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Embeddings of Ree unitals in a projective plane over a field [☆]

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We show that the Ree unital $\mathcal{R}(q)$ has an embedding in a projective plane over a field F if and only if $q = 3$ and $\mathbb{F}_8$ is a subfield of F. In this case, the embedding is unique up to projective linear transformations. Besides elementary calculations, our proof uses the classification of the maximal subgroups of the simple Ree groups.

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1. Introduction

A t-(n, k, λ) *design,* or equivalently a *Steiner system* $S_\lambda(t, k, n)$, is a finite simple incidence structure consisting of n points and a number of blocks, such that every block is incident with k points and every t-subset of points is incident with exactly λ blocks. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a design and $\Pi = (\mathcal{P}', \mathcal{B}', I')$ a projective plane. The map $\varrho : \mathcal{P} \cup \mathcal{B} \to \mathcal{P}' \cup \mathcal{B}'$ is an *embedding* of $\mathcal{U}$, provided it is injective, $\varrho(\mathcal{P}) \subseteq \mathcal{P}'$, $\varrho(\mathcal{B}) \subseteq \mathcal{B}'$, and

$$\forall P \in \mathcal{P}, \forall B \in \mathcal{B}: \quad PIB \Leftrightarrow \varrho(P) I' \varrho(B).$$

The embedding ϱ is *admissible* (or *equivariant*), if for any automorphism α of $\mathcal{U}$, there is a collineation β of Π such that $\varrho(P^\alpha) = \varrho(P)^\beta$ holds for all $P \in \mathcal{P}$.

An *abstract unital* or a *unital design* of order n is a 2-$(n^3 + 1, n + 1, 1)$ design. The problem of the embeddings of abstract unitals in projective planes is a classical one with many old unsolved questions, see [3–5,10–13,15]. The classical *hermitian unital* $\mathcal{H}(q)$ of order q is constructed from the set of absolute points and non-absolute lines of the desarguesian plane $\mathsf{PG}(2, q^2)$. The abstract hermitian unital $\mathcal{H}(q)$ has a natural embedding in $\mathsf{PG}(2, q^2)$, which is unique up to projective equivalence, see [11,14]. Moreover, this embedding is admissible.

Another class of abstract unitals of order $q = 3^{2n+1}$ was discovered by Lüneburg [13]. Let $\mathsf{Ree}(q) = {}^2G_2(q)$ be the *Ree group* of order $(q^3+1)q^3(q-1)$, $q = 3^{2n+1}$, see [1,9]. Then $\mathsf{Ree}(q)$ has a 2-transitive action on $q^3 + 1$ points, namely by conjugation on the set of all Sylow 3-subgroups. The pointwise stabilizer of two points P, Q is cyclic of order $q - 1$ and thus contains a unique involution t. It follows that $\mathsf{Ree}(q)$ has a unique conjugacy class of involutions, and any involution t fixes exactly $q + 1$ points. The blocks of the *Ree unital* $\mathcal{R}(q)$ are the sets of fixed points of the involutions of $\mathsf{Ree}(q)$. $\mathcal{R}(q)$ admits the $\mathsf{Ree}(q)$ as a 2-transitive automorphism group; the full automorphism group is larger, for $n \geq 1$, admitting also the field automorphism, see [3]. The smallest Ree unital $\mathcal{R}(3)$ and the smallest Ree group $\mathsf{Ree}(3) \cong \mathsf{P\Gamma L}(2, 8) \cong \mathsf{PSL}(2, 8) \rtimes C_3$ are a little different from the general case, see [2,6]. For $q > 3$, $\mathsf{Ree}(q)$ is simple.

Lüneburg [13] showed that the Ree unital $\mathcal{R}(q)$ has no admissible embeddings in projective planes of order q^2 (desarguesian or not). For $q = 3$, Grüning [6] proved that the smallest Ree unital $\mathcal{R}(3)$ has no embedding in any projective plane of order 9. Montinaro [15] extended these results by showing that for $q \neq 3$ and $n \leq q^4$, $\mathcal{R}(q)$ has no admissible embedding in a projective plane of order n. Moreover, if $\mathcal{R}(3)$ is embedded in a projective plane Π of order $n \leq 3^4$ in an admissible way, then either $\Pi \cong \mathsf{PG}(2, 8)$, or $n = 2^6$, see [15, Theorem 5].

In this paper, we completely characterize the embeddings of $\mathcal{R}(3)$ in a projective plane over a field, extending Montinaro's result.

Theorem 1. *Let F be a field and $\varphi : \mathcal{R}(3) \to \mathsf{PG}(2, F)$ an embedding. Then the following hold:*

(i) $\mathbb{F}_8$ *is a subfield of* F*, and the image of* φ *is contained in a subplane of order* 8*.*
(ii) The embedding is unique up to $\mathrm{Aut}(\mathcal{R}(3))$ *and* $\mathsf{PGL}(3, F)$*.*
(iii) The embedding is admissible.

Our main result is the following:

Theorem 2. *Let* n *be a positive integer, and* $q = 3^{2n+1}$*. Suppose that* Π *is a projective plane such that for each embedding* $\varphi : \mathcal{R}(3) \to \Pi$*, the image* $\varphi(\mathcal{R}(3))$ *is contained in a pappian subplane. Then the Ree unital* $\mathcal{R}(q)$ *has no embedding in* Π*. In particular,* $\mathcal{R}(q)$ *has no embedding in a projective plane over a field.*

These results suggest that the problem of projective embeddings of the Ree unitals can be reduced to the question whether the smallest Ree unital has an embedding in a non-desarguesian projective plane. This question is surprisingly hard, even if we assume that the embedding is admissible.

The structure of the paper is as follows. In section 2, we present the embedding of $\mathcal{R}(3)$ in $\mathsf{PG}(2, 8)$, and some technical lemmas to ease the calculations in $\mathsf{PG}(2, 8)$. In section 3, we study sets of five points of $\mathsf{PG}(2, 8)$, determining ten external lines of a conic. Such *external pentagons* correspond to *super O'Nan configurations* of $\mathcal{R}(3)$; their properties are listed in section 4. In sections 5 and 6, we prove Theorems 1 and 2.

2. Preliminaries

The embedding of $\mathcal{R}(3)$ in $\mathsf{PG}(2, 8)$ deserves special attention. The construction was first given by Grüning [6], who attributes the idea to F.C. Piper. The embedding is slightly simpler to present in the dual setting. Let $\mathcal{K}$ be a non-singular conic in $\mathsf{PG}(2, 8)$. The tangents of $\mathcal{K}$ have a common point N, which is called the *nucleus* of $\mathcal{K}$ (see [8]). The set $\mathcal{O} = \mathcal{K} \cup \{N\}$ is a hyperoval, that is, a set of 10 points such that each line intersects it in 0 or 2 points. The point P is *external*, if $P \notin \mathcal{O}$. The line ℓ is *external,* if $\ell \cap \mathcal{O} = \emptyset$. There are 63 external points, 28 external lines, and each external point is incident with 4 external lines. In other words, the external points and the external lines form a (dual) unital $\mathcal{U}$ of order 3. Let $G = \mathsf{P\Gamma O}(3, 8)$ be the group of projective semilinear transformations of $\mathsf{PG}(2, 8)$, preserving $\mathcal{O}$. G is isomorphic to

$$\mathsf{P\Gamma L}(2, 8) \cong \mathsf{PSL}(2, 8) \rtimes C_3,$$

and acts 2-transitively on the set of external lines. Hence, $\mathcal{U}$ has a 2-transitive automorphism group and $\mathcal{R}(3) \cong \mathcal{U}$ by [2]. We call the isomorphism $\varphi : \mathcal{R}(3) \to \mathcal{U}$ a *dual embedding* of $\mathcal{R}(3)$ into $\mathsf{PG}(2, 8)$ with respect to the conic $\mathcal{K}$.

To make the computation in $\mathbb{F}_8$ more transparent, we fix a root $\gamma \in \mathbb{F}_8$ of the polynomial $X^3 + X + 1 = 0$ in $\mathbb{F}_8$. Then

$$\gamma^4 = \gamma^2 + \gamma, \quad \gamma^5 = \gamma^2 + \gamma + 1, \quad \gamma^6 = \gamma^2 + 1, \quad \gamma^7 = 1.$$

The trace map of $\mathbb{F}_8$ over $\mathbb{F}_2$ is defined as

$$\mathrm{Tr}(x) = x + x^2 + x^4.$$

We fix the coordinate frame (X, Y, Z) in $\mathsf{PG}(2,8)$ and extend the action of the Frobenius automorphism $\Phi : x \mapsto x^2$ to the points and lines of $\mathsf{PG}(2,8)$. In this way, we obtain a projective semilinear transformation of order 3. For $c \in \mathbb{F}_8$, the map

$$\tau_c : (x, y, z) \to (x + cz, y + c^2 z, z)$$

is an elation with axis $Z = 0$.

Lemma 3.

(i) *If $c \neq 0$ then the line $Y = mX + bZ$ is τ_c-invariant if and only if $m = c$.*
(ii) *Φ and τ_c $(c \in \mathbb{F}_8)$ preserve the conic $\mathcal{K} : X^2 + YZ = 0$ of $\mathsf{PG}(2,8)$ and the line $\ell_\infty : Z = 0$ at infinity.*
(iii) *The line $Y = mX + bZ$ is external to $\mathcal{K}$ if and only if $\mathrm{Tr}(b/m^2) = 1$.*
(iv) *Let Γ denote the group*

$$\Gamma = \{\tau_c \mid \mathrm{Tr}(c) = 0\}$$

of elations. Γ is elementary abelian of order 4. By conjugation, Φ permutes the nontrivial elements of Γ.
(v) *The group $G_0 = \langle \Gamma, \Phi \rangle$ of semilinear transformations is isomorphic to A_4.*

Proof. τ_c maps $Y = mX + bZ$ to $Y = mX + (b + c^2 + cm)Z$. This implies (i). (ii) is trivial. (iii) follows from the fact that in a finite field $\mathbb{F}_q$ of even order, the quadratic form $X^2 + mX + b$ is reducible if and only if $\mathrm{Tr}(b/m^2) = 0$. (iv) and (v) are immediate. □

Finally, we present a lemma on commuting involutions in $\mathsf{PSL}(2,q)$ and $\mathsf{Ree}(q)$.

Lemma 4. *Let n be a positive integer and $q = 3^{2n+1}$. Let $G = \mathsf{PSL}(2,q)$ or $G = \mathsf{Ree}(q)$. Then for two involutions b, c of G there are involutions $b_0 = b$, $b_1, \dots, b_k = c$ such that $b_i b_{i+1} = b_{i+1} b_i$ holds for all $i = 0, \dots, k-1$.*

Proof. Observe first that in both cases, G has a unique class I of involutions. Moreover, G acts primitively on I by conjugation. For $G = \mathsf{Ree}(q)$, this follows from [9, Theorem C]. For $G = \mathsf{PSL}(2,q)$, the centralizer of an involution in G is isomorphic to the dihedral group D_{q+1} of order $q+1$. By Dickson's Theorem, D_{q+1} is maximal in G; see also [16, Chapter 3.6, Exercise 7(i)].

Let Γ be the graph with vertex set I and $b, c \in I$ connected by an edge if and only if $bc = cb$. G induces an automorphism group of Γ that acts primitively on the vertices. As the connected components of Γ are blocks of imprimitivity, and Γ is not the empty graph, Γ has to be connected. Fix $b, c \in I$ and take a path $b_0 = b$, $b_1, \ldots, b_k = c$ from b to c. Then, $b_i b_{i+1} = b_{i+1} b_i$ holds for all $i = 0, \ldots, k-1$. $\square$

3. External pentagons in $\mathsf{PG}(2,8)$

Let p be a prime, and $q = p^e$ be a prime power. Let $\mathcal{K}$ be a non-singular conic in $\mathsf{PG}(2,q)$. For any line ℓ, we have $|\mathcal{K} \cap \ell| \leq 2$. We call ℓ *secant, tangent* or *external* to $\mathcal{K}$, according if $|\mathcal{K} \cap \ell|$ is 2, 1, or 0. If q is even, then all tangents pass through a common point, the *nucleus* of $\mathcal{K}$. (See [8].) The group of collineations preserving $\mathcal{K}$ is $\mathsf{P\Gamma O}(3,q)$. One has the isomorphisms $\mathsf{PGO}(3,q) \cong \mathsf{PGL}(2,q)$ and $\mathsf{P\Gamma O}(3,q) \cong \mathsf{P\Gamma L}(2,q)$.

Definition 5. Let $\mathcal{K}$ be a conic in $\mathsf{PG}(2,q)$. We say that the points $P_0, \ldots, P_4$ in general position form an *external pentagon* with respect to $\mathcal{K}$, if P_iP_j are external lines of $\mathcal{K}$, $0 \leq i < j \leq 4$. The external pentagon is said to have type A_4, if there is a group G_0 of collineations preserving $\mathcal{K}$ and $\{P_0, \ldots, P_4\}$, such that $G_0 \cong A_4$.

Notice that the points of an external pentagon are not in $\mathcal{K}$, and if q is even, then they are also distinct from the nucleus of $\mathcal{K}$.

Lemma 6. *Let $\mathcal{P} = \{P_0, \ldots, P_4\}$ be an external pentagon of type A_4 in $\mathsf{PG}(2,q)$ with respect to the conic $\mathcal{K}$. Then q is even and the following hold:*

(i) G_0 fixes one point of $\mathcal{P}$ and acts 2-transitively on the remaining four.
(ii) If $q = 8$, then $G_0 \cong A_4$ is the full stabilizer of the sets $\mathcal{K}$ and $\mathcal{P}$ in the collineation group of $\mathsf{PG}(2,8)$.

Proof. (i) The Sylow 2-subgroup $T = G_0'$ of G_0 is normal in G_0, and $T \leq \mathsf{PGO}(3,q)$. Hence, T acts faithfully on $\mathcal{P}$, with a unique fixed point, say, P_0. As T is normal in G_0, G_0 fixes P_0. Moreover, T acts regularly and G_0 acts 2-transitively on $\{P_1, \ldots, P_4\}$. Let H be the stabilizer of P_0 in $\mathsf{P\Gamma O}(3,q)$. If q is odd then either $H \cong D_{2(q+1)} \rtimes C_e$ (if no tangent of $\mathcal{K}$ is incident with P), or $H \cong D_{2(q-1)} \rtimes C_e$ (if 2 tangents are incident with P). In both cases, the dihedral subgroup $D_{2(q\pm1)}$ contains a unique central involution α, which is contained in each elementary abelian subgroup of order 4. Hence, no subgroup of H of order 12 can be isomorphic to A_4, a contradiction.

(ii) Assume $q = 8$. Then $H \cong \mathbb{F}_8^+ \rtimes C_3$ has order 24 and we have to show that H does not leave $\mathcal{P}$ invariant. Let t_0 be the tangent line through P_0 to $\mathcal{K}$. The elementary abelian part A of H consists of elations with respect to t_0, that is, for any 2-element $\alpha \in A$, the set of fixed point of α is t_0. As for $i \geq 1$, $P_i \notin t_0$, and the A-orbit of P_i has length 8. $\square$

Definition 7. The *fundamental pentagon* $\mathcal{F}$ of $\mathsf{PG}(2,8)$ is the set of points $A(1,1,0)$, $C_1(0,1,1)$ and

$$C_2 = (\gamma, \gamma^6, 1), \quad C_3 = (\gamma^2, \gamma^5, 1), \quad C_4 = (\gamma^4, \gamma^3, 1).$$

Lemma 8. *The fundamental pentagon $\mathcal{F}$ is an external pentagon with respect to the conic $\mathcal{K} : X^2 + YZ = 0$. Moreover, $\mathcal{F}$ is of A_4 type with collineation group $G_0 = \langle \Gamma, \Phi \rangle$.*

Proof. The following facts can be checked by calculations: A is fixed by G_0. Φ fixes C_1 and permutes C_1, C_2, C_3. The Γ-orbit of C_1 is $\{C_1, \dots, C_4\}$. The lines $AC_1 : Y = X + Z$ and $C_1C_2 : Y = \gamma X + Z$ are external. $\square$

Lemma 9. *Let $\mathcal{K}$ be a conic in $\mathsf{PG}(2, 8)$ and $\mathcal{P}$ an external pentagon of type A_4 with respect to $\mathcal{K}$. The projective coordinate frame can be chosen such that $\mathcal{K}$ has equation $X^2 + YZ = 0$ and $\mathcal{P}$ is the fundamental pentagon.*

Proof. We can assume the equation $\mathcal{K} : X^2 + YZ = 0$ and that $A(1, 1, 0)$ is the point of $\mathcal{P}$ which is fixed by G_0. Then G_0 is a subgroup of

$$H = \{\tau_c \mid c \in \mathbb{F}_8\} \rtimes \langle \Phi \rangle,$$

which is the stabilizer of $\mathcal{K}$ and A. The 2-subgroup $\{\tau_c \mid c \in \mathbb{F}_8\}$ has two Φ-invariant, irreducible proper subgroups: $Z(H) = \langle \tau_1 \rangle$ and Γ. Hence, $\langle \Gamma, \Phi \rangle$ is the unique subgroup H which is isomorphic to A_4, and $G_0 = \langle \Gamma, \Phi \rangle$ follows. Let C_1 denote the point of $\mathcal{P} \setminus \{A\}$ that is fixed by Φ. As AC_1 is an external line, C_1 must have coordinates $(x, x + b, 1)$ with $x, b \in \mathbb{F}_2$ and $\mathrm{Tr}(b) = 1$. This means that either $C_1 = (0, 1, 1)$ or $C_1 = (1, 0, 1)$. Applying τ_1 to $\mathcal{P}$, we can assume $C_1 = (0, 1, 1)$. Straightforward computation shows that $\{C_1, \dots, C_4\}$ is the Γ-orbit of C_1, and $\{A, C_1, \dots, C_4\}$ is indeed the fundamental pentagon. $\square$

Remark 10. *Lemma 3(ii) and Lemma 9 imply that with fixed conic $\mathcal{K}$ of $\mathsf{PG}(2, 8)$, the number of A_4-type external pentagons is $|\mathsf{P\Gamma O}(3, 8) : G_0| = 126$.*

For the rest of this section, we use the notation of Lemma 3 for $\mathcal{K}$, Γ, G_0, and τ_c.

Proposition 11. *Let $\mathcal{F} = \{A, C_1, \dots, C_4\}$ be the fundamental pentagon in $\mathsf{PG}(2, 8)$. For any even permutation $ijk\ell$ of $\{1, 2, 3, 4\}$, let $d_{ijk\ell}$ denote the line connecting the points $AC_i \cap C_jC_\ell$ and $AC_j \cap C_kC_\ell$. The following hold:*

- *(i) G_0 permutes the lines $d_{ijk\ell}$ regularly. In particular, the lines $d_{ijk\ell}$ are distinct.*
- *(ii) The lines $d_{ijk\ell}$ are external to $\mathcal{K}$.*
- *(iii) For any coset Γg of Γ, the four lines d_π, $\pi \in \Gamma g$, share a common point at infinity $Z = 0$.*
- *(iv) The lines AC_4, d_{1234} and d_{3241} are concurrent.*

Proof. Obviously, G_0 acts on the lines $d_{ijk\ell}$. We have

$$AC_1 \cap C_2C_4 = (\gamma^6, \gamma^2, 1), \quad AC_2 \cap C_3C_4 = (1, \gamma^4, 1),$$

with connecting line $d_{1234} : Y = \gamma^6 X + \gamma^3 Z$. The intersection $d_{1234} \cap \ell_\infty = (1, \gamma^6, 0)$ is not fixed by any element of order 3 of G. Thus, the stabilizer of d_{1234} in G is contained in Γ. Since $\mathrm{Tr}(\gamma^6) = 1$, the stabilizer of d_{1234} in Γ is trivial by Lemma 3(i). This proves (i), and also (ii), since d_{1234} is an external line. (iii) follows from the fact that Γ fixes the points at infinity. Computing the equations and the determinant

$$\det \begin{bmatrix} 1 & 1 & \gamma^6 \\ \gamma^6 & 1 & \gamma^3 \\ \gamma^3 & 1 & \gamma^4 \end{bmatrix} = 0,$$

we obtain (iv). □

4. Super O'Nan configurations in $\mathcal{R}(3)$

In a 2-design, an *O'Nan (or Pasch) configuration* consists of four pairwise intersecting blocks, no three of which pass through the same point. Brouwer [2] observed that in $\mathcal{R}(3)$, each O'Nan configuration is contained in a *super O'Nan configuration,* that is, in a set of five pairwise intersecting blocks in general position. In this section, we collect some facts on super O'Nan configurations of $\mathcal{R}(3)$.

Lemma 12 *([2]). The number of super O'Nan configurations in $\mathcal{R}(3)$ is 126.* □

For the rest of this section, we fix a dual embedding φ^* of $\mathcal{R}(3)$ in $\mathsf{PG}(2,8)$ with respect to the conic $\mathcal{K} : X^2 + YZ = 0$. Notice that the blocks a, b of $\mathcal{R}(3)$ intersect if and only if the points $\varphi^*(a), \varphi^*(b)$ determine an external line of $\mathcal{K}$. This implies that $\{b_0, \dots, b_4\}$ is a super O'Nan configuration of $\mathcal{R}(3)$ if and only if $\{\varphi^*(b_0), \dots, \varphi^*(b_4)\}$ is an external pentagon with respect to $\mathcal{K}$.

Lemma 13. $\mathsf{Ree}(3)$ *acts transitively on the set of super O'Nan configurations of $\mathcal{R}(3)$. The stabilizer of a super O'Nan configuration is isomorphic to A_4. It fixes one of the blocks b_i and acts 2-transitively on the remaining four.*

Proof. Remark 10 and Lemma 12 imply that each external pentagon of $\mathsf{PG}(2,8)$ is of A_4 type. Our claim follows from Lemma 9. □

Proposition 14. *Let $\mathcal{B}$ be a super O'Nan configuration of $\mathcal{R}(3)$ with stabilizer subgroup $S \cong A_4$. We can label the blocks of $\mathcal{B}$ by $a, c_1, \dots, c_4$ such that for any even permutation $ijk\ell$, the blocks $(a \cap c_i)(c_j \cap c_\ell)$ and $(a \cap c_j)(c_k \cap c_\ell)$ have a unique intersection $D_{ijk\ell}$. Moreover, the following hold:*

(i) *The points $D_{ijk\ell}$ are distinct.*
(ii) *Let T be the Sylow 2-subgroup of S. For any coset Tg of S, the four points D_π, $\pi \in Tg$, form a block.*
(iii) *The points $a \cap c_4$, D_{1234} and D_{3241} are contained in a block.*

Proof. We use Lemma 13, Proposition 11 and the dual embedding φ^* of $\mathcal{R}(3)$ in $\mathsf{PG}(2,8)$. $\square$

5. Embeddings of $\mathcal{R}(3)$

Proof of Theorem 1. Let $\{a, c_1, \ldots, c_4\}$ be a super O'Nan configuration of $\mathcal{R}(3)$, as given in Proposition 14. Let us choose the projective coordinate frame of $\mathsf{PG}(2,F)$ such that $\varphi(c_1) : X+Y+Z=0$, $\varphi(c_2) : X=0$, $\varphi(c_3) : Y=0$, $\varphi(c_4) : Z=0$. There are elements $u, v \in F \setminus \{0,1\}$, $u \neq v$, such that $\varphi(a) : uX+vY+Z=0$. We have

$$\varphi(D_{1234}) = (v^2-v, v-u, v^2-uv), \qquad \varphi(D_{2143}) = (u-uv, u-1, v-u),$$
$$\varphi(D_{3412}) = (v, uv-u, -uv), \qquad \varphi(D_{4321}) = (v-u, u^2-u, u-uv).$$

By Proposition 14(ii), these are collinear points, thus,

$$\det \begin{bmatrix} v^2-v & v-u & v^2-uv \\ u-uv & u-1 & v-u \\ v & uv-u & -uv \end{bmatrix} = u(u-1)v(v-1)(v^2-uv+2u-3v) = 0,$$

and

$$\det \begin{bmatrix} v^2-v & v-u & v^2-uv \\ u-uv & u-1 & v-u \\ v-u & u^2-u & u-uv \end{bmatrix} = u(u-1)v(v-1)(uv-u^2+2u-3v+1) = 0.$$

The difference of these two equations is

$$u(u-1)v(v-1)((u-v)^2-1) = 0,$$

which implies $u = v \pm 1$. Substituting back, we obtain either $2v-2=0$ or $2=0$, which are not possible unless $\mathrm{char}(F)=2$. In this case, all equations so far reduce to $u+v+1=0$. Computing $\varphi(a \cap c_4) = (v, v+1, 0)$ and

$$\varphi(D_{3241}) = (v+1, 1, v^2),$$

we have

$$\det \begin{bmatrix} v & v+1 & 0 \\ v^2+v & 1 & v \\ v+1 & 1 & v^2 \end{bmatrix} = v(v+1)(v^3+v^2+1) = 0.$$

This shows that $u, v \in \mathbb{F}_8$, and for any even permutation $ijk\ell$, $\varphi(D_{ijk\ell})$ is contained in the subplane $\mathsf{PG}(2,8)$. Hence, at least 22 points of $\varphi(\mathcal{R}(3))$ are contained in $\mathsf{PG}(2,8)$. If Q is one of the remaining 6 points, then there are at least two blocks through Q with equation over $\mathbb{F}_8$, and therefore Q is in $\mathsf{PG}(2,8)$ as well. The computation shows that up to the action of the Frobenius map Φ, the embedding φ is uniquely determined by the images of the blocks $c_1, \ldots, c_4$. In particular, φ must be an embedding with respect to a dual conic $\mathcal{K}^*$. All subplanes of order 8, and all dual conics of a given subplane of order 8 are projectively equivalent in $\mathsf{PG}(2,F)$. Hence, we obtain (ii). Montinaro's [15, Theorem 5] implies (iii). □

Corollary 15. *Let F be a field and $\varphi : \mathcal{R}(3) \to \mathsf{PG}(2,F)$ an embedding. Let $S = \{1, a_1, \ldots, a_7\}$ be a Sylow 2-subgroup of $\mathsf{Ree}(3)$. Then the lines $\varphi(a_1), \ldots, \varphi(a_7)$ are concurrent.*

Proof. Consider the dual embedding $\varphi^* : \mathcal{R}(3) \to \mathsf{PG}(2,F)$. By Theorem 1, $\varphi^*(\mathcal{R}(3))$ is contained in a subplane $\mathsf{PG}(2,8)$. As before, we identify the blocks of $\mathcal{R}(3)$ and the involutions of $\mathsf{Ree}(3)$. For a block a, $\varphi^*(a)$ is an external point of the conic $\mathcal{K}$. Moreover, a determines a unique collineation $\hat{a}$ of order 2. The following are equivalent:

(1) Two involutions $a_1, a_2 \in \mathsf{Ree}(3)$ commute.
(2) The involutions $\hat{a}_1, \hat{a}_2 \in \mathsf{PGL}(3,F)$ commute.
(3) The line $\varphi^*(a_1)\varphi^*(a_2)$ is tangent to $\mathcal{K}$.
(4) $\hat{a}_1$, $\hat{a}_2$ fix the same point of $\mathcal{K}$.

This implies that $\varphi^*(a_1), \ldots, \varphi^*(a_7)$ are contained in a tangent of $\mathcal{K}$, which is our claim in the dual setting. □

6. The nonexistence of embeddings of $\mathcal{R}(q)$, $q \geq 27$

In this section, we write $q = 3^{2n+1}$, and $G = \mathsf{Ree}(q)$. We have $2n+1 = |\mathrm{Out}(\mathsf{Ree}(q))|$ and for any divisor α of $2n+1$ there is an outer automorphism ψ_α of $\mathsf{Ree}(q)$ of order α. (See [9, Lemma 6.2].) Write $q_0 = q^{1/\alpha} = 3^{2n_0+1}$ and $G_0 = C_{\mathsf{Ree}(q)}(\psi_\alpha)$. We have $G_0 \cong \mathsf{Ree}(q_0)$.

In order to be self-contained, we recall Kleidman's classification [9, Theorem C] of the maximal subgroups of G, see also [7]. If $q \geq 27$ and H is a maximal subgroup of G, then one of the following cases occurs:

(M1) H is a 1-point stabilizer, isomorphic to the semidirect product of a group of order q^3 with the cyclic group of order $q-1$.
(M2) $H \cong \mathsf{Ree}(q_0)$, where $q_0 = q^{1/\alpha}$, α prime.
(M3) $H \cong C_2 \times \mathsf{PSL}(2,q)$ is the centralizer of an involution.
(M4) $H \cong (C_2^2 \times D_{(q+1)/2}) \rtimes C_3$ is the normalizer of a subgroup of order 4.

(M5) $H \cong C_{q+\sqrt{3q}+1} \rtimes C_6$.
(M6) $H \cong C_{q-\sqrt{3q}+1} \rtimes C_6$.

If $q = 3$ then (M2) and (M6) do not occur. Moreover, $H \cong \mathsf{PSL}(2,8)$, or $H \cong (C_2^3 \rtimes C_7) \rtimes C_3$ is the normalizer of a Sylow 2-subgroup, that contains the subgroups (M3) and (M4).

In $\mathsf{Ree}(q)$, the stabilizer of two points is cyclic of order $q-1$. Hence, the intersection of two Sylow 3-subgroups is trivial. This implies that any Sylow 3-subgroup S_0 of G_0 is contained in a unique Sylow 3-subgroup S of G, and S is left invariant by ψ_α. Conversely, let S be a ψ_α-invariant Sylow 3-subgroup of G. The normalizer $N_G(S)$ is a parabolic subgroup of G, isomorphic to the semidirect product of a group of order q^3 with the cyclic group of order $q-1$. The centralizer of the field automorphism in $N_G(S)$ has order $q_0^3(q_0-1)$. This shows that $S_0 = S \cap G_0$ is a Sylow 3-subgroup in G_0.

Proposition 16. *Let $q = 3^{2n+1}$, $q_0 = 3^{2n_0+1}$ such that $2n_0+1$ divides $2n+1$. Then $\mathcal{R}(q)$ has a subdesign $\mathcal{D} \cong \mathcal{R}(q_0)$. Moreover, the stabilizer of $\mathcal{D}$ in $\mathsf{Ree}(q)$ is isomorphic to $\mathsf{Ree}(q_0)$. In particular, $\mathcal{R}(3)$ is a subdesign of $\mathcal{R}(q)$ with stabilizer subgroup $\mathsf{Ree}(3)$.*

Proof. Remember that the points and blocks of $\mathcal{R}(q)$ can be identified with the Sylow 3-subgroups, and the involutions of $\mathsf{Ree}(q)$, respectively. Hence, any automorphism of $G = \mathsf{Ree}(q)$ induces an automorphism of $\mathcal{R}(q)$. The involutions fixed by ψ_α and the Sylow 3-subgroups left invariant by ψ_α form a subdesign $\mathcal{D}$ of $\mathcal{R}(q)$. As explained above, ψ_α-invariant involutions and Sylow 3-subgroups of G correspond to involutions and Sylow 3-subgroups of G_0. Hence, $\mathcal{D} \cong \mathcal{R}(q_0)$. Let T_0 be the stabilizer of $\mathcal{D}$ in G; clearly $G_0 \le T_0$. Looking at the list of maximal subgroups of G in [9, Theorem C], we see that either $T_0 = \mathsf{Ree}(q)$, or T_0 is contained in a subgroup isomorphic to $\mathsf{Ree}(q_1)$ with $q_1 = q^{1/\beta}$, β prime. Repeating this argument, we conclude that T_0 itself is isomorphic to a Ree group $\mathsf{Ree}(q_*)$, where $\mathbb{F}_{q_*}$ is a subfield of $\mathbb{F}_q$. As T_0 preserves the set of involutions of G_0, the only possibility is $q_0 = q_*$. $\square$

We are now in the position to prove Theorem 2.

Proof of Theorem 2. Let us suppose that an embedding $\varphi : \mathcal{R}(q) \to \Pi$ exists. Let I denote the set of involutions of $\mathsf{Ree}(q)$. In three steps, we show that all lines $\varphi(a)$ $(a \in I)$ are concurrent, a contradiction.

Claim 1: For any Sylow 2-subgroup $S = \{1, a_1, \ldots, a_7\}$ of $\mathsf{Ree}(q)$, the lines $\varphi(a_1), \ldots, \varphi(a_7)$ are concurrent.

By Proposition 16, $\mathcal{R}(q)$ has a subdesign $\mathcal{D} \cong \mathcal{R}(3)$, whose stabilizer is the subgroup $\mathsf{Ree}(3)$ of $\mathsf{Ree}(q)$. Both $\mathsf{Ree}(q)$ and $\mathsf{Ree}(3)$ have an elementary abelian Sylow 2-subgroup of order 8. Hence, we can assume w.l.o.g. that $S \le \mathsf{Ree}(3)$. By the assumption, $\varphi(\mathcal{D})$ is contained in a pappian subplane Π_0. We apply Corollary 15 to the restriction $\varphi : \mathcal{D} \to \Pi_0$ to prove the claim.

Claim 2: For any $a \in I$, there is a point $P_a \in \varphi(a)$ of $\mathsf{PG}(2, F)$ such that $P_a \in \varphi(b)$ for all $b \in I$ with $ab = ba$.

Fix $a \in I$. The centralizer $C_G(a)$ is $\langle a \rangle \times T$, with $T \cong \mathsf{PSL}(2, q)$. T has a unique class J of involutions. For arbitrary *commuting involutions* $b, c \in J$, $\langle a, b, c \rangle$ is contained in a Sylow 2-subgroup of G. By claim 1, the lines $\varphi(b), \varphi(c)$ intersect on $\varphi(a)$. Fix $b \in J$ and define $P_a = \varphi(a) \cap \varphi(b)$. By Lemma 4, for *any* $c \in J$, there are elements $b_0 = b, b_1, \ldots, b_k = c \in J$ such that $b_i b_{i+1} = b_{i+1} b_i$ for all $i = 0, \ldots, k-1$. For all indices i, $\varphi(a) \cap \varphi(b_i) = \varphi(a) \cap \varphi(b_{i+1})$. Hence, $\varphi(a)$, $\varphi(b)$ and $\varphi(c)$ are concurrent. If c is an involution of $C_G(a)$, not in $J \cup \{a\}$, then $ac \in J$ and $\varphi(a)$, $\varphi(b)$ and $\varphi(ac)$ are concurrent. Also, the lines $\varphi(a)$, $\varphi(c)$ and $\varphi(ac)$ are concurrent, that shows $P_a \in \varphi(c)$.

Claim 3: All lines $\varphi(a)$ $(a \in I)$ of the embedding are concurrent.

If $a, b \in I$ commute then $P_a = P_b$. Fix arbitrary elements $a, b \in I$. By Lemma 4, there are elements $a_0 = a, a_1, \ldots, a_k = b \in I$ such that $a_i a_{i+1} = a_{i+1} a_i$ for all $i = 0, \ldots, k-1$. Then $P_a = P_{a_0} = \ldots = P_{a_k} = P_b$, that finishes the proof. $\square$

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