty} \mathbb{E} A(K \setminus K_n^r) \cdot n^{2/3} = \sqrt[3]{\frac{2A^2(K)}{3}} \Gamma\left(\frac{5}{3}\right) c_1(K, r), \qquad (2)$$

where the constant

$$c_1(K, r) = \int_{\partial K} \left(\kappa(x) - \frac{1}{r}\right)^{1/3} dx,$$

seems to resemble to the affine arc-length although it is unclear what is its exact geometric meaning. The formulas (1) and (2) are generalizations of the corresponding results of Rényi and Sulanke for the classical case, see Section 3 of [8]. We also note that in (1) and (2) the condition that $r > r_M$ is important. If K is a disc-polygon of radius r itself, then the order of magnitude of $\mathbb{E} f_0(K_n^r)$ and $\mathbb{E} A(K \setminus K_n^r)$ are different, see [9].

In the case when $K = B$, it is proved in [8, Theorem 1.3 on p. 902] that for $r = 1$, it holds that

$$\lim_{n \to \infty} \mathbb{E} f_0(B_n^1) = \frac{\pi^2}{2},$$

$$\lim_{n \to \infty} \mathbb{E} A(B \setminus B_n^1) \cdot n = \frac{\pi^3}{2}.$$

Significantly less is known about the higher moments of these quantities.
In the classical case, Reitzner [15, 16] proved the following upper bounds for the variance of the number of j-dimensional faces and the volume of the random polytopes in smooth convex bodies in dimension d:

$$\mathrm{Var}(\mathrm{Vol}(\overline{K}_n)) \ll n^{-(d+3)/(d+1)},$$
$$\mathrm{Var}(f_j(\overline{K}_n)) \ll n^{(d-1)/(d+1)}.$$

Here $\overline{K}_n$ denotes the (classical) convex hull of n i.i.d. random points selected from the d-dimensional convex body $K \subset \mathbb{R}^d$ with C_+^2 smooth boundary. The symbol $\mathrm{Vol}(\cdot)$ denotes the volume of Lebesgue measurable sets in $\mathbb{R}^d$ and $f_j(\cdot)$ is the number of j-dimensional faces. The implied constants depend only on K and the dimension. The upper bounds also imply strong laws of large numbers for these quantities.
Reitzner [17] gave matching lower bounds for the variance in case K is smooth:

$$\mathrm{Var}(\mathrm{Vol}(\overline{K}_n)) \gg n^{-(d+3)/(d+1)},$$
$$\mathrm{Var}(f_j(\overline{K}_n)) \gg n^{(d-1)/(d+1)}.$$

Using these lower bounds Reitzner [17] established central limit theorems for the number of j-dimensional faces and the volume for smooth convex bodies. Pardon [13] proved central limit theorems for the missed area and number of vertices of uniform random polygons in arbitrary convex discs in the plane.
Upper and lower bounds, laws of large numbers and central limit theorems were extended for the case when $K \subset \mathbb{R}^d$ is a polytope by Bárány and Reitzner [4]. Bárány and Steiger [3] proved asymptotic upper bounds and strong laws for the missed area and the number of vertices in the classical random model for arbitrary convex discs in $\mathbb{R}^2$ without any smoothness condition.
Fodor and Vígh [10] proved asymptotic upper bounds for the variance of the vertex number and the missed area of uniform random disc-polygons in C_+^2 smooth convex discs: For any $r > r_M$ it holds that

$$\mathrm{Var}(f_0(K_n^r)) \ll n^{1/3}, \tag{3}$$
$$\mathrm{Var}(A(K_n^r)) \ll n^{-5/3}, \tag{4}$$

where the implied constants depend only on K and r. In the case when $K = B^2$, they proved [10] that

$$\mathrm{Var}(f_0(K_n^r)) \approx 1, \tag{5}$$
$$\mathrm{Var}(A(K_n^r)) \ll n^{-2}. \tag{6}$$

where the implied constants are universal. Note that the lower bound in (5) follows from the fact that the expected number of vertices $\mathbb{E}f_0(K_n)$ is a non-integer constant. Formulas (3), (4) and (5) imply laws of large number for the corresponding quantities, see [10].

In this paper we prove matching lower bounds for the variance of the vertex number and the missed area for the case when $r > r_M$. Our main results are summarized in the following theorem.

THEOREM 1. [1] *Let K be a convex disc whose boundary is of class C_+^2. For any $r > r_M$ it holds that*

$$\text{Var}(f_0(K_n^r)) \gg n^{\frac{1}{3}}, \tag{7}$$

$$\text{Var}(A(K_n^r)) \gg n^{-\frac{5}{3}}, \tag{8}$$

where the implied constants depend only on K and r.

We achieve these results using a modified version of the method of Reitzner [16] which had already been used in adapted forms in several different settings, see, for example, [7, 23, 24]. In the disc-polygonal setting, the difficulty lies in the intrinsic geometry of the model.

The lower bounds in Theorem 1 may open the road towards quantitative central limit theorems, similarly as in [5, 23, 24] using normal approximation bounds from Stein's method.

The layout of the paper is the following: in Section 2 we collect some necessary preparatory material. Section 3 contains the proof of (8) and Section 4 contains the outlines of the changes necessary for the proof of (7).

2 Preparations

Without loss of generality, we may assume that $r = 1$ and prove Theorem 1 for the case when $r_M < 1$, since the general statements follow by a scaling argument. For simplicity, we write K_n for K_n^1.

For $p \in \mathbb{R}^2$, the set $K \setminus (B^\circ + p)$ is called a *disc-cap (of radius 1)* of K. We use the notations from [8]. Let x and y be two points in K. The two unit circles that pass through these points, determine two disc-caps of K, denoted by $D_-(x,y)$ and $D_+(x,y)$, respectively, such that $A(D_-(x,y)) \leq A(D_+(x,y))$. Briefly, we write $A_-(x,y) = A(D_-(x,y))$ and $A_+(x,y) = A(D_+(x,y))$. Lemma 4.3 in [8] states that if the boundary of K is of class C_+^2 ($r_M < 1$), then there exists a $\delta > 0$ (depending only on K) with the property that for any $x, y \in \text{int } K$ it holds that $A_+(x,y) > \delta$.

We need some further technical statements about general disc-caps. Denote the (unique) outer unit normal to K at the boundary point x by $u_x \in S^1$, and the unique boundary point with outer unit normal $u \in S^1$ by $x_u \in \partial K$. It is proved in Lemma 4.1 of [8] that if K is a convex disc with C_+^2 boundary and $\kappa_m > 1$, and $D = K \setminus (B^\circ + p)$ is a non-empty disc-cap, then there exists a unique point $x_0 \in \partial K \cap \partial D$ such that $p = x_0 - (1+t)u_{x_0}$ for some $t \geq 0$. The point x_0 is the *vertex* and the number t is the *height of D*.

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[1] We note that Theorem 1 is contained in the Master thesis of B. Grünfelder [12], and also in his Hungarian OTDK student competition paper.

Let us denote the disc-cap with vertex $x_u \in \partial K$ and height t by $D(u,t)$. To simplify the notation, we write $A(u,t) = A(D(u,t))$, and let $\ell(u,t)$ denote the arc-length of $\partial D(u,t) \cap (\partial B + x_u - (1+t)u)$. The latter also exists since for each $u \in S^1$, there exists a maximal positive constant $t^*(u)$ such that $(B + x_u - (1+t)u) \cap K \neq \emptyset$ for all $t \in [0, t^*(u)]$.

The following limit relations for $A(u,t)$ and $\ell(u,t)$ are proved (in a more precise form) in [8, p. 905, Lemma 4.2]:

$$\ell(u_x, t) \approx t^{1/2}, \qquad A(u_x, t) \approx t^{3/2}, \tag{9}$$

as $t \to 0^+$, where the implied constants depend only on K.

Let D be a disc-cap of K with vertex x. For a line $e \subset \mathbb{R}^2$ perpendicular to u_x, let e_+ denote the closed half plane that contains x. Then there exists a maximal cap $C_-(D) = K \cap e_+$ that is fully contained in D, and a minimal cap $C_+(D) = e'_+ \cap K$ containing D. We recall [10, Claim 1 on p. 1146], that gives a relation between classical caps and disc-caps, as follows: There exists a constant $\hat{c}$ depending only on K such that if the height of the disc-cap D is sufficiently small, then

$$C_-(D) - x \supset \hat{c}(C_+(D) - x). \tag{10}$$

The relation (10) means that the area of a disc-cap can be bounded by two classical caps such that one of them is an enlarged image of the other one by a constant.

Let x_i, x_j $(i \neq j)$ be two points of X_n, and let $B(x_i, x_j)$ be one of the unit discs containing x_i and x_j on its boundary. The arc $\partial B(x_i, x_j) \cap K$ forms an edge of K_n if the entire set X_n is contained in $B(x_i, x_j)$. It may happen that the pair x_i, x_j determines two edges of K_n if the above condition holds for both unit discs that contain x_i and x_j on its boundary.

3 Proof of (8) in Theorem 3

The proof is based on the ideas of Reitzner [16]: we give small (disc-)caps which contribute to the variance geometrically independently and show that the variance in these caps is already sufficiently large.

For every $x \in \partial K$ and $t \in (0,1)$ consider the disc-cap $D(x,t)$ of vertex x and height t. Let the Euclidean cap of vertex x and height t be $C(x,t)$. Let the line cutting off the cap $C(x,t)$ be $H(x,t)$. Clearly, $D(x,t) \supset C(x,t)$.

In the following we use large values of n, thus by an inequality of type $\ll$ we always assume that t is sufficiently small.

Denote the intersections of ∂K and the line $H(x,t)$ by w_1 and w_2, and let $w_0 = x$. For the triangle $\Delta = [w_0, w_1, w_2]$ we have

$$\Delta \subset C(x,t) \subset D(x,t).$$

Let us define for $j = 0, 1, 2$ the small triangles

$$\Delta_j = \Delta_j(x,t) = w_j + \frac{1}{20}([w_0, w_1, w_2] - w_j),$$

i.e. we shrink the Δ from each of its vertices by a factor of $1/20$. It follows from (10) that $A(\Delta_j(x,t)) \approx t^{\frac{3}{2}}$, since the order of magnitude of the height of triangle Δ is t and of its base is $\sqrt{t}$.

Let x, t and the points $z_1 \in \Delta_1(x,t)$ and $z_2 \in \Delta_2(x,t)$ be fixed. For $z_0 \in \Delta_0(x,t)$ let $\hat{A}(z_0)$ denote the area of the non-convex triangular region $\widetilde{\Delta}(z_0)$ we obtain by joining z_0 with z_1 and z_2 by circular arcs of radius 1 that are outside of the triangle $z_0 z_1 z_2$, and also joining z_1 and z_2 such that the arc intersects the interior of $z_0 z_1 z_2$.
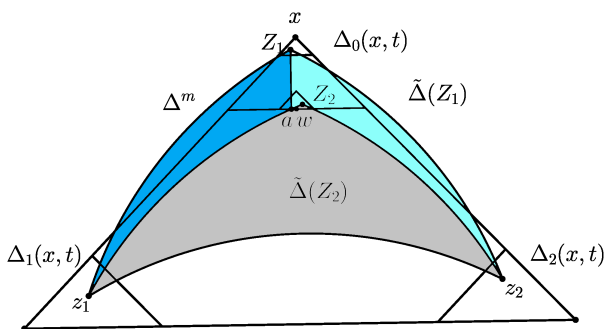

Figure 1: Splitting Δ^m

LEMMA 1. *Let Z be a uniform random point in $\Delta_0(x,t)$. Then*

$$\mathrm{Var}(\hat{A}(Z)) \gg t^3.$$

Proof. Let w denote the midpoint of the side opposite to x in the triangle $\Delta_0(x,t)$. Let

$$\Delta_0^{(1)}(x,t) = x + \frac{1}{3}(\Delta_0(x,t) - x)$$

and

$$\Delta_0^{(2)}(x,t) = w + \frac{1}{3}(\Delta_0(x,t) - w),$$

i.e. in the triangle $\Delta_0(x,t)$ we take two smaller triangles which are the shrunk images of $\Delta_0(x,t)$ by a factor of $1/3$ from x and w. The area of $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ is one-ninth of that of Δ_0, respectively.

For every $Z_1 \in \Delta_0^{(1)}$ and $Z_2 \in \Delta_0^{(2)}$, it holds that $\widetilde{\Delta}(Z_1) \supset \widetilde{\Delta}(Z_2)$, therefore $\hat{A}(Z_1) > \hat{A}(Z_2)$. Let $\Delta^m = \widetilde{\Delta}(Z_1) \setminus \widetilde{\Delta}(Z_2)$. We need $A(\Delta^m)$.

Cut Δ^m by a segment trough Z_1 perpendicular to the line H, and denote the other intersection point of this segment with $\partial \Delta^m$ by a. Then $d(Z_1, a) \approx t$.
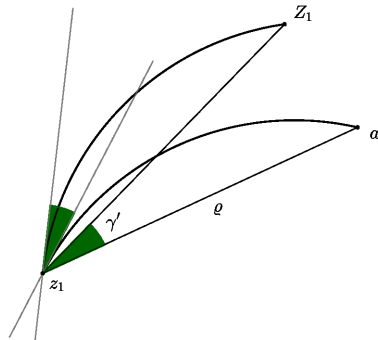
Figure 2: Angle of circular arcs and chords

Suppose that after the cut Z_2 is contained in the set that has z_2 on its boundary, see Figure 1.

Consider the Euclidean triangle $[Z_1, a, z_1]$, and let γ denote the angle at the vertex z_1. The radius of the circumscribed circle of this triangle is of order $\sqrt{t}$. Thus, since the side opposite to the angle γ is of order t, the law of sines gives that $\sin \gamma \approx \sqrt{t}$. By the smallness of t, the angle γ has the same order of magnitude as $\sin \gamma$.

After that, we translate a along $\partial \Delta^m$ into a point a' such that $d(z_1, a') = d(z_1, Z_1)$ holds. (In case a' has reached Z_2 and the distances are still not equal, we translate Z_1 closer to z_1.) By this, for the angle at z_1 we have $\gamma' \geq \gamma$.

Since $d(z_1, a') = d(z_1, Z_1)$, the angle between the two circular arcs of radius 1 corresponding to the two segments is γ' as well, see Figure 2. By the angle between two circular arc we mean the angle between their tangent lines.

Consider the sector-like shape determined by z_1, a' and Z_1, whose legs are circular arcs of radius 1. This is a part of a circular disc of radius $\varrho = d(z_1, Z_1)$, by which, rotating around the centre, we can cover the whole disc. The area of a shape of this property is proportional to the central angle and the square of the radius. Here we have $\varrho \approx \sqrt{t}$ and the order of magnitude of the angle is at least $\sqrt{t}$, therefore the area of this shape is at least of order $t^{3/2}$.

We have estimated the area $\hat{A}(Z_1) - \hat{A}(Z_2)$ we are looking for from below, examining its subset. This gives us the following lower estimate:

$$\hat{A}(Z_1) - \hat{A}(Z_2) \gg t^{3/2}. \tag{11}$$

Let Z, Z_1' and Z_2' be i.i.d uniform random points in $\Delta_0(x, t)$. Using (11), we

obtain the desired lower bound:

$$\mathrm{Var}(\hat{A}(Z)) = \frac{1}{2}\mathbb{E}\left[(\hat{A}(Z_1') - \hat{A}(Z_2'))^2\right] \geq$$
$$\geq \frac{1}{2}\mathbb{E}\left[\left(\hat{A}(Z_1') - \hat{A}(Z_2')\right)^2 \mathbb{1}(Z_1' \in \Delta_1, Z_2' \in \Delta_2)\right] \gg$$
$$\gg t^3 \mathbb{E}\left[\mathbb{1}(Z_1' \in \Delta_1, Z_2' \in \Delta_2)\right] \gg t^3.$$

$\square$

We may assume that $n > n_0$ for some suitable n_0, since it is sufficient to prove the lower bound of variance for large n. In the following, we consider disc-caps of height t_n for

$$t_n = n^{-\frac{2}{3}}. \tag{12}$$

Choose a maximal set of points $y_1, \dots, y_m$ on ∂K such that $|y_i - y_j| \geq 2\sqrt{c_2}\sqrt{t_n}$ for any $i, j \in \{1, \dots, m\}$ for some constant c_2 that we specify later. Then

$$m \gg n^{\frac{1}{3}}. \tag{13}$$

Assume that c_2 is so large that the disc-caps $D(y_j, t_n)$ are pairwise disjoint, and consider the previously defined triangles $\Delta(y_j, t_n)$ in them, for $j \in [m]$. For each $\Delta(y_j, t_n)$, also construct the small triangles $\Delta_i(y_j, t_n)$, for $i = 0, 1, 2$. Let E_j be the event that each of the small triangles $\Delta_i(y_j, t_n)$ contains exactly one of the random points $x_1, \dots, x_n$ and that $D(y_j, c_2 t_n)$ contains no other random point. By (9) we have

$$A(D(y_j, c_2 t_n)) \ll \frac{1}{n},$$

and for $i = 0, 1, 2$

$$A(\Delta_i(y_j, t_n)) \gg \frac{1}{n}.$$

We have for every $j \in [m]$

$$\mathbb{P}(E_j) \gg \binom{n}{3}\left(\frac{1}{n}\right)^3\left(1 - \frac{1}{n}\right)^{n-3} \gg 1,$$

thus

$$\mathbb{E}\left(\sum_{j=1}^{m} \mathbb{1}(E_j)\right) = \sum_{j=1}^{m} \mathbb{P}(E_j) \gg m. \tag{14}$$

In the case the event E_j occurs, let the random point in $\Delta_0(y_j, t_n)$ be denoted by Z_j.

Lemma 2. *Assume that $J \subset [m]$ and E_j occurs for every $j \in J$. Then*

$$A(K_n) = A\left(\mathrm{conv}_r\left(X_n \setminus \{Z_1, \dots Z_j\}\right)\right) + \sum_{j \in J} \hat{A}(Z_j).$$

Proof. Our goal is to show that if for indices $j, k \in J$, $j \neq k$, Z_j and Z_k are the random points in triangles $\Delta_0(y_j, t_n)$ and $\Delta_0(y_k, t_n)$, then Z_j and Z_k are vertices of K_n and there is no edge between them. This means that the contributions of Z_i and Z_j to the area of K_n are geometrically "independent". For this, we need that for every j, the circular arc of radius 1 determined by the two random points in triangles $\Delta_0(y_j, t_n)$ and $\Delta_i(y_j, t_n)$ $(i \in \{1, 2\})$, meets the boundary of K without intersecting any other disc-cap $D(y_k, t_n)$.

For simplicity take a fixed disc-cap of height t and vertex x, and the triangles Δ_0 and Δ_1 in it. Orient the cap in such a way that the outer normal at x points in the positive direction of the y-axis. The intersection with ∂K has minimal y-coordinate in case the random point in Δ_0 lies at the bottom corner, nearest to Δ_1, and the random point in Δ_1 is in the point farthest from the boundary, see Figure 3. Let us denote these points by a_0 and a_1, respectively.

In the case when K is a circle of radius r, we can exactly compute the y-coordinate of the intersection for a fixed r. The depth will be smaller than $c_3 t$ for a suitable constant c_3 depending only on K. This c_3 is bounded as r is strictly smaller than 1. Now we can specify the constant c_2 to be as large such that the disc-caps $D(y_k, t_n)$ are far apart enough and the observed intersection point is not contained in any other disc-cap. The constant c_2 depends only on K. Therefore the statement of the Lemma is true for circles.

For a general convex disc K with C_+^2 boundary, we estimate the y-coordinate of the intersection point as follows. Consider the osculating circle of K at the point x with radius $(R_0(x) =)R_0 < 1$. There exists an $\varepsilon > 0$, such that in any neighbourhood of radius less than ε of x, for the circles of radii $R_0 + \varepsilon$ and $R_0 - \varepsilon$ having the same tangent line as K at x, it is true that K is locally inside of the larger circle and the smaller circle is inside of K.

The line $H(x, t)$ meets these circles in p_1, p_2 and q_1, q_2, where the points with the same indices are close to each other. Then for $i = 1, 2$,

$$
\begin{aligned}
d(p_i, q_i) &= \sqrt{(2R_0 + 2\varepsilon - t)t} - \sqrt{(2R_0 - 2\varepsilon - t)t} \\
&= \sqrt{t} \left(\sqrt{2R_0 + 2\varepsilon - t} - \sqrt{2R_0 - 2\varepsilon - t} \right) \\
&= \sqrt{t} \left(\frac{4\varepsilon}{\sqrt{2R_0 + 2\varepsilon - t} + \sqrt{2R_0 - 2\varepsilon - t}} \right).
\end{aligned} \tag{15}
$$

Since $t \to 0$, we may assume that $t < \varepsilon$, thus we can estimate (15) from above as follows.

$$
\sqrt{t} \left(\frac{4\varepsilon}{\sqrt{2R_0 + 2\varepsilon - t} + \sqrt{2R_0 - 2\varepsilon - t}} \right) \leq \frac{4}{\sqrt{2R_0}} \varepsilon \sqrt{t} \leq c_4 \varepsilon \sqrt{t},
$$

where $c_4 = \max_{x \in \partial K} 4/\sqrt{2R_0(x)}$.

Let β denote the length of the side of Δ_0 opposite to x. Then the distance between the intersection of ∂K with the line H and a_1 is also β. The order of magnitude of β is $\sqrt{t}$. We may assume that $\beta = c_5 \sqrt{t}$ for some constant c_5.
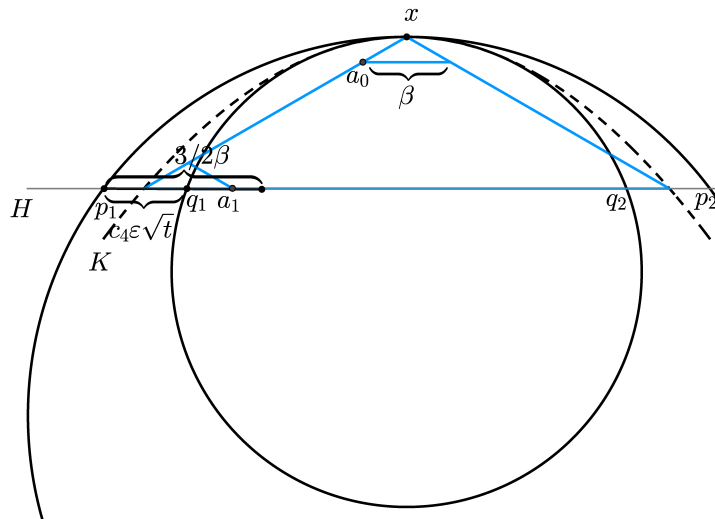
Figure 3: K bounded between circles

Consider the unique point of $H \cap K$ which is at distance $3/2\beta$ from p_1. The circular arc of radius 1 determined by this point and a_0, meets the outer circle at a computable, bounded depth depending on $(R_0 + \varepsilon)$. In the case $c_4 \cdot \varepsilon < c_5/2$, the circular arc incident with a_0 and a_1 meets the outer circle less deeply. For this, we need that

$$\varepsilon < \frac{c_5}{2c_4}.$$

Note that constant on the right-hand side of the inequality depends only on K. Since $t \to 0$, ε can be chosen smaller than that. The only restriction on ε is that K has to be locally between the two circles, that is, the intersection point $c_6 t$ deep is in this range. Therefore, we can choose ε arbitrary small.

Thus for every K, the circular arc determined by the points a_0 and a_1 meets the boundary of K in a depth of order t. Therefore this arc will not intersect any other disc-cap $D(y_j, t_n)$. $\qquad\square$

Let $\mathcal{F}$ denote the σ-algebra generated by the events E_j. Consider the conditional variance on $\mathcal{F}$. By the Law of Total Variance,

$$\operatorname{Var} A(K_n) = \mathbb{E} \operatorname{Var}(A(K_n) \mid \mathcal{F}) + \operatorname{Var} \mathbb{E}(A(K_n) \mid \mathcal{F})$$
$$\geq \mathbb{E} \operatorname{Var}(A(K_n) \mid \mathcal{F}). \tag{16}$$

For the set of indices $J \subseteq [m]$, let $E(J)$ be the event that the event E_j occurs for exactly the indices $j \in J$, and does not occur for the other indices in $[m]$.

Then the conditional variance on $\mathcal{F}$ can be expanded in the following form:

$$\operatorname{Var}(A(K_n) \mid \mathcal{F}) = \mathbb{E} \sum_{J \subseteq [m]} \operatorname{Var}(A(K_n) \mid E(J)) \cdot \mathbb{1}(E(J)). \qquad (17)$$

By Lemma 2, in case the event $E(J)$ occurs, the area of K_n can be written as the sum of $\hat{A}(Z_j)$ and the area of the hull of the remaining points:

$$\operatorname{Var}(A(K_n) \mid E(J)) =$$
$$= \operatorname{Var}\left[A\left(\operatorname{conv}_r\left(X_n \setminus \{Z_1, \dots Z_j\}\right)\right) + \sum_{j \in J} \hat{A}(Z_j) \mid E(J)\right]. \qquad (18)$$

Let the points of $X_n \setminus \{Z_1, \dots Z_j\}$ be fixed. Then the first term on the right-hand side of (18), the area of the r-hull of the remaining points is constant, thus we can omit it. The other terms are independent since the contributions of the points Z_j are geometrically independent, therefore we can take the variance term-by-term.

This holds for every fixed set of points $X_n \setminus \{Z_1, \dots Z_j\}$, thus we have the following inequality. The variance Var_{Z_j} means that the variance is taken only for the corresponding variable.

$$\operatorname{Var}(A(K_n) \mid E(J)) \geq \sum_{j \in J} \operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right),$$

Substituting it in (17), we obtain that

$$\sum_{J \subseteq [m]} \operatorname{Var}(A(K_n) \mid E(J)) \cdot \mathbb{1}(E(J)) \geq$$
$$\geq \sum_{J \subseteq [m]} \mathbb{1}(E(J)) \sum_{j \in J} \operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right). \qquad (19)$$

Rearrange the right-hand side of (19) such that the sum is according to the index of $\operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right)$:

$$\sum_{J \subseteq [m]} \mathbb{1}(E(J)) \sum_{j \in J} \operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right) =$$
$$= \sum_{j=1}^{m} \operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right) \sum_{\{J : j \in J\}} \mathbb{1}(E(J)) =$$
$$= \sum_{j=1}^{m} \operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right) \mathbb{1}(E_j).$$

Thus, we have

$$\operatorname{Var}(A(K_n) \mid \mathcal{F}) \geq \sum_{j=1}^{m} \operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right) \mathbb{1}(E_j). \qquad (20)$$

By Lemma 1 together with (12)-(14), (16), and (20),

$$\operatorname{Var} A(K_n) \geq \mathbb{E}\left(\sum_{j=1}^m \operatorname{Var}_{Z_j}\left(\hat{A}(Z_j) \mid E_j\right) \mathbb{1}(E_j)\right) \gg \mathbb{E}\left(\sum_{j=1}^m t_n^3 \mathbb{1}(E_j)\right) \gg$$

$$\gg n^{-2}\left(\mathbb{E}\sum_{j=1}^m \mathbb{1}(E_j)\right) \gg n^{-2} m \gg n^{-\frac{5}{3}},$$

which is (8) of Theorem 1, thus we have finished the proof.

4 The variance of the number of vertices

We give the outline of the necessary changes in the previous argument to prove (7) of Theorem 1, the lower bound of the variance of the number of vertices, see also Reitzner [17].

Let the disc-caps $D(y_j, t_n)$ be defined as before, and also the triangles $\Delta(y_n, t_n)$ together with the small triangles $\Delta_i(y_n, t_n)$, where $j \in [m]$ and $i \in \{0, 1, 2\}$. Let F_j denote the event that $\Delta_1(y_n, t_n)$ and $\Delta_2(y_n, t_n)$ each contain exactly one of the random points $x_1, \ldots, x_n$, $\Delta_0(y_n, t_n)$ contains exactly two of $x_1, \ldots, x_n$, and there is no other random point in the disc-cap $D(y_j, c_2 t_n)$. The probability of F_j is

$$\mathbb{P}(F_j) \gg \binom{n}{4}\left(\frac{1}{n}\right)^4\left(1 - \frac{1}{n}\right)^{n-4} \gg 1,$$

thus, similar to (14), we have

$$\mathbb{E}\left(\sum_{j=1}^m \mathbb{1}(F_j)\right) \gg m. \tag{21}$$

If the event F_j occurs, denote the random points in $\Delta_0(y_j, t_n)$ by Y_j and Z_j. It follows from the proof of Lemma 2, that in this case each one of the triangles $\Delta_i(y_n, t_n)$ ($i \in \{0, 1, 2\}$) contains a vertex of K_n. Either only one of the points Y_j and Z_j is a vertex of the K_n, or both of them are. Therefore the disc-cap $D(y_j, t_n)$ contains either 3 or 4 vertices, both of these events have positive probability. It follows, taking the variance for only these two points, that

$$\operatorname{Var}_{Y_j, Z_j}\left(\hat{f}_0([Y_j, Z_j, z_{j,1}, z_{j,2}]) \mid F_j\right) \gg 1, \tag{22}$$

where $\hat{f}_0([Y_j, Z_j, z_{j,1}, z_{j,2}]$ denotes the number of vertices contained in the j-th disc-cap.

Let $\mathcal{G}$ be the σ-algebra generated by the events F_j. Similar to the proof of Lemma 2, it holds here as well, that the number of vertices in one of the disc-caps does not affect how many vertices are there in an other disc-cap $D(y_j, t_n)$.

Using this fact, together with (13),(16) and (21)-(22), we get

$$
\begin{aligned}
\operatorname{Var} f_0(K_n) &\geq \mathbb{E}\operatorname{Var}(f_0(K_n) \mid \mathcal{G}) \\
&\geq \mathbb{E}\left(\sum_{j=1}^{m} \operatorname{Var}_{Y_j,Z_j}\left(\hat{f}_0([Y_j, Z_j, z_{j,1}, z_{j,2}]) \mid F_j\right) \mathbb{1}(F_j)\right) \\
&\gg \mathbb{E}\left(\sum_{j=1}^{m} \mathbb{1}(F_j)\right) \gg m \gg n^{\frac{1}{3}},
\end{aligned}
$$

which is the statement of (7) of Theorem 1, thus we have finished the proof.

Acknowledgments

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